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A Necessary Condition for the Instantaneous Shrinking Property of Solutions to a Semilinear Heat Equation

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Abstract. In the paper, we prove that if the support of the solution of a semilinear heat equation has the instantaneous shrinking property, then the initial data must vanish at infinity. We apply the weak comparison principle to prove this conclusion. We also give an example to illustrate our results.

Keywords. Comparison principles, instantaneous shrinking of the support, strong absorption, semilinear heat equation

Mathematics Subject Classification (2000). Primary 35B05, secondary 35K15

1. Introduction

In this paper, we consider the following semilinear heat equation

$$u_t - \triangle u + |u|^{q-1}u = 0, \quad (x,t) \in \mathbb{R}^N \times (0, +\infty), \tag{1}$$

where the exponent q will be constant in the range $0 < q < 1$(the term $|u|^{q-1}u$ usually represents a strong absorption term). We also assume initial data

$$u(x,0) = u_0(x) \geq 0, \quad u_0(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{2}$$

Such equation appears in the subject of plasma physics, biomathematics and other applied fields. Under the assumption (2), there exists a unique nonnegative classical solution $u(x,t)$ of (1). As a result of the strong absorption term, the solution of (1) may occur a phenomenon called instantaneous shrinking of the support which is defined as follows.

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Definition 1.1. Suppose $u(x,t) \in C(\mathbb{R}^N \times [0,T]), 0 < T \leq +\infty$, $u(x,t) \geq 0$, set

$$\zeta(t;u) = \sup\{|x|; u(x,t) > 0\}.$$

We say that *instantaneous shrinking of the support* (briefly, ISS) occurs for $u(x,t)$ if $\zeta(0;u) = +\infty$ and there exists $\tau > 0$ such that $\zeta(t;u) < +\infty$ for all $t \in (0,\tau]$.

By the definition, if ISS occurs for $u(x,t)$, then for any small $t > 0$ the support of $u(x,t)$ is bounded although the initial data $u_0(x)$ may be positive everywhere.

The ISS phenomenon has been studied by many authors. The results in [1] yield the following conclusion: If the initial data $u_0(x)$ satisfies

$$\lim_{|x|\to\infty} u_0(x) = 0, \tag{3}$$

then ISS occurs for the solution of (1), (2). This result gives a sufficient condition of the initial data which ensures the occurrence of ISS property. However, in this paper, we will prove that (3) is also a necessary condition for the emergence of the ISS property for the solution of (1), (2). The following theorem is our main result.

Theorem 1.2. *Let $u(x,t)$ be the solution of* (1), (2). *Then ISS occurs for $u(x,t)$ if and only if $u_0(x)$ satisfies* (3).

Since the results in [1] yield the sufficient part of Theorem 1.2, hence in order to prove Theorem 1.2, we only need to prove the following theorem.

Theorem 1.3. *Let $u(x,t)$ be the solution of* (1), (2), *and assume that $u_0(x)$ satisfies either*

$$\liminf_{|x|\to\infty} u_0(x) = C > 0, \tag{4}$$

or

$$\liminf_{|x|\to\infty} u_0(x) = 0, \quad \limsup_{|x|\to\infty} u_0(x) = C > 0. \tag{5}$$

Then there exists $\tau > 0$ such that $\zeta(t;u) = +\infty$ for all $t \in [0,\tau]$. Therefore ISS does not occur for $u(x,t)$ in both case (4) *and* (5).

It is well-known that the solution of heat equation has the infinite propagation property even the support of initial data is compact, hence the ISS property of (1) and (2) is due to the presence of the strong absorption term. Comparing equation (1), here we give some remarks about the following equation with variable coefficient:

$$u_t - \triangle u + b(x,t)|u|^{q-1}u = 0, \quad (x,t) \in \mathbb{R}^N \times (0,+\infty), \tag{6}$$

where $b(x,t) \geq 0$, $0 < q < 1$. In this case, the appearance of the ISS property for the solution of (6) and (2) depends not only on the behavior of $u_0(x)$, but also on the relationship between $b(x,t)$ and $u_0(x)$. For example the results in [7] ($N = 1$) say that if $u_0(x)$ and $b(x,t)$ satisfying

$$u_0(x) \leq \frac{c_0}{(1+|x|)^{\alpha}} \quad (c_0 > 0, \alpha > 0)$$
$$b(x,t) \geq \frac{b_0}{(1+|x|)^{\beta}} \quad (b_0 > 0, \beta > 0),$$

and if $\beta < \alpha(1-q)$, then ISS occurs for $u(x,t)$ which solves (6) and (2). See also [5]. Furthermore, for the variable coefficient equation with strong absorption such as (6), we emphasize that even $u_0(x)$ does not vanish at infinity, i.e., $u_0(x)$ does not satisfy (3), the phenomenon of ISS property can still occur under an extra condition of $b(x,t)$. Here we point out that the paper [8] gets the ISS property when $u_0(x)$ is taken in $L^p(\mathbb{R}^N)(1 < p < \infty)$ spaces, and what is even more surprising is that [9, 10] also proves the ISS property when $u_0(x)$ is growing rapidly at infinity, in which $u_0(x)$ can be taken the form $A_0 \exp(\alpha|x|^2)$ where $A_0 > 0, \alpha > 0$.

With the effect of the strong absorption term, the solution of (1) and (2) also has the extinction behavior, which means the solution will vanish after finite time. See [2, 3, 4] and the references therein for more discussion about the extinction behavior.

Using comparison principles is one of the main methods to discuss the ISS property, hence in Section 2 we give some comparison principles which will be used in the proof of Theorem 1.3, but it is interesting to notice that the classical comparison principle is not enough for our proof to Theorem 1.3, so we prove a weaker comparison principle in Section 2. In Section 3, we prove Theorem 1.3. Section 4 deals with the ISS property under the condition $N = 1$, $u_0(x) = |\sin x|$.

2. Comparison principles

This section gives some useful comparison principles. For simplicity, throughout Lemma 2.1–Lemma 2.3, we set

$$Lu := u_t - \triangle u + f(u),$$

where $f : \mathbb{R} \to \mathbb{R}$ be a continuous, non-decreasing function. The proofs of Lemma 2.1–Lemma 2.3 are analogous to that of linear parabolic equations, so we omit their proofs.

Lemma 2.1. *Let Ω be an open bounded set in $\mathbb{R}^N$, $T < +\infty$, $Q_T = \Omega \times (0,T]$, and $\partial_p Q_T := \overline{Q_T} \backslash Q_T$ be the parabolic boundary of Q_T. Assume that $u, v \in C(\overline{Q_T}) \cap C^{2,1}_{x,t}(Q_T)$ satisfy the following condition:*

$$\begin{cases} Lv \le Lu, & (x,t) \in Q_T \\ v(x,t) \le u(x,t), & (x,t) \in \partial_p Q_T. \end{cases}$$

Then $v(x,t) \le u(x,t)$ everywhere in $\overline{Q_T}$.

Lemma 2.2. *Suppose that $u, v \in C(\mathbb{R}^N \times [0,+\infty)) \cap C^{2,1}_{x,t}(\mathbb{R}^N \times (0,+\infty))$ are bounded and satisfy the following condition:*

$$\begin{cases} Lv \le Lu, & (x,t) \in \mathbb{R}^N \times (0,+\infty) \\ v(x,0) \le u(x,0), & x \in \mathbb{R}^N. \end{cases}$$

Then $v(x,t) \le u(x,t)$ everywhere in $\mathbb{R}^N \times [0,+\infty)$.

Lemma 2.3. *Assume that $u, v \in C((\mathbb{R}^N \backslash B_n) \times [0,+\infty)) \cap C^{2,1}_{x,t}((\mathbb{R}^N \backslash \overline{B_n}) \times (0,+\infty))$ are bounded and satisfy the following condition:*

$$\begin{cases} Lv \le Lu, & (x,t) \in (\mathbb{R}^N \backslash \overline{B_n}) \times (0,+\infty) \\ v(x,t) \le u(x,t), & (x,t) \in \partial B_n \times [0,+\infty) \\ v(x,0) \le u(x,0), & x \in \mathbb{R}^N \backslash \overline{B_n}, \end{cases}$$

where $B_n := B_n(0)$ be the open ball with center 0 and radius n. Then $v(x,t) \le u(x,t)$ everywhere in $(\mathbb{R}^N \backslash B_n) \times [0,+\infty)$.

For a given function, it sometimes does not have continuous first order or second order derivatives, therefore we cannot apply Lemma 2.1–Lemma 2.3 for such a function. In order to generalize the comparison principles for these functions, following the idea in [6, 7], we first introduce the following concept of a weak solution.

Definition 2.4. Let Ω be an open bounded set in $\mathbb{R}^N$, $T < +\infty$, $Q_T = \Omega \times (0,T]$. We shall say a function $u(x,t)$ is a *weak subsolution* of equation $u_t - \triangle u + f(u) = 0$ in Q_T if it is continuous in $\overline{Q_T}$ and satisfy the inequality

$$\begin{aligned} I(u,\varphi,\overline{\Omega}\times[0,t_1]) := & \int_\Omega u(x,t_1)\varphi(x,t_1)\mathrm{d}x - \int_\Omega u(x,0)\varphi(x,0)\mathrm{d}x \\ & - \int_0^{t_1}\int_\Omega u\varphi_t \mathrm{d}x\mathrm{d}t - \int_0^{t_1}\int_\Omega u\triangle\varphi \mathrm{d}x\mathrm{d}t \\ & + \int_0^{t_1}\int_{\partial\Omega} u\frac{\partial\varphi}{\partial n}\mathrm{d}S\mathrm{d}t + \int_0^{t_1}\int_\Omega f(u)\varphi\mathrm{d}x\mathrm{d}t \\ \le\; & 0 \end{aligned}$$

for all nonnegative test functions $\varphi \in C^{2,1}_{x,t}(\overline{\Omega}\times[0,t_1])$ such that $\varphi(x,t)|_{\partial\Omega\times[0,t_1]} = 0$ and for all $t_1 \le T$. We define a *weak solution* of equation $u_t - \triangle u + f(u) = 0$ in Q_T as a continuous function $u(x,t)$ which satisfies $I(u,\varphi,\overline{\Omega}\times[0,t_1]) = 0$ for all test functions $\varphi(x,t)$ and for all $t_1 \le T$.

According to Definition 2.4, it is easy to check that a classical solution must be a weak solution. Suppose now $u(x,t)$ is a nonnegative classical solution of equation $u_t - \triangle u + u^q = 0$ in Q_T, and let $u_\varepsilon(x,t)$ $(0 < \varepsilon < 1)$ be the classical solution of following problem:

$$\begin{cases} (u_\varepsilon)_t - \triangle u_\varepsilon + (u_\varepsilon)^q - \varepsilon^q = 0, & (x,t) \in Q_T, \\ u_\varepsilon(x,t) = u(x,t) + \varepsilon, & (x,t) \in \partial_p Q_T. \end{cases} \tag{7}$$

As for $u_\varepsilon(x,t)$, we have the following lemma which admits that $u(x,t)$ can be approximated by positive solution of the approximated problem.

Lemma 2.5. *$u_\varepsilon(x,t)$ satisfies the following properties:*

(i) *$u + \varepsilon \le u_\varepsilon \le M + 1$, where $M = \max_{\overline{Q_T}} u(x,t)$, therefore $u_\varepsilon(x,t)$ has a positive infimum.*

(ii) *For small $\varepsilon > 0$, we have $u_\varepsilon \le u + \varepsilon^q (M+1)^{1-q} q^{-1}$.*

(iii) *$u_\varepsilon(x,t) \to u(x,t)$ uniformly in $\overline{Q_T}$ as $\varepsilon \to 0^+$.*

Proof. (i) Let $L_1 u := u_t - \triangle u + u^q$, then we have

$$\begin{cases} L_1(u+\varepsilon) \le \varepsilon^q = L_1 u_\varepsilon \le L_1(M+1), & (x,t) \in Q_T \\ u + \varepsilon = u_\varepsilon(x,t) \le M+1, & (x,t) \in \partial_p Q_T, \end{cases}$$

where we have used the inequality $(a+b)^p \le a^p + b^p (a, b \ge 0, 0 < p < 1)$. By Lemma 2.1, we then conclude that $u + \varepsilon \le u_\varepsilon \le M + 1$.

(ii) Let $z(x,t) = u_\varepsilon(x,t) - u(x,t)$, then $u_\varepsilon^q - u^q = q\int_0^1 (\theta u_\varepsilon + (1-\theta)u)^{q-1} \mathrm{d}\theta \cdot z(x,t)$. Set $b_\varepsilon(x,t) = q\int_0^1 (\theta u_\varepsilon + (1-\theta)u)^{q-1}\mathrm{d}\theta$. Since $u_\varepsilon \ge \varepsilon$, we see that $b_\varepsilon(x,t)$ is well-defined in $\overline{Q_T}$ and $q(M+1)^{q-1} \le b_\varepsilon(x,t) \le \varepsilon^{q-1}$. Now define $L_2 u := u_t - \triangle u + q(M+1)^{q-1} u$. Then for small $\varepsilon > 0$ we get

$$\begin{cases} L_2 z \le \varepsilon^q = L_2(\varepsilon^q (M+1)^{1-q} q^{-1}), & (x,t) \in Q_T \\ z(x,t) = \varepsilon \le \varepsilon^q (M+1)^{1-q} q^{-1}, & (x,t) \in \partial_p Q_T. \end{cases}$$

Owing to Lemma 2.1, we deduce that $z(x,t) \le \varepsilon^q (M+1)^{1-q} q^{-1}$, then $u_\varepsilon \le u + \varepsilon^q (M+1)^{1-q} q^{-1}$.

(iii) Combining property (1) and (2), we immediately see that $u_\varepsilon(x,t) \to u(x,t)$ uniformly in $\overline{Q_T}$ as $\varepsilon \to 0^+$. □

Now we generalize Lemma 2.1 for weak solution as follows.

Lemma 2.6. *Let Ω, T, Q_T be the same with Lemma 2.1. Assume that $u(x,t)$ is a nonnegative classical solution of $u_t - \triangle u + u^q = 0$ in Q_T and $v(x,t)$ is a nonnegative weak subsolution of the same equation in Q_T. If $v(x,t) \le u(x,t)((x,t) \in \partial_p Q_T)$, then $v(x,t) \le u(x,t)$ everywhere in $\overline{Q_T}$.*

Proof. We argue by contradiction. If there exists $(x_0,t_0) \in Q_T$ such that $v(x_0,t_0) - u(x_0,t_0) > 0$, then by the continuity of u and v, we choose $\delta > 0$ such that $v(x,t_0) - u(x,t_0) > 0$ for all $x \in B_\delta(x_0) \subset \Omega$. Let now $\psi(x)$ be a nonnegative $C_0^\infty(\Omega)$ function with supp $\psi \subset B_\delta(x_0)$, then we have

$$\int_\Omega \psi(x)[v(x,t_0) - u(x,t_0)]\mathrm{d}x > 0.$$

We will show that

$$\int_\Omega \psi(x)[v(x,t_0) - u(x,t_0)]\mathrm{d}x \le 0, \tag{8}$$

and then a contradiction occurs.

Let $\varphi(x,t) \in C_{x,t}^{2,1}(\overline{\Omega} \times [0,t_0])$ be a nonnegative function, $\varphi(x,t)|_{\partial\Omega\times[0,t_0]} = 0$. By Lemma 2.5, $u_\varepsilon(x,t)$ is a classical solution of (7) , and then $u_\varepsilon(x,t)$ is also a weak solution of $(u_\varepsilon)_t - \triangle u_\varepsilon + (u_\varepsilon)^q - \varepsilon^q = 0$ in Q_T. Then from the definition of a weak solution we have

$$I(u_\varepsilon, \varphi, \overline{\Omega} \times [0,t_0]) = 0. \tag{9}$$

On the other hand, $v(x,t)$ is a weak subsolution of (1.1), hence we have

$$I(v, \varphi, \overline{\Omega} \times [0,t_0]) \le 0. \tag{10}$$

(9) and (10) yield

$$\begin{aligned}\int_\Omega [v(x,t_0) - u_\varepsilon(x,t_0)]\varphi(x,t_0)\mathrm{d}x \le &\int_\Omega [v(x,0) - u_\varepsilon(x,0)]\varphi(x,0)\mathrm{d}x \\ &+ \int_0^{t_0}\int_\Omega (v-u_\varepsilon)(\varphi_t + \triangle\varphi - c_\varepsilon(x,t)\varphi)\mathrm{d}x\mathrm{d}t \\ &- \int_0^{t_0}\int_{\partial\Omega} (v-u_\varepsilon)\frac{\partial\varphi}{\partial n}\mathrm{d}S\mathrm{d}t - \varepsilon^q \int_0^{t_0}\int_\Omega \varphi\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{11}$$

where we have used the equality $v^q - u_\varepsilon^q = q(v - u_\varepsilon)\int_0^1 (\theta v + (1-\theta)u_\varepsilon)^{q-1}\mathrm{d}\theta :=$ $c_\varepsilon(x,t)(v - u_\varepsilon)$, with $c_\varepsilon(x,t) = q\int_0^1 (\theta v + (1-\theta)u_\varepsilon)^{q-1}\mathrm{d}\theta$, notice that $c_\varepsilon(x,t)$ is well defined since $u_\varepsilon \ge \varepsilon > 0$. In view of $v(x,0) \le u(x,0) \le u_\varepsilon(x,0)$ and $\varphi \ge 0$, from (11) we get that

$$\begin{aligned}&\int_\Omega [v(x,t_0) - u_\varepsilon(x,t_0)]\varphi(x,t_0)\mathrm{d}x \\ &\le \int_0^{t_0}\int_\Omega (v-u_\varepsilon)(\varphi_t + \triangle\varphi - c_\varepsilon(x,t)\varphi)\mathrm{d}x\mathrm{d}t - \int_0^{t_0}\int_{\partial\Omega} (v-u_\varepsilon)\frac{\partial\varphi}{\partial n}\mathrm{d}S\mathrm{d}t.\end{aligned} \tag{12}$$

Now we choose smooth functions $\{c_{\varepsilon k}(x,t)\}_{k=1}^{\infty}$ which satisfy

$$c_{\varepsilon k}(x,t) \to c_\varepsilon(x,t) \quad \text{in } L^2(Q_T) \text{ as } k \to \infty. \tag{13}$$

Consider the following boundary value problem:

$$\begin{cases} \varphi_t + \triangle\varphi - c_{\varepsilon k}(x,t)\varphi = 0, & (x,t) \in \Omega \times [0,t_0) \\ \varphi(x,t) = 0, & (x,t) \in \partial\Omega \times [0,t_0] \\ \varphi(x,t_0) = \psi(x), & x \in \Omega. \end{cases} \tag{14}$$

(14) is a linear parabolic equation with smooth coefficients, and by the classical theory ([9]), (14) has a unique $C^{2,1}(\overline{\Omega} \times [0,t_0])$ solution $\varphi_{\varepsilon k}(x,t)$. By maximum principles, we see that $0 \le \varphi_{\varepsilon k}(x,t) \le \max|\psi(x)|$ and

$$\frac{\partial \varphi_{\varepsilon k}}{\partial n} \le 0, \quad (x,t) \in \partial\Omega \times [0,t_0]. \tag{15}$$

Inserting $\varphi(x,t) = \varphi_{\varepsilon k}(x,t)$ into (12), using (15) and the condition $v(x,t) \le u(x,t)((x,t) \in \partial_p Q_T)$, we have

$$\begin{aligned} \int_\Omega [v(x,t_0) - u_\varepsilon(x,t_0)]\psi(x)\mathrm{d}x &\le \int_0^{t_0}\int_\Omega (v-u_\varepsilon)(c_{\varepsilon k} - c_\varepsilon)\varphi_{\varepsilon k}\mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^{t_0}\int_{\partial\Omega} (v-u_\varepsilon)\frac{\partial\varphi_{\varepsilon k}}{\partial n}\mathrm{d}S\mathrm{d}t \\ &\le \int_0^{t_0}\int_\Omega (v-u_\varepsilon)(c_{\varepsilon k} - c_\varepsilon)\varphi_{\varepsilon k}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{16}$$

Taking the limit in (16) with respect to $k \to +\infty$, then (13) implies that

$$\int_\Omega [v(x,t_0) - u_\varepsilon(x,t_0)]\psi(x)\mathrm{d}x \le 0.$$

Thanks to Lemma 2.5, let $\varepsilon \to 0^+$ in the above inequality, we finally arrive at (8), and the proof is complete. □

3. Proof of Theorem 1.3

This section gives the proof of Theorem 1.3. The main tool in the proof is comparison principles that are established in Section 2. We prove Theorem 1.3 in three cases.

Proof. `Case 1:` $u_0(x)$ satisfies (4).

In this case, fixed $\varepsilon \in (0,1)$, we construct an auxiliary function

$$v(x,t) = \varepsilon C(1-\lambda t)_+^{\omega}, \tag{17}$$

where $\omega = \frac{1}{1-q}$, $\lambda > 0$ to be fixed later, $a_+ = \max\{a, 0\}$. Note that $\omega > 1$, and by direct calculation we find that

$$v_t - \triangle v + v^q = (1-\lambda t)^{\frac{q}{1-q}} \Big(-\frac{\lambda}{1-q} \cdot \varepsilon C + (\varepsilon C)^q \Big) \leq 0, \quad (x,t) \in \mathbb{R}^N \times \big(0, \tfrac{1}{\lambda}\big], \quad (18)$$

if we choose $\lambda = \lambda(C, \varepsilon, q)$ large enough.

Since $\liminf_{|x|\to\infty} u_0(x) = C > 0$, it follows that there exists $n > 0$ such that $u_0(x) > \varepsilon C$ if $|x| \geq n$. Thus we have

$$v(x, 0) = \varepsilon C < u_0(x), \quad |x| \geq n. \quad (19)$$

On the other hand, by the continuity of v and u, we can find $\tau_1 = \tau_1(n) > 0$ such that

$$v(x,t) \leq u(x,t), \quad |x| = n,\, t \in [0, \tau_1]. \quad (20)$$

By Lemma 2.3, it follows $v(x,t) \leq u(x,t)$, $(x,t) \in (\mathbb{R}^N \backslash B_n) \times [0, \tau]$, from (18)–(20), where we choose $\tau < \min\{\frac{1}{\lambda}, \tau_1\}$. Hence (17) and the above inequality imply that $u(x,t) > 0$, $(x,t) \in (\mathbb{R}^N \backslash B_n) \times [0,\tau]$, thus ISS does not occur for $u(x,t)$.

Instead of continuing to prove Theorem 1.3, we here turn to prove the following proposition.

Proposition 3.1. *Given $x_0 \in \mathbb{R}$, $c_0 > 0$, $\delta > 0$. Let $f(x)$ be a function satisfying*

$$f(x) = \begin{cases} \frac{c_0}{\delta}(x - x_0) + c_0, & x \in [x_0 - \delta, x_0) \\ -\frac{c_0}{\delta}(x - x_0) + c_0, & x \in [x_0, x_0 + \delta] \\ 0, & else, \end{cases}$$

and $u(x,t)$ be the solution of equation (1) ($N = 1$) *with initial data $u_0(x) = f(x)$. Then there exists $\tau_0 = \tau_0(c_0) > 0$ such that*

$$u(x_0, t) > 0, \quad \textit{for all } t \in [0, \tau_0).$$

Remark 3.2. The graph of $f(x)$ is a "tent" of height $c_0 = f(x_0)$ and width 2δ, lying above the interval $[x_0 - \delta, x_0 + \delta]$. The most important point in this proposition is that τ_0 only depends on the height of the "tent" function $f(x)$, and does not depend on the width, thus τ_0 does not depend on the slope of the "tent" function.

Proof. Fix $\varepsilon \in (0,1)$. Let us consider the function

$$v(x,t) = \varepsilon f(x) \left(1 - \frac{t}{(\varepsilon f(x))^{1-q}} \right)_+^{\omega}, \quad (21)$$

where $\omega = \frac{1}{1-q}$, $a_+ = \max\{a, 0\}$. Notice that $v(x,t)$ does not always have continuous second order partial derivatives. For simplicity, let $A(x,t) = 1 - \frac{t}{(\varepsilon f(x))^{1-q}}$ and

$$\begin{aligned} E^+ &= \{(x,t) \in (x_0-\delta, x_0+\delta) \times (0,+\infty);\ A(x,t) > 0\} \\ E^0 &= \{(x,t) \in (x_0-\delta, x_0+\delta) \times (0,+\infty);\ A(x,t) = 0\} \\ E^- &= \{(x,t) \in (x_0-\delta, x_0+\delta) \times (0,+\infty);\ A(x,t) < 0\}. \end{aligned}$$

Then $v = 0$ in the set E^-, thus

$$v_t - v_{xx} + v^q = 0, \quad (x,t) \in E^-. \tag{22}$$

In the set $E^+ \backslash (\{x_0\} \times (0,+\infty))$, we compute

$$\begin{aligned} -v_{xx} = &-\varepsilon f'' A^\omega - \big[\omega(1-q) - \omega(1-q)^2\big]\varepsilon^q t A^{\omega-1} f^{q-2} (f')^2 \\ &- \varepsilon^{2q-1}\omega(\omega-1)(1-q)^2 t^2 A^{\omega-2} f^{2q-3}(f')^2 - \varepsilon^q \omega(1-q) t A^{\omega-1} f^{q-1} f''. \end{aligned}$$

Since $f'' = 0$ a.e., $\omega = \frac{1}{1-q}$, omitting the nonpositive terms in the above equality gives that

$$-v_{xx} \le 0 \quad \text{a.e. in } E^+. \tag{23}$$

It concludes from (21) and (23) that

$$\begin{aligned} v_t - v_{xx} + v^q &\le -\varepsilon^q \omega f^q A^{\omega-1} + \varepsilon^q f^q A^{\omega q} \\ &= \varepsilon^q f^q A^{\omega q}\Big(-\frac{1}{1-q} + 1\Big) < 0 \quad \text{a.e. in } E^+. \end{aligned} \tag{24}$$

Note that (22) and (24) imply that $v(x,t)$ is a weak subsolution of equation $u_t - \triangle u + u^q = 0$ in $(x_0-\delta, x_0+\delta) \times (0,+\infty)$, and recall that

$$v(x,0) = \varepsilon f(x) \le u(x,0), \quad x \in\ (x_0-\delta, x_0+\delta)$$

$$0 = v(x_0 \pm \delta, t) \le u(x_0 \pm \delta, t), \quad t \ge 0.$$

Using these facts and Lemma 2.6, we have

$$v(x,t) \le u(x,t), \quad (x,t) \in [x_0-\delta, x_0+\delta] \times [0,+\infty).$$

By the definition of $v(x,t)$, we choose $\tau_0 = (\varepsilon c_0)^{1-q}$, then $u(x_0,t) > 0$ for all $t \in [0, \tau_0)$. $\square$

We now continue to prove Theorem 1.3.

`Case 2:` $u_0(x)$ satisfies (5), and the dimension $N = 1$.

Since $\limsup_{x\to\infty} u_0(x) = C > 0$, then we can choose a sequence $\{x_k\}_{k=-\infty}^{+\infty}$ such that $\lim_{k\to\pm\infty} x_k = \pm\infty$ and $u_0(x_k) > \frac{C}{2}$ $(k = 0, \pm1, \pm2, \ldots)$. In view of the continuity of $u_0(x)$, we know that there exists $\{\delta_k\}_{k=-\infty}^{+\infty}$ with $\delta_k > 0$ $(k = 0, \pm1, \pm2, \ldots)$ and the intervals $(x_k - \delta_k, x_k + \delta_k)$ $(k = 0, \pm1, \pm2, \ldots)$ are disjoint and $u_0(x) \geq \frac{C}{2}$ for all $x \in (x_k - \delta_k, x_k + \delta_k)$ $(k = 0, \pm1, \pm2, \ldots)$.

Given k, we now define

$$f_k(x) = \begin{cases} \frac{C}{2\delta_k}(x - x_k) + \frac{C}{2}, & x \in [x_k - \delta_k, x_k) \\ -\frac{C}{2\delta_k}(x - x_k) + \frac{C}{2}, & x \in [x_k, x_k + \delta_k] \\ 0, & \text{else.} \end{cases}$$

From the definition of $f_k(x)$, we easily see that

$$u_0(x) \geq f_k(x), \quad x \in \mathbb{R}. \tag{25}$$

Then (25) and Lemma 2.2 together imply that

$$u(x,t) \geq v_k(x,t), \quad (x,t) \in \mathbb{R} \times [0, +\infty),$$

where $v_k(x,t)$ is the solution of (1) with initial data f_k. By the previous proposition and above inequality, we know

$$u(x_k, t) > 0, \quad t \in \left[0, \left(\varepsilon \frac{C}{2}\right)^{1-q}\right), \; k = 0, \pm1, \ldots,$$

which implies that ISS does not occur for $u(x,t)$ in Case 2.

`Case 3:` $u_0(x)$ satisfies (5), and the dimension $N > 1$.

Since $\limsup_{|x|\to\infty} u_0(x) = C > 0$ and $u_0(x)$ is continuous in $\mathbb{R}^N$, there exists $\{x_k\}_{k=1}^{\infty} = \{(x_k^1, x_k^2, \ldots, x_k^N)\}_{k=1}^{\infty}$ and $\{\delta_k\}_{k=1}^{\infty} = \{(\delta_k^1, \delta_k^2, \ldots, \delta_k^N)\}_{k=1}^{\infty}$ satisfying $\lim_{k\to\infty} |x_k| = \infty$, $\delta_k^i > 0$ $(k = 1, 2, \ldots, \infty,\ i = 1, 2, \ldots, N)$, $u_0(x) \geq \frac{C}{2}$ for all $x \in B_{|\delta_k|}(x_k)$, and the balls $B_{|\delta_k|}(x_k)$ $(k = 1, 2, \ldots, \infty)$ are disjoint. Then for each k and i, we define

$$f_k^i(s) = \begin{cases} \left(\frac{C}{2}\right)^{\frac{1}{N}} \frac{1}{\delta_k^i}(s - x_k^i) + \left(\frac{C}{2}\right)^{\frac{1}{N}}, & s \in [x_k^i - \delta_k^i, x_k^i) \\ -\left(\frac{C}{2}\right)^{\frac{1}{N}} \frac{1}{\delta_k^i}(s - x_k^i) + \left(\frac{C}{2}\right)^{\frac{1}{N}}, & s \in [x_k^i, x_k^i + \delta_k^i] \\ 0, & \text{else.} \end{cases}$$

We now set $f_k(x) = f_k(x^1, x^2, \ldots, x^N) = f_k^1(x^1) f_k^2(x^2) \cdots f_k^N(x^N)$. From the definition we can check that $u_0(x) \geq f_k(x),\ x \in \mathbb{R}^N$.

Fix $\varepsilon \in (0,1)$, let $\omega = \frac{1}{1-q}$, consider the function

$$v_k(x,t) = \varepsilon f_k(x) \left(1 - \frac{t}{(\varepsilon f_k(x))^{1-q}}\right)_+^{\omega}. \tag{26}$$

Proceeding similarly as in the proof of Proposition 3.1, we deduce that

$$u(x,t) \geq v_k(x,t),\ (x,t) \in [x_k^1 - \delta_k^1, x_k^1 + \delta_k^1] \times \cdots \times [x_k^N - \delta_k^N, x_k^N + \delta_k^N] \times [0, +\infty),$$

thus this inequality and (26) imply that there exists $\tau = \left(\varepsilon \frac{C}{2}\right)^{1-q}$ such that

$$u(x_k, t) > 0, \quad t \in [0, \tau),\ k = 1, 2, \ldots, \infty. \tag{27}$$

Then (27) demonstrates that ISS does not occur for $u(x, t)$ in Case 3.

Combining Case 1–Case 3, we complete the proof of Theorem 1.3. □

4. An example

This section gives an example in the case $N = 1$, $u_0(x) = |\sin x|$. By Theorem 1.3, we know that ISS does not occur for the solution of (1) and (2) with this $u_0(x)$, furthermore, we have the following proposition.

Proposition 4.1. *Assume that $N = 1$, $u(x, t)$ be the solution of* (1) *with $u(x, 0) = u_0(x) = |\sin x|$. Then there exists $\varepsilon \in (0, 1)$ and $\tau > 0$ such that*

$$u(x, t) \geq \varepsilon |\sin x| \left(1 - \frac{t}{(\varepsilon |\sin x|)^{1-q}}\right)_+^{\omega}, \quad (x, t) \in \mathbb{R} \times [0, \tau],$$

where $\omega = \frac{1}{1-q}$. Particularly, one has $u(k\pi + \frac{\pi}{2}, t) > 0$ for all $t \in (0, \tau]$, $k = 0, \pm 1, \pm 2, \ldots$.

Proof. Consider in $(0, \pi) \times (0, +\infty)$ the function

$$v(x, t) = \varepsilon \sin x \left(1 - \frac{t}{(\varepsilon \sin x)^{1-q}}\right)_+^{\omega}, \tag{28}$$

where $\omega = \frac{1}{1-q}$, $\varepsilon \in (0, 1)$ to be fixed. We simply set $A(x, t) = 1 - \frac{t}{(\varepsilon \sin x)^{1-q}}$ and

$$\begin{aligned} E^+ &= \{(x, t) \in (0, \pi) \times (0, +\infty);\ A(x, t) > 0\} \\ E^0 &= \{(x, t) \in (0, \pi) \times (0, +\infty);\ A(x, t) = 0\} \\ E^- &= \{(x, t) \in (0, \pi) \times (0, +\infty);\ A(x, t) < 0\}. \end{aligned}$$

Hence we have

$$v_t - v_{xx} + v^q = 0, \quad (x, t) \in E^-. \tag{29}$$

In the set E^+, a direct calculation gives

$$v_t = -\omega \varepsilon^q \sin^q x A^{\omega - 1} \tag{30}$$

and

$$\begin{aligned} -v_{xx} &= \varepsilon \sin x A^{\omega} - [\omega(1-q) - \omega(1-q)^2] \varepsilon^q t A^{\omega-1} \sin^{q-2} x \cos^2 x \\ &\quad - \varepsilon^{2q-1} \omega(\omega - 1)(1-q)^2 t^2 A^{\omega-2} \sin^{2q-3} x \cos^2 x \\ &\quad + \varepsilon^q \omega (1-q) t A^{\omega-1} \sin^q x \\ &\leq \varepsilon \sin x A^{\omega} + \varepsilon^q \omega (1-q) t A^{\omega-1} \sin^q x, \end{aligned} \tag{31}$$

where we have dropped the nonpositive terms in the last inequality.

Thus (28), (30) and (31) imply that

$$\begin{aligned} v_t - v_{xx} + v^q &\le \varepsilon^q \sin^q x A^{\omega q}\big[-\omega + \varepsilon^{1-q}\sin^{1-q} xA + t + 1\big] \\ &\le \varepsilon^q \sin^q x A^{\omega q}\Big[-\frac{q}{1-q} + \varepsilon^{1-q} + t\Big], \end{aligned} \tag{32}$$

where we have used the inequality $0 < A(x,t) < 1((x,t) \in E^+)$. Now we first choose $\varepsilon = \varepsilon(q) \in (0,1)$ such that $-\frac{q}{1-q} + \varepsilon^{1-q} < 0$, then choose $\tau = \tau(q) > 0$ satisfying $\tau(q) < \varepsilon^{1-q}$ and $-\frac{q}{1-q} + \varepsilon^{1-q} + \tau \le 0$. It follows from (32) that

$$v_t - v_{xx} + v^q \le 0, \quad (x,t) \in E^+ \cap ((0,\pi)\times(0,\tau]). \tag{33}$$

Owing to (29) and (33), we see that $v(x,t)$ is a weak subsolution of (1) in $(0,\pi)\times(0,\tau]$. On the other hand, since $u(x,t)$ is the nonnegative solution of (1) with $u_0(x) = |\sin x|$, we have

$$\begin{aligned} v(x,0) &= \varepsilon \sin x < \sin x = u(x,0), && 0 < x < \pi \\ v(0,t) &= 0 \le u(0,t), && 0 \le t \le \tau \\ v(\pi,t) &= 0 \le u(\pi,t), && 0 \le t \le \tau. \end{aligned}$$

Utilizing these facts and Lemma 2.6, we deduce that $v(x,t) \le u(x,t)$, $(x,t) \in [0,\pi]\times[0,\tau]$. Using the same method, we conclude that

$$v(x,t) \le u(x,t), \quad (x,t) \in [2k\pi,(2k+1)\pi]\times[0,\tau],\ k = 0,\pm1,\pm2,\ldots. \tag{34}$$

If we consider function

$$\widetilde{v}(x,t) = \varepsilon(-\sin x)\left(1 - \frac{t}{(-\varepsilon\sin x)^{1-q}}\right)_+^{\omega},$$

where ε and ω are the same as in the function $v(x,t)$. Then a similar discussion yields that

$$\widetilde{v}(x,t) \le u(x,t), \quad (x,t) \in [(2k+1)\pi,(2k+2)\pi]\times[0,\tau],\ k = 0,\pm1,\pm2,\ldots. \tag{35}$$

Combining now (34) and (35), we find that

$$u(x,t) \ge \varepsilon|\sin x|\left(1 - \frac{t}{(\varepsilon|\sin x|)^{1-q}}\right)_+^{\omega}, \quad (x,t) \in \mathbb{R}\times[0,\tau],$$

hence the proposition is proved. □

Remark 4.2. By the proposition, we know that for all $\delta \in [0,\frac{\pi}{2})$, there exists $\tau = \tau(\delta,\varepsilon) > 0$ such that $u(k\pi + \frac{\pi}{2} - \delta, t) > 0$, for all $t \in (0,\tau]$ $(k = 0,\pm1,\pm2,\ldots)$.

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