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# Existence of Three Nontrivial Smooth Solutions for Nonlinear Resonant Neumann Problems Driven by the $p$-Laplacian

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**Abstract.** We consider a nonlinear Neumann elliptic problem driven by the $p$-Laplacian and with a reaction term which asymptotically at $\pm\infty$ exhibits resonance with respect to the principal eigenvalue $\lambda_0 = 0$. Using variational methods combined with tools from Morse theory, we show that the resonant problem has at least three nontrivial smooth solutions, two of which have constant sign (one positive, the other negative).

**Keywords.** $p$-Laplacian, resonance, critical groups, local minimizers, contractible sets
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## 1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a $C^2$-boundary $\partial Z$. In this paper, we study the following nonlinear Neumann problem:

$$\begin{cases} -\Delta_p x(z) = f(z, x(z)) & \text{a.e. on } Z \\ \dfrac{\partial x}{\partial n} = 0 & \text{on } \partial Z. \end{cases} \tag{1}$$

Here $\Delta_p$ stands for the $p$-Laplacian differential operator defined by

$$\Delta_p u(z) = \operatorname{div}\big(\|\nabla u(z)\|^{p-2}\nabla u(z)\big),$$

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where $p \in (1, +\infty)$ and $n(z)$ denotes the outward unit normal on $\partial Z$.

The aim of this work is to establish the existence of at least three nontrivial smooth solutions, when resonance occurs at $\pm\infty$ with respect to the principal eigenvalue $\lambda_0 = 0$. Recently, three solutions theorems, were proved in the context of the Dirichlet $p$-Laplacian. We mention the works of Carl–Perera [4], Liu [12], Liu–Liu [13], Papageorgiou–Papageorgiou [17], and Zhang–Chen–Li [21]. No such results exist for the Neumann $p$-Laplacian. The existing multiplicity results in this direction, do not allow for resonance to occur and impose additional restrictive conditions, such as that $p > N$ (low dimensional problem), see Bonanno–Candito [3], Faraci [6], Ricceri [18], Wu–Tan [20], or impose symmetry conditions on the nonlinearity $f(z, \cdot)$, see Motreanu–Papageorgiou [16], or produce only two nontrivial smooth solutions, see Filippakis–Gasiński–Papageorgiou [7]. We should mention the recent work of Gasiński–Papageorgiou [10], where the authors produce two nontrivial, smooth solutions, when the potential function $F(z, \zeta) = \int_0^\zeta f(z, s)\, ds$ admits asymptotic $L^\infty$-limits as $\zeta \to \pm\infty$ (strong resonance at $\pm\infty$).

Our approach here combines variational methods based on the critical point theory together with techniques from Morse theory.

## 2. Mathematical Background

In this section we briefly recall some basic notion and facts from critical point theory and from Morse theory, which we shall use in the sequel. We also recall some needed properties about the spectrum of the Neumann $p$-Laplacian and we fix our notation.

Let $X$ be a Banach space and let $X^*$ be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(X^*, X)$. Let $\varphi \in C^1(X)$. We say that $\varphi$ satisfies *the Cerami condition at level* $c \in \mathbb{R}$ (*the* $C_c$*-condition* for short), if the following holds: every sequence $\{x_n\}_{n \geqslant 1} \subseteq X$, such that

$$\varphi(x_n) \longrightarrow c \quad \text{and} \quad (1 + \|x_n\|)\varphi'(x_n) \longrightarrow 0 \ \text{ in } X^*,$$

has a strongly convergent subsequence. We say that $\varphi$ satisfies *the Cerami condition* (*the* $C$*-condition* for short), if it satisfies the $C_c$-condition at every level $c \in \mathbb{R}$.

It was shown by Bartolo–Benci–Fortunato [1], that the deformation theorem and consequently the minimax theory for the critical values of a function $\varphi \in C^1(X)$, remains valid if instead of the usual Palais–Smale condition (see e.g. Gasiński–Papageorgiou [9]), we employ the $C$-condition. The two compactness-type conditions coincide when $\varphi \in C^1(X)$ is bounded below (see Gasiński–Papageorgiou [8, p. 127, Proposition 2.1.2]).

The following notion is helpful in verifying the $C$-condition.

**Definition 2.1.** Let $X$ be a reflexive Banach space and let $A\colon X \longrightarrow X^*$ be an operator. We say that $A$ is of *type* $(S)_+$, if for every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that

$$x_n \longrightarrow x \text{ weakly in } X \quad \text{and} \quad \limsup_{n\to+\infty} \langle A(x_n), x_n - x\rangle \leqslant 0,$$

one has $x_n \longrightarrow x$ in $X$.

For every $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we define the sublevel set of $\varphi$ at $c$, by $\varphi^c = \{x \in X : \varphi(x) \leqslant c\}$; the critical set of $\varphi$, by $K^\varphi = \{x \in X : \varphi'(x) = 0\}$; the critical set of $\varphi$ at the level $c$, by $K_c^\varphi = \{x \in K^\varphi : \varphi(x) = c\}$.

If $(Y_1, Y_2)$ is a topological pair with $Y_2 \subseteq Y_1 \subseteq X$, then for every integer $k \geqslant 0$, we denote by $H_k(Y_1, Y_2)$ the $k$-th relative singular homology group of $(Y_1, Y_2)$ with integer coefficients. The critical groups of $\varphi$ at an isolated critical point $x_0 \in X$ with $\varphi(x_0) = c$ are defined by

$$C_k(\varphi, x_0) = H_k\big(\varphi^c \cap U, \varphi^c \cap U \setminus \{x_0\}\big) \quad \forall k \geqslant 0,$$

where $U$ is a neighbourhood of $x_0$, such that $K^\varphi \cap \varphi^c \cap U = \{x_0\}$ (see Chang [5], Mawhin–Willem [14]). The excision property of singular homology implies that the above definition is independent of the particular neighbourhood $U$ we use.

Suppose that $\varphi \in C^1(X)$ satisfies the $C$-condition and that $\inf \varphi(K^\varphi) > -\infty$. Choose $c < \inf \varphi(K^\varphi)$, the critical groups of $\varphi$ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \forall k \geqslant 0$$

(see Bartsch–Li [2, p. 424, Definition 3.4]). Recall that the deformation theorem is still valid, if the $C$-condition is assumed (see Bartolo–Benci–Fortunato [1]). By virtue of the deformation theorem, it follows that the above definition is independent of $c < \inf \varphi(K^\varphi)$.

In the analysis of problem (1), we shall use the following spaces:

$$C_n^1(\overline{Z}) = \left\{u \in C^1(\overline{Z}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial Z\right\} \quad \text{and} \quad W_n^{1,p}(Z) = \overline{C_n^1(\overline{Z})}^{\|\cdot\|},$$

with $\|\cdot\|$ being the usual norm of $W^{1,p}(Z)$.

The Banach space $C_n^1(\overline{Z})$ is an ordered Banach space with positive cone:

$$C_+ = \big\{u \in C_n^1(\overline{Z}) : u(z) \geqslant 0 \text{ for all } z \in \overline{Z}\big\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int} C_+ = \big\{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{Z}\big\}.$$

Finally let us recall a few basic facts about the spectrum of the negative $p$-Laplacian ($p \in (1,+\infty)$), denoted by $\big(-\Delta_p, W^{1,p}_n(Z)\big)$. So we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p x(z) = \lambda |x(z)|^{p-2}x(z) & \text{a.e. on } Z \\ \dfrac{\partial x}{\partial n} = 0 & \text{on } \partial Z. \end{cases} \tag{2}$$

A number $\lambda \in \mathbb{R}$ is an eigenvalue of $\big(-\Delta_p, W^{1,p}_n(Z)\big)$, if problem (2) has a nontrivial solution, called an eigenfunction corresponding to the eigenvalue $\lambda$. It is easy to see that an eigenvalue satisfies $\lambda \geqslant 0$. In fact, $\lambda_0 = 0$ is an eigenvalue with corresponding eigenspace $\mathbb{R}$ and it is isolated (i.e., there is $\varepsilon > 0$, such that $(0,\varepsilon) \cap \sigma(p) = \emptyset$, with $\sigma(p)$ denoting the set of eigenvalues of $\big(-\Delta_p, W^{1,p}_n(Z)\big)$). By virtue of the Ljusternik–Schnirelmann theory, we have a whole strictly increasing sequence of eigenvalues $\{\lambda_k\}_{k\geqslant 1}$, $\lambda_k \longrightarrow +\infty$. These are the so called *LS-eigenvalues* of $\big(-\Delta_p, W^{1,p}_n(Z)\big)$. When $p = 2$ (linear eigenvalue problem), the $LS$-eigenvalues are all the eigenvalues of $\big(-\Delta, H^1_n(Z)\big)$.

In what follows, we use the notation $r^{\pm} = \max\{\pm r, 0\}$ for all $r \in \mathbb{R}$. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Also by $\|\cdot\|_r$ we denote the norm of $L^r(Z)$ and by $\|\cdot\|$ the norm of the Sobolev space $W^{1,p}(Z)$ or of $\mathbb{R}^N$ – it will always be clear from the context, which one we use. Finally,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p \\ +\infty & \text{if } N \leqslant p \end{cases}$$

is the critical Sobolev exponent.

In the next section using a variational argument, we produce two nontrivial smooth solutions of constant sign.

## 3. The solutions of constant sign

The hypotheses on the nonlinearity $f(z,\zeta)$ are the following:

$\underline{H(f)}$: $f\colon Z \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function, such that:

(i) for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z,\zeta)$ is measurable;

(ii) for almost all $z \in Z$, the function $\zeta \longmapsto f(z,\zeta)$ is continuous and $f(z,0) = 0$;

(iii) there exist a function $a \in L^\infty(Z)_+$ and $c > 0$, such that for almost all $z \in Z$ and all $\zeta \in \mathbb{R}$, we have

$$|f(z,\zeta)| \leqslant a(z) + c|\zeta|^{p-1};$$

(iv) if $F(z,\zeta)=\int_0^\zeta f(z,s)\,ds$, then

$$\lim_{|\zeta|\to+\infty}\frac{F(z,\zeta)}{|\zeta|^p}=0 \quad \text{uniformly for a.a. } z\in Z$$

and

$$\lim_{|\zeta|\to+\infty}\big(f(z,\zeta)\zeta-pF(z,\zeta)\big)=-\infty \quad \text{uniformly for a.a. } z\in Z;$$

(v) there exist $d<0<a$, such that

$$\operatorname*{ess\,sup}_Z f(z,a)<0<\operatorname*{ess\,inf}_Z f(z,d)$$

and, if $r=\max\{-d,a\}$, then we can find $\xi_r>0$, such that for almost all $z\in Z$, the function $\zeta\longmapsto f(z,\zeta)+\xi_r|\zeta|^{p-2}\zeta$ is nondecreasing on $[-r,r]$;

(vi) there exist $\delta_0>0$, $\tau\in(1,p)$, $q\in(p,p^*)$ and $c_1,c_2,c_3>0$, such that for almost all $z\in Z$, we have

$$\begin{aligned} &f(z,\zeta)\zeta>0 && \forall\zeta:\ 0<|\zeta|\leqslant\delta_0\\ &f(z,\zeta)\zeta\geqslant -c_1|\zeta|^p && \forall\zeta\in\mathbb{R}\end{aligned}$$

and

$$\tau F(z,\zeta)-f(z,\zeta)\zeta\geqslant c_2|\zeta|^p-c_3|\zeta|^q \quad \forall\zeta\in\mathbb{R}.$$

**Remark 3.1.** Condition $H(f)$(iv) implies that problem (1) is resonant at $\pm\infty$ with respect to the principal eigenvalue $\lambda_0=0$ from the right. Hence the Euler functional of the problem is indefinite.

**Example 3.2.** The following function $f(\zeta)$ satisfies the hypotheses $H(f)$. For the sake of simplicity, we drop the $z$-dependence:

$$f(\zeta)=\begin{cases}\eta|\zeta|^{\tau-2}\zeta-c|\zeta|^{p-2}\zeta+|\zeta|^{q-2}\zeta & \text{if } |\zeta|\leqslant 1\\ |\zeta|^{\mu-2}\zeta-\dfrac{2\zeta}{|\zeta|} & \text{if } |\zeta|>1,\end{cases}$$

where $c>2$, $\eta=c-2$ and $1<\mu,\tau<p<q<p^*$. In this case $a=1$ and $d=-1$.

We consider the Euler functional $\varphi\colon W_n^{1,p}(Z)\longrightarrow\mathbb{R}$ for problem (1), defined by

$$\varphi(x)=\frac{1}{p}\|\nabla x\|_p^p-\int_Z F(z,x(z))\,dz \quad \forall x\in W_n^{1,p}(Z).$$

Evidently $\varphi\in C^1(W_n^{1,p}(Z))$.

First, let us quote a version of the Brezis–Nirenberg theorem due to Motreanu–Motreanu–Papageorgiou [15, Proposition 2.5] on $W_n^{1,p}(Z)$ versus $C_n^1(\overline{Z})$ local minimizers of $\varphi$.

**Proposition 3.3.** *If hypotheses* $H(f)$ *hold and* $x_0 \in W^{1,p}_n(Z)$ *is a local* $C^1_n(\overline{Z})$*-minimizer of* $\varphi$*, i.e., there exists* $\varrho_1 > 0$ *such that*

$$\varphi(x_0) \leqslant \varphi(x_0 + h) \quad \forall h \in C^1_n(\overline{Z}),\ \|h\|_{C^1_n(\overline{Z})} \leqslant \varrho_1,$$

*then* $x_0 \in C^1_n(\overline{Z})$*, and it is a local* $W^{1,p}_n(Z)$*-minimizer of* $\varphi$*, i.e., there exists* $\varrho_2 > 0$ *such that*

$$\varphi(x_0) \leqslant \varphi(x_0 + h) \quad \forall h \in W^{1,p}_n(Z),\ \|h\|_{W^{1,p}_n(Z)} \leqslant \varrho_2.$$

For the proof we refer to Motreanu–Motreanu–Papageorgiou [15, Proposition 2.5] (our hypotheses on $f$ are much stronger then the ones needed for Proposition 3.3).

In the next proposition, using variational tools, we establish the existence of two solutions of constant sign.

**Proposition 3.4.** *If hypotheses* $H(f)$ *hold, then problem* (1) *has two smooth solutions of constant sign:*

$$x_0 \in \mathit{int}\, C_+ \quad \text{and} \quad v_0 \in -\mathit{int}\, C_+,$$

*which are local minimizers of* $\varphi$*.*

*Proof.* First we produce a positive solution. For this purpose, for $\varepsilon \in (0,1)$, we introduce the following truncation:

$$\widehat{g}^+_\varepsilon(z,\zeta) = \begin{cases} 0 & \text{if } \zeta < 0 \\ \varepsilon\zeta^{p-1} + f(z,\zeta) & \text{if } \zeta \in [0,a] \\ \varepsilon a^{p-1} + f(z,a) & \text{if } \zeta > a. \end{cases} \tag{3}$$

Evidently this is a Carathéodory function. We set $\widehat{G}^+_\varepsilon(z,\zeta) = \int_0^\zeta \widehat{g}^+_\varepsilon(z,s)\,ds$ and then consider the functional $\widehat{\varphi}^+_\varepsilon \colon W^{1,p}_n(Z) \longrightarrow \mathbb{R}$, defined by

$$\widehat{\varphi}^+_\varepsilon(x) = \frac{1}{p}\|\nabla x\|_p^p + \frac{\varepsilon}{p}\|x\|_p^p - \int_Z \widehat{G}^+_\varepsilon(z,x(z))\,dz \quad \forall x \in W^{1,p}_n(Z).$$

Clearly $\widehat{\varphi}^+_\varepsilon \in C^1(W^{1,p}_n(Z))$. Also it is coercive and sequentially weakly lower semicontinuous. Therefore, by the Weierstrass theorem, we can find $x_0 \in W^{1,p}_n(Z)$, such that

$$\widehat{m}^+_\varepsilon = \inf_{W^{1,p}_n(Z)} \widehat{\varphi}^+_\varepsilon = \widehat{\varphi}^+_\varepsilon(x_0). \tag{4}$$

We may always assume that $\delta_0 < \min\{-d, a, 1\}$ (see hypothesis $H(f)$(vi)). By virtue of hypothesis $H(f)$(vi) and since $q > p$, we can find $\delta_1 \in (0,\delta_0)$ such that

$$\tau F(z,\zeta) \geqslant f(z,\zeta)\zeta > 0 \quad \text{for a.a. } z \in Z,\ \text{all } 0 < |\zeta| \leqslant \delta_1. \tag{5}$$

From (5) and reasoning as in the case of the Ambrosetti–Rabinowitz condition (see, e.g., Gasiński–Papageorgiou [8, p. 298]), we can have that

$$F(z,\zeta) \geqslant c_4|\zeta|^\tau \quad \text{for a.a.} z \in Z, \text{ all } |\zeta| \leqslant \delta_1, \tag{6}$$

for some $c_4 > 0$. Using (3) and (6), for $\xi \in (0, \delta_1]$, we have

$$\widehat{\varphi}_\varepsilon^+(\xi) = -\int_Z F(z,\xi)\,dz \leqslant -c_4\xi^\tau|Z|_N < 0,$$

so $\widehat{m}_\varepsilon = \widehat{\varphi}_\varepsilon^+(x_0) < 0 = \widehat{\varphi}_\varepsilon^+(0)$, i.e., $x_0 \neq 0$. From (4), we have $(\widehat{\varphi}_\varepsilon^+)'(x_0) = 0$, so

$$A(x_0) + \varepsilon|x_0|^{p-2}x_0 = N_{\widehat{g}_\varepsilon^+}(x_0), \tag{7}$$

where $A\colon W_n^{1,p}(Z) \longrightarrow W_n^{1,p}(Z)^*$ is the nonlinear map, defined by

$$\langle A(u), v\rangle = \int_Z \|\nabla u\|^{p-2}(\nabla u, \nabla v)_{\mathbb{R}^N}\,dz \quad \forall u, v \in W_n^{1,p}(Z) \tag{8}$$

and

$$N_{\widehat{g}_\varepsilon^+}(u)(\cdot) = \widehat{g}_\varepsilon^+(\cdot, u(\cdot)) \quad \forall u \in W_n^{1,p}(Z).$$

On (7) we act with $-x_0^- \in W_n^{1,p}(Z)$ and obtain $\|\nabla x_0^-\|_p^p + \varepsilon\|x_0^-\|_p^p = 0$, so

$$x_0^- = 0, \quad \text{i.e., } x_0 \geqslant 0,\ x_0 \neq 0. \tag{9}$$

Also, on (7) we act with $(x_0-a)^+ \in W_n^{1,p}(Z)$ and using also hypothesis $H(f)$(v), we obtain

$$\|\nabla x_0\|_p^p + \varepsilon\int_{\{x_0>a\}}(x_0^{p-1} - a^{p-1})(x_0 - a)\,dz \leqslant \int_{\{x_0>a\}} f(z,a)(x_0 - a)\,dz < 0,$$

so

$$\big|\{x_0 > a\}\big|_N = 0, \quad \text{i.e., } x_0 \leqslant a. \tag{10}$$

Because of (9) and (10), equation (7) becomes $A(x_0) = N_f(x_0)$, where $N_f(u)(\cdot) = f(\cdot, u(\cdot))$ for all $u \in W_n^{1,p}(Z)$ (see (3)). From (10), as in Motreanu–Papageorgiou [16, pp. 24–25, proof of Proposition 12], using the nonlinear Green's identity (see, e.g., Gasiński–Papageorgiou [9, p. 211, Theorem 2.4.54]), we have

$$\begin{cases} -\Delta_p x_0(z) = f(z, x_0(z)) & \text{a.e. on } Z \\ \dfrac{\partial x_0}{\partial n} = 0 & \text{on } \partial Z. \end{cases}$$

Nonlinear regularity theory (see, e.g., Gasiński–Papageorgiou [9]), implies that $x_0 \in C_+ \setminus \{0\}$. According to hypothesis $H(f)$(v), we have

$$-\Delta_p x_0(z) + \xi_r x_0(z)^{p-1} = f(z, x_0(z)) + \xi_r x_0(z)^{p-1} \geqslant 0 \quad \text{a.e. on } Z,$$

so $\Delta_p x_0(z) \leqslant \xi_r x_0(z)^{p-1}$ a.e. on $Z$ and from Vázquez [19, p. 192, Theorem 1], we have

$$x_0 \in \operatorname{int} C_+. \tag{11}$$

Next for $\delta > 0$, let $u_\delta(z) = x_0(z) + \delta$. Then $u_\delta \in \operatorname{int} C_+$. We have

$$-\Delta_p u_\delta(z) + \xi_r u_\delta(z)^{p-1} = -\Delta_p x_0(z) + \xi_r x_0(z)^{p-1} + \lambda(\delta),$$

where $\lambda \in C\big((0,+\infty);\mathbb{R}_+\big)$, $\lambda(\delta) \longrightarrow 0$ as $\delta \to 0^+$. Then, using hypothesis $H(f)$(v), we have

$$-\Delta_p u_\delta(z)+\xi_r u_\delta(z)^{p-1} = f(z,x_0(z))+\xi_r x_0(z)^{p-1}+\lambda(\delta) \leqslant f(z,a)+\xi_r a^{p-1}+\lambda(\delta).$$

Let $\beta_a = \operatorname{ess\,sup}_Z f(z,a) < 0$ (see hypothesis $H(f)$(v)). Choose $\delta > 0$ small enough, such that $\beta_a + \lambda(\delta) \leqslant 0$. Hence

$$-\Delta_p u_\delta(z) + \xi_r u_\delta(z)^{p-1} \leqslant \xi_r a^{p-1} = -\Delta_p a + \xi_r a^{p-1}.$$

Acting on this inequality with $(u_\delta - a)^+$ and assuming that $\big|\{u_\delta > 0\}\big|_N > 0$, we obtain

$$0 < \int_{\{u_\delta>a\}} \|\nabla u_\delta\|^p \, dz + \xi_r \int_{\{u_\delta>a\}} \big(u_\delta^{p-1} - a^{p-1}\big)(u_\delta - a)\, dz \leq \ 0,$$

a contradiction, hence $|\{u_\delta > 0\}|_N = 0$ and so $u_\delta(z) \leqslant a$ for all $z \in \overline{Z}$, so $x(z) < a$ for all $z \in \overline{Z}$ and thus

$$a - x_0 \in \operatorname{int} C_+. \tag{12}$$

From (11) and (12), it follows that we can find $r > 0$, small enough, such that, if $\overline{B}_r^{C_n^1}(x_0) = \big\{x \in C_n^1(\overline{Z}) : \|x - x_0\|_{C_n^1(\overline{Z})} \leqslant r\big\}$, then

$$\widehat{\varphi}_\varepsilon^+|_{\overline{B}_r^{C_n^1}(x_0)} = \varphi|_{\overline{B}_r^{C_n^1}(x_0)}.$$

This implies that $x_0 \in \operatorname{int} C_+$ is a local $C_n^1(\overline{Z})$-minimizer of $\varphi$. Invoking Proposition 3.3, $x_0 \in \operatorname{int} C_+$ is a local $W_n^{1,p}(Z)$-minimizer of $\varphi$.

Similarly, for $\varepsilon \in (0,1)$, we introduce

$$\widehat{g}_\varepsilon^-(z,\zeta) = \begin{cases} \varepsilon|d|^{p-2}d + f(z,d) & \text{if } \zeta < d \\ \varepsilon|\zeta|^{p-2}\zeta + f(z,\zeta) & \text{if } \zeta \in [d,0] \\ 0 & \text{if } \zeta > 0. \end{cases} \tag{13}$$

We set $\widehat{G}_\varepsilon^-(z,\zeta)=\int_0^\zeta \widehat{g}_\varepsilon^-(z,s)ds$ and introduce the $C^1$-functional $\widehat{\varphi}_\varepsilon^- : W_n^{1,p}(Z) \longrightarrow \mathbb{R}$, defined by

$$\widehat{\varphi}_\varepsilon^-(x) = \frac{1}{p}\|\nabla x\|_p^p + \frac{\varepsilon}{p}\|x\|_p^p - \int_Z \widehat{G}_\varepsilon^-(z,x(z))\, dz \quad \forall x \in W_n^{1,p}(Z).$$

Working with the functional $\widehat{\varphi}_\varepsilon^-$ instead of $\widehat{\varphi}_\varepsilon^+$ and using this time (13), we obtain $v_0 \in -\operatorname{int} C_+$, a second nontrivial constant sign smooth solution of (1), which is also a local minimizer of $\varphi$. □

## 4. The critical groups at 0 and $\infty$

Clearly $x = 0$ is a critical point of $\varphi$. We may assume that it is an isolated critical point $\varphi$ or otherwise we have a whole sequence of distinct nontrivial solutions of (1) and so we are done.

**Proposition 4.1.** *If hypotheses $H(f)$ hold, then $C_k(\varphi, 0) = 0$ for all $k \geqslant 0$.*

*Proof.* Recall that for some $\delta_1 \in (0, \delta_0)$, we have

$$F(z,\zeta) \geqslant c_4|\zeta|^\tau \quad \text{for a.a. } z \in Z, \text{ all } |\zeta| \leqslant \delta_1 \tag{14}$$

(see (6)). Combining (14) with hypothesis $H(f)$(vi), we have

$$F(z,\zeta) \geqslant c_5|\zeta|^\tau + c_6|\zeta|^p - c_7|\zeta|^q \quad \text{for a.a.} z \in Z, \text{ all } \zeta \in \mathbb{R}, \tag{15}$$

for some $c_5, c_6, c_7 > 0$. Let $x \in W_n^{1,p}(Z) \setminus \{0\}$ and $t > 0$. Then

$$\begin{aligned} \varphi(tx) &= \frac{t^p}{p}\|\nabla x\|_p^p - \int_Z F(z,tx)\,dz \\ &\leqslant \frac{t^p}{p}\|\nabla x\|_p^p - c_5 t^\tau \|x\|_\tau^\tau - c_6 t^p \|x\|_p^p + c_7 t^q \|x\|_q^q \end{aligned} \tag{16}$$

(see (15)). Because $1 < \tau < p < q$, from (16), it follows that we can find $t_x = t_x(x) \in (0,1)$ small enough, such that

$$\varphi(tx) < 0 \quad \forall t \in (0, t_x). \tag{17}$$

*Claim* 1. There exists $r_1 > 0$, such that

$$\frac{d}{dt}\varphi(tx)|_{t=1} > 0 \quad \forall x \in W_n^{1,p}(Z),\ 0 < \|x\| \leqslant r_1,\ \varphi(x) = 0. \tag{18}$$

For the proof, let $x \in W_n^{1,p}(Z)$ be such that $\varphi(x) = 0$. Then, using the facts that $\varphi(x) = 0$, $\tau < p$ and hypothesis $H(f)$(vi), we have

$$\begin{aligned} \frac{d}{dt}\varphi(tx)|_{t=1} &= \langle \varphi'(x), x\rangle \\ &= \|\nabla x\|_p^p - \int_Z f(z,x)x\,dz \\ &= \Big(1 - \frac{\tau}{p}\Big)\|\nabla x\|_p^p + \int_Z \big(\tau F(z,x) - f(z,x)x\big)\,dz \\ &\geqslant \Big(1 - \frac{\tau}{p}\Big)\|\nabla x\|_p^p + c_2\|x\|_p^p - c_3\|x\|_q^q \\ &\geqslant c_8\|x\|^p - c_9\|x\|^q, \end{aligned} \tag{19}$$

for some $c_8, c_9 > 0$. Because $p < q$, from (19), we infer that we can find $r_1 \in (0,1)$ small enough, such that (18) hold. This proves Claim 1.

*Claim* 2. We have

$$\varphi(tx) \leqslant 0 \quad \forall t \in [0,1],\ x \in (\varphi^0 \cap \overline{B}_{r_1}) \setminus \{0\}. \tag{20}$$

For the proof, consider $x \in W_n^{1,p}(Z)$, such that $0 < \|x\| \leqslant r_1$ and $\varphi(x) \leqslant 0$ (i.e., $x \in (\varphi^0 \cap \overline{B}_{r_1}) \setminus \{0\}$). We proceed by contradiction. So, suppose we can find $t_0 \in (0,1)$, such that $\varphi(t_0 x) > 0$. Due to the continuity of $\varphi$, we can find $t_1 \in (t_0, 1]$, such that $\varphi(t_1 x) = 0$. We set $t_2 = \min\big\{t \in [t_0,1] :\ \varphi(tx) = 0\big\}$. Then

$$\varphi(tx) > 0 \quad \forall t \in [t_0, t_2]. \tag{21}$$

Let $u = t_2 x$. Then $\varphi(u) = 0$ and $0 < \|u\| = t_2\|x\| \leqslant r$. Hence, on account of Claim 1 (see (18)), we have

$$\frac{d}{dt}\varphi(tu)|_{t=1} > 0. \tag{22}$$

On the other hand, due to (21), for every $t \in [t_0, t_2)$, we have $0 = \varphi(u) = \varphi(t_2 x) < \varphi(tx)$. Consequently,

$$\frac{d}{dt}\varphi(tu)|_{t=1} = \frac{d}{dt}\varphi(tx)|_{t=t_2} = \lim_{t \nearrow t_2} \frac{\varphi(tx) - \varphi(t_2 x)}{t - t_2} \leqslant 0. \tag{23}$$

Comparing (22) and (23), we reach a contradiction. This implies that (20) is true and so Claim 2 is proved.

Choose $r \in (0, r_1)$ small enough, such that $\overline{B}_r \cap K^{\varphi} = \{0\}$. Let $h\colon [0,1] \times (\varphi^0 \cap \overline{B}_r) \longrightarrow \varphi^0 \cap \overline{B}_r$ be defined by $h(t,x) = (1-t)x$. Because of Claim 2 (see (20)), $h$ is well defined and it is a continuous deformation. Therefore $\varphi^0 \cap \overline{B}_r$ is contractible in itself.

*Claim* 3. For every $x \in \overline{B}_r$ with $\varphi(x) > 0$ there exists a unique $\overline{t}_x \in (0,1)$, such that $\varphi(\overline{t}_x x) = 0$.

For the proof, fix $x \in \overline{B}_r$ with $\varphi(x) > 0$. The existence of $\overline{t}_x \in (0,1)$ follows from the fact that $\varphi(x) > 0$, from (17) and from the continuity of $t \longmapsto \varphi(tx)$ on $[0,1]$. We need to show the uniqueness of $\overline{t}_x \in (0,1)$. To this end, suppose that there exist $0 < \overline{t}_x^1 < \overline{t}_x^2 < 1$, such that $\varphi(\overline{t}_x^1 x) = \varphi(\overline{t}_x^2 x) = 0$. From Claim 2 (see (20)), we have $\varphi(t\overline{t}_x^2 x) \leqslant 0$ for all $t \in [0,1]$, so $\overline{t}_x^1$ is a local maximum of the function $[0,1] \ni t \longmapsto \varphi(tx)$, hence $\frac{d}{dt}\varphi(t\overline{t}_x^1 x)|_{t=1} = 0$, which contradicts Claim 1 (see (18)). This proves Claim 3.

From Claim 3, it follows that for every $x \in \overline{B}_r$ with $\varphi(x) > 0$, we have

$$\begin{cases} \varphi(tx) < 0 & \forall t \in (0, \overline{t}_x) \\ \varphi(tx) > 0 & \forall t \in (\overline{t}_x, 1]. \end{cases} \tag{24}$$

Next let $\eta\colon \overline{B}_r \setminus \{0\} \longrightarrow (0,1]$ be defined by

$$\eta(x) = \begin{cases} 1 & \text{if } x \in \overline{B}_r \setminus \{0\} \text{ and } \varphi(x) \leqslant 0 \\ \bar{t}_x & \text{if } x \in \overline{B}_r \setminus \{0\} \text{ and } \varphi(x) > 0, \end{cases} \tag{25}$$

with $\bar{t}_x \in (0,1)$ being as in Claim 3. This is a well defined map. Moreover, due to Claim 1 (see (18)), (24) and the implicit function theorem, we have that $\eta$ is continuous. Then we define the map $\xi\colon \overline{B}_r \setminus \{0\} \longrightarrow (\varphi^0 \cap \overline{B}_r) \setminus \{0\}$ by

$$\xi(x) = \begin{cases} \eta(x)x & \text{if } x \in \overline{B}_r \setminus \{0\} \text{ and } \varphi(x) \geqslant 0 \\ x & \text{if } x \in \overline{B}_r \setminus \{0\} \text{ and } \varphi(x) < 0. \end{cases}$$

From (25), we see that $\eta(x) = 1$ when $\varphi(x) = 0$. Therefore, $\xi$ is well defined and due to the continuity of $\eta$, it is continuous. Moreover, note that $\xi|_{(\varphi^0 \cap \overline{B}_r)\setminus\{0\}} = id|_{(\varphi^0 \cap \overline{B}_r)\setminus\{0\}}$, so $\xi$ is a retraction of $\overline{B}_r \setminus \{0\}$ onto $(\varphi^0 \cap \overline{B}_r) \setminus \{0\}$.

Recall that $\overline{B}_r \setminus \{0\}$ is contractible and retracts of contractible sets, are contractible too. Therefore $(\varphi^0 \cap \overline{B}_r) \setminus \{0\}$ is contractible in itself. We established earlier that $\varphi^0 \cap \overline{B}_r$ is contractible in itself. It follows that

$$H_k\big(\varphi^0 \cap \overline{B}_r, (\varphi^0 \cap \overline{B}_r) \setminus \{0\}\big) = 0 \quad \forall k \geqslant 0$$

(see Dugundji–Granas [11, p. 389]), so $C_k(\varphi, 0) = 0$ for all $k \geqslant 0$. ☐

**Proposition 4.2.** *If hypotheses $H(f)$ hold, then $C_1(\varphi, \infty) \neq 0$.*

*Proof.* First we show that the functional $\varphi$ satisfies the $C$-condition. To this end, let $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_n(Z)$ be a sequence, such that

$$|\varphi(x_n)| \leqslant M_1 \quad \forall n \geqslant 1, \tag{26}$$

for some $M_1 > 0$ and

$$(1 + \|x_n\|)\varphi'(x_n) \longrightarrow 0. \tag{27}$$

*Claim 1.* The sequence $\{x_n\}_{n\geqslant 1} \subseteq W^{1,p}_n(Z)$ is bounded.

We argue by contradiction. So, suppose that the claim is not true. Then, we may assume that $\|x_n\| \longrightarrow +\infty$. Let $y_n = \frac{x_n}{\|x_n\|}$ for all $n \geqslant 1$. Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so by passing to a suitable subsequence if necessary, we may assume that

$$\begin{aligned} y_n &\longrightarrow y \quad \text{weakly in } W^{1,p}_n(Z) \\ y_n &\longrightarrow y \quad \text{in } L^p(Z). \end{aligned} \tag{28}$$

From (26), we have $\frac{\varphi(x_n)}{\|x_n\|^p} \leqslant \frac{M_1}{\|x_n\|^p}$ for all $n \geqslant 1$, so

$$\frac{1}{p}\|\nabla y_n\|_p^p - \int_Z \frac{F(z,x_n)}{\|x_n\|^p}\,dz \leqslant \frac{M_1}{\|x_n\|^p} \quad \forall n \geqslant 1. \tag{29}$$

By virtue of hypotheses $H(f)$(iii) and (iv), for a given $\varepsilon > 0$ we can find $c_\varepsilon > 0$, such that

$$|F(z,\zeta)| \leqslant \frac{\varepsilon}{p}|\zeta|^p + c_\varepsilon \quad \text{for a.a. } z \in Z, \text{ all } \zeta \in \mathbb{R}. \tag{30}$$

We use (30) in (29) and we obtain

$$\frac{1}{p}\|\nabla y_n\|_p^p \leqslant \frac{M_1}{\|x_n\|^p} + \frac{\varepsilon}{p}\|y_n\|_p^p + \frac{c_\varepsilon}{\|x_n\|^p},$$

so, using also (28), we have $\|\nabla y_n\|_p^p \leqslant \varepsilon\|y\|_p^p$. Because $\varepsilon > 0$ was arbitrary, we let $\varepsilon \searrow 0$ and obtain $\|\nabla y\|_p = 0$, so $y = \xi \in \mathbb{R}$.

If $\xi = 0$, then $\|\nabla y_n\|_p \longrightarrow 0$, hence $\|y_n\| \longrightarrow 0$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geqslant 1$.

If $\xi \neq 0$, then without any loss of generality we may assume that $\xi > 0$ (the proof is similar, if we assume that $\xi < 0$). Then we have $x_n(z) \longrightarrow +\infty$ for a.a. $z \in Z$. Hence, from hypothesis $H(f)$(iv), we have

$$\lim_{n\to+\infty} \big(f(z,x_n(z))x_n(z) - pF(z,x_n(z))\big) = -\infty \quad \text{for a.a. } z \in Z.$$

By Fatou's lemma, we have

$$\lim_{n\to+\infty} \int_Z \big(f(z,x_n)x_n - pF(z,x_n)\big)\,dz = -\infty. \tag{31}$$

On the other hand, from (27), we have $|\langle \varphi'(x_n), x_n\rangle| \leqslant \varepsilon_n'$, with $\varepsilon_n' \searrow 0$, so

$$\left|\|\nabla x_n\|_p^p - \int_Z f(z,x_n)x_n\,dz\right| \leqslant \varepsilon_n' \quad \forall n \geqslant 1. \tag{32}$$

Also from (26), we have

$$\left|\|\nabla x_n\|_p^p - \int_Z pF(z,x_n)\,dz\right| \leqslant pM_1 \quad \forall n \geqslant 1. \tag{33}$$

From (32) and (33), it follows that

$$\int_Z \big(f(z,x_n)x_n - pF(z,x_n)\big)\,dz \geqslant -M_2 \quad \forall n \geqslant 1, \tag{34}$$

for some $M_2 > 0$. Comparing (31) and (34), we reach a contradiction. This proves Claim 1.

On account of Claim 1, we may assume that

$$\begin{aligned} x_n &\longrightarrow x \qquad \text{weakly in } W_n^{1,p}(Z) \\ x_n &\longrightarrow x \qquad \text{in } L^p(Z). \end{aligned} \tag{35}$$

From (27), we have

$$\left| \langle A(x_n), x_n - x \rangle - \int_Z f(z, x_n)(x_n - x)\, dz \right| \leqslant \frac{\varepsilon_n}{1 + \|x_n\|} \|x_n - x\| = \varepsilon_n'',$$

with $\varepsilon_n'' \searrow 0$, where $A$ is defined by (8). Evidently $\int_Z f(z, x_n)(x_n - x)\, dz \longrightarrow 0$ (see (35)). Hence $\lim_{n\to+\infty} \langle A(x_n), x_n - x\rangle = 0$. Using the fact that $A$ is of type $(S)_+$ (see Definition 2.1), we have

$$x_n \longrightarrow x \quad \text{in } W_n^{1,p}(Z),$$

so $\varphi$ satisfies the $C$-condition.

Next we consider the following direct sum decomposition $W_n^{1,p}(Z) = \mathbb{R} \oplus V$, with $V = \{x \in W_n^{1,p}(Z) : \int_Z x(z)\, dz = 0\}$. By virtue of hypothesis $H(f)$(iv), for a given $\beta > 0$, we can find $M_3 = M_3(\beta) > 0$, such that

$$f(z, \zeta)\zeta - pF(z, \zeta) \leqslant -\beta \quad \text{for a.a. } z \in Z, \text{ all } |\zeta| \geqslant M_3. \tag{36}$$

Suppose that $\zeta \geqslant M_3$. Then

$$\begin{aligned} \frac{d}{d\zeta} \frac{F(z, \zeta)}{\zeta^p} &= \frac{f(z, \zeta)\zeta^p - p\zeta^{p-1} F(z, \zeta)}{\zeta^{2p}} \\ &= \frac{f(z, \zeta)\zeta - pF(z, \zeta)}{\zeta^{p+1}} \\ &\leqslant -\frac{\beta}{\zeta^{p+1}} \quad \text{for a.a. } z \in Z \end{aligned}$$

(see (36)). Integrating this last inequality on the interval $[y, u]$, with $M_3 \leqslant y \leqslant u$, we obtain

$$\frac{F(z, u)}{u^p} - \frac{F(z, y)}{y^p} \leqslant \frac{\beta}{p}\left( \frac{1}{u^p} - \frac{1}{y^p} \right) \quad \text{for a.a. } z \in Z.$$

We let $u \to +\infty$. Using hypothesis $H(f)$(iv), we have $\frac{F(z,y)}{y^p} \geqslant \frac{\beta}{p} \frac{1}{y^p}$, so

$$F(z, y) \geqslant \frac{\beta}{p} \quad \text{for a.a. } z \in Z, \text{ all } y \geqslant M_3. \tag{37}$$

Therefore, if $\xi \in \mathbb{R}$, $\xi > 0$, from (37) and since $\beta > 0$ was arbitrary, we infer that

$$\varphi(\xi) = -\int_Z F(z, \xi)\, dz \longrightarrow -\infty \quad \text{as } \xi \to +\infty.$$

Similarly, we show that $\varphi(\xi) \longrightarrow -\infty$ as $\xi \to -\infty$. So, we conclude that

$$\varphi(\xi) \longrightarrow -\infty \quad \text{as } |\xi| \to +\infty. \tag{38}$$

Now, using (30), for any $v \in V$, we have

$$\varphi(v) = \frac{1}{p}\|\nabla v\|_p^p - \int_Z F(z, v(z))\, dz \geqslant \frac{1}{p}\|\nabla v\|_p^p - \frac{\varepsilon}{p}\|v\|_p^p - c_\varepsilon |Z|_N. \tag{39}$$

From the Poincaré–Wirtinger inequality (see, e.g., Gasiński–Papageorgiou [8, p. 224]), we have

$$\widehat{c}\|v\|_p^p \leqslant \|\nabla v\|_p^p \quad \forall v \in V, \tag{40}$$

for some $\widehat{c} > 0$. Using (40) in (39), we obtain

$$\varphi(v) \geqslant \frac{1}{p}\left(1 - \frac{\varepsilon}{\widehat{c}}\right)\|\nabla v\|_p^p - c_\varepsilon |Z|_N. \tag{41}$$

Choosing $\varepsilon \in (0, \widehat{c})$ and since $\|\nabla v\|_p$ is an equivalent norm on $V$, from (41), we infer that $\varphi|_V$ is coercive. Because of this fact and (38), we can apply Proposition 3.8 of Bartsch–Li [2] and conclude that $C_1(\varphi, \infty) \neq 0$. □

## 5. Three Nontrivial Smooth Solutions

In this section, we prove the full multiplicity theorem (three nontrivial smooth solutions) for problem (1).

**Theorem 5.1.** *If hypotheses* $H(f)$ *hold, then problem* (1) *has at least three nontrivial smooth solutions* $x_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ *and* $u_0 \in C_n^1(\overline{Z})$.

*Proof.* From Proposition 3.4, we already have two constant sign solutions $x_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$.

Moreover, we know that both are local minimizers of $\varphi$. Hence

$$C_k(\varphi, x_0) = C_k(\varphi, v_0) = \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0 \tag{42}$$

(see Chang [5, p. 33] and Mawhin–Willem [14, p. 175]). From Proposition 4.2, we know that $C_1(\varphi, \infty) \neq 0$. Hence, there is a critical point $u_0$ of $\varphi$, such that

$$C_1(\varphi, u_0) \neq 0 \tag{43}$$

(see Bartsch–Li [2, Proposition 3.5]).

Also, from Proposition 4.1, we know that

$$C_k(\varphi, 0) = 0 \quad \forall k \geqslant 0. \tag{44}$$

From (42), (43) and (44), we infer that $u_0 \notin \{0, x_0, v_0\}$, hence it is a third nontrivial critical point of $\varphi$. We have

$$A(u_0) = N_f(u_0). \tag{45}$$

From (45), as before, using the nonlinear Green's identity, we deduce that $u_0$ is a solution of (1) and $u_0 \in C_n^1(\overline{Z})$ (nonlinear regularity theory). □

**Remark 5.2.** The third solution in the proof of Theorem 5.1 can be obtained also in another way. Namely, because we already have two solutions $x_0$ and $v_0$ being local minimizers of $\varphi$ and $\varphi$ is unbounded from below, so using the mountain pass theorem we can obtain a third solution $u_0$ with no use of Proposition 4.2. Because $C_1(\varphi, u_0) \neq 0$ so $u_0$ is nontrivial (see Proposition 4.1). Nevertheless, Proposition 4.2 is of independent interest.

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