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A Quasilinear Eigenvalue Problem with Robin Conditions on the Non-Smooth Domain of Finite Measure

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Abstract. In this paper, we consider a nonlinear eigenvalue problem involving the p-Laplacian with Robin boundary conditions on a domain of finite measure. We show the existence, simplicity and isolation of principal eigenvalue and regularity results for the corresponding eigenfunction. Furthermore we establish the link between the Dirichlet and Neumann problems by means of the Robin boundary conditions with variable parameter.

Keywords. Nonlinear eigenvalue problem, p-Laplacian, Robin boundary conditions, Non-smooth domains
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1. Introduction

The aim of this paper is to study the existence and main properties of the principal eigenvalue and corresponding eigenfunction of the following nonlinear boundary value problem:

$$\begin{cases} -\Delta_p u = \lambda\, |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + |u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here Ω is a domain in $\mathbb{R}^N (N > 1)$ of finite measure, Δ_p denotes the p-Laplacian operator defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p \in (1, \infty)$, ν is the outward pointing unit normal to the boundary $\partial\Omega$, $\lambda \in \mathbb{R}$ is the eigenvalue parameter. As we want to deal with domains having non smooth boundary, the condition on $\partial\Omega$

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in (1) has to be interpreted in a weak sense that we shall specify later, see Definition 3.1.

The problem (1) appears in mathematical models for subjects such as glaciology, nonlinear diffusion, filtration problem [32], power-low materials [20], non-Newtonian fluids [8], reaction-diffusion problems, flow through porous media, nonlinear elasticity, petroleum extraction, torsional creep problems, etc. For a discussion and some physical background, we refer the reader to [16]. The nonlinear boundary condition describes the flux through the boundary $\partial\Omega$ which depends on the solution itself. For a physical motivation of such conditions, see for example [31].

On the other hand, the properties of the principal eigenvalue are of prime interest since, for example, they are closely associated with the dynamics of the corresponding evolution equations (e.g., global bifurcation, stability) or with the description of the solution set of corresponding perturbed problems, see, e.g., [33]. These properties are: existence, positivity, simplicity, uniqueness up to eigenfunctions which do not change sign and isolation, which hold, e.g., in the case of the p-Laplacian operator in a bounded domain with smooth boundary, see [2, 3, 6, 11].

Eigenvalue problems involving the Laplacian or the p-Laplacian on bounded domains have been the topic of many other studies. We cite the works [10, 12, 13, 17–19, 21–23]. The Dirichlet and/or the Neumann eigenvalue problem involving the p-Laplacian

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{and/or} \quad |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

has attracted considerable attention. Recently in [26] the author studied nonlinear eigenvalue problems for the p-Laplacian operator subject to different types of boundary conditions on a bounded domain.

The "smoothness" of the boundary $\partial\Omega$ is an important assumption in papers mentioned above. In this paper we want to extend these results to more general domains which can occur in applications. In particular, we want to emphasize that our results cover a wide class of domains with non Lipschitz boundary and which are allowed to be unbounded in $\mathbb{R}^N$.

To this end we have to choose different functional framework than usual. Namely, we seek weak solutions in a suitable subspace V_p of $W_p^1(\Omega)$ which reflects the influence of boundary conditions. This allows us to deal with more general domains by using an inequality due to Maz'ja [28], see Section 2 for the details. On the other hand, if Ω is bounded and has a Lipschitz boundary, then the standard space $V_p = W_p^1(\Omega)$ is a suitable functional framework.

We also consider parameter dependent boundary conditions,

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \mu |u|^{p-2}u = 0 \quad \text{on } \partial\Omega,$$

with $\mu \in [0, +\infty)$ and establish the link between the Dirichlet and the Neumann boundary conditions letting μ to approach 0 and ∞, respectively.

The paper is organized as follows. In Section 2, we establish necessary preliminaries and introduce the functional framework. In Section 3, we use a variational method to show the existence and simplicity of principal eigenvalue λ_1 of (1). We also show the regularity of corresponding principal eigenfunction. In Section 4, we show the isolation of principal eigenvalue. Finally, in Section 5, we study the dependence of $\lambda_1 = \lambda_1(\mu)$ on the parameter $\mu \geq 0$ and link the Dirichlet and the Neumann problem (2).

2. Preliminaries

Let Ω be a domain of finite measure $|\Omega|$. Let $W_p^1(\Omega)$ denote the usual Sobolev space and for $u \in W_p^1(\Omega)$ let us put $\|u\|_{1,p} = (\int_\Omega |\nabla u|^p dx)^{1/p}$. We will write $\|\cdot\|_p$ for the L_p-norm. We then set

$$V_p = W^1_{p,p}(\Omega, \partial\Omega)$$

for a Banach space defined as a completion of $W_p^1(\Omega) \cap C(\overline{\Omega})$ with respect to the norm

$$\|u\|_{V_p} = \left(\|u\|_{1,p}^p + \|u|_{\partial\Omega}\|^p_{L_p(\partial\Omega)} \right)^{\frac{1}{p}},$$

where

$$\|u|_{\partial\Omega}\|_{L_p(\partial\Omega)} = \left(\int_{\partial\Omega} |u|^p d\mathcal{H}_{N-1} \right)^{\frac{1}{p}}.$$

Note that the $(N-1)$-dimensional Hausdorff measure coincides with the usual surface Lebesque measure if $\partial\Omega$ is Lipschitz. These spaces have been introduced by Maz'ja (see [28, Section 3.6]).

Remark 2.1. Since the L_p-norms are uniformly convex, also the V_p-norm is uniformly convex. In particular by Milman's Theorem, V_p is a reflexive space (see [36, Section 5.2]). For $v \in V_p$ let $v^+ = \frac{|v|+v}{2}$ and $v^- = \frac{|v|-v}{2}$ be the positive and negative part of v, respectively. Then $|v|, v^+, v^- \in V_p$ follows directly from the definition of V_p.

Further discussion focuses on the fact that for rather general domains Ω the space V_p is more suitable functional framework than $W_p^1(\Omega)$ and that, in general, we can have $V_p \neq W_p^1(\Omega)$. Indeed, Maz'ja's result ([28, Corollary Section 3.6.3, p.189]) says that there exists a constant $c > 0$ (merely depending on the dimension and not on the set Ω) such that

$$\|u\|_{\frac{N}{N-1}} \leq c\big(\|u\|_{1,1} + \|u|_{\partial\Omega}\|_{L_1(\partial\Omega)}\big)$$

for all $u \in W_1^1(\Omega) \cap C(\overline{\Omega})$. Following the calculations from [15, Sec. 4] (replacing above u by $|u|^p$, applying Hölder's and Young's inequalities with usual notation $p' = \frac{p}{p-1}$) we subsequently get

$$\begin{aligned}
\|u\|^p_{\frac{Np}{N-1}} &= \||u|^p\|_{\frac{N}{N-1}} \\
&\leq c\Big(\||u|^p\|_{1,1} + \||u|^p|_{\partial\Omega}\|_{L_1(\partial\Omega)}\Big) \\
&= c\Big(p\,\||u|^{p-1}\nabla u\|_1 + \|u|_{\partial\Omega}\|^p_{L_p(\partial\Omega)}\Big) \\
&\leq c\Big(p\,|\Omega|^{\frac{1}{Np'}}\,\|u\|^{p-1}_{\frac{Np}{N-1}}\|u\|_{1,p} + \|u|_{\partial\Omega}\|^p_{L_p(\partial\Omega)}\Big) \\
&\leq c\Bigg(p\,|\Omega|^{\frac{1}{Np'}}\,\frac{\varepsilon^{p'}}{p'}\,\|u\|^p_{\frac{Np}{N-1}} + |\Omega|^{\frac{1}{Np'}}\,\frac{1}{\varepsilon^p}\,\|u\|^p_{1,p} + \|u|_{\partial\Omega}\|^p_{L_p(\partial\Omega)}\Bigg)
\end{aligned}$$

and hence, taking ε small enough, there exists $C > 0$ such that

$$\|u\|_{\frac{Np}{N-1}} \leq C\|u\|_{V_p} \tag{3}$$

for all $u \in V_p$. In particular, we have continuous embedding $V_p \hookrightarrow L_p(\Omega)$ for $q \leq \frac{Np}{N-1}$, since the volume of Ω is finite.

Note that for a domain with an outward pointing exponential cusp the V_p-norm is stronger than the W_p^1-norm and thus by the open mapping theorem the space V_p is a proper subspace of $W_p^1(\Omega)$. That the norm is strictly stronger follows from [1, Theorem 5.32], , asserting that for a domain with a sufficiently sharp outward pointing cusp $W_p^1(\Omega) \not\subseteq L_q(\Omega)$ for all $q > p$, contradicting (3) if we assume that $V_p = W_p^1(\Omega)$. Similar situation occurs when Ω is an unbounded domain with finite volume (see [1, Theorem 5.30]).

Let us consider the embedding $V_p \hookrightarrow L_p(\Omega)$. There are domains for which this embedding is not injective (see [7, Example 4.3, pp.357 and 358] for an example illustrating this phenomenon in the case $p = 2$). In other words, due to the influence of the boundary $\partial\Omega$ there exists a function $w \in V_p$ such that $w \neq o_{V_p}$ but $w = o_{L_p(\Omega)}$. Here, o_{V_p} and $o_{L_p(\Omega)}$ denote the zero elements in V_p and $L_p(\Omega)$, respectively. Notice that this cannot happen if the trace of a function from V_p is locally defined in a usual sense up to a set of $(N-1)$-dimensional Hausdorff measure zero. The domains for which the embedding $V_p \hookrightarrow L_p(\Omega)$ is injective are usually called *admissible*, cf. [15]. In particular, it follows from above discussion that any domain with Lipschitz boundary is admissible.

In order to apply the results from [15] and get L_∞ estimates for principal eigenfunction as well as to perform some of our proofs, we require our domain to be admissible. However, it follows from [7, Section 4] that this is not essential restriction on the domain Ω (cf. also [9] and [14, Section 3]). An example of a bounded domain which is not admissible is constructed in [7, pp.357 and 358]. One can see that the domains of this kind are rather special. On the other hand, most of the domains which appear in applications do not possess such complicated structure. In particular, due to our approach we can go "far beyond" the class of Lipschitz domains and to extend some known results for much wider class of boundary value problems arising in the real world applications.

To be more specific, our approach covers for instance the domains with cusps. We can consider bounded planar domain

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : 0 < x < 1,\, 0 < y < e^{\frac{x}{x-1}}\right\}.$$

The boundary $\partial\Omega$ is smooth between the points $(0,0), (0,1)$ and $(1,0)$, it has Lipschitz edges at $(0,0)$ and $(0,1)$, but there is an exponential cusp at $(1,0)$ (cf. [1, Theorem 5.32]). In particular, the last fact implies that $V_p \neq W_p^1(\Omega)$. On the other hand, since the trace of a function from V_p is locally well-defined up to a set of 1-Hausdorff measure zero (i.e., up to the point (1,0)), the set Ω is admissible (cf. [7, Section 4]).

Our results cover also unbounded domains of finite measure like

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : -\infty < x < +\infty,\, 0 < y < e^{-x^2}\right\}.$$

Finally, we recall the assertions of Maz'ja (see [28, Section 4.11, Corollaries 2 and 3]) from which it follows that the embedding $V_p \hookrightarrow\hookrightarrow L_q(\Omega)$ is compact for $q < \frac{Np}{N-1}$. In particular, we have

$$V_p \hookrightarrow\hookrightarrow L_p(\Omega). \tag{4}$$

3. Existence and simplicity of principal eigenvalue

Since we deal with the domains having non smooth boundaries, the expressions in (1) do not make sense in general. To make our exposition precise, we give the definition of a weak solution of (1).

Definition 3.1. We say that $(u,\lambda) \in V_p \times \mathbb{R}$ is a *weak solution* to (1) if for all $\phi \in V_p$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi\, dx + \int_{\partial\Omega} |u|^{p-2}u\phi\, d\mathcal{H}_{N-1} = \lambda \int_\Omega |u|^{p-2}u\phi\, dx.$$

In this case, such a pair (u,λ), with u nontrivial, is called an *eigenpair*, λ is an *eigenvalue* and u is called an associated *eigenfunction*.

Let us formulate problem (1) variationally. For that purpose we introduce the C^1 functionals $\mathcal{I}$ and $\mathcal{J} : V_p \to \mathbb{R}$ defined by

$$\mathcal{I}(u) = \int_\Omega |\nabla u|^p \, dx + \int_{\partial\Omega} |u|^p d\mathcal{H}_{N-1} \quad \text{and} \quad \mathcal{J}(u) = \int_\Omega |u|^p dx.$$

In particular, we have $\mathcal{I}(u) = \|u\|_{V_p}^p$. It follows from previous definitions that a real value λ is an eigenvalue of problem (1) if and only if there exists $u \in V_p \setminus \{0\}$ such that $\mathcal{I}'(u) = \lambda \mathcal{J}'(u)$. Here $\mathcal{I}'(u)$ and $\mathcal{J}'(u)$ denote the Fréchet derivatives of $\mathcal{I}$ and $\mathcal{J}$ at u, respectively. At this point let us introduce the set

$$\mathcal{M} = \big\{u \in V_p \, : \, \mathcal{J}(u) = 1\big\}.$$

Obviously, we have $\mathcal{M} \neq \emptyset$. Moreover the set $\mathcal{M}$ is a C^1 manifold in V_p.

Theorem 3.2. . *There exists the principal* (*i.e., the least*) *eigenvalue* λ_1 *of* (1). *Moreover,* $\lambda_1 > 0$ *and any principal eigenfunction* (*i.e., any eigenfunction corresponding to* λ_1) *belongs to* $L_\infty(\Omega) \cap C^{1,\delta}(\Omega)$ *for some* $\delta \in (0,1)$ *and it is of definite sign in* Ω.

Proof of Theorem 3.2. We proceed in standard way. We use the compact embedding (4) and show that $\mathcal{I}$ achieves its infimum on $\mathcal{M}$,

$$\lambda_1 = \inf\{\mathcal{I}(u) : u \in \mathcal{M}\}.$$

Let (u_n) be a minimizing sequence for λ_1, i.e.,

$$\mathcal{J}(u_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{I}(u_n) = \lambda_1.$$

Obviously (u_n) is bounded in V_p. By the reflexivity of V_p it has a weakly convergent subsequence, so by renumbering it we can assume that there exists $\varphi_1 \in V_p$ such that, $u_n \rightharpoonup \varphi_1$ in V_p. The compact embedding $V_p \hookrightarrow\hookrightarrow L_p(\Omega)$, see (4), implies $u_n \to \varphi_1$ in $L_p(\Omega)$, i.e., $\mathcal{J}(\varphi_1) = 1$. In particular, $\varphi_1 \not\equiv 0$. The weak lower semicontinuity of the norm in V_p yields $\lambda_1 \le \mathcal{I}(\varphi_1) = \int_\Omega |\nabla\varphi_1|^p \, dx + \int_{\partial\Omega} |\varphi_1|^p d\mathcal{H}_{N-1} = \|\varphi_1\|_{V_p}^p \le \liminf_{n\to\infty} \|u_n\|_{V_p}^p = \liminf_{n\to\infty} \mathcal{I}(u_n) = \lambda_1$, i.e.,

$$\lambda_1 = \mathcal{I}(\varphi_1) = \int_\Omega |\nabla\varphi_1|^p \, dx + \int_{\partial\Omega} |\varphi_1|^p d\mathcal{H}_{N-1}. \tag{5}$$

It follows from (5) and $\mathcal{J}(\varphi_1) = 1$ that $\lambda_1 > 0$ and by the Lagrange multiplier method we get that λ_1 is the least (principal) eigenvalue of (1) with corresponding principal eigenfunction φ_1. Moreover, if u is an eigenfunction corresponding to λ_1 then $|u|$ is also an eigenfunction corresponding to λ_1. It follows from Daners and Drábek [15, Theorem 2.7], that $|u| \in L_\infty(\Omega)$. Regularity result of Tolksdorf [34] implies that $|u| \in C^{1,\delta}(\Omega)$ for some $\delta \in (0,1)$. If u changes sign, there is a point in Ω at which $|u|$ vanishes. But then it violates the Harnack inequality of Trudinger [35]. This proves Theorem 3.2. $\square$

Remark 3.3. Let $u \in V_p$ be a principal eigenfunction satisfying $u > 0$ in Ω. Then $u \geq 0$ a.e. on $\partial\Omega$ (in the sense of the Hausdorff measure). Indeed, it follows from the Definition 3.1 with $\phi = -u^-$ and $\lambda = \lambda_1$ that $\int_{\partial\Omega} |u^-|^p d\mathcal{H}_{N-1} = 0$, i.e., $u \geq 0$ a.e. on $\partial\Omega$.

By convexity argument, as shown in [4], we show that the eigenfunctions corresponding to λ_1 are unique (up to a multiplicative constant). Our proof is based on the observation, made in [24, Proposition 4], that for nonnegative functions u, the functional $\mathcal{I}(u)$ is convex in u^p. The proof is included here for completeness even if it is known in the literature.

Theorem 3.4. . *The principal eigenvalue λ_1 is simple.*

Proof of Theorem 3.4. In the light of Theorem 3.2 it is sufficient to prove that positive eigenfunctions $u, v \in \mathcal{M}$ associated with λ_1 coincide in Ω. Indeed, let $w = \zeta^{\frac{1}{p}}$ with $\zeta = \frac{u^p+v^p}{2}$. Then $w \in V_p$ and we have $\int_\Omega w^p dx = \frac{1}{2}\big(\int_\Omega u^p dx + \int_\Omega v^p dx\big) = 1$. Hence, $w \in \mathcal{M}$. Let $\theta(x) = \frac{u^p}{u^p+v^p} \in (0,1), x \in \Omega$. Now we calculate

$$\nabla w = \zeta^{-1+\frac{1}{p}} \left(\frac{u^{p-1}\nabla u + v^{p-1}\nabla v}{2} \right),$$

so that, by the convexity of the map $s \mapsto |s|^p$, we have

$$\begin{aligned}
|\nabla w|^p &= \zeta^{1-p} \left| \frac{1}{2}\big(u^{p-1}\,\nabla u + v^{p-1}\,\nabla v\big) \right|^p \\
&= \zeta \left| \frac{1}{2} \left(\frac{u^p}{\zeta} \cdot \frac{\nabla u}{u} + \frac{v^p}{\zeta} \cdot \frac{\nabla v}{v} \right) \right|^p \\
&= \zeta \left| \theta(x)\, \frac{\nabla u}{u} + (1-\theta(x))\, \frac{\nabla v}{v} \right|^p \\
&\leq \zeta \left(\theta(x) \left| \frac{\nabla u}{u} \right|^p + (1-\theta(x)) \left| \frac{\nabla v}{v} \right|^p \right) \\
&= \frac{1}{2} \left(u^p \left| \frac{\nabla u}{u} \right|^p + v^p \left| \frac{\nabla v}{v} \right|^p \right) \\
&= \frac{1}{2}\big(|\nabla u|^p + |\nabla v|^p\big).
\end{aligned} \tag{6}$$

We note that equality occurs in (6) if and only if $\frac{\nabla u}{u} = \frac{\nabla v}{v}$ in Ω. Thus,

$$\int_\Omega |\nabla w|^p dx \leq \frac{1}{2}\bigg(\int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla v|^p dx \bigg).$$

Also we have $\int_{\partial\Omega} w^p d\mathcal{H}_{N-1} = \frac{1}{2}\big(\int_{\partial\Omega} u^p d\mathcal{H}_{N-1} + \int_{\partial\Omega} v^p d\mathcal{H}_{N-1}\big)$. Hence

$$\mathcal{I}(w) \leq \frac{1}{2}\big(\mathcal{I}(u) + \mathcal{I}(v)\big). \tag{7}$$

On the other hand, u and v are both minimizers of $\mathcal{I}$ on $\mathcal{M}$, then we have

$$\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} u^p d\mathcal{H}_{N-1} \le \int_\Omega |\nabla w|^p dx + \int_{\partial\Omega} w^p d\mathcal{H}_{N-1}$$

and

$$\int_\Omega |\nabla v|^p dx + \int_{\partial\Omega} v^p d\mathcal{H}_{N-1} \le \int_\Omega |\nabla w|^p dx + \int_{\partial\Omega} w^p d\mathcal{H}_{N-1}.$$

Hence

$$\mathcal{I}(u) + \mathcal{I}(v) \le 2\mathcal{I}(w). \tag{8}$$

Due to (7) and (8) we have that actually equality holds in (7), and we conclude

$$\int_\Omega |\nabla w|^p dx = \frac{1}{2}\Big(\int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla v|^p dx \Big).$$

With respect to (6) we then have $\frac{\nabla u}{u} = \frac{\nabla v}{v}$ in Ω. But this implies that $\nabla(\frac{u}{v}) = 0$ in Ω, so that $u = cv$ for some $c \in \mathbb{R}$. The facts $\mathcal{J}(u) = \mathcal{J}(v) = 1$ and u, v positive in Ω imply $c = 1$. This completes the proof. $\square$

Remark 3.5. It follows from above that the set of all eigenfunctions associated with the principal eigenvalue λ_1 is one-dimensional vector space spanned by a single function $\varphi_1 \in \mathcal{M}$ satisfying $\varphi_1 > 0$ in Ω and $\varphi_1 \ge 0$ on $\partial\Omega$.

4. Isolation of the principal eigenvalue

The proofs in this section are standard for the Dirichlet problem. Since we have to take care also about the boundary values, we include them here for completeness.

Theorem 4.1. *The principal eigenvalue λ_1 is isolated, that is, there exists $\eta > 0$ such that in the interval $(\lambda_1, \lambda_1 + \eta)$ there are no other eigenvalues of* (1).

For the proof we need the following lemmas. At first, we recall the "Picone identity" proved in [5, p.820, Theorem 1.1].

Lemma 4.2. *Let $v > 0, u \ge 0$ be differentiable in Ω. Denote*

$$L(u,v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \nabla u\,,$$
$$R(u,v) = |\nabla u|^p - |\nabla v|^{p-2}\nabla\Big(\frac{u^p}{v^{p-1}}\Big)\nabla v.$$

Then

(i) $L(u,v) = R(u,v)$

(ii) $L(u,v) \ge 0$ *a.e. in* Ω.

Next, we show that any eigenfunction associated with an eigenvalue λ_0 such that $\lambda_0 \neq \lambda_1$ has to change the sign in Ω.

Proposition 4.3. *Any eigenfunction corresponding to the eigenvalue λ_0, $\lambda_0 \neq \lambda_1$ changes sign in Ω.*

Proof of Proposition 4.3. We assume the contrary. Let v_0 be an eigenfunction corresponding to the eigenvalue λ_0 of (1) and assume that $v_0 \geq 0$ (the case $v \leq 0$ being completely analogous). By [15, Theorem 2.7], we have $v_0 \in L_\infty(\Omega)$ and hence according to [34] we have $v_0 \in C^{1,\delta}(\Omega)$ for some $\delta \in (0,1)$. Then Harnack's inequality implies that $v_0(x) > 0$ for all $x \in \Omega$. Note that given $\epsilon > 0$, $\frac{\varphi_1^p}{(v_0+\epsilon)^{p-1}} \in V_p$ follows from $\varphi_1, v_0 \in V_p \cap L_\infty(\Omega)$ and from Ω being of finite measure. Hence, for any $\epsilon > 0$ we can apply Lemma 4.2 to the pair $\varphi_1, v_0 + \epsilon$ and use the fact that φ_1 is a weak solution of (1) with $\lambda = \lambda_1$, where $\frac{\varphi_1^p}{(v_0+\epsilon)^{p-1}}$ can be taken as a test function.

We thus have

$$\begin{aligned}
0 &\leq \int_\Omega L(\varphi_1, v_0+\epsilon)\,dx \\
&= \int_\Omega R(\varphi_1, v_0+\epsilon)\,dx \\
&= \int_\Omega \left[|\nabla\varphi_1|^p - |\nabla v_0|^{p-2}\nabla\left(\frac{\varphi_1^p}{(v_0+\epsilon)^{p-1}}\right)\nabla v_0\right]dx \\
&= \lambda_1\int_\Omega \varphi_1^p\,dx - \int_{\partial\Omega}\varphi_1^p d\mathcal{H}_{N-1} - \int_\Omega |\nabla v_0|^{p-2}\nabla\left(\frac{\varphi_1^p}{(v_0+\epsilon)^{p-1}}\right)\nabla v_0\,dx \\
&= \lambda_1\int_\Omega \varphi_1^p\,dx - \int_{\partial\Omega}\varphi_1^p d\mathcal{H}_{N-1} - \lambda_0\int_\Omega v_0^{p-1}\frac{\varphi_1^p}{(v_0+\epsilon)^{p-1}}\,dx \\
&\quad + \int_{\partial\Omega} v_0^{p-1}\frac{\varphi_1^p}{(v_0+\epsilon)^{p-1}}\,d\mathcal{H}_{N-1} \\
&= \int_\Omega \varphi_1^p\left(\lambda_1 - \lambda_0\frac{v_0^{p-1}}{(v_0+\epsilon)^{p-1}}\right)dx - \int_{\partial\Omega}\varphi_1^p\left(1 - \frac{v_0^{p-1}}{(v_0+\epsilon)^{p-1}}\right)d\mathcal{H}_{N-1}\,.
\end{aligned}$$

Letting $\epsilon \to 0$ we apply the Lebesgue dominated convergence theorem to obtain $0 \leq (\lambda_1 - \lambda_0)\int_\Omega \varphi_1^p\,dx = (\lambda_1 - \lambda_0)$, which is a contradiction since $\lambda_0 > \lambda_1$ by the variational characterization of the principal eigenvalue. □

Lemma 4.4. *Let $k > 0$ be fixed and $\lambda > 0$, $\lambda \neq \lambda_1$ be an eigenvalue of* (1) *such that $\lambda \leq k$. Denote $\Omega^+ = \{x \in \Omega\colon v(x) > 0\}$ and $\Omega^- = \{x \in \Omega\colon v(x) < 0\}$, where v is an eigenfunction associated with λ. Then there exists a positive constant $c = c(k,\Omega) > 0$ such that*

$$|\Omega^+| \geq c \quad \textit{and} \quad |\Omega^-| \geq c.$$

Proof of Lemma 4.4. We give the proof for Ω^-, the other case being analogous. It follows from the Definition 3.1 with $\phi = v^-$, that $\int_\Omega |\nabla v^-|^p\, dx + \int_{\partial\Omega} |v^-|^p\, d\mathcal{H}_{N-1} = \lambda \int_\Omega |v^-|^p\, dx$. By Hölder inequality we have

$$\|v^-\|^p_{V_p} \le k\, \|v^-\|^p_{\frac{Np}{N-1}}\, |\Omega^-|^{\frac{1}{N}}.$$

On the other hand, by (3) we have

$$\|v^-\|^p_{\frac{Np}{N-1}} \le C\|v^-\|^p_{V_P}.$$

Note that $\|v^-\|_{V_p} > 0$ due to Proposition 4.3. Hence $|\Omega^-| \ge (k\,C)^{-N} = c > 0$. This proves the lemma. $\square$

Proof of Theorem 4.1. We proceed similarly as in the case of the Dirichlet problem. Suppose that λ_1 is not isolated. Then there exists a sequence of eigenvalues of (1), $(\lambda_n)_{n\ge 2}$, with $\lambda_n \to \lambda_1$. Let u_n be an eigenfunction associated to λ_n. Then $\lambda_1 < \lambda_n$, every u_n changes sign in Ω (see Proposition 4.3), and without loss of generality we may assume that $\|u_n\|_p = 1$. Since $\|u_n\|^p_{V_p} = \lambda_n \int_\Omega |u_n|^p\, dx$ we can assume that (u_n) is bounded in V_p. The reflexivity of V_p yields the weak convergence $u_n \rightharpoonup \tilde{u}$ in V_p for some $\tilde{u} \in V_p$ (at least for some subsequence of (u_n)). The compact embedding $V_p \hookrightarrow\hookrightarrow L_p(\Omega)$ implies the strong convergence $u_n \to \tilde{u}$ in $L_p(\Omega)$. Moreover $\mathcal{J}(\tilde{u}) = 1$. On the other hand

$$\mathcal{I}(\tilde{u}) := \|\tilde{u}\|^p_{V_p} \le \liminf_{n\to\infty} \|u_n\|^p_{V_p} = \lambda_1 = \inf_{v\in\mathcal{M}} \mathcal{I}(v),$$

and hence $\lambda_1 = \mathcal{I}(\tilde{u})$. By Theorem 3.2, we can assume $\tilde{u} > 0$. Since $u_n \to \tilde{u}$ in $L_p(\Omega)$, then $u_n \to \tilde{u}$ in measure, and then (u_n) has a subsequence (still denoted u_n) which converges to $\tilde{u}$ a.e. in Ω, then $|\Omega_n^-| \to 0$, where $\Omega_n^- = \{x \in \Omega\colon u_n(x) < 0\}$. But this contradicts Lemma 4.4. $\square$

5. The principal eigencurve

In order to link all three basic types of homogeneous boundary conditions (Dirichlet, Neumann and Robin), we consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \mu |u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where μ is a variable parameter. We study the dependence of $\lambda_1 = \lambda_1(\mu)$ and $\varphi_1 = \varphi_1(\mu)$ on $\mu \in [0, \infty)$.

We define

$$\mathcal{I}_\mu(u) = \int_\Omega |\nabla u|^p \, dx + \mu \int_{\partial\Omega} |u|^p d\mathcal{H}_{N-1} \quad \text{and} \quad \mathcal{J}(u) = \int_\Omega |u|^p dx,$$

and

$$\lambda_1 : \mu \to \lambda_1(\mu) = \inf \{ \mathcal{I}_\mu(u) : u \in \mathcal{M} \},$$

where $\mathcal{M} = \{u \in V_p : \mathcal{J}(u) = 1\}$. The graph of $\lambda_1 = \lambda_1(\mu)$ is called the first *eigencurve* of (9).

By similar arguments as in Sections 3 and 4, we can prove the existence, simplicity and isolation of principal eigenvalue $\lambda_1(\mu)$ of (9) for any fixed μ. We have $\varphi_1(\mu) \in \mathcal{M}$, $\varphi_1(\mu) \in L_\infty(\Omega) \cap C^{1,\delta}(\Omega)$ for some $\delta \in (0,1)$ and $\varphi_1(\mu) > 0$ in Ω, $\varphi_1(\mu) \geq 0$ on $\partial\Omega$.

Lemma 5.1. *The function $\lambda_1 = \lambda_1(\mu)$ is concave in $(0, \infty)$.*

Proof of Lemma 5.1. It is easy to see that, $\mu \to \mathcal{I}_\mu(u)$ is an affine and so concave function. As the infimum of the collection of concave functions is concave, it follows that $\lambda_1 = \lambda_1(\mu)$ is concave function. □

Lemma 5.2. *The function $\lambda_1 = \lambda_1(\mu)$ is continuous in $(0, \infty)$, i.e., for any $\mu_0 \in (0, \infty)$, $\mu \to \mu_0$ implies $\lambda_1(\mu) \to \lambda_1(\mu_0)$.*

Proof of Lemma 5.2. Continuity of $\lambda_1 = \lambda_1(\mu)$ follows from Lemma 5.1 and the fact that a concave function is continuous on its open domain of definition. □

Lemma 5.3. *The function $\varphi_1 = \varphi_1(\mu)$ is continuous in $(0, \infty)$, i.e., for any $\mu_0 \in (0, \infty)$, $\mu \to \mu_0$ implies $\varphi_1(\mu) \to \varphi_1(\mu_0)$ in V_p.*

Proof of Lemma 5.3. Let $\mu \to \mu_0 \in (0, \infty)$. We introduce an equivalent norm on V_p by

$$\|u\| = \left(\|u\|_{1,p}^p + \mu_0 \, \|u|_{\partial\Omega}\|_{L_p(\partial\Omega)}^p \right)^{\frac{1}{p}}.$$

As $\varphi_1(\mu)$ is bounded in V_p, there exists $\tilde{\varphi} \in V_p$ such that, $\varphi_1(\mu) \rightharpoonup \tilde{\varphi}$ weakly in V_p, and $\varphi_1(\mu) \to \tilde{\varphi}$ strongly in $L_p(\Omega)$, i.e., $\mathcal{J}(\tilde{\varphi}) = 1$.

On the other hand, by Lemma 5.2, we know that $\lambda_1(\mu) \to \lambda_1(\mu_0)$, hence

$$\lim_{\mu \to \mu_0} \|\varphi_1(\mu)\|^p = \lim_{\mu \to \mu_0} \left(\lambda_1(\mu) + (\mu - \mu_0) \int_{\partial\Omega} \varphi_1(\mu)^p d\mathcal{H}_{N-1} \right) = \lim_{\mu \to \mu_0} \lambda_1(\mu) = \|\varphi_1(\mu_0)\|^p.$$

The weak lower semicontinuity of the norm in V_p then yields

$$\lambda_1(\mu_0) \leq \|\tilde{\varphi}\|^p \leq \liminf_{\mu \to \mu_0} \|\varphi_1(\mu)\|^p = \lim_{\mu \to \mu_0} \|\varphi_1(\mu)\|^p = \|\varphi_1(\mu_0)\|^p = \lambda_1(\mu_0),$$

i.e., $\lambda_1(\mu_0) = \|\tilde{\varphi}\|^p$. Hence, $\tilde{\varphi} = \varphi_1(\mu_0)$ by simplicity of $\lambda_1(\mu_0)$. Moreover, the uniform convexity of V_p then implies the strong convergence $\varphi_1(\mu) \to \varphi_1(\mu_0)$ in V_p. □

The next lemma asserts that $\lambda_1 = \lambda_1(\mu)$ is actually differentiable.

Lemma 5.4. *For any $\mu_0 \in (0, \infty)$, we have*

$$\frac{d\lambda_1}{d\mu}(\mu_0) = \int_{\partial\Omega} \varphi_1(\mu_0)^p d\mathcal{H}_{N-1}.$$

In particular, the function $\lambda_1 = \lambda_1(\mu)$ is non decreasing.

Proof of Lemma 5.4. Let us recall the normalization $\varphi_1(\mu) \in \mathcal{M}$. For any positive μ and μ_0, by using the variational characterization of $\lambda_1(\mu)$ and $\lambda_1(\mu_0)$, respectively, we have

$$\lambda_1(\mu_0) = \mathcal{I}_{\mu_0}(\varphi_1(\mu_0)) \le \mathcal{I}_{\mu_0}(\varphi_1(\mu)) \quad \text{and} \quad \lambda_1(\mu) = \mathcal{I}_{\mu}(\varphi_1(\mu)) \le \mathcal{I}_{\mu}(\varphi_1(\mu_0)).$$

Thus,

$$(\mu - \mu_0) \int_{\partial\Omega} \varphi_1(\mu)^p d\mathcal{H}_{N-1} \le \lambda_1(\mu) - \lambda_1(\mu_0) \le (\mu - \mu_0) \int_{\partial\Omega} \varphi_1(\mu_0)^p d\mathcal{H}_{N-1}.$$

Dividing by $(\mu - \mu_0)$ and letting $\mu \to \mu_0$, the result follows from Lemma 5.3. $\square$

Continuity of $\lambda_1(\mu)$ and $\varphi_1(\mu)$ at 0 requires a special attention. Let us assume that Ω is a bounded domain with Lipschitz boundary. Note that $(0 =)$ $\lambda_1(0) = \lambda_1^N$, where λ_1^N denotes the principal eigenvalue of the p-Laplacian subject to the Neumann boundary conditions with corresponding eigenfunction $\varphi_1(0) = \text{const}$.

It follows from the Trace Theorem (see [25, 29]) that there exists $c > 0$ such that for any $W_p^1(\Omega) \cap C(\overline{\Omega})$ we have $\|u|_{\partial\Omega}\|_{L_p(\partial\Omega)} \le c\|u\|_p$. In particular, $V_p = W_p^1(\Omega)$. Moreover, given $\mu_0 > 0$, it follows from the definition of $\varphi_1(\mu)$ that for all $\mu \in (0, \mu_0)$ the norms $\|\varphi_1(\mu)\|_{V_p}$ are uniformly bounded. Hence, there exists $\phi_1 \in W_p^1(\Omega)$ such that for $\mu \to 0$, we may assume that $\varphi_1(\mu) \rightharpoonup \phi_1$ weakly in $W_p^1(\Omega)$ and $\varphi_1(\mu) \to \phi_1$ strongly in $L_p(\Omega)$. In particular, we have $\phi_1 \ge 0$ a.e. in Ω and $\phi_1 \ne 0$ by normalization of $\varphi_1(\mu)$. Due to Lemma 5.2 we also have $\lambda(\mu) \to \tilde{\lambda} \ge 0$. Letting $\mu \to 0$ in

$$\begin{aligned}&\int_\Omega |\nabla\varphi_1(\mu)|^{p-2}\nabla\varphi_1(\mu)\nabla\phi \, dx + \mu \int_{\partial\Omega} |\varphi_1(\mu)|^{p-2}\varphi_1(\mu)\phi \, d\mathcal{H}_{N-1} \\ &= \lambda(\mu) \int_\Omega |\varphi_1(\mu)|^{p-2}\varphi_1(\mu)\phi \, dx,\end{aligned}$$

we conclude that ϕ_1 is a weak solution of the Neumann problem

$$\begin{cases} -\Delta_p\phi_1 = \tilde{\lambda}\, |\phi_1|^{p-2}\phi_1 & \text{in } \Omega \\ |\nabla\phi_1|^{p-2}\dfrac{\partial\phi_1}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

This fact together with $\phi_1 \geq 0$ a.e. in Ω imply that $\phi_1 = \varphi_1(0) = \text{const} \neq 0$, $\tilde{\lambda} = \lambda(0) = 0$. Substituting $\phi = \varphi_1(\mu)$ into the integral identity above and letting $\mu \to 0$, we get $\|\varphi_1(\mu)\|_{1,p} \to 0$ and so, eventually, $\varphi_1(\mu) \to \varphi_1(0)$ strongly in $W_p^1(\Omega)$.

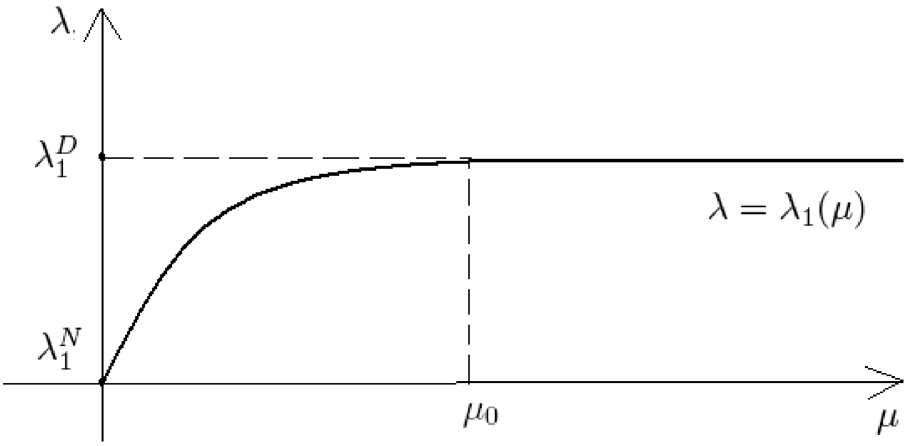

Figure 1. Graph of $\mu \to \lambda_1(\mu)$ when $\lambda_1(\mu_0)=\lambda_1^D$.

Finally, we proceed to prove that $\lambda_1(\mu) \to \lambda_1^D$ and $\varphi_1(\mu) \to \varphi_1^D$ as $\mu \to \infty$, where λ_1^D denotes the principal eigenvalue of the p-Laplacian subject to the Dirichlet boundary conditions and φ_1^D is the corresponding eigenfunction.

We observe that the function $\lambda_1 = \lambda_1(\mu)$ is bounded above by λ_1^D. Indeed, let us assume that there is a $\mu_0 > 0$ such that $\lambda_1(\mu_0)=\lambda_1^D$. By the uniqueness result for the positive normalized eigenfunction of the homogeneous Dirichlet problem, $\varphi_1(\mu_0)$ is its principal eigenfunction which belongs to $\mathring{W}_p^1(\Omega)$. It then follows from Lemma 5.4 that $\frac{d\lambda_1}{d\mu}(\mu_0) = 0$. Since $\lambda_1 = \lambda_1(\mu)$ is concave and non decreasing function, we actually have $\frac{d\lambda_1}{d\mu}(\mu) = 0$, i.e., $\lambda_1(\mu) = \lambda_1^D$ for all $\mu \geq \mu_0$, see Figure 1.

Let the boundary of Ω be regular at a point $x_0 \in \partial\Omega$ and the Hopf Maximum Principle applies at x_0. For example, this is the case, when $\partial\Omega$ satisfies the interior sphere condition at x_0. Then $\varphi_1(\mu_0)$ would violate the Robin boundary condition at x_0 and the equality $\lambda_1(\mu_0)=\lambda_1^D$ never holds with a finite μ_0. Due to the continuity of function $\lambda_1 = \lambda_1(\mu)$, we thus have $\lambda_1(\mu) < \lambda_1^D$ for all $\mu \in [0, \infty)$.

To summarize above discussion, we present the following

Theorem 5.5. . *The function $\lambda_1 = \lambda_1(\mu)$ is concave, increasing, continuously differentiable and bounded above. The function $\varphi_1 = \varphi_1(\mu)$ (as a function from $\mathbb{R}$ into V_p) is continuous. Moreover, if the Hopf maximum principle holds at some point $x_0 \in \partial\Omega$, the following asymptotic properties as $\mu \to \infty$ hold true* (*see Figure* 2)*:*

$$\lambda_1(\mu) < \lambda_1^D, \quad \mu \in [0, \infty), \qquad \lim_{\mu\to\infty} \lambda_1(\mu) = \lambda_1^D,$$

and

$$\lim_{\mu\to\infty} \|\varphi_1(\mu) - \varphi_1^D\|_{V_p} = 0\,.$$

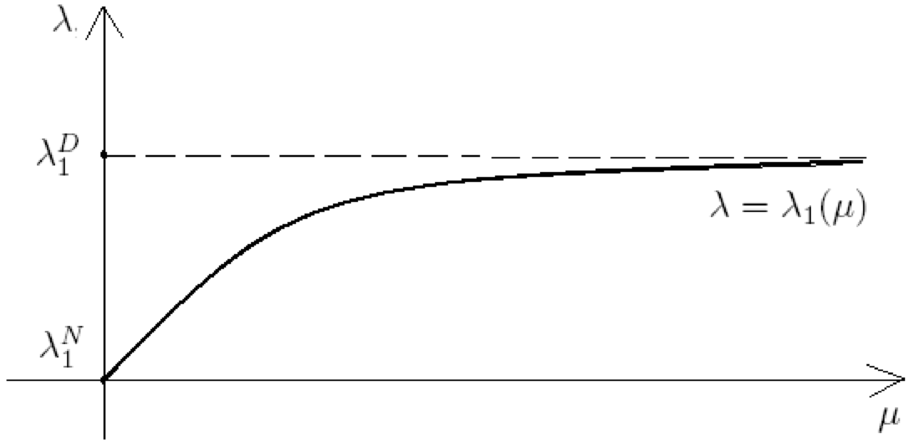

Figure 2. Graph of $\mu \to \lambda_1(\mu)$.

Proof of Theorem 5.5. It remains to prove the asymptotic properties of $\lambda_1(\mu)$ and $\varphi_1(\mu)$. By Lemma 5.3, $(\varphi_1(\mu))$ is bounded in V_p as $\mu \to \infty$. Then there exists $\varphi_\infty \in V_p$ such that, $\varphi_1(\mu) \rightharpoonup \varphi_\infty$ weakly in V_p as $\mu \to \infty$ and $\varphi_1(\mu) \to \varphi_\infty$ strongly in $L_p(\Omega)$ as $\mu \to \infty$, i.e.,

$$\mathcal{J}(\varphi_\infty) = 1, \quad \text{and} \quad \varphi_\infty > 0 \quad \text{in}\, \Omega.$$

The weak lower semicontinuity of the norm in V_p yields

$$\begin{aligned}
\lambda_1^D &\le \|\varphi_\infty\|_{1,p} \\
&\le \mathcal{I}(\varphi_\infty) \\
&= \int_\Omega |\nabla \varphi_\infty|^p \, dx + \int_{\partial\Omega} \varphi_\infty^p d\mathcal{H}_{N-1} \\
&\le \liminf_{\mu\to\infty} \int_\Omega |\nabla \varphi_1(\mu)|^p \, dx + \int_{\partial\Omega} \varphi_1(\mu)^p d\mathcal{H}_{N-1} \\
&= \liminf_{\mu\to\infty} \mathcal{I}(\varphi_1(\mu)) \\
&\le \liminf_{\mu\to\infty} \mathcal{I}_\mu(\varphi_1(\mu)) \\
&= \liminf_{\mu\to\infty} \lambda_1(\mu) \\
&\le \limsup_{\mu\to\infty} \lambda_1(\mu)
\end{aligned}$$

$$\leq \lambda_1^D.$$

i.e., $\lim_{\mu\to\infty} \lambda_1(\mu) = \lambda_1^D$, and $\varphi_\infty = \varphi_1^D$. Moreover, it follows from the above inequalities that $\|\varphi_1(\mu)\|_{V_p} \to \|\varphi_1^D\|_{V_p}$ as $\mu \to \infty$. The uniform convexity of V_p then implies the strong convergence $\varphi_1(\mu) \to \varphi_1^D$ in V_p as $\mu \to \infty$. □

Remark 5.6. Our results could be formulated and proved for the nonlinear eigenvalue problem of the form

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \mu b(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $b(x) \in L_\infty(\partial\Omega)$ and there exists a constant $b_0 > 0$ such that $b(x) \geq b_0$; $a(x) \in L_{\frac{q}{q-p}}(\Omega)$ with some q satisfying $p < q < \frac{Np}{N-1}$ or $a(x) \in L_\infty(\Omega)$, and moreover, $|\{x \in \Omega :\ a(x) > 0\}| > 0$. Then all our proofs can be recovered using multiple use of Hölder's inequality.

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