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A Characterization of Some Weighted Norm Inequalities for Maximal Operators

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Abstract. It is found class of pairs of exponents $(p(\cdot), q(\cdot))$ such that for pairs of Banach function spaces $(L^{p(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$ weak Minkowski's inequality holds. Also some conditions which ensure the boundednness of maximal operator $M_\varphi : L^{p(\cdot)}(\mathbb{R}^n) \longrightarrow L^{q(\cdot)}_w(\mathbb{R}^n)$ are found, when for the pair of Banach function spaces $(L^{p(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$ holds weak Minkowski's inequality.

Keywords. Maximal operator, variable L^p spaces

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1. Introduction

Let $p(\cdot) : \mathbb{R}^n \mapsto [1, +\infty)$ be a measurable function. Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the space of functions f such that for some $\lambda > 0$

$$\int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty,$$

with norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0; \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

The Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent and the corresponding variable Sobolev spaces $W^{k,p(\cdot)}$ are of interest for their applications to modelling problems in physics, and to the study of variational integrals and partial differential equations with non-standard growth condition (see [1,21]).

We suppose that the continuous increasing function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfies the following condition:

$$\varphi\left(\sum_i t_i \right) \le C \sum_i \varphi(t_i), \quad t_i \in \mathbb{R}_+,\ \varphi(0) = 0.$$

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The generalized maximal operator M_φ is defined on the space of locally integrable functions on $\mathbb{R}^n$ by the formula

$$M_\varphi f(x) = \sup \frac{1}{\varphi(|Q|)} \int_Q |f(t)|\, dt,$$

where the supremum is taken over all cubes Q containing x ($|Q|$ denotes the Lebesgue measure of the set Q).

If $\varphi(t) = t$, then M_φ is the Hardy–Littlewood maximal operator which will be denoted by M. If $\varphi(t) = t^{1-\frac{\alpha}{n}}$, $0 < \alpha < n$, then M_φ is the fractional maximal operator which will be denoted by M_α.

Assume that $p_- = \operatorname{essinf}_{x\in\mathbb{R}^n} p(x)$ and $p_+ = \operatorname{esssup}_{x\in\mathbb{R}^n} p(x)$. Let $\mathcal{P}(\mathbb{R}^n)$ be the class of all functions $p(\cdot)\,(1 < p_- \le p_+ < \infty)$ for which the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. This class has been a focus of intense study in recent years. We refer to the papers [5]–[20], where several results on maximal, potential and singular operators in variable Lebesgue spaces were obtained. Note that an explicit description in terms of Muckenhoupt type condition of general weights for which the maximal operator M_φ is bounded in the $L^{p(\cdot)}$ space still remains an open problem.

A certain subclass of general weights was considered in [12], where for the case of bounded domain Ω in the Euclidian space, the boundedness of the maximal operator M in the space $L^{p(\cdot)}(\Omega, \rho)$ was proved (under usual log-Hölder condition). This subclass may be characterized as a class of radial type weights which satisfy the Zygmund–Bari–Stechkin condition. Weight inequalities with power-type weights for operator M in $L^{p(\cdot)}$ spaces have been established in [13]. Muckenhoupt-type condition governing the one-weight inequality for M in variable exponent Lebesgue spaces was derived in [11]. Necessary and sufficient condition on weight par (w, v) guaranteeing the boundedness of M_α from L^r_w to $L^{q(\cdot)}_v$ (r is constant) were found in [9]. Note also that two-weight criteria for $M_\alpha : L^{p(\cdot)}_w(J) \to L^{p(\cdot)}_v(J)$ (J is interval) were found in [14].

In this paper we give a necessary and sufficient condition on weight w guaranteeing the boundedness of the maximal operator M_φ from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}_w(\mathbb{R}^n)$ provided that $p(\cdot), q(\cdot) \in \mathcal{L}(\mathbb{R}^n)$ (see definition bellow) and $p(x) \le q(x)$, $x \in \mathbb{R}^n$ (Theorem 4.7).

Finally, C will denote positive constant depending only on the dimension, but whose value may change at each appearance.

2. Weak Minkovski's inequality in Banach function spaces

Let (Ω, μ) be a complete σ-finite measure space. By S we denote the collection of all real-valued measurable function on Ω. A Banach subspace E in S is said to be Banach function space (BFS) if:

1) the norm $\|f\|_E$ is defined for every measurable function f and $f \in E$ if and only if $\|f\|_E < \infty$, $\|f\|_E = 0$ if and only if $f = 0$ a.e.;
2) $\||f|\|_E = \|f\|_E$ for all $f \in E$;
3) if $0 \le f \le g$ a.e., then $\|f\|_E \le \|g\|_E$;
4) if $0 \le f_n \uparrow f$ a.e., then $\|f_n\|_E \uparrow \|f\|_E$;
5) if X is measurable subset of Ω such that $\mu(X) < \infty$ and χ_X is characteristic function of X, then $\|\chi_X\|_E < \infty$;
6) for every measurable set X, $\mu(X) < \infty$, there is a constant $C_X < \infty$ such that $\int_X f(t)dt \le C_X\|f\|_E$.

Given a Banach function space E we can always consider its associate space E' consisting of those $g \in S$ that $f \cdot g \in L^1$ for every $f \in E$ with the norm $\|g\|_{E'} = \sup\{\|f \cdot g\|_{L^1} : \|f\|_E \le 1\}$. E' is a BFS on Ω and a closed norming subspace of conjugate space E^*.

Let w be a weight ($w(x) > 0$ a.e. on Ω). By E_w we denote BFS with norm $\|f\|_{E_w} = \|fw\|_E$.

Let $\Im$ be some fixed family of sequences $\mathcal{Q} = \{Q_i\}$ of disjoint measurable subsets of Ω, $\mu(Q_i) > 0$ such that $\Omega = \cup_{Q_i \in \mathcal{Q}} Q_i$. We ignore the difference in notation caused by a null set.

Everywhere below by $l_{\mathcal{Q}}$ we denote a Banach sequential space (BSS), meaning that axioms 1)−6) are satisfied with respect to the count measure. Let $\{e_k\}$ be standard unit vectors in $l_{\mathcal{Q}}$.

Definition 2.1. Let $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q} \in \Im}$ be a family of BSSs. A BFS E is said to satisfy a *uniformly upper* (*lower*) *l-estimate* if there exists a constant $C > 0$ such that for every $f \in E$ and $\mathcal{Q} \in \Im$ we have

$$\|f\|_E \le C \left\| \sum_{Q_i \in \mathcal{Q}} \|f\chi_{Q_i}\|_E \cdot e_i \right\|_{l_{\mathcal{Q}}} \qquad \left(\left\| \sum_{Q_i \in \mathcal{Q}} \|f\chi_{Q_i}\|_E \cdot e_i \right\|_{l_{\mathcal{Q}}} \le C\|f\|_E \right).$$

In case $\Omega = [0, +\infty)$ Definition 2.1 was introduced in [16]. The notions of uniformly upper (lower) l-estimates, when $l_{\mathcal{Q}_1} = l_{\mathcal{Q}_2}$ for all $\mathcal{Q}_1, \mathcal{Q}_2 \in \Im$ were introduced by Berezhnoi (see [2]). Note that if the space E satisfies uniformly upper (lower) l-estimate, then the space E_w also satisfies uniformly upper (lower) l-estimate for any weight w.

If BFS E simultaneously satisfies uniformly upper and lower l-estimates, then for any $f \in E$ and $\mathcal{Q} \in \Im$

$$\frac{1}{C}\|f\|_E \le \left\| \sum_{Q_i \in \mathcal{Q}} \frac{\|f\chi_{Q_i}\|_E}{\|\chi_{Q_i}\|_E} \chi_{Q_i} \right\|_E \le C\|f\|_E. \tag{1}$$

Note also that if E satisfies upper (lower) $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q} \in \Im}$ estimates, then E' satisfies lower (upper) $l' = \{l'_{\mathcal{Q}}\}_{\mathcal{Q} \in \Im}$ estimates (see [16, Theorem 2] for case

$\Omega = [0, +\infty)$, generalization for arbitrary Ω may be achieved in the similar way and we omit its proof).

Let $E = E_t, F = F_s$ be Banach function spaces on (Ω, μ). Under the spaces with mixed norm $E[F]$, $F[E]$ we mean the spaces consisting of all $k(t, s) \in S(\Omega \times \Omega, \mu \times \mu)$ such that $\|k(t, \cdot)\|_F \in E$ and$\|k(\cdot, s)\|_E \in F$ with norms

$$\|k\|_{E[F]} = \big\| \, \|k(t, \cdot)\|_F \, \big\|_E, \quad \|k\|_{F[E]} = \big\| \, \|k(\cdot, s)\|_E \, \big\|_F.$$

It is known that $F[E]$ and $F[E]$ are Banach function spaces on $\Omega \times \Omega$. In general case the spaces $E[F]$ and $F[E]$ are not isomorphic. According to the theorem of Kolmogorov–Nagumo, if $E[F]$ and $F[E]$ are isomorphic, then E and F are isomorphic respectively to some $L^p_{w_1}$ and $L^p_{w_2}$ spaces $1 \le p < \infty$ or both E, F are AM spaces (see [4]).

Definition 2.2. A pair of BFSs (E, F) is said to have the property $\mathbf{M}(\Im)$ ($(E, F) \in \mathbf{M}(\Im)$) if there exists a constant C such that

$$\Big\| \sum_{Q_i \in \mathcal{Q}} f(t)\chi_{Q_i}(t) g(s)\chi_{Q_i}(s) \Big\|_{F[E]} \le C \| \sum_{Q_i \in \mathcal{Q}} f(t)\chi_{Q_i}(t) g(s)\chi_{Q_i}(s) \|_{E[F]}$$

for any $\mathcal{Q} \in \Im$ and every $f \in E, g \in F$.

Definition 2.2 was introduced in [16]. For any BFS F we have continuous embedding $L^1[F] \subset F[L^1]$ (generalized Minkowski's inequality), and the property $\mathbf{M}(\Im)$ may be interpreted as the weak Minkowski's inequality for pair (E, F). Note that a pair (E, F) of BFSs possesses the property $\mathbf{M}(\Im)$ if and only if there exists family $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q} \in \Im}$ of BSSs for which the space F has uniformly upper l-estimate and the space E has uniformly lower l-estimate ([16,Theorem 2]).

Definition 2.3. A pair (E, F) of BFSs is said to have the property $\mathbf{G}(\Im)$ ($(E, F) \in \mathbf{G}(\Im)$) if there exists a constant C such that

$$\sum_{Q_i \in \mathcal{Q}} \|f\chi_{Q_i}\|_E \cdot \|g\chi_{Q_i}\|_{F'} \le C \|f\|_E \cdot \|g\|_{F'}$$

for any sequence $\mathcal{Q} = \{Q_i\}$, $\mathcal{Q} \in \Im$ and every $f \in E$, $g \in F'$.

In case $\Omega = \mathbb{R}^n$ Definition 2.3 was introduced by Berezhnoi (see [3]). Let us remark that pair (L_p, L_q) satisfies property $\mathbf{G}(\Im)$ if $p \le q$. Conditions, when the pair of BFSs (E, F) satisfies property $\mathbf{G}(\Im)$ in terms of ℓ-concavity and ℓ-convexity (in this case $\ell_{\mathcal{Q}_1} = \ell_{\mathcal{Q}_2}$ for any $\mathcal{Q}_1, \mathcal{Q}_2 \in \Im$) can be found in [2]. Here (E, F) is a pair of symmetric spaces (Lebesgue, Lorentz, Marcinkewicz). Note also that $(E, F) \in \mathbf{G}(\Im)$ if and only if $(E, F) \in \mathbf{M}(\Im)$ ([16,Theorem 2]).

3. Weak Minkowski's inequality in variable exponent Lebesgue spaces

First we consider the cases $\Omega = [0,1]$ and $\Omega = \mathbb{R}$. By $\Im_\Omega$ we denote the family of all sequences $\{Q_i\}$ of disjoint intervals from Ω.

Let $\mathcal{L}([0,1])$ denote the set of exponents $p(\cdot) : [0,1] \to [1,+\infty)$ with log-Hölder condition

$$|(p(x)-p(y))\log|x-y|| \le C, \quad x,\, y \in [0,1],\, x \ne y. \tag{2}$$

Proposition 3.1 ([16]). *Let $p(\cdot),\, q(\cdot) \in \mathcal{L}([0,1])$ and $p(x) \le q(x),\ x \in [0,1]$. Then the pair of BFSs $(L^{p(\cdot)}([0,1]), L^{q(\cdot)}([0,1]))$ satisfies the property $\boldsymbol{M}(\Im_{[0,1]})$.*

By $\mathcal{AC}$ we denote the set of exponents $p(\cdot) : \mathbb{R} \to [1,+\infty)$ of the form $p(x) = p + \int_{-\infty}^{x} l(u)du$, where $\int_{-\infty}^{+\infty} |l(u)|du < +\infty$.

Proposition 3.2. *If $p(\cdot), q(\cdot) \in \mathcal{AC}$ and $p(x) \le q(x)$, $x \in \mathbb{R}$, then $(L^{p(\cdot)}(\mathbb{R}), L^{q(\cdot)}(\mathbb{R})) \in \boldsymbol{M}(\Im_{\mathbb{R}})$.*

Proof. Let $p(x) = p + \int_{-\infty}^{x} l_1(u)du$ and $q(x) = q + \int_{-\infty}^{x} l_2(u)du$. Let $\bar{l}(\cdot)$ be a positive function such that $|l_1(x)|,\, |l_2(x)| \le \bar{l}(x)$ on $\mathbb{R}$ and $\int_{-\infty}^{+\infty} \bar{l}(u)du = a < +\infty$. Note that

$$|p(x_2)-p(x_1)|,\, |q(x_2)-q(x_1)| \le \left| \int_{x_1}^{x_2} \bar{l}(x)dx \right|.$$

Let $M(x) = \frac{1}{a}\int_{-\infty}^{x} \bar{l}(u)du$, and $M^{-1}(\cdot) : (0,1) \to \mathbb{R}$ is the inverse function of M. Define $\overline{p}(\cdot),\, \overline{q}(\cdot) : [0;1] \to [1;+\infty)$ by $\overline{p}(\cdot) = p(M^{-1}(\cdot))$, $\overline{q}(\cdot) = q(M^{-1}(\cdot))$. Define the weights

$$w_1(\cdot) = \big((M^{-1})'(\cdot)\big)^{\frac{1}{\overline{p}(\cdot)}}, \quad w_2(\cdot) = \big((M^{-1})'(\cdot)\big)^{\frac{1}{\overline{q}(\cdot)}}.$$

Note that if $y_1, y_2 \in [0,1]$, then $|\overline{p}(y_1)-\overline{p}(y_2)|,\, |\overline{q}(y_1)-\overline{q}(y_2)| \le a|y_1-y_2|$. By Proposition 3.1 there exists family $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\Im_{[0,1]}}$ of BSSs for which the space $L^{\overline{p}(\cdot)}([0,1])$ satisfies uniformly lower l-estimate and the space $L^{\overline{q}(\cdot)}([0,1])$ satisfies uniformly upper l-estimate. Consequently the space $L^{\overline{p}(\cdot)}_{w_1}([0,1])$ satisfies uniformly lower l-estimate and the space $L^{\overline{q}(\cdot)}_{w_2}([0;1])$ satisfies uniformly upper l-estimate (here we are using the fact that for any BFS E and any weight we have $\|f\|_{E_w} = \|fw\|_E$).

Let $M(\Im_{\mathbb{R}})$ denote the family of all sequences $M(\mathcal{Q}) = \{M(Q_i)\}$ where $\mathcal{Q} = \{Q_i\} \in \Im_{\mathbb{R}}$. Note that $M(\Im_{\mathbb{R}}) \subset \Im_{[0,1]}$. For $\mathcal{Q} \in \Im_{\mathbb{R}}$ let $\widetilde{l}_{\mathcal{Q}} = l_{\mathcal{Q}'}$ where $M^{-1}(Q_i') = Q_i$.

Note that the spaces $L^{p(\cdot)}(\mathbb{R})$ and $L^{\overline{p}(\cdot)}_{w_1}([0,1])$, also the spaces $L^{q(\cdot)}(\mathbb{R})$ and $L^{\overline{q}(\cdot)}_{w_2}([0;1])$ are isomorphic. Then, for the family $\widetilde{l} = \{\widetilde{l}_{\mathcal{Q}}\}_{\mathcal{Q}\in\Im_{\mathbb{R}}}$ of BSSs, $L^{p(\cdot)}(\mathbb{R})$ satisfies uniformly lower $\widetilde{l}$-estimate and $L^{q(\cdot)}(\mathbb{R})$ satisfies uniformly upper $\widetilde{l}$-estimate. Consequently $(L^{p(\cdot)}(\mathbb{R}), L^{q(\cdot)}(\mathbb{R})) \in \mathbf{M}(\Im_{\mathbb{R}})$. □

Analogously we have

Proposition 3.3. *Let for* $p(\cdot)$, $q(\cdot) : \mathbb{R} \to [1, +\infty)$ *there exists* $M(\cdot) : \mathbb{R} \to (0,1)$ *of the form* $M(x) = \int_{-\infty}^{x} l(t)dt$, $x \in \mathbb{R}$, $l(t) > 0$ *a.e., such that* $p(M^{-1}(\cdot))$ *and* $q(M^{-1}(\cdot))$ *satisfy* (2). *Then* $(L^{p(\cdot)}(\mathbb{R}), L^{q(\cdot)}(\mathbb{R})) \in \boldsymbol{M}(\Im_{\mathbb{R}})$.

By $\mathcal{L}(\mathbb{R}^n)$ we denote the set of exponents $p(\cdot) : \mathbb{R}^n \to [1, +\infty)$ with the properties: $1 \le p_- \le p_+ < \infty$ and $p(\cdot)$ satisfies the log-Hölder condition

$$|p(x) - p(y)| \le \frac{C}{\log\left(e + \frac{1}{|x-y|}\right)} \quad \text{for } |x-y| < \frac{1}{4} \tag{3}$$

$$|p(x) - p_\infty| \le \frac{C}{\log(e + |x|)} \quad \text{for some } p_\infty \in \mathbb{R}. \tag{4}$$

Assume that $1 < p_- \le p_+ < \infty$. In [7] L. Diening proved that if $p(\cdot)$ satisfies the condition (3) and $p(\cdot)$ is a constant outside some compact set, then $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The second condition on $p(\cdot)$, namely the behavior of $p(\cdot)$ at infinity, was improved independently by D. Cruz–Uribe, A. Fiorenza and C. Neugebauer [5], and A. Nekvinda [20]. It was shown in [5] that if $p(\cdot)$, satisfies (3) and (4), then $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. In [20] (4) is replaced by a slightly more general integral condition.

By $\Im$ we denote the family of all sequences $\{Q_i\}$ of disjoint cubes from $\mathbb{R}^n$.

Proposition 3.4. *If* $p(\cdot), q(\cdot) \in \mathcal{L}(\mathbb{R}^n)$ *and* $p(x) \le q(x)$, $x \in \mathbb{R}^n$, *then* $(L^{p(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n)) \in \boldsymbol{M}(\Im)$.

Proof. Let us consider the map $g : \mathbb{R}^n \to Q_0$ in the following form:

$$g(x) = g(x_1, \dots, x_n) = \left(\tfrac{1}{\pi}\arctan x_1, \dots, \tfrac{1}{\pi}\arctan x_n\right),$$

where $Q_0 = (-\frac{1}{2}, \frac{1}{2})^n$. By $g(\Im)$ we denote the family of all sequences $g(\mathcal{Q}) = \{g(Q_i)\}$ where $\mathcal{Q} = \{Q_i\} \in \Im$.

Let $\overline{p}(\cdot) = p(g^{-1}(\cdot))$, $\overline{q}(\cdot) = q(g^{-1}(\cdot))$. Note that there exist weights w_1, w_2 on Q_0 such that the spaces $L^{\overline{p}(\cdot)}_{w_1}(Q_0)$ and $L^{p(\cdot)}(\mathbb{R}^n)$, also the spaces $L^{\overline{q}(\cdot)}_{w_2}(Q_0)$ and $L^{q(\cdot)}(\mathbb{R}^n)$ are isomorphic.

We need to show that

$$|\log|g(Q)|(\overline{p}(x) - \overline{p}(y))| \le C \tag{5}$$

and

$$|\log|g(Q)|(\overline{q}(x) - \overline{q}(y))| \le C \tag{6}$$

for all cubes $Q \subset \mathbb{R}^n$ and $x, y \in g(Q)$.

Note that from (5), (6) it holds that the pair of BFSs $(L^{\overline{p}(\cdot)}(Q_0), L^{\overline{q}(\cdot)}(Q_0))$

has property $\mathbf{M}(g(\Im))$ and consequently the pair of BFSs $(L^{\overline{p}(\cdot)}_{w_1}(Q_0), L^{\overline{q}(\cdot)}_{w_2}(Q_0))$ has property $\mathbf{M}(g(\Im))$. The proof is exactly the same as Proposition 3.1 and we can omit it. Thus it follows that the pair of BFSs $(L^{p(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$ has property $\mathbf{M}(\Im)$.

The proof of (5) and (6) are similar, so we will prove only (5). We start with large cubes. Let Q be a cube with property $|g(Q)| \geq \frac{1}{4}$. Then (5) is obvious.

Now, let $Q = [a_1, a_1 + h] \times \cdots \times [a_n, a_n + h]$, where $h > 0$ and $|g(Q)| < \frac{1}{4}$. Firstly assume that $a_i \geq 0$ for all i. Denote $a_0 = \max_i a_i$. It is clear that $1 > \frac{1}{\pi}(\arctan(a_i + h) - \arctan a_i) \geq \frac{1}{\pi}(\arctan(a_0 + h) - \arctan a_0) > 0$ for all i and consequently

$$|\log g(Q)| \leq C\big|\log\left(\pi^{-1}(\arctan(a_0 + h) - \arctan a_0)\right)\big|. \tag{7}$$

For the proof of (5) we will consider three cases.

Case 1. Let $a_0 > 100$ and $h \in \left[\frac{k}{a_0}, \frac{k+1}{a_0}\right)$ for some $k \in \mathbb{N}$. Note that for $x \geq 0$ $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$ and

$$\arctan \frac{1}{a_0} - \arctan \frac{1}{a_0 + h} = \arctan' \xi \left(\frac{1}{a_0} - \frac{1}{a_0 + h}\right) \leq C\frac{k}{a_0(a_0^2 + k)}.$$

From this we have

$$\begin{aligned} D &= \left|\log\left(\frac{1}{\pi}(\arctan(a_0 + h) - \arctan a_0)\right)\right| \\ &= \left|\log\left(\frac{1}{\pi}(\arctan \frac{1}{a_0 + h} - \arctan \frac{1}{a_0})\right)\right| \\ &\asymp \left|\log \frac{k}{a_0(a_0^2 + k)}\right|. \end{aligned}$$

If $k > 2a_0^2$, then $\frac{k}{a_0^2+k} \asymp C$ and $D \asymp \log a_0$. If $1 \leq k \leq 2a_0^2$, then $\frac{1}{3a_0^3} \leq \frac{k}{a_0(a_0^2+k))} \leq \frac{2}{a_0}$, and consequently $D \asymp \log a_0$. From (4) and (7) we obtain (5).

Case 2. Let $a_0 > 100$ and $h \in \left(0, \frac{1}{a_0}\right)$. Then it follows by (3), (4) that

$$|(\widetilde{p}(x) - \widetilde{p}(y))\log a_0| \leq C \tag{8}$$

and

$$|(\widehat{p}(x) - \widetilde{p}(y))\log h| \leq C, \quad x, y \in g(Q). \tag{9}$$

Since $\frac{1}{a_0}, \frac{1}{a_0+h} < \frac{1}{100}$ we have

$$\arctan \frac{1}{a_0} - \arctan \frac{1}{a_0 + h} = \arctan' \xi \left(\frac{1}{a_0} - \frac{1}{a_0 + h}\right) \asymp \frac{h}{a_0^2}. \tag{10}$$

Combining estimates (7), (8), (9), (10) we get (5).

Case 3. Let $a_0 \le 100$ and $h \le \frac{1}{2}$. Note that $\arctan(a_0+h) - \arctan a_0 = h\arctan'\xi$. By using (3), (7) we obtain (5). If $a_0 \le 100$ and $h > \frac{1}{2}$, then $g(Q) \ge C$ and (5) is obvious.

Let the origin be an interior point of Q. Denote by $\widetilde{Q}$ the smallest cube containing Q and centered at the origin. Denote $\widetilde{Q}_i = \widetilde{Q} \cap \mathbb{R}^n_i$ $i = 1,\dots,2^n$, where $\mathbb{R}^n_i$ are octants of $\mathbb{R}^n$. We have $g(Q) \asymp g(\widetilde{Q})$ and $g(\widetilde{Q}) = \cup_{i=1}^{2^n} g(\widetilde{Q}_i)$. Using above method for $g(\widetilde{Q}_i)$, $i = 1,\dots,2^n$ we obtain (3.4) for Q. Analogously we may consider remaining cases. □

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. For any $\mathcal{Q} \in \Im$ we define the space $l^{\mathcal{Q},p(\cdot)}$ by

$$l^{\mathcal{Q},p(\cdot)} := \Big\{ \bar{t} = \{t_Q\}_{Q\in\mathbb{Q}} : \sum_{Q\in\mathcal{Q}} |t_Q|^{p_Q} < \infty \Big\},$$

equipped with the Luxemburg's norm, where $\frac{1}{p_Q} = \frac{1}{|Q|}\int_Q \frac{1}{p(x)}dx$. Analogously we define the space $l^{\mathcal{Q},p'(\cdot)}$ where $\frac{1}{p(t)} + \frac{1}{p'(t)} = 1$, $t \in \mathbb{R}^n$.

Note that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ then for simple functions we have uniformly lower and upper $l = \{l^{\mathcal{Q},p(\cdot)}\}_{\mathcal{Q}\in\Im}$ estimates.

Theorem 3.5. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then uniformly*

$$\Big\| \sum_{Q\in\mathcal{Q}} t_Q \chi_Q \Big\|_{p(\cdot)} \asymp \Big\| \sum_{Q\in\mathcal{Q}} t_Q \|\chi_Q\|_{p(\cdot)} \Big\|_{l^{\mathcal{Q},p(\cdot)}} \tag{11}$$

and

$$\Big\| \sum_{Q\in\mathcal{Q}} t_Q \chi_Q \Big\|_{p'(\cdot)} \asymp \Big\| \sum_{Q\in\mathcal{Q}} t_Q \|\chi_Q\|_{p'(\cdot)} \Big\|_{l^{\mathcal{Q},p'(\cdot)}}. \tag{12}$$

Theorem 3.5 is another version of necessary part of Diening's Theorem 4.2 in [6] (proof may be found in [17]). We will prove that conditions(11) and (12) in general do not imply $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Our proof relies on the example constructed by Lerner in [19].

Indeed, let $E = \bigcup_{k\ge1}(e^{k^3}, e^{k^3 e^{1/k^2}})$ and $p_0(x) = \int_{|x|}^{\infty} \frac{1}{t\log t}\chi_E(t)dt$. There exist $\alpha > 1$ and β_0 ($\frac{1}{\alpha} < \beta_0 < 1$) such that $p_0(\cdot) + \alpha \in \mathcal{P}(\mathbb{R})$ and $\beta_0(p_0(\cdot)+\alpha)$ $\overline{\in}\mathcal{P}(\mathbb{R})$ (see [19], Theorem 1.7). By Proposition 3.2 we have $(L^{p(\cdot)}(\mathbb{R}), L^{p(\cdot)}(\mathbb{R})) \in \mathbf{M}(\Im)$ where $p(\cdot) = p_0(\cdot) + \alpha$, and consequently there exists family $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\Im}$ of BSSs for which $L^{p(\cdot)}(\mathbb{R})$ satisfies uniformly lower and upper l-estimate. From (11) we have $l_{\mathcal{Q}} \cong l^{\mathcal{Q},p(\cdot)}$ and consequently

$$\|f\|_{p(\cdot)} \asymp \Big\| \sum_{Q\in\mathcal{Q}} \|f\chi_Q\|_{p(\cdot)} \Big\|_{l^{\mathcal{Q},p(\cdot)}}. \tag{13}$$

Note that for all $1 > \beta > \frac{1}{p_-}$

$$\left\| f^{\frac{1}{\beta}} \right\|_{\beta p(\cdot)}^{\beta} = \|f\|_{p(\cdot)} \tag{14}$$

and

$$\left\| \{t_Q\} \right\|_{l^{\mathcal{Q},p(\cdot)}} = \left\| \left\{ |t_Q|^{\frac{1}{\beta}} \right\} \right\|_{l^{\mathcal{Q},\beta p(\cdot)}}^{\beta} . \tag{15}$$

From (13), (14), (15) we have

$$\|g\|_{\beta p(\cdot)} \asymp \left\| \sum_{Q\in\mathcal{Q}} \|g\chi_Q\|_{\beta p(\cdot)} \right\|_{l^{\mathcal{Q},\beta p(\cdot)}}$$

for $g \in L^{\beta p(\cdot)}(\mathbb{R})$ and the space $L^{\beta p(\cdot)}(\mathbb{R})$ satisfies uniformly lower and upper l^{β}-estimates where $l^{\beta}_{\mathcal{Q}} = l^{\mathcal{Q},\beta p(\cdot)}$.

Note that $\frac{1}{(\beta p(\cdot))_Q} + \frac{1}{((\beta p(\cdot))')_Q} = 1$ and $(l^{\mathcal{Q},\beta p(\cdot)})' = l^{\mathcal{Q},(\beta p(\cdot))'}$. Thus the space $(L^{\beta p(\cdot)}(\mathbb{R}))'$ satisfies uniformly lower and upper $(l^{\beta})'$-estimates where $(l^{\beta})'_{\mathcal{Q}} = l^{\mathcal{Q},(\beta p(\cdot))'}$ and (11), (12) are valid for any $\beta p(\cdot)$, $(\beta p(\cdot))'$ where $1 > \beta > \frac{1}{p_-}$. Consequently for exponent $\beta_0 p(\cdot)$ (11) and (12) are valid but, $\beta_0 p(\cdot) \notin \mathcal{P}(\mathbb{R})$.

4. Some applications

The result from Section 3 can be applied to study boundedness of some classical operators of analysis in $L^{p(\cdot)}$ spaces.

In [6] L. Diening showed that $p \in \mathcal{P}(\mathbb{R}^n)$ if and only if there exists $C > 0$ such that for any family of pairwise disjoint cubes π and any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\left\| \sum_{Q\in\pi} (|f_Q|)\chi_Q \right\|_{p(\cdot)} \le C\|f\|_{p(\cdot)}, \tag{16}$$

where $f_Q = \frac{1}{|Q|}\int_Q f$.

We say that dx satisfies the condition $A_{p(\cdot)}$ $(dx \in A_{p(\cdot)})$ if $1 < p_- \le p_+ < \infty$ and there exists $C > 0$ such that for any cube Q and any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$|f|_Q\|\chi_Q\|_{p(\cdot)} \le C\|f\chi_Q\|_{p(\cdot)}.$$

It is easy to see that $dx \in A_{p(\cdot)}$ if and only if $\sup_Q A(Q) < \infty$, where $A(Q) = \frac{1}{|Q|}\|\chi_Q\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)}$ and $p'(x) = \frac{p(x)}{p(x)-1}$. (Note that this condition could be considered as a full analogue of the classical Muckenhoupt A_p condition in the context of variable Lebesgue spaces.)

It is natural to ask whether (16) can be replaced by $dx \in A_{p(\cdot)}$. In [15] was proved that if $p(\cdot)$ is a constant outside some ball, then $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ if and only if $dx \in A_{p(\cdot)}$. A. Lerner proved in [19] the following

Theorem 4.1. *Let $dx \in A_{p(\cdot)}$, and let $E \subset \mathbb{R}^n$ be a measurable set of positive measure. Then there exists a constant $C > 0$ depending on $p(\cdot), n$ and E such that for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|(Mf)\chi_E\|_{p(\cdot)} \le C\|f\chi_E\|_{p(\cdot)}.$$

Given a function $p(\cdot)$, we say that M is weak type $(p(\cdot), p(\cdot))$ if there exists $C > 0$ such that for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\sup_{\alpha>0} \alpha\|\chi_{\{x:\, Mf(x)>\alpha\}}\|_{p(\cdot)} \le C\|f\|_{p(\cdot)}.$$

It is easy to see that weak $(p(\cdot), p(\cdot))$ property of M implies $dx \in A_{p(\cdot)}$. A. Lerner [19] proved following

Theorem 4.2. *Let $p(\cdot)$ be a radially decreasing function with $1 < p_- \le p_+ < \infty$. Then M is of weak $(p(\cdot), p(\cdot))$ type if and only if $dx \in A_{p(\cdot)}$.*

In [18] is proved following

Theorem 4.3. *Let $n \ge 2$. There exists a exponent $p(\cdot)$ such that:*

1) *$dx \in A_{p(\cdot)}$;*
2) *$p(\cdot) \notin \mathcal{P}(\mathbb{R}^n)$;*
3) *operator M is not weak $(p(\cdot), p(\cdot))$ type;*
4) *$dx \notin A_{\alpha p(\cdot)}$ for any α where $\frac{1}{p_-} < \alpha < 1$;*
5) *$dx \notin A_{p(\cdot)-\alpha}$ for any α where $0 < \alpha < p_- - 1$.*

Note 4.4. Note that the following propositions are equivalent:

1) $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and
2) $\alpha p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ for some $\frac{1}{p_-} < \alpha < 1$ (see [6]).

Note also that the example for $p(\cdot) \notin \mathcal{P}(\mathbb{R}^n)$ and $dx \in A_{p(\cdot)}$ was given in [8].

The next theorem can be viewed as an analogue of Muckenhoupt's characterization of the weighted L^p_w boundedness of M in terms of the A_p condition.

Theorem 4.5. *Let $1 < p_- \le p_+ < \infty$ and $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n)) \in \boldsymbol{M}(\Im)$. The following assertions are equivalent:*

1) *M is weak type $(p(\cdot), p(\cdot))$;*
2) *$\|Mf\|_{p(\cdot)} \le C\|f\|_{p(\cdot)}$;*
3) *$dx \in A_{p(\cdot)}$.*

Proof. It is easy to see that 1)$\Rightarrow$ 3), 2)$\Rightarrow$ 3) and 2)$\Rightarrow$ 1). Let $dx \in A_{p(\cdot)}$. Using Hölder's inequality we get

$$\frac{1}{|Q|}\int_Q |f(x)|dx \le C\frac{\|f\chi_Q\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}}.$$

Combining this with (1) we obtain (16). □

An important role in our results will play the following

Theorem 4.6 ([3]). *Let a pair* (X,Y) *of BFSs on* $\mathbb{R}^n$ *satisfies the property* $\boldsymbol{G}(\Im)$. *If the operator* M *is bounded from* X_w *into* X_w, *then* M_φ *is bounded from* X_w *into* Y_v *if and only if*

$$\sup_Q \frac{1}{\varphi(|Q|)} \|v\chi_Q\|_Y \left\|w^{-1}\chi_Q\right\|_{X'} < \infty. \tag{17}$$

We are in a position to prove following result.

Theorem 4.7. *Let* $p,q \in \mathcal{L}(\mathbb{R}^n)$, $p_- > 1$ *and* $q(t) \le p(t)$, $t \in \mathbb{R}^n$. *Then* M_φ *is bounded from* $L^{p'(\cdot)}(\mathbb{R}^n)$ *into* $L_w^{q'(\cdot)}(\mathbb{R}^n)$ *if and only if*

$$\sup_Q \frac{1}{\varphi(|Q|)} \|w\chi_Q\|_{L^{q'(\cdot)}} \|\chi_Q\|_{L^{p(\cdot)}} < \infty, \tag{18}$$

where $\frac{1}{p(t)} + \frac{1}{p'(t)} = 1, \frac{1}{q(t)} + \frac{1}{q'(t)} = 1,\ t \in \mathbb{R}^n$.

Proof. In first note that it is possible in Theorem 4.7 $q'_+ = \infty$. The pair of BFSs $(L^{q(t)}(\mathbb{R}^n),\ L^{p(t)}(\mathbb{R}^n))$ satisfies property $\mathbf{M}(\Im)$ and consequently the pair of BFSs $(L^{p'(\cdot)}(\mathbb{R}^n),\ L^{q'(\cdot)}(\mathbb{R}^n))$ satisfies the property $\mathbf{M}(\Im)$. According to condition (17) we have that M_φ is bounded from $L^{p'(\cdot)}(\mathbb{R}^n)$ into $L_w^{q'(\cdot)}(\mathbb{R}^n)$ if and only if (18) is fulfilled. □

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