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Compact Embeddings of Bessel-Potential-Type Spaces into Generalized Hölder Spaces Involving *k*-Modulus of Smoothness

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Abstract. We present conditions which are necessary and sufficient for compact embeddings of Bessel potential spaces $H^\sigma X(\mathbb{R}^n)$, modelled upon a rearrangement-invariant Banach function spaces $X(\mathbb{R}^n)$, into generalized Hölder spaces involving k-modulus of smoothness. To this end, we derive a characterization of compact subsets of generalized Hölder spaces. We apply our results to the case when $X(\mathbb{R}^n)$ is a Lorentz–Karamata space $L_{p,q;b}(\mathbb{R}^n)$. Applications cover both superlimiting and limiting cases.

Keywords. Bessel potentials, (fractional) Sobolev-type spaces, rearrangement-invariant Banach function spaces, generalized Hölder-type spaces, embedding theorems, slowly varying functions, Lorentz–Karamata spaces

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1. Introduction and main results

The paper is a continuation of [16], where optimal embeddings of Bessel-potential-type spaces $H^\sigma X(\mathbb{R}^n)$ with order of smoothness $\sigma \in (0, n)$, modelled upon rearrangement-invariant Banach function spaces (r.i.BFS) $X(\mathbb{R}^n)$, into generalized Hölder spaces $\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$, $0 < r \leq +\infty$, $\mu \in \mathcal{L}^k_r$, involving the k-modulus of smoothness, were investigated. (The class $\mathcal{L}^k_r$ consists of all continuous functions $\mu : (0, 1) \to (0, +\infty)$ that satisfy (2.5) and (2.6) bellow. We refer to Section 2

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for definitions of the spaces in question). To this end, in [16] we have used a sharp estimate of the k-modulus of smoothness of the convolution of a function f from an r.i.BFS $X(\mathbb{R}^n)$ with the Bessel potential kernel g_σ, $\sigma \in (0,n)$. Such an estimate states that if g_σ belongs to the associate space of $X(\mathbb{R}^n)$, then

$$\omega_k(f*g_\sigma,t)\precsim\int_0^{t^n} s^{\frac{\sigma}{n}-1}f^*(s)\,ds \quad \text{for all } t\in(0,1) \text{ and every } f\in X(\mathbb{R}^n), \tag{1.1}$$

provided that $k \geq [\sigma]+1$ (f^* denotes the non-increasing rearrangement of f). Estimate (1.1) and its reverse form (cf. Theorem 3.8 below) have enabled us to characterize the continuous embeddings of spaces in question. Namely, we have proved the following theorem.

Theorem 1.1 (cf. [16, Theorem 1.1])**.** *Let $\sigma \in (0,n)$ and let $X = X(\mathbb{R}^n) = X(\mathbb{R}^n,\mu_n)$ be an r.i.BFS such that $\|g_\sigma\|_{X'} < +\infty$. Put $k := [\sigma]+1$, assume that $r \in (0,+\infty]$ and $\mu \in \mathcal{L}^k_r$. Then*

$$H^\sigma X(\mathbb{R}^n) \hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) \tag{1.2}$$

if and only if

$$\left\| t^{-\frac{1}{r}}(\mu(t))^{-1}\int_0^{t^n} \tau^{\frac{\sigma}{n}-1}f^*(\tau)\,d\tau \right\|_{r;(0,1)} \precsim \|f\|_X \quad \textit{for all } f\in X. \tag{1.3}$$

Note that the implication (1.2) $\Longrightarrow$ (1.3) in Theorem 1.1 remains true if $k \in \mathbb{N}$. Theorem 1.1 has the following corollary.

Corollary 1.2 (cf. [16, Corollary 1.2])**.** *Let $\sigma \in (0,n)$ and let $X = X(\mathbb{R}^n)$ be an r.i.BFS. Put $k := [\sigma]+1$, assume that $r \in (0,+\infty]$ and $\mu \in \mathcal{L}^k_r$. Then embedding* (1.2) *holds if and only if* $\|g_\sigma\|_{X'} < +\infty$ *and* (1.3) *is satisfied.*

Moreover, in [16] we have applied Theorem 1.1 to the case when $X(\mathbb{R}^n)$ is a Lorentz–Karamata space $L_{p,q;b}(\mathbb{R}^n)$ and we have considered both the superlimiting case when $p > \frac{n}{\sigma}$ and the limiting case when $p = \frac{n}{\sigma}$. For example, choosing in the superlimiting case $\sigma = k+1$, $p = q = \frac{n}{k}$, with $k \in \mathbb{N}$, $k < n-1$, and $b(t) = (1+|\log t|)^\alpha$, $t > 0$, $\alpha < 1-\frac{k}{n}$, we have obtained that (cf. [16, (5.16)])

$$W^{k+1}L^{n/k}(\log L)^\alpha(\mathbb{R}^n) \hookrightarrow \Lambda^{2,\lambda(\cdot)}_{\infty,n/k}(\overline{\mathbb{R}^n}), \tag{1.4}$$

where $\lambda(t) := t(1+|\log t|)^{-\alpha}$, $t > 0$. Embedding (1.4) is the continuous embedding of the Sobolev–Orlicz space $W^{k+1}L^{n/k}(\log L)^\alpha(\mathbb{R}^n)$ (the Sobolev space modelled upon the Orlicz space $L^{n/k}(\log L)^\alpha(\mathbb{R}^n) = L_\Phi(\mathbb{R}^n)$, where the Young function Φ satisfies $\Phi(t) \approx [t(1+|\log t|)^\alpha]^{\frac{n}{k}}$, $t > 0$) into the Zygmund-type space $\Lambda^{2,\lambda(\cdot)}_{\infty,n/k}(\overline{\mathbb{R}^n})$. (Note that in the notation $\Lambda^{2,\lambda(\cdot)}_{\infty,n/k}(\overline{\mathbb{R}^n})$ the upper index 2 indicates that the second order modulus of smoothness is used to define this space.) This means that there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R}^n)} + \left(\int_0^1 \left(\frac{\omega_2(u,t)}{t(1+|\log t|)^{-\alpha}} \right)^{\frac{n}{k}} \frac{dt}{t} \right)^{\frac{k}{n}} \leq C\|u\|_{W^{k+1}L^{n/k}(\log L)^\alpha(\mathbb{R}^n)}$$

for all $u \in W^{k+1}L^{n/k}(\log L)^\alpha(\mathbb{R}^n)$. Such embedding is optimal and it does not follow from known results on embeddings of Sobolev–Orlicz spaces. In particular, if $\alpha = 0$, we have arrived to the continuous embedding of the Sobolev space $W^{k+1,n/k}(\mathbb{R}^n) = W^{k+1}L^{n/k}(\mathbb{R}^n)$, with $k \in \mathbb{N}$, $k < n-1$, into the Zygmund-type space $\Lambda^{2,Id(\cdot)}_{\infty,n/k}(\overline{\mathbb{R}^n})$ (Id stands for the identity map on $(0,+\infty)$). The last mentioned embedding shows that the Brézis–Wainger result* on "almost" Lipschitz continuity of functions from the Sobolev space $W^{k+1,n/k}(\mathbb{R}^n)$ is a consequence of a better embedding whose target is a Zygmund-type space. Similarly, choosing $\sigma = k$, $p = q = \frac{n}{k}$, with $k \in \mathbb{N}$, $k < n$, and $b(t) = (1+|\log t|)^\alpha$, $t > 0$, $\alpha > 1 - \frac{k}{n}$ in the limiting case, we have obtained optimal continuous embeddings of the Sobolev–Orlicz space $W^kL^{n/k}(\log L)^\alpha(\mathbb{R}^n)$ into Hölder-type spaces. Namely (cf. [16, (7.9)]),

$$W^kL^{n/k}(\log L)^\alpha(\mathbb{R}^n) \hookrightarrow \Lambda^{1,\lambda(\cdot)}_{\infty,n/k}(\overline{\mathbb{R}^n}), \tag{1.5}$$

where $\lambda(t) = (1+|\log t|)^{1-\alpha}$, $t \in (0,1)$.

The aim of this paper is the characterization of compact embeddings of the Bessel potential space $H^\sigma X(\mathbb{R}^n)$ with the order of smoothness $\sigma \in (0,n)$, modelled upon r.i.BFS $X(\mathbb{R}^n)$, into generalized Hölder spaces $\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$, where $0 < r \leq +\infty$ and Ω is a bounded domain in $\mathbb{R}^n$. To this end, it is essential to characterize totally bounded subsets of the space $\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$, Ω having minimally smooth boundary. The result is given in the following theorem.

Theorem 1.3. *Let $k \in \mathbb{N}$, $r \in (0,+\infty)$, $\mu \in \mathcal{L}^k_r$ and let Ω be a bounded domain in $\mathbb{R}^n$ with minimally smooth boundary. Then $\mathcal{S} \subset \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$ is totally bounded if and only if $\mathcal{S}$ is bounded in $\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$ and*

$$\sup_{u\in\mathcal{S}} \left\| t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_k(u,t) \right\|_{r;(0,\xi)} \to 0 \quad \text{as } \xi \to 0_+. \tag{1.6}$$

Our main result (which is an analogue of Theorem 1.1) reads as follows.

Theorem 1.4. *Let $\sigma \in (0,n)$ and let $X = X(\mathbb{R}^n) = X(\mathbb{R}^n,\mu_n)$ be an r.i.BFS such that $\|g_\sigma\|_{X'} < \infty$. Put $k := [\sigma]+1$, assume that $r \in (0,+\infty)$, $\mu \in \mathcal{L}^k_r$ and that Ω is a bounded domain in $\mathbb{R}^n$. Then*

$$H^\sigma X(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \quad ^\dagger \tag{1.7}$$

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*Note that the Brézis–Wainger embedding has the target space $\Lambda^{1,\mu(\cdot)}_{\infty,\infty}(\overline{\mathbb{R}^n})$ with $\mu(t) = t(1+|\log t|)^{1-\frac{k}{n}}$, $t \in (0,1)$, cf. [5, Corollary 5].

†This means that the mapping $u \mapsto u|_\Omega$ from $H^\sigma X(\mathbb{R}^n)$ into $\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$ is compact.

if and only if

$$\sup_{\|f\|_X \le 1} \left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \int_0^{t^n} \tau^{\frac{\sigma}{n}-1} f^*(\tau)\, d\tau \right\|_{r;(0,\xi)} \to 0 \qquad \text{as } \xi \to 0_+. \tag{1.8}$$

Note that the implication (1.7) $\Longrightarrow$ (1.8) in Theorem 1.4 remains true if $k \in \mathbb{N}$ (cf. Remark 5.1 below). The counterpart of Corollary 1.2 reads as follows.

Corollary 1.5. *Let $\sigma \in (0, n)$ and let $X = X(\mathbb{R}^n)$ be an r.i.BFS. Put $k := [\sigma] + 1$, assume that $r \in (0, +\infty)$, $\mu \in \mathcal{L}_r^k$ and that Ω is a bounded domain in $\mathbb{R}^n$. Then* (1.7) *holds if and only if $\|g_\sigma\|_{X'} < +\infty$ and* (1.8) *is satisfied.*

As in [16], we apply our main result (Theorem 1.4) to the case when $X(\mathbb{R}^n)$ is a Lorentz–Karamata space $L_{p,q;b}(\mathbb{R}^n)$. The corresponding compact embeddings are characterized in Theorems 1.6 and 1.8 (and Corollaries 1.7 and 1.9) below. The former theorem concerns the superlimiting case $p > \frac{n}{\sigma}$ while the latter one is devoted to the limiting case $p = \frac{n}{\sigma}$.

Theorem 1.6. *Let $\sigma \in (0, n)$, $p \in (\frac{n}{\sigma}, +\infty)$, $q \in [1, +\infty]$, $b \in SV(0, +\infty)$, $r \in (0, +\infty)$, $k = [\sigma] + 1$ and $\mu \in \mathcal{L}_r^k$. Assume that Ω is a bounded domain in $\mathbb{R}^n$. Let $\lambda : (0, 1) \to (0, +\infty)$ be defined by*

$$\lambda(x) := x^{\sigma - \frac{n}{p}} \big(b(x^n)\big)^{-1} \quad \text{for all} \quad x \in (0, 1). \tag{1.9}$$

(*Note that $\lambda \in \mathcal{L}_r^k$ for any $r \in (0, +\infty]$; recall that b is continuous* (*cf.* (2.1)).)

(i) *If $1 \le q \le r < +\infty$, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \tag{1.10}$$

if and only if

$$\lim_{x \to 0_+} \left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \right\|_{r;(x,1)} \lambda(x) = 0. \tag{1.11}$$

(ii) *If $0 < r < q \le +\infty$ and $q > 1$, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \tag{1.12}$$

if and only if

$$\int_0^1 \left(\left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \right\|_{r;(x,1)} \lambda(x) \right)^u \frac{dx}{x} < +\infty, \tag{1.13}$$

where $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$.

Corollary 1.7. *Assume that all the assumptions of Theorem* 1.6 *are satisfied. Let $\mu \in \mathcal{L}_r^{[\sigma - n/p]+1}$. If $1 \le q \le r < +\infty$, then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{[\sigma - n/p]+1,\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \tag{1.14}$$

if and only if (1.11) *is satisfied.*

Theorem 1.8. *Let* $\sigma \in (0,n)$, $p = \frac{n}{\sigma}$, $q \in (1,+\infty]$, $r \in (0,+\infty)$, $k = [\sigma]+1$, $\mu \in \mathcal{L}_r^k$ *and let* $b \in SV(0,+\infty)$ *be such that* $\big\|t^{-\frac{1}{q'}}(b(t))^{-1}\big\|_{q';(0,1)} < +\infty$. *Assume that* Ω *is a bounded domain in* $\mathbb{R}^n$. *Let* λ_{qr} *be defined by*

$$\lambda_{qr}(x) := b^{\frac{q'}{r}}(x^n)\left(\int_0^{x^n} b^{-q'}(t)\,\frac{dt}{t}\right)^{\frac{1}{q'}+\frac{1}{r}}, \quad x \in (0,1). \tag{1.15}$$

(*Note that* $\lambda_{qr} \in \mathcal{L}_r^k$; *recall that* b *is continuous* (*cf.* (2.1)).)

(i) *If* $1 < q \le r < +\infty$, *then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \tag{1.16}$$

if and only if

$$\lim_{x\to 0_+} \frac{\big\|t^{-\frac{1}{r}}(\mu(t))^{-1}\big\|_{r;(x,1)}}{\big\|t^{-\frac{1}{r}}(\lambda_{qr}(t))^{-1}\big\|_{r;(x,1)}} = 0. \tag{1.17}$$

(ii) *If* $0 < r < q \le +\infty$ *and* $q > 1$, *then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \tag{1.18}$$

if and only if

$$\int_0^{\frac{1}{2}} \left(\frac{\big\|t^{-\frac{1}{r}}(\mu(t))^{-1}\big\|_{r;(x,1)}}{\big\|t^{-\frac{1}{r}}(\lambda_{qr}(t))^{-1}\big\|_{r;(x,1)}}\right)^u \left(\int_0^{x^n} t^{-1} b^{-q'}(t)\,dt\right)^{-1} b^{-q'}(x^n)\,\frac{dx}{x} < +\infty, \tag{1.19}$$

where $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$.

Corollary 1.9. *Assume that all the assumptions of Theorem* 1.8 *are satisfied. Let* $\mu \in \mathcal{L}_r^1$. *If* $1 < q \le r < +\infty$, *then*

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{1,\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \tag{1.20}$$

if and only if (1.17) *is satisfied.*

When Ω is a bounded domain in $\mathbb{R}^n$, Theorem 1.6 and Corollary 1.7 yield the compactness result corresponding to (1.4): If $k \in \mathbb{N}$, $k < n-1$ and $\alpha < 1 - \frac{k}{n}$, then

$$W^{k+1}L^{n/k}(\log L)^\alpha(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{2,\mu(\cdot)}_{\infty,n/k}(\overline{\Omega}),$$

where $\mu(t) := t(1+|\log t|)^{-\beta}$, $t > 0$, $\beta \in \mathbb{R}$, holds if and only if $\beta < \alpha$. Similarly, Theorem 1.8 and Corollary 1.9 provide the compactness result corresponding to (1.5): If $k \in \mathbb{N}$, $k < n$ and $\alpha > 1 - \frac{k}{n}$, then

$$W^k L^{n/k}(\log L)^\alpha(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{1,\mu(\cdot)}_{\infty,n/k}(\overline{\Omega}),$$

where $\mu(t) = (1 + |\log t|)^{1-\beta}$, $t \in (0,1)$, $\beta \in \mathbb{R}$, holds if and only if $\beta \in (1 - \frac{k}{n}, \alpha)$.

The paper is organized as follows. Section 2 contains notation, definitions and basic properties. In Section 3 we summarize some results, which we shall need in subsequent sections. This section also involves key estimate (1.1) and its reverse form proved in [14]. In Section 4 we prove Theorem 1.3, while Section 5 is devoted to the proofs of Theorem 1.4 and Corollary 1.5. Finally, in Section 6 we present proofs of Theorems 1.6, 1.8 and Corollaries 1.7, 1.9. Note that any of Sections 4–6 contains some important remarks and comments on the results proved there.

2. Notation, definitions and basic properties

As usual, $\mathbb{R}^n$ denotes the Euclidean n-dimensional space. Throughout the paper μ_n is the n-dimensional Lebesgue measure in $\mathbb{R}^n$ and Ω is a μ_n-measurable subset of $\mathbb{R}^n$. We denote by χ_Ω the characteristic function of Ω and put $|\Omega|_n = \mu_n(\Omega)$. The family of all extended scalar-valued (real or complex) μ_n-measurable functions on Ω is denoted by $\mathcal{M}(\Omega)$ while $\mathcal{M}^+(\Omega)$ stands for the subset of $\mathcal{M}(\Omega)$ consisting of all functions which are non-negative μ_n-a.e. on Ω. When Ω is an interval $(a,b) \subseteq \mathbb{R}$, we denote these sets by $\mathcal{M}(a,b)$ and $\mathcal{M}^+(a,b)$, respectively. By $\mathcal{M}^+(a,b;\downarrow)$ we mean the subset of $\mathcal{M}^+(a,b)$ containing all non-increasing functions on the interval (a,b). The symbol $\mathcal{W}(a,b)$ stands for the class of weight functions on $(a,b) \subseteq \mathbb{R}$ consisting of all μ_n-measurable functions which are positive and finite μ_n-a.e. on (a,b). The *non-increasing rearrangement* of $f \in \mathcal{M}(\Omega)$ is the function f^* defined by $f^*(t) := \inf\{\lambda \geq 0 : |\{x \in \Omega : |f(x)| > \lambda\}|_n \leq t\}$ for all $t \geq 0$. By f^{**} we denote the maximal function of f^* given by $f^{**}(t) := t^{-1}\int_0^t f^*(\tau)\, d\tau$, $t > 0$.

Given a rearrangement-invariant Banach function space (r.i.BFS) X, its associate space is denoted by X'. For general facts about rearrangement-invariant Banach function spaces we refer to [3].

Let X and Y be two (quasi-)Banach spaces. We say that X *coincides* with Y (and write $X = Y$) if X and Y are equal in the algebraic and topological sense (their (quasi-)norms are equivalent). The symbol $X \hookrightarrow Y$ means that $X \subset Y$ and the natural embedding of X in Y is continuous.

By c, C, c_1, C_1, c_2, C_2, etc. we denote positive constants independent of appropriate quantities. For two non-negative expressions (i.e., functions or functionals) $\mathcal{A}$, $\mathcal{B}$, the symbol $\mathcal{A} \precsim \mathcal{B}$ (or $\mathcal{A} \succsim \mathcal{B}$) means that $\mathcal{A} \leq c\,\mathcal{B}$ (or $c\,\mathcal{A} \geq \mathcal{B}$). If $\mathcal{A} \precsim \mathcal{B}$ and $\mathcal{A} \succsim \mathcal{B}$, we write $\mathcal{A} \approx \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent. Throughout the paper we use the abbreviation LHS($*$) (RHS($*$)) for the left- (right-) hand side of the relation ($*$). We adopt the convention that $\frac{a}{+\infty} = 0$ and $\frac{a}{0} = +\infty$ for all $a > 0$. If $p \in (0, +\infty]$, the conjugate number p'

is given by $\frac{1}{p}+\frac{1}{p'}=1$. Note that p' is negative, if $p\in(0,1)$. In the whole paper $\|.\|_{p;(c,d)}$, $p\in(0,+\infty]$, denotes the usual L^p-(quasi-)norm on the interval $(c,d)\subseteq\mathbb{R}$.

For $\rho\in(0,+\infty)$ and $x\in\mathbb{R}^n$, $B(x,\rho)=B_n(x,\rho)$ stands for the open ball in $\mathbb{R}^n$ of radius ρ and centre x. By β_n we denote the volume of the unit ball in $\mathbb{R}^n$.

Let either $a=1$ or $a=+\infty$. We say that a positive and μ_n-measurable function b is *slowly varying* on $(0,a)$, and write $b\in SV(0,a)$, if, for each $\varepsilon>0$, $t^{\varepsilon}b(t)$ is equivalent to a non-decreasing function on $(0,a)$ and $t^{-\varepsilon}b(t)$ is equivalent to a non-increasing function on $(0,a)$. Here we follow the definition of $SV(0,+\infty)$ given in [17]; for other definitions see, for example, [4,8,11,21]. The family $SV(0,+\infty)$ includes not only powers of iterated logarithms and the broken logarithmic functions of [12] but also such functions as $t\to\exp\left(|\log t|^a\right)$, $a\in(0,1)$. (The last mentioned function has the interesting property that it tends to infinity more quickly than any positive power of the logarithmic function).

We can see from Lemma 3.1 (i) below that any $b\in SV(0,+\infty)$ is equivalent to a $\widetilde{b}\in SV(0,+\infty)$ which is continuous on $(0,+\infty)$. Consequently, without loss of generality, we shall assume that

$$\text{all slowly varying functions in question are continuous on } (0,+\infty). \tag{2.1}$$

More properties and examples of slowly varying functions can be found in [25, Chapter V, p. 186] and [4,11,17,21].

Let $p,q\in(0,+\infty]$, $b\in SV(0,+\infty)$ and let Ω be a μ_n-measurable subset of $\mathbb{R}^n$. The *Lorentz–Karamata* (LK) space $L_{p,q;b}(\Omega)$ is defined to be the set of all functions $f\in\mathcal{M}(\Omega)$ such that

$$\|f\|_{p,q;b;\Omega}:=\left\|t^{\frac{1}{p}-\frac{1}{q}}\,b(t)\,f^*(t)\right\|_{q;(0,+\infty)}<+\infty. \tag{2.2}$$

If $\Omega=\mathbb{R}^n$, we simply write $\|\cdot\|_{p,q;b}$ instead of $\|\cdot\|_{p,q;b;\mathbb{R}^n}$.

Particular choices of b give well-known spaces. If $m\in\mathbb{N}$, $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_m)\in\mathbb{R}^m$ and

$$b(t)=\boldsymbol{\ell}^{\boldsymbol{\alpha}}(t):=\prod_{i=1}^{m}\ell_i^{\alpha_i}(t)\quad\text{ for all } t>0$$

(where $\ell(t)=\ell_1(t):=1+|\log t|$, $\ell_i(t):=\ell_1(\ell_{i-1}(t))$ if $i>1$), then the LK-space $L_{p,q;b}(\Omega)$ is the generalized Lorentz–Zygmund space $L_{p,q,\boldsymbol{\alpha}}$ introduced in [10] and endowed with the (quasi-)norm $\|f\|_{p,q;\boldsymbol{\alpha};\Omega}$, which in turn becomes the Lorentz–Zygmund space $L^{p,q}(\log L)^{\alpha_1}(\Omega)$ of Bennett and Rudnick [2] when $m=1$. If $\boldsymbol{\alpha}=(0,\dots,0)$, we obtain the Lorentz space $L^{p,q}(\Omega)$ endowed with the (quasi-) norm $\|\cdot\|_{p,q;\Omega}$, which is just the Lebesgue space $L^p(\Omega)$ equipped with the (quasi-)norm $\|\cdot\|_{p;\Omega}$ when $p=q$; if $p=q$ and $m=1$, we obtain the Zygmund space $L^p(\log L)^{\alpha_1}(\Omega)$ endowed with the (quasi-)norm $\|\cdot\|_{p;\alpha_1;\Omega}$.

The *Bessel kernel* g_σ, $\sigma > 0$, is defined as that function on $\mathbb{R}^n$ whose Fourier transform is $\widehat{g_\sigma}(\xi) = (2\pi)^{-\frac{n}{2}}(1+|\xi|^2)^{-\frac{\sigma}{2}}$, $\xi \in \mathbb{R}^n$, where the Fourier transform $\hat{f}$ of a function f is given by $\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx$. Some basic properties of the Bessel kernel g_σ can be found, e.g., in [24].

Let $\sigma > 0$ and let $X = X(\mathbb{R}^n) = X(\mathbb{R}^n, \mu_n)$ be an r. i. Banach function space endowed with the norm $\|\cdot\|_X$. The *Bessel potential* space $H^\sigma X(\mathbb{R}^n)$ is defined by

$$H^\sigma X(\mathbb{R}^n) := \big\{u : u = f * g_\sigma,\ f \in X(\mathbb{R}^n)\big\} \tag{2.3}$$

and is equipped with the norm

$$\|u\|_{H^\sigma X} := \|f\|_X. \tag{2.4}$$

Note that, given $f \in X$, the convolution $u = f * g_\sigma$ is well defined and finite μ_n-a.e. on $\mathbb{R}^n$ since the measure space $(\mathbb{R}^n, \mu_n)$ is resonant and so (cf. [3, Theorem II.6.6]) $X \hookrightarrow L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$.

If $p \in (1, +\infty]$, $q \in [1, +\infty]$ and $b \in SV(0, +\infty)$, then the space $L_{p,q;b}(\mathbb{R}^n)$ coincides with an r. i. Banach function space $X(\mathbb{R}^n)$ (the (quasi-)norm (2.2) is equivalent to the norm $\|t^{\frac{1}{p}-\frac{1}{q}}\ b(t)\ f^{**}(t)\|_{q;(0,+\infty)}$, which follows from the estimate $f^* \le f^{**}$ and Lemma 3.2 (i) with $r = q$, $w(t) = t^{\frac{1}{p}-\frac{1}{q}-1}b(t)$, $v(t) = t^{\frac{1}{p}-\frac{1}{q}}b(t)$ and $a = +\infty$). Consequently, if $\sigma > 0$, $p \in (1, +\infty]$, $q \in [1, +\infty]$ and $b \in SV(0, +\infty)$, $H^\sigma L_{p,q;b}(\mathbb{R}^n) := H^\sigma X(\mathbb{R}^n)$ is the usual Bessel potential space modelled upon the Lorentz–Karamata space $L_{p,q;b}(\mathbb{R}^n)$, which is equipped with the (quasi-)norm

$$\|u\|_{\sigma;p,q;b} := \|f\|_{p,q;b}.$$

When $m \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and $b = \boldsymbol{\ell}^{\boldsymbol{\alpha}}$, we obtain the logarithmic Bessel potential space $H^\sigma L_{p,q;\boldsymbol{\alpha}}(\mathbb{R}^n)$, endowed with the (quasi-)norm $\|u\|_{\sigma;p,q;b}$ and considered in [10]. Note that if $\boldsymbol{\alpha} = (0, \ldots, 0)$, $H^\sigma L_{p,p;\boldsymbol{\alpha}}(\mathbb{R}^n)$ is simply the (fractional) Sobolev space $H^{\sigma,p}(\mathbb{R}^n)$ of order σ.

When $k \in \mathbb{N}$, $p, q \in (1, +\infty)$ and $b \in SV(0, +\infty)$, then

$$H^k L_{p,q;b}(\mathbb{R}^n) = \big\{u : D^\alpha u \in L_{p,q;b}(\mathbb{R}^n) \text{ if } |\alpha| \le k\big\}$$

and

$$\|u\|_{k;p,q;b} \approx \sum_{|\alpha|\le k} \|D^\alpha u\|_{p,q;b} \quad \text{for all } u \in H^k L_{p,q;b}(\mathbb{R}^n)$$

according to [13, Lemma 4.5] and [22, Theorem 5.3].

Let Ω be a domain in $\mathbb{R}^n$. We denote by $B(\Omega)$ the set of all scalar-valued functions (real or complex) which are bounded on Ω and we equip this set with the norm

$$\|f\|_{B(\Omega)} := \sup\{|f(x)| : x \in \Omega\}.$$

The subspace of $B(\Omega)$ of all continuous functions on Ω is denoted by $C_B(\Omega)$ and it is equipped with the $B(\Omega)$-norm. By $C(\overline{\Omega})$ we mean the subspace of $C_B(\Omega)$ of all uniformly continuous functions on Ω.

Let Ω be a domain in $\mathbb{R}^n$ and let $k \in \mathbb{N}$. For each $h \in \mathbb{R}^n$, put $\Omega(kh) := \{x \in \Omega : x + th \in \Omega, 0 \le t \le k\}$. The first difference operator Δ_h is defined on scalar functions $f \in B(\Omega)$ by $\Delta_h f(x) = f(x+h) - f(x)$ for all $x \in \Omega(h)$, and higher order differences are defined inductively by

$$\Delta_h^{k+1} f(x) = \Delta_h(\Delta_h^k f)(x), \quad x \in \Omega((k+1)h).$$

The *k-modulus of smoothness* of a function f in $C_B(\Omega)$ is given by

$$\omega_{k,\Omega}(f,t) := \sup_{|h| \le t} \left\| \Delta_h^k f | B(\Omega(kh)) \right\| \quad \text{for all } t \ge 0.$$

If $k = 1$, we write $\omega_\Omega(f,t)$ instead of $\omega_{1,\Omega}(f,t)$.

It is clear that the k-modulus of smoothness depends on a given domain Ω. In what follows we shall sometimes omit the subscript Ω at the k-modulus of smoothness since it will be always clear from the context which k-modulus of smoothness we have in mind.

Let $k \in \mathbb{N}$, $r \in (0, +\infty]$ and let $\mathcal{L}_r^k$ be the class of all continuous functions $\mu : (0,1) \to (0, +\infty)$ such that

$$\left\| t^{-\frac{1}{r}} \frac{1}{\mu(t)} \right\|_{r;(0,1)} = +\infty \tag{2.5}$$

and

$$\left\| t^{-\frac{1}{r}} \frac{t^k}{\mu(t)} \right\|_{r;(0,1)} < +\infty. \tag{2.6}$$

When $r = +\infty$, we simply write $\mathcal{L}^k$ instead of $\mathcal{L}_r^k$.

Let $k \in \mathbb{N}$, $r \in (0, +\infty]$, $\mu \in \mathcal{L}_r^k$ and let Ω be a domain in $\mathbb{R}^n$. The *generalized Hölder* space $\Lambda_{\infty,r}^{k,\mu(\cdot)}(\overline{\Omega})$ consists of all functions $f \in C_B(\Omega)$ for which the quasi-norm

$$\|f|\Lambda_{\infty,r}^{k,\mu(\cdot)}(\overline{\Omega})\| := \|f|B(\Omega)\| + \left\| t^{-\frac{1}{r}} \frac{\omega_k(f,t)}{\mu(t)} \right\|_{r;(0,1)} \tag{2.7}$$

is finite. Standard arguments show that the space $\Lambda_{\infty,r}^{k,\mu(\cdot)}(\overline{\Omega})$ is complete (cf. [20, Theorem 3.1.4]).

Conditions (2.5) and (2.6) are natural (see [16]) and, if $r = +\infty$, we can assume without loss of generality in the definition of $\Lambda_{\infty,r}^{k,\mu(\cdot)}(\overline{\Omega})$ that all the elements μ of $\mathcal{L}_r^k$ are continuous non-decreasing functions on the interval $(0,1)$ satisfying $\lim_{t\to 0_+} \mu(t) = 0$ (see again [16]).

If $\mu(t) = t$, $t \in (0,1)$, then $\Lambda_{\infty,\infty}^{1,\mu(\cdot)}(\overline{\Omega})$ coincides with the space $Lip(\Omega)$ of the Lipschitz functions. If $\mu(t) \equiv t^\alpha$, $\alpha \in (0,1)$, then the space $\Lambda_{\infty,r}^{1,\mu(\cdot)}(\overline{\Omega})$ coincides with the space $C^{0,\alpha,r}(\overline{\Omega})$ considered in [1, p. 232].

3. Auxiliary results and key estimate

In this section we summarize results, which we shall need in subsequent sections. Theorem 3.8 mentioned below gives sharp estimates for the k-modulus of smoothness of the convolution of a function f from an r.i.BFS $X(\mathbb{R}^n)$ with the Bessel potential kernel g_σ, $0 < \sigma < n$, with $k \geq [\sigma]+1$. Such estimates are essential in what follows. The case $0 < \sigma < 1$ and $k=1$ has been considered in [15, Theorem 1].

Some properties of slowly varying functions are given in the next lemma.

Lemma 3.1 (cf. [16, Lemma 3.1])**.** *Let $b \in SV(0,+\infty)$.*

(i) *If $\alpha > 0$ and $q \in (0,+\infty]$, then for all $t>0$,*

$$\left\|\tau^{\alpha-\frac{1}{q}} b(\tau)\right\|_{q;(0,t)} \approx t^\alpha b(t) \quad \text{and} \quad \left\|\tau^{-\alpha-\frac{1}{q}} b(\tau)\right\|_{q;(t,\infty)} \approx t^{-\alpha} b(t).$$

(ii) *If $\alpha > 0$, then*

$$\lim_{t\to 0_+} t^\alpha b(t) = 0 \quad \text{and} \quad \lim_{t\to 0_+} t^{-\alpha} b(t) = +\infty.$$

We shall need the following weighted Hardy inequalities, for which we refer to [23, Theorems 5.9, 5.10 and 9.3]. The case $r \in (0,1)$ and $q=+\infty$ can be proved as the case $1 \leq r < +\infty$ and $q = +\infty$ in [19, Theorem 1.3.1/2].

Lemma 3.2. *Let $q \in [1,+\infty]$, $r \in (0,+\infty]$, $a \in (0,+\infty]$ and $v, w \in \mathcal{W}(0,a)$.*

(i) *If $1 \leq q \leq r \leq +\infty$, then*

$$\left\|w(t)\int_0^t h(s)\,ds\right\|_{r;(0,a)} \leq C\,\|v(t)\,h(t)\|_{q;(0,a)} \quad \text{for all } h \in \mathcal{M}^+(0,a) \tag{3.1}$$

if and only if

$$A := \sup_{x\in(0,a)} \|w(t)\|_{r;(x,a)} \left\|(v(t))^{-1}\right\|_{q';(0,x)} < +\infty.$$

Moreover, the best possible constant C in (3.1) satisfies the estimate $C \approx A$ and the constants involved in this equivalence are independent of a.

(ii) *If $0 < r < q \leq +\infty$ and $q > 1$, then (3.1) holds if and only if*

$$B := \left(\int_0^a \left[\|w(t)\|_{r;(x,a)} \left\|(v(t))^{-1}\right\|_{q';(0,x)}^{\frac{q'}{r'}}\right]^u v^{-q'}(x)\,dx\right)^{\frac{1}{u}} < +\infty,$$

where $\frac{1}{u} = \frac{1}{r} - \frac{1}{q}$. Moreover, the best possible constant C in (3.1) satisfies $C \approx B$ and the constants involved in this equivalence are independent of a.

Now, we recall some more properties of moduli of smoothness. For each fixed f in $C_B(\Omega)$, $\omega_k(f,\cdot)$ is a non-negative non-decreasing function on $[0,+\infty)$. Putting $\widetilde{\omega}_k(f,t) := \frac{1}{t^k}\omega_k(f,t)$ for all $t>0$, one can prove that $\widetilde{\omega}_k(f,\cdot)$ is equivalent to a non-increasing function on $(0,+\infty)$. If $f\in C_B(\Omega)$, then

$$\omega_k(f,t) \le 2^k \|f|B(\Omega)\|, \quad t>0. \tag{3.2}$$

We refer to [3, pp. 331–333, 431], [6, pp. 40–50] and [7, 18] for more details about k-modulus of smoothness.

Let $r\in\mathbb{N}$ and let $f\in C_B(\Omega)$. One can estimate $\omega_r(f,\cdot)$ by means of moduli of smoothness of lower order:

$$\omega_r(f,\cdot) \le 2^{r-k}\omega_k(f,\cdot), \quad 1\le k\le r. \tag{3.3}$$

Marchaud has shown that (3.3) has a weak inverse (when $\Omega=\mathbb{R}^n$). Namely, one can dominate $\omega_k(f,\cdot)$ by means of an integral of $\omega_r(f,\cdot)$, $k<r$. See, for example, [3, Theorem V.4.4] or [6, Theorem II.8.1]. If a domain Ω has minimally smooth boundary (see [3, p. 430] or [24, p. 189]), then the Marchaud inequality still holds. We refer to [7, 18] for more details.

Theorem 3.3 (Marchaud)**.** *Let Ω be a domain in $\mathbb{R}^n$ with minimally smooth boundary. Let $k,r\in\mathbb{N}$, $k<r$. Then there is a positive constant c such that*

$$\omega_k(f,t) \le c\,t^k\left(\|f|B(\Omega)\| + \int_t^{+\infty}\frac{\omega_r(f,s)}{s^k}\frac{ds}{s}\right) \tag{3.4}$$

for all $t>0$ and all f in $C_B(\Omega)$.

When $\Omega=\mathbb{R}^n$, the term $\|f|B(\Omega)\|$ can be dropped (see [3]).

One can easily see from (3.4) that

$$\omega_k(f,t) \le c\left(\frac{2^r}{k}+1\right)t^k\|f|B(\Omega)\| + c\,t^k\int_t^1\frac{\omega_r(f,s)}{s^k}\frac{ds}{s}$$

for all $t\in(0,1)$ and all f in $C_B(\Omega)$.

The next lemma is a straightforward extension of [14, Lemma 4.2].

Lemma 3.4. *Let Ω be a domain in $\mathbb{R}^n$ with minimally smooth boundary. Let $k\in\mathbb{N}$ and let $\mathcal{S}$ be a bounded subset of $C_B(\Omega)$ such that $\lim_{t\to 0^+}\sup_{u\in\mathcal{S}}\omega_k(u,t)=0$. Then $\lim_{t\to 0^+}\sup_{u\in\mathcal{S}}\omega(u,t)=0$. In particular, if $u\in C_B(\Omega)$ satisfies $\lim_{t\to 0^+}\omega_k(u,t)=0$, then $\lim_{t\to 0^+}\omega(u,t)=0$, which means that $u\in C(\overline{\Omega})$.*

Proof. Suppose that $k\ge 2$, otherwise the result is trivial. Since $\mathcal{S}$ is a bounded subset of $C_B(\Omega)$, there is $K>0$ such that $\sup_{u\in\mathcal{S}}\|u|B(\Omega)\|<K$. Let $u\in\mathcal{S}$. By Marchaud's inequality (3.4), there exists $c>0$ such that

$$\omega(u,t) \le c\,t\left(\|u|B(\Omega)\| + \int_t^{+\infty}\frac{\omega_k(u,s)}{s}\frac{ds}{s}\right), \quad t>0. \tag{3.5}$$

Let $\varepsilon > 0$. Since $\lim_{t\to 0^+} \sup_{u\in\mathcal{S}} \omega_k(u,t) = 0$, there is $\delta_1 > 0$ such that $\sup_{u\in\mathcal{S}} \omega_k(u,t) < \frac{\varepsilon}{2c}$ for all $t \in (0,\delta_1)$. Hence, by (3.5) and (3.2),

$$\begin{aligned}\omega(u,t) &\le c\,t \int_t^{\frac{\delta_1}{2}} \frac{\omega_k(u,s)}{s}\,\frac{ds}{s} + c\,t \int_{\frac{\delta_1}{2}}^{+\infty} \frac{\omega_k(u,s)}{s}\,\frac{ds}{s} + c\,t\|u|B(\Omega)\| \\ &\le c\,\omega_k\Big(u, \frac{\delta_1}{2}\Big) + c\,t\,\Big(2^k \frac{2}{\delta_1} + 1\Big)\|u|B(\Omega)\| \\ &< \frac{\varepsilon}{2} + c\,t\Big(2^k \frac{2}{\delta_1} + 1\Big)K \quad \text{for all } t \in \Big(0, \frac{\delta_1}{2}\Big).\end{aligned}$$

Thus, taking $\delta := \min\left\{\frac{\delta_1}{2}, \frac{\varepsilon\,\delta_1}{c\,(2^{k+2}+2\delta_1)K}\right\}$, we obtain $\sup_{u\in\mathcal{S}} \omega(u,t) \le \varepsilon$ for all $t \in (0,\delta)$, and the result follows. □

The next lemma shows that if we define the generalized Hölder space $\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$ as a subspace of $C(\overline{\Omega})$ rather than a subspace of $C_B(\Omega)$, then both definitions coincide provided that Ω is a domain in $\mathbb{R}^n$ with minimally smooth boundary.

Lemma 3.5. *Let Ω be a domain in $\mathbb{R}^n$ with minimally smooth boundary. Let $k \in \mathbb{N}$, $r \in (0,+\infty]$ and let $\mu \in \mathcal{L}^k_r$. Then*

$$\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega}).$$

Proof. Let $f \in \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$. Then there is $M \in (0,+\infty)$ such that

$$\left\| t^{-\frac{1}{r}} \frac{\omega_k(f,t)}{\mu(t)} \right\|_{r;(0,1)} < M.$$

Since also, for all $t \in (0,1)$,

$$\left\| \tau^{-\frac{1}{r}} \frac{\omega_k(f,\tau)}{\mu(\tau)} \right\|_{r;(t,1)} \ge \omega_k(f,t) \left\| \tau^{-\frac{1}{r}} \frac{1}{\mu(\tau)} \right\|_{r;(t,1)},$$

we obtain that

$$\omega_k(f,t) \lesssim \left\| \tau^{-\frac{1}{r}} \frac{1}{\mu(\tau)} \right\|^{-1}_{r;(t,1)} \quad \text{for all } \ t \in (0,1).$$

Together with (2.5), this implies that $\omega_k(f,t) \to 0$ as $t \to 0_+$. Now, Lemma 3.4 implies that f is uniformly continuous on Ω. □

We shall need the following result.

Lemma 3.6. *Let $k \in \mathbb{N}$, $r \in (0, +\infty]$ and $\lambda, \mu \in \mathcal{L}_r^k$. Let Ω be a domain in $\mathbb{R}^n$ and let $\xi \in (0, \frac{1}{2}]$. If*

$$A := \frac{\sup_{x\in(0,\xi)} \left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \right\|_{r;(x,\xi)}}{\left\| t^{-\frac{1}{r}} (\lambda(t))^{-1} \right\|_{r;(x,1)}} < +\infty, \tag{3.6}$$

then

$$\left\| t^{-\frac{1}{r}} \frac{\omega_{k,\Omega}(f,t)}{\mu(t)} \right\|_{r;(0,\xi)} \le C \left\| t^{-\frac{1}{r}} \frac{\omega_{k,\Omega}(f,t)}{\lambda(t)} \right\|_{r;(0,1)} \quad \textit{for all } f \in C_B(\Omega). \tag{3.7}$$

Moreover, if C is the best constant in (3.7)*, then $C \le A$.*

Proof. When $r \in (0, +\infty)$, the result follows from [17, Lemma 2.6]) because $(\omega_{k,\Omega}(f, \cdot))^r$ is a non-decreasing function.

If $r = +\infty$, we can assume without loss of generality that λ and μ are continuous and non-decreasing functions on the interval $(0, 1)$ (cf. [16, the end of Section 2]). Then (3.6) implies that $(\mu(x))^{-1} \le A(\lambda(x))^{-1}$ for $x \in (0, \xi)$ and (3.7) is clear. □

We continue with some important results from [14,16] which are fundamental for what follows.

Remark 3.7 ([14, Remark 3.2])**.** Let $\sigma \in (0, n)$ and let $X = X(\mathbb{R}^n)$ be an r.i.BFS such that $\|g_\sigma\|_{X'} < +\infty$. Then

$$\int_0^1 s^{\frac{\sigma}{n}-1} f^*(s)\, ds < +\infty \quad \text{for all } f \in X$$

(which implies that a function $f \in X(\mathbb{R}^n)$ belongs to the Lorentz space $L^{\frac{n}{\sigma},1}(B)$ for any ball $B \subset \mathbb{R}^n$).

The next theorem gives sharp estimates for the k-modulus of smoothness of the convolution of a function f from an r.i.BFS $X(\mathbb{R}^n)$ with the Bessel potential kernel g_σ, $0 < \sigma < n$, when $k \ge [\sigma] + 1$. The case $0 < \sigma < 1$ and $k = 1$ has been considered in [15, Theorem 1].

Theorem 3.8 ([14, Theorem 3.1])**.** *Let $\sigma \in (0, n)$ and let $X = X(\mathbb{R}^n)$ be an r.i.BFS such that $\|g_\sigma\|_{X'} < +\infty$. Then $f * g_\sigma \in C(\overline{\mathbb{R}^n})$ for all $f \in X$ and*

$$\omega_k(f * g_\sigma, t) \precsim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \quad \textit{for all } t \in (0,1) \textit{ and every } f \in X, \tag{3.8}$$

where $k \ge [\sigma] + 1$.

Moreover, estimate (3.8) *is sharp in the sense that given* $k \in \mathbb{N}$, *there are* (*small enough*) $\delta \in (0,1)$ *and* (*big enough*) $\alpha > 0$ *such that*

$$\omega_k(\overline{f} * g_\sigma, t) \gtrsim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \quad \text{for all } t \in (0,1) \text{ and every } f \in X, \tag{3.9}$$

where

$$\overline{f}(x) := f^*(\beta_n |x|^n) \chi_{C_\alpha(0,\delta)}(x), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

$C_\alpha(0,\delta) := C_\alpha \cap B(0,\delta)$ *with* $C_\alpha := \{y \in \mathbb{R}^n : y_1 > 0,\ y_1^2 > \alpha \sum_{i=2}^n y_i^2\}$.

Remark 3.9. We shall investigate the compactness of the embedding

$$H^\sigma X(\mathbb{R}^n) \hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega}), \tag{3.10}$$

where Ω will be a bounded domain in $\mathbb{R}^n$. Note that, by (3.10), the restriction to Ω of a function $u \in H^\sigma X(\mathbb{R}^n)$ belongs to the space $\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$.‡ Note also that $u = f * g_\sigma$ for some $f \in X(\mathbb{R}^n)$. Under the assumptions of Theorem 3.8, $u \in C(\overline{\mathbb{R}^n})$, which implies that $u \in C(\overline{\Omega})$. To calculate $\|u|\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})\|$, we need the k-modulus of smoothness $\omega_k(u,t) = \omega_{k,\Omega}(u,t)$, $t \ge 0$, of the function u. Recall also that the k-modulus of smoothness $\omega(f * g_\sigma, \cdot)$ involved in Theorem 3.8 is the k-modulus of smoothness with respect to the whole $\mathbb{R}^n$, that is, $\omega_{k,\mathbb{R}^n}(f * g_\sigma, \cdot)$.

To characterize the compactness of the embedding (3.10), we shall need analogues of estimates (3.8) and (3.9) with ω_k replaced by $\omega_{k,\Omega}$. Since

$$\omega_{k,\Omega}(f * g_\sigma, t) \le \omega_{k,\mathbb{R}^n}(f * g_\sigma, t), \quad t \ge 0,$$

estimate (3.8) implies that

$$\omega_{k,\Omega}(f * g_\sigma, t) \lesssim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \qquad \text{for all } t \in (0,1) \text{ and every } f \in X, \tag{3.11}$$

where $k \ge [\sigma] + 1$. To get an analogue of (3.9), take $x_0 = (x_{01}, \dots, x_{0n}) \in \Omega$ and $\delta_1 \in (0,1]$ so that $B(x_0, \delta_1) \subset \Omega$. Then, for given $k \in \mathbb{N}$,

$$\omega_{k,\Omega}(\overline{f} * g_\sigma, t) \gtrsim \int_0^{t^n} s^{\frac{\sigma}{n}-1} f^*(s)\, ds \quad \text{for all } t \in (0,1) \text{ and every } f \in X, \tag{3.12}$$

where

$$\overline{f}(x) := f^*(\beta_n |x - x_0|^n) \chi_{C_\alpha(x_0,\delta_2)}(x), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{3.13}$$

$C_\alpha(x_0,\delta_2) := (x_0 + C_\alpha) \cap B(x_0,\delta_2)$ with $C_\alpha := \{y \in \mathbb{R}^n : y_1 > 0, y_1^2 > \alpha \sum_{i=2}^n y_i^2\}$, $\delta_2 := \min\{\delta, \delta_1\}$ and $\delta \in (0,1)$ is given by Theorem 3.8. Indeed, take $t \in (0, \frac{\delta_2}{k})$

‡In the whole paper we use the symbol u both for the function u and its restriction to Ω.

and put $\bar{t} = (-t, 0, \ldots, 0) \in \mathbb{R}^n$ and $\overline{u} = \overline{f} * g_\sigma$. Then, instead of [14, (4.12)], we now have

$$\begin{aligned}
\omega_{k,\Omega}(\overline{u}, t) &\geq \left|(\Delta_{\bar{t}}^k \overline{u})(x_0)\right| \\
&= \left|\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \overline{u}(x_0 + i\bar{t})\right| \\
&= \left|\int_{C_\alpha(x_0,\delta_2)} f^*(\beta_n |y - x_0|^n) \left(\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} g_\sigma(x_0 + i\bar{t} - y)\right) dy\right| \\
&= \left|\int_{C_\alpha(0,\delta_2)} f^*(\beta_n |y|^n) \left(\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} g_\sigma(y - i\bar{t})\right) dy\right|,
\end{aligned}$$

with $C_\alpha(0, \delta_2) = -x_0 + C_\alpha(x_0, \delta_2)$, and the same arguments as those used in part (ii) of the proof of Theorem 3.8 yield (3.12).

In what follows we shall omit again the subscript Ω at k-modulus of smoothness (since it will be always clear from the context which k-modulus of smoothness we have in mind).

The next lemma is a consequence of Theorem 3.8 and [15, Lemma 6].

Lemma 3.10. *Let $X = X(\mathbb{R}^n)$ be an r.i.BFS, let Ω be a domain in $\mathbb{R}^n$ and let $\sigma > 0$. Then $H^\sigma X(\mathbb{R}^n) \hookrightarrow C(\overline{\Omega})$ if and only if $\|g_\sigma\|_{X'} < +\infty$.*

4. Proof of Theorem 1.3

`Sufficiency`. Since $\mathcal{S}$ is bounded in $\Lambda_{\infty,r}^{k,\mu(\cdot)}(\overline{\Omega})$, Lemma 3.5 implies that $\mathcal{S}$ is also bounded in $C(\overline{\Omega})$.

Let $\varepsilon \in (0, 1)$. By (1.6), there is $\delta \in (0, 1)$ such that

$$\sup_{u\in\mathcal{S}} \left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \omega_k(u, t) \right\|_{r;(0,\delta)} < \frac{\varepsilon}{4}. \tag{4.1}$$

Now, by (3.2) and (2.7), for this δ, there is a positive constant $c(\delta)$ such that, for all $u \in \mathcal{S}$,

$$\left\| u | \Lambda_{\infty,r}^{k,\mu(\cdot)}(\overline{\Omega}) \right\| \leq c(\delta) \| u | B(\Omega) \| + \left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \omega_k(u, t) \right\|_{r;(0,\delta)}. \tag{4.2}$$

By (4.1), for all ξ with $|\xi| < \delta$,

$$1 > \sup_{u\in\mathcal{S}} \left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \omega_k(u, t) \right\|_{r;(\xi,\delta)} \geq \sup_{u\in\mathcal{S}} \omega_k(u, \xi) \left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \right\|_{r;(\xi,\delta)}.$$

Hence,

$$\sup_{u\in\mathcal{S}} \omega_k(u,\xi) \precsim \left\| t^{-\frac{1}{r}}(\mu(t))^{-1} \right\|_{r;(\xi,\delta)}^{-1} \quad \text{if } |\xi| < \delta$$

and (2.5) implies that $\sup_{u\in\mathcal{S}} \omega_k(u,\xi) \to 0$ as $\xi \to 0_+$. Thus, by Lemma 3.4, $\sup_{u\in\mathcal{S}} \omega(u,\xi) \to 0$ as $\xi \to 0_+$, which means that $\mathcal{S}$ is equicontinuous. Therefore, the Ascoli–Arzelà theorem implies that $\mathcal{S}$ is totally bounded in $C(\overline{\Omega})$. Consequently, there exists a finite $\frac{\varepsilon}{2\,c(\delta)}$-net $\{u_1,\ldots,u_N\} \subset \mathcal{S}$ such that

$$\min_{m\in\{1,\ldots,N\}} \|u - u_m|B(\Omega)\| < \frac{\varepsilon}{2\,c(\delta)} \quad \text{for all } u \in \mathcal{S}. \tag{4.3}$$

Using estimates (4.2), (4.3) and (4.1), we obtain for any $u \in \mathcal{S}$ that

$$\begin{aligned}
\min_{m\in\{1,\ldots,N\}} \left\| u - u_m | \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \right\| &\le c(\delta) \min_{m\in\{1,\ldots,N\}} \|u - u_m|B(\Omega)\| \\
&\quad + \sup_{m\in\{1,\ldots,N\}} \left\| t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_k(u-u_m,t) \right\|_{r;(0,\delta)} \\
&\le c(\delta) \min_{m\in\{1,\ldots,N\}} \|u - u_m|B(\Omega)\| \\
&\quad + \sup_{m\in\{1,\ldots,N\}} \left\| t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_k(u,t) \right\|_{r;(0,\delta)} \\
&\quad + \sup_{m\in\{1,\ldots,N\}} \left\| t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_k(u_m,t) \right\|_{r;(0,\delta)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon,
\end{aligned}$$

which proves that $\mathcal{S}$ is totally bounded in $\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$.

`Necessity.` Suppose that $\mathcal{S}$ is totally bounded in $\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$. Then $\mathcal{S}$ is bounded in $\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$. On the other hand, given $\varepsilon > 0$, there exists a finite $\frac{\varepsilon}{2}$-net $\{u_1,\ldots,u_N\} \subset \mathcal{S}$ such that

$$\min_{m\in\{1,\ldots,N\}} \left\| u - u_m | \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \right\| < \frac{\varepsilon}{2} \quad \text{for all } u \in \mathcal{S}. \tag{4.4}$$

Because $r \in (0,+\infty)$, for each $m \in \{1,\ldots,N\}$ there is $\delta_m > 0$ such that

$$\left\| t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_k(u_m,t) \right\|_{r;(0,\delta_m)} < \frac{\varepsilon}{2}. \tag{4.5}$$

Let $\delta := \min_{m\in\{1,\ldots,N\}} \delta_m$. Since, for all $u \in \mathcal{S}$, any $m \in \{1,\ldots,N\}$ and all $t \in (0,1)$, $\omega_k(u,t) \le \omega_k(u-u_m,t) + \omega_k(u_m,t)$, (4.4) and (4.5) imply that

$$\begin{aligned}
\left\| t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_k(u,t) \right\|_{r;(0,\delta)} &\le \min_{m\in\{1,\ldots,N\}} \left\| t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_k(u-u_m,t) \right\|_{r;(0,1)} \\
&\quad + \sup_{m\in\{1,\ldots,N\}} \left\| t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_k(u_m,t) \right\|_{r;(0,\delta)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } u \in \mathcal{S}.
\end{aligned}$$

Therefore, $\sup_{u\in\mathcal{S}} \|t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_k(u,t)\|_{r;(0,\delta)} \le \varepsilon$ and (1.6) follows. □

Remark 4.1. (i) In Theorem 1.3 the implication

$$\begin{aligned} &\mathcal{S} \subset \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \ \text{ is bounded and (1.6) holds} \\ &\implies \mathcal{S} \text{ is totally bounded in } \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega}) \end{aligned} \tag{4.6}$$

remains true even if $r = +\infty$. (This can be seen from the proof of Theorem 1.3.)

(ii) If $r = +\infty$ in Theorem 1.3, then the reverse implication to (4.6) holds provided that we assume $\mathcal{S} \subset \Lambda^{k,\mu(\cdot),0}_{\infty,\infty}(\overline{\Omega})$. Here $\Lambda^{k,\mu(\cdot),0}_{\infty,\infty}(\overline{\Omega})$ is a subspace of $\Lambda^{k,\mu(\cdot)}_{\infty,\infty}(\overline{\Omega})$ consisting of those functions u which satisfy

$$\lim_{\delta\to 0_+} \left\|(\mu(t))^{-1}\,\omega_k(u,t)\right\|_{\infty;(0,\delta)} = 0.$$

(This follows from the necessity part of the proof of Theorem 1.3.)

(iii) Summarizing what we have said, we arrive at the following result: *Let $\mu \in \mathcal{L}^k$ and let Ω be a bounded domain in $\mathbb{R}^n$ with minimally smooth boundary. Then $\mathcal{S} \subset \Lambda^{k,\mu(\cdot),0}_{\infty,\infty}(\overline{\Omega})$ is totally bounded in $\Lambda^{k,\mu(\cdot)}_{\infty,\infty}(\overline{\Omega})$ if and only if $\mathcal{S}$ is bounded in $\Lambda^{k,\mu(\cdot)}_{\infty,\infty}(\overline{\Omega})$ and*

$$\sup_{u\in\mathcal{S}} \left\|(\mu(t))^{-1}\,\omega_k(u,t)\right\|_{\infty;(0,\xi)} \to 0 \quad \text{as } \xi \to 0_+.$$

5. Proofs of Theorem 1.4 and Corollary 1.5

Proof of Theorem 1.4.

`Sufficiency`. Since Ω is bounded, there is a ball B_1 such that $\Omega \subset B_1$. By (1.8), there is $\delta \in (0,1)$ such that

$$\sup_{\|f\|_X\le 1} \left\| t^{-\frac{1}{r}}(\mu(t))^{-1} \int_0^{t^n} \tau^{\frac{\sigma}{n}-1} f^*(\tau)\,d\tau \right\|_{r;(0,\delta)} \le 1. \tag{5.1}$$

As $g^*_\sigma(t) \approx t^{\frac{\sigma}{n-1}}$ for all $t \in (0,1)$ (cf. [9]), $\int_0^1 \tau^{\frac{\sigma}{n}-1} f^*(\tau)\,d\tau \precsim \|g_\sigma\|_{X'}\,\|f\|_X$ for all $f \in X$. This estimate and (2.6) imply that, for all $f \in X$,

$$\left\| t^{-\frac{1}{r}}(\mu(t))^{-1} \int_0^{t^n} \tau^{\frac{\sigma}{n}-1} f^*(\tau)\,d\tau \right\|_{r;[\delta,1)} \precsim \left\| t^{-\frac{1}{r}}(\mu(t))^{-1} \right\|_{r;[\delta,1)} \|g_\sigma\|_{X'}\,\|f\|_X \precsim \|f\|_X.$$

Together with estimate (5.1), this yields

$$\sup_{\|f\|_X\le 1} \left\| t^{-\frac{1}{r}}(\mu(t))^{-1} \int_0^{t^n} \tau^{\frac{\sigma}{n}-1} f^*(\tau)\,d\tau \right\|_{r;(0,1)} \precsim 1\,.$$

Therefore, by Theorem 1.1, $H^\sigma X(\mathbb{R}^n) \hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$. Consequently, the unit ball of $H^\sigma X$ is bounded in $\Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{B_1})$.

Let $f \in X$ be such that $\|f\|_X \le 1$. Then, by (3.11) of Remark 3.9,

$$\begin{aligned}&\sup_{\|f\|_X\le 1}\left\|t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_{k,B_1}(f*g_\sigma,t)\right\|_{r;(0,\xi)}\\ &\lesssim \sup_{\|f\|_X\le 1}\left\|t^{-\frac{1}{r}}(\mu(t))^{-1}\int_0^{t^n}\tau^{\frac{\sigma}{n}-1}f^*(\tau)\,d\tau\right\|_{r;(0,\xi)},\end{aligned}$$

which, together with (1.8), gives

$$\sup_{\|f\|_X\le 1}\left\|t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_{k,B_1}(f*g_\sigma,t)\right\|_{r;(0,\xi)} \to 0 \quad \text{as } \xi\to 0_+.$$

With respect to (2.3) and (2.4), this and Theorem 1.3 imply that the unit ball of the space $H^\sigma X$ is totally bounded in $\Lambda^{k;\mu(\cdot)}_{\infty,r}(\overline{B_1})$. Since $\Lambda^{k;\mu(\cdot)}_{\infty,r}(\overline{B_1}) \hookrightarrow \Lambda^{k;\mu(\cdot)}_{\infty,r}(\overline{\Omega})$, the result follows.

`Necessity.` Suppose that (1.7) holds. Since Ω is open, there is a ball B_0 such that $B_0 \subset \Omega$. Then (1.7) also holds with Ω replaced by B_0. Let $f \in X$, $\|f\|_X \le 1$, and define $\overline{f}$ by (3.13). Since $(\overline{f})^* \le f^*$, we have that $\|\overline{f}\|_X \le 1$. Moreover, by (3.12), for all $\xi \in (0,1)$,

$$\begin{aligned}\sup_{\|h\|_X\le 1}\left\|t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_{k,B_0}(h*g_\sigma,t)\right\|_{r;(0,\xi)} &\ge \left\|t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_{k,B_0}(\overline{f}*g_\sigma,t)\right\|_{r;(0,\xi)}\\ &\gtrsim \left\|t^{-\frac{1}{r}}(\mu(t))^{-1}\int_0^{t^n}\tau^{\frac{\sigma}{n}-1}f^*(\tau)\,d\tau\right\|_{r;(0,\xi)}.\end{aligned}$$

Hence,

$$\begin{aligned}&\sup_{\|h\|_X\le 1}\left\|t^{-\frac{1}{r}}(\mu(t))^{-1}\omega_{k,B_0}(h*g_\sigma,t)\right\|_{r;(0,\xi)}\\ &\gtrsim \sup_{\|f\|_X\le 1}\left\|t^{-\frac{1}{r}}(\mu(t))^{-1}\int_0^{t^n}\tau^{\frac{\sigma}{n}-1}f^*(\tau)\,d\tau\right\|_{r;(0,\xi)}.\end{aligned}$$

This, (1.7) with Ω replaced by B_0, (2.3), (2.4) and Theorem 1.3, imply (1.8). □

Remark 5.1. The implication (1.7) $\Longrightarrow$ (1.8) in Theorem 1.4 remains true if $k \in \mathbb{N}$ (cf. Theorem 3.8, Remark 3.9 and the necessity part in the proof of Theorem 1.4).

Proof of Corollary 1.5.
Using Theorem 1.4 and Lemma 3.10, we obtain the result. □

Remark 5.2. (i) In Theorem 1.4 the implication (1.8) $\Longrightarrow$ (1.7) remains true even if $r = +\infty$. (This can be seen from Remark 4.1 (i) and the proof of Theorem 1.4.)

(ii) We see from Remark 4.1 (iii) that if we assume additionally in Theorem 1.4 that $r = +\infty$ and the space $X(\mathbb{R}^n)$ and $\mu \in \mathcal{L}^k$ are such that

$$H^\sigma X(\mathbb{R}^n) \hookrightarrow \Lambda^{k,\mu(\cdot),0}_{\infty,\infty}(\overline{\Omega}), \tag{5.2}$$

then (1.7) is equivalent to (1.8).

(iii) For example, (5.2) is satisfied provided that

$$\text{the Schwartz space } \mathscr{S}(\mathbb{R}^n) \text{ is dense in } H^\sigma X(\mathbb{R}^n) \tag{5.3}$$

$$H^\sigma X(\mathbb{R}^n) \hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,\infty}(\overline{\Omega}) \tag{5.4}$$

$$\lim_{t\to 0_+} \frac{t^k}{\mu(t)} = 0. \tag{5.5}$$

Indeed, given $u \in H^\sigma X(\mathbb{R}^n)$ and $\varepsilon > 0$, there is $v \in \mathscr{S}(\mathbb{R}^n)$ such that $\|u - v\|_{H^\sigma X} < \varepsilon$. Moreover, $\omega_k(v, t) \le ct^k$ for all $t \in (0, 1)$, where $c = c(v)$ is a positive constant. Thus, using also (5.4), we obtain

$$\begin{aligned} \|(\mu(t))^{-1}\omega_k(u,t)\|_{\infty;(0,\delta)} &\le \|(\mu(t))^{-1}\omega_k(u-v,t)\|_{\infty;(0,\delta)} + \|(\mu(t))^{-1}\omega_k(v,t)\|_{\infty;(0,\delta)} \\ &\precsim \|u-v\|_{H^\sigma X} + c\left\|\frac{t^k}{\mu(t)}\right\|_{\infty;(0,\delta)} \\ &\le \varepsilon + c\left\|\frac{t^k}{\mu(t)}\right\|_{\infty;(0,\delta)} \qquad \text{for all } \delta \in (0,1). \end{aligned}$$

Together with (5.5), this implies (5.2).

For instance, (5.3) holds if

$$\text{the Schwartz space } \mathscr{S}(\mathbb{R}^n) \text{ is dense in } X(\mathbb{R}^n). \tag{5.6}$$

Indeed, this is a consequence of (2.3), (2.4), the fact that the mapping $h \mapsto g_\sigma * h$ maps $\mathscr{S}(\mathbb{R}^n)$ on $\mathscr{S}(\mathbb{R}^n)$, and (5.6). In particular, (5.6) is satisfied provided that the r.i.BFS $X(\mathbb{R}^n)$ has absolutely continuous norm (cf. [10, Remark 3.13]).

6. Proofs of Theorems 1.6, 1.8 and Corollaries 1.7, 1.9

To prove Theorems 1.6 and 1.8, we are going to apply Theorem 1.4. If the space X is a Lorentz–Karamata space $L_{p,q;b}(\mathbb{R}^n)$ with $p \in (1, +\infty)$, $q \in [1, +\infty]$ and $b \in SV(0, \infty)$, then X coincides with an r. i. Banach function space and

$$X' = L_{p',q';1/b}(\mathbb{R}^n)$$

(see [21, Theorem 3.1] and replace γ_b by b and $\gamma_{\frac{1}{b}}$ by $\frac{1}{b}$ there). To verify the assumption $\|g_\sigma\|_{X'} < +\infty$ in Theorem 1.4, we shall use the next lemma.

Lemma 6.1 ([15, Lemma 7]). *Let* $\sigma \in (0,n)$, $p \in (1,+\infty)$, $q \in [1,+\infty]$ *and* $b \in SV(0,+\infty)$. *If* $X = L_{p,q;b}(\mathbb{R}^n)$, *then* $g_\sigma \in X'$ *if and only if either*

$$p > \frac{n}{\sigma}$$

or

$$p = \frac{n}{\sigma} \quad \text{and} \quad \left\| t^{-\frac{1}{q'}} (b(t))^{-1} \right\|_{q';(0,1)} < +\infty.$$

To characterize (1.8) when X is the Lorentz–Karamata space $L_{p,q;b}(\mathbb{R}^n)$, we shall need the following lemma.

Lemma 6.2. *Let* $\sigma \in (0,n)$, $p \in [\frac{n}{\sigma},+\infty)$, $q \in [1,+\infty]$, $b \in SV(0,+\infty)$, $\xi \in (0,1)$, $r \in (0,+\infty]$ *and let* $\mu \in \mathcal{W}(0,1)$. *Then*

$$\sup_{\|f\|_{p,q;b}\le 1} \left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \int_0^{t^n} \tau^{\frac{\sigma}{n}-1} f^*(\tau)\, d\tau \right\|_{r;(0,\xi)} \to 0 \quad \text{as } \xi \to 0_+$$

if and only if

$$\sup_{N_\xi(h)\le 1} \left\| t^{-\frac{1}{r}} \left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1} \int_0^{t} h(\tau)\, d\tau \right\|_{r;(0,\xi)} \to 0 \quad \text{as } \xi \to 0_+, \tag{6.1}$$

where $N_\xi(h) := \left\| t^{\frac{1}{p}+\frac{1}{q'}-\frac{\sigma}{n}} b(t) h(t) \right\|_{q;(0,\xi)}$ *for all* $h \in \mathcal{M}^+(0,\xi)$ *and* $\xi \in (0,1)$.

Proof. The proof is the same as that of [15, Lemma 9], where the case $\sigma \in (0,1)$ was considered. □

Proof of Theorem 1.6.
Put $X = L_{p,q;b}(\mathbb{R}^n)$. By Lemma 6.1, $\|g_\sigma\|_{X'} < +\infty$. Consequently, by Theorem 1.4 and Lemma 6.2, $H^\sigma X \hookrightarrow\hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})$ if and only if (6.1) is satisfied.

(i) If $1 \le q \le r < +\infty$, then Lemma 3.2(i) states that (6.1) holds if and only if

$$\sup_{x\in(0,\xi)} \left\| t^{-\frac{1}{r}} \left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1} \right\|_{r;(x,\xi)} \left\| t^{\frac{\sigma}{n}-\frac{1}{p}-\frac{1}{q'}} (b(t))^{-1} \right\|_{q';(0,x)} \to 0 \quad \text{as } \xi \to 0_+. \tag{6.2}$$

Since $\frac{\sigma}{n} - \frac{1}{p} > 0$, Lemma 3.1 (i) shows that

$$\left\| t^{\frac{\sigma}{n}-\frac{1}{p}-\frac{1}{q'}} (b(t))^{-1} \right\|_{q';(0,x)} \approx x^{\frac{\sigma}{n}-\frac{1}{p}} (b(x))^{-1} \quad \text{for all } x > 0.$$

Thus, (6.2) is equivalent to

$$\sup_{x\in(0,\xi)} \left\| t^{-\frac{1}{r}} \left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1} \right\|_{r;(x,\xi)} x^{\frac{\sigma}{n}-\frac{1}{p}} (b(x))^{-1} \to 0 \quad \text{as } \xi \to 0_+. \tag{6.3}$$

Now, we are going to prove that (6.3) is equivalent to (1.11). First, suppose that (1.11) holds. Then, given $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\left\| t^{-\frac{1}{r}} \big(\mu\big(t^{\frac{1}{n}}\big)\big)^{-1} \right\|_{r;(x,1)} x^{\frac{\sigma}{n}-\frac{1}{p}} (b(x))^{-1} < \varepsilon \quad \text{for all } x \in (0,\delta).$$

Let $\xi \in (0,\delta)$. Then

$$\left\| t^{-\frac{1}{r}} \big(\mu\big(t^{\frac{1}{n}}\big)\big)^{-1} \right\|_{r;(x,\xi)} x^{\frac{\sigma}{n}-\frac{1}{p}} (b(x))^{-1} < \varepsilon \quad \text{for all } x \in (0,\xi).$$

Consequently,

$$\sup_{x\in(0,\xi)} \left\| t^{-\frac{1}{r}} \big(\mu\big(t^{\frac{1}{n}}\big)\big)^{-1} \right\|_{r;(x,\xi)} x^{\frac{\sigma}{n}-\frac{1}{p}} (b(x))^{-1} \le \varepsilon \quad \text{for all } \xi \in (0,\delta),$$

which gives (6.3).

Conversely, suppose that (6.3) holds. Then, given $\varepsilon > 0$, there exists $\Delta \in (0,1)$ such that

$$\sup_{x\in(0,\Delta)} \left\| t^{-\frac{1}{r}} \big(\mu\big(t^{\frac{1}{n}}\big)\big)^{-1} \right\|_{r;(x,\Delta)} x^{\frac{\sigma}{n}-\frac{1}{p}} (b(x))^{-1} < \frac{\varepsilon}{2}.$$

Since $\lambda(t) \to 0$ as $t \to 0_+$ (cf. Lemma 3.1(ii)) and $\mu \in \mathcal{L}_r^k$, we can find $\delta \in (0,\Delta)$ such that

$$\left\| t^{-\frac{1}{r}} \big(\mu\big(t^{\frac{1}{n}}\big)\big)^{-1} \right\|_{r;(\Delta,1)} \delta^{\frac{\sigma}{n}-\frac{1}{p}} (b(\delta))^{-1} < \frac{\varepsilon}{2}.$$

Therefore, since λ is equivalent to a non-decreasing function, for all $x \in (0,\delta)$,

$$\begin{aligned} \left\| t^{-\frac{1}{r}} \big(\mu\big(t^{\frac{1}{n}}\big)\big)^{-1} \right\|_{r;(x,1)} x^{\frac{\sigma}{n}-\frac{1}{p}} (b(x))^{-1} &\le \left\| t^{-\frac{1}{r}} \big(\mu\big(t^{\frac{1}{n}}\big)\big)^{-1} \right\|_{r;(x,\Delta)} x^{\frac{\sigma}{n}-\frac{1}{p}} (b(x))^{-1} \\ &\quad + \left\| t^{-\frac{1}{r}} \big(\mu\big(t^{\frac{1}{n}}\big)\big)^{-1} \right\|_{r;(\Delta,1)} x^{\frac{\sigma}{n}-\frac{1}{p}} (b(x))^{-1} \\ &\precsim \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and (1.11) follows.

(ii) If $0 < r < q \le +\infty$ and $q > 1$, then Lemma 3.2 (ii) states that (6.1) holds if and only if

$$\int_0^\xi \left\| t^{-\frac{1}{r}} \big(\mu\big(t^{\frac{1}{n}}\big)\big)^{-1} \right\|_{r;(x,\xi)}^u (V(x))^{\frac{u}{r'}} \Big(x^{\frac{\sigma}{n}-\frac{1}{p}} (b(x))^{-1} \Big)^{q'} \frac{dx}{x} \to 0 \quad \text{as } \xi \to 0_+, \tag{6.4}$$

where

$$V(x) = \left\| t^{\frac{\sigma}{n}-\frac{1}{p}-\frac{1}{q'}} (b(t))^{-1} \right\|_{q';(0,x)}^{q'}.$$

Since $\frac{\sigma}{n} - \frac{1}{p} > 0$, Lemma 3.1(i) and (1.9) imply that

$$V(x) \approx \left(x^{\frac{\sigma}{n}-\frac{1}{p}}(b(x))^{-1}\right)^{q'} = \left(\lambda\left(x^{\frac{1}{n}}\right)\right)^{q'} \quad \text{for all } x \in (0,1).$$

This and the identity $q'(\frac{u}{r'}+1) = u$ show that condition (6.4) can be rewritten as

$$\int_0^\xi \left\| t^{-\frac{1}{r}}\left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1} \right\|_{r;(x,\xi)}^u \left(\lambda\left(x^{\frac{1}{n}}\right)\right)^u \frac{dx}{x} \to 0 \quad \text{as } \xi \to 0_+. \tag{6.5}$$

Finally, since singularities of functions in question are only at the origin, (6.5) is equivalent to (1.13). □

Remark 6.3. Compact embeddings of spaces $H^\sigma L_{p,q;b}(\mathbb{R}^n)$ with $\sigma \in (0,1)$ into generalized Hölder spaces $\Lambda^{1,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$ in the superlimiting case (that is, when $p > \frac{n}{\sigma}$) were characterized in [15, Theorem 7]. Theorem 1.6 extends this result to the case when $\sigma \in (0,n)$ and when $\Lambda^{1,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$ is replaced by $\Lambda^{[\sigma]+1,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$. Its formulation is slightly different because in [15] *the definition of the class* $\mathcal{L}^1_r$ *was more restrictive* (in particular, the function $\mu \in \mathcal{L}^1_r$ was equivalent to an increasing function on the interval $(0,1]$).

Remark 6.4. As in Remark 5.1, we see that in Theorem 1.6 the implications (1.10) $\Longrightarrow$ (1.11) and (1.12) $\Longrightarrow$ (1.13) remain true if $k \in \mathbb{N}$.

Remark 6.5 ([16, Remark 5.5 (i)])**.** Let $\sigma \in (0,n)$, $p \in (\frac{n}{\sigma}, +\infty)$, $q \in [1,+\infty]$, $b \in SV(0,+\infty)$, $r \in [1,+\infty]$, $k = [\sigma]+1$, and let λ be given by (1.9). Then

$$\Lambda^{k,\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) = \Lambda^{[\sigma-\frac{n}{p}]+1,\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}).$$

Proof of Corollary 1.7. Suppose that (1.11) holds. Then, by [16, Corollary 1.4],

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{[\sigma-\frac{n}{p}]+1,\lambda(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}). \tag{6.6}$$

Since Ω is bounded, there is an open ball B_1 such that $\Omega \subset B_1$ and (6.6) implies that

$$H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{[\sigma-\frac{n}{p}]+1,\lambda(\cdot)}_{\infty,r}(\overline{B_1}). \tag{6.7}$$

On the other hand, (1.11) is equivalent to

$$\sup_{x\in(0,\xi)} \left\| t^{-\frac{1}{r}}(\mu(t))^{-1} \right\|_{r;(x,\xi)} x^{\sigma-\frac{n}{p}}\left(b(x^n)\right)^{-1} \to 0 \quad \text{as } \xi \to 0_+ \tag{6.8}$$

(see the proof of Theorem 1.6). Moreover,

$$\left\| t^{-\frac{1}{r}}(\lambda(t))^{-1} \right\|_{r;(x,1)}^{-1} \approx x^{\sigma-\frac{n}{p}}\left(b(x^n)\right)^{-1} \quad \text{for all } x \in \left(0,\frac{1}{2}\right) \tag{6.9}$$

(cf. Lemma 3.1(i) and the fact that $-\sigma + \frac{n}{p} < 0$).

Therefore, by Lemma 3.6 with $\xi = \frac{1}{2}$ and $k = \left[\sigma - \frac{n}{p}\right] + 1$, we obtain that

$$\left\| t^{-\frac{1}{r}} \frac{\omega_k(f,t)}{\mu(t)} \right\|_{r;(0,\frac{1}{2})} \precsim \left\| t^{-\frac{1}{r}} \frac{\omega_k(f,t)}{\lambda(t)} \right\|_{r;(0,1)} \quad \text{for all } f \in \Lambda_{\infty,r}^{[\sigma-\frac{n}{p}]+1,\lambda(\cdot)}(\overline{B_1}). \tag{6.10}$$

Making use of (3.2) and the fact that $\mu \in \mathcal{L}_r^{[\sigma-\frac{n}{p}]+1}$, we arrive at

$$\begin{aligned} \left\| t^{-\frac{1}{r}} \frac{\omega_k(f,t)}{\mu(t)} \right\|_{r;(\frac{1}{2},1)} &= \left\| t^{-\frac{1}{r}} \frac{t^k}{\mu(t)} \frac{\omega_k(f,t)}{t^k} \right\|_{r;(\frac{1}{2},1)} \\ &\precsim \|f|B(B_1)\| \left\| t^{-\frac{1}{r}} \frac{t^k}{\mu(t)} \right\|_{r;(\frac{1}{2},1)} \\ &\precsim \|f|B(B_1)\| \quad \text{for all } f \in B(B_1). \end{aligned}$$

Together with (6.10), this shows that

$$\left\| t^{-\frac{1}{r}} \frac{\omega_k(f,t)}{\mu(t)} \right\|_{r;(0,1)} \precsim \|f|B(B_1)\| + \left\| t^{-\frac{1}{r}} \frac{\omega_k(f,t)}{\lambda(t)} \right\|_{r;(0,1)}$$

for all $f \in \Lambda_{\infty,r}^{[\sigma-\frac{n}{p}]+1,\lambda(\cdot)}(\overline{B_1})$. Consequently,

$$\Lambda_{\infty,r}^{[\sigma-\frac{n}{p}]+1,\lambda(\cdot)}(\overline{B_1}) \hookrightarrow \Lambda_{\infty,r}^{[\sigma-\frac{n}{p}]+1,\mu(\cdot)}(\overline{B_1}). \tag{6.11}$$

Now we are going to prove that embedding (6.11) is compact. Let $\mathcal{S}$ be the closed unit ball in $\Lambda_{\infty,r}^{[\sigma-\frac{n}{p}]+1,\lambda(\cdot)}(\overline{B_1})$. Let $\xi \in (0, \frac{1}{2})$ and $f \in \mathcal{S}$. By (6.8), (6.9) and Lemma 3.6 with $k = [\sigma - \frac{n}{p}] + 1$, we obtain that

$$\begin{aligned} \left\| t^{-\frac{1}{r}} \frac{\omega_k(f,t)}{\mu(t)} \right\|_{r;(0,\xi)} &\le \left(\sup_{x\in(0,\xi)} \frac{\left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \right\|_{r;(x,\xi)}}{\left\| t^{-\frac{1}{r}} (\lambda(t))^{-1} \right\|_{r;(x,1)}} \right) \left\| t^{-\frac{1}{r}} \frac{\omega_k(f,t)}{\lambda(t)} \right\|_{r;(0,1)} \\ &\le \sup_{x\in(0,\xi)} \frac{\left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \right\|_{r;(x,\xi)}}{\left\| t^{-\frac{1}{r}} (\lambda(t))^{-1} \right\|_{r;(x,1)}}. \end{aligned}$$

Together with (6.8) and (6.9), this implies that

$$\sup_{f\in\mathcal{S}} \left\| t^{-\frac{1}{r}} \frac{\omega_k(f,t)}{\mu(t)} \right\|_{r;(0,\xi)} \to 0 \quad \text{as } \xi \to 0_+.$$

Thus, by Theorem 1.3,

$$\Lambda_{\infty,r}^{[\sigma-\frac{n}{p}]+1,\lambda(\cdot)}(\overline{B_1}) \hookrightarrow\hookrightarrow \Lambda_{\infty,r}^{[\sigma-\frac{n}{p}]+1,\mu(\cdot)}(\overline{B_1}). \tag{6.12}$$

Since $\Omega \subset B_1$, (1.14) follows from (6.7) and (6.12).

Conversely, if (1.14) holds, Remark 6.4 implies that (1.11) is satisfied. $\square$

Proof of Theorem 1.8. Put $X = L_{p,q;b}(\mathbb{R}^n)$. By Lemma 6.1, $\|g_\sigma\|_{X'} < +\infty$. Consequently, by Theorem 1.4 and Lemma 6.2, $H^\sigma X \hookrightarrow\hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\overline{\Omega})$ if and only if (6.1) is satisfied.

(i) If $1 < q \le r < +\infty$, by Lemma 3.2 (i), (6.1) holds if and only if

$$\sup_{x\in(0,\xi)} \left\|t^{-\frac{1}{r}}\left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1}\right\|_{r;(x,\xi)} \left\|t^{-\frac{1}{q'}}(b(t))^{-1}\right\|_{q';(0,x)} \to 0 \quad \text{as } \xi \to 0_+. \tag{6.13}$$

We show that (6.13) is equivalent to

$$\lim_{x\to 0_+} \left\|t^{-\frac{1}{r}}\left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1}\right\|_{r;(x,1)} \left\|t^{-\frac{1}{q'}}(b(t))^{-1}\right\|_{q';(0,x)} = 0. \tag{6.14}$$

Indeed, assume that (6.13) holds. Then, given $\varepsilon > 0$, there is $\Delta \in (0,1)$ such that

$$\sup_{x\in(0,\Delta)} \left\|t^{-\frac{1}{r}}\left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1}\right\|_{r;(x,\Delta)} \left\|t^{-\frac{1}{q'}}(b(t))^{-1}\right\|_{q';(0,x)} < \frac{\varepsilon}{2}.$$

As $\left\|t^{-\frac{1}{q'}}(b(t))^{-1}\right\|_{q';(0,1)} < +\infty$, $q' < +\infty$, and $\mu \in \mathcal{L}^k_r$, we can find $\delta \in (0,\Delta)$ satisfying

$$\left\|t^{-\frac{1}{r}}\left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1}\right\|_{r;(\Delta,1)} \left\|t^{-\frac{1}{q'}}(b(t))^{-1}\right\|_{q';(0,\delta)} < \frac{\varepsilon}{2}.$$

Therefore, for all $x \in (0,\delta)$,

$$\begin{aligned}
&\left\|t^{-\frac{1}{r}}\left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1}\right\|_{r;(x,1)} \left\|t^{-\frac{1}{q'}}(b(t))^{-1}\right\|_{q';(0,x)}\\
&\precsim \left\|t^{-\frac{1}{r}}\left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1}\right\|_{r;(x,\Delta)} \left\|t^{-\frac{1}{q'}}(b(t))^{-1}\right\|_{q';(0,x)}\\
&\quad + \left\|t^{-\frac{1}{r}}\left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1}\right\|_{r;(\Delta,1)} \left\|t^{-\frac{1}{q'}}(b(t))^{-1}\right\|_{q';(0,x)}\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

and (6.14) follows. The converse implication is a consequence of the estimate

$$\begin{aligned}
&\sup_{x\in(0,\xi)} \left\|t^{-\frac{1}{r}}\left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1}\right\|_{r;(x,\xi)} \left\|t^{-\frac{1}{q'}}(b(t))^{-1}\right\|_{q';(0,x)}\\
&\le \sup_{x\in(0,\xi)} \left\|t^{-\frac{1}{r}}\left(\mu\left(t^{\frac{1}{n}}\right)\right)^{-1}\right\|_{r;(x,1)} \left\|t^{-\frac{1}{q'}}(b(t))^{-1}\right\|_{q';(0,x)}.
\end{aligned}$$

Since

$$\left\|t^{-\frac{1}{r}}(\lambda_{qr}(t^{\frac{1}{n}}))^{-1}\right\|_{r;(x,1)} \approx \left\|t^{-\frac{1}{q'}}(b(t))^{-1}\right\|^{-1}_{q';(0,x)} \quad \text{for all } x \in \left(0,\frac{1}{2}\right), \tag{6.15}$$

(6.14) is equivalent to (1.17) and the proof of part (i) is complete.

(ii) If $0 < r < q \le +\infty$ and $q > 1$, then Lemma 3.2 (ii) shows that (6.1) holds if and only if

$$\int_0^{\xi} \left\| t^{-\frac{1}{r}} \big(\mu(t^{\frac{1}{n}})\big)^{-1} \right\|_{r;(x,\xi)}^{u} (V(x))^{\frac{u}{r'}}\, b^{-q'}(x)\, \frac{dx}{x} \to 0 \quad \text{as } \xi \to 0_+, \tag{6.16}$$

where

$$V(x) = \left\| t^{-\frac{1}{q'}} (b(t))^{-1} \right\|_{q';(0,x)}^{q'}. \tag{6.17}$$

Using the identity $\frac{u}{r'} = \frac{u}{q'} - 1$ and (6.15), we see that (6.16) is equivalent to

$$\int_0^{\xi} \frac{\left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \right\|_{r;(x,\xi)}^{u}}{\left\| t^{-\frac{1}{r}} (\lambda_{qr}(t))^{-1} \right\|_{r;(x,1)}^{u}} \left(\int_0^{x^n} t^{-1} b^{-q'}(t)dt \right)^{-1} b^{-q'}(x^n) \frac{dx}{x} \to 0 \ \text{ as } \xi \to 0_+. \tag{6.18}$$

Finally, since singularities of functions in question are only at the origin, (6.18) is satisfied if and only if (1.19) holds. □

Remark 6.6. As in Remark 5.1, we see that in Theorem 1.8 the implications (1.16) $\Longrightarrow$ (1.17) and (1.18) $\Longrightarrow$ (1.19) remain true if $k \in \mathbb{N}$.

Remark 6.7 ([16, Remark 7.2 (i)])**.** Let $\sigma \in (0, n)$, $p = \frac{n}{\sigma}$, $q \in (1, +\infty]$, $r \in [1, +\infty]$, $k = [\sigma]+1$, let $b \in SV(0, +\infty)$ be such that $\left\| t^{-\frac{1}{q'}} (b(t))^{-1} \right\|_{q';(0,1)} < +\infty$ and let λ_{qr} be given by (1.15). Then $\Lambda^{k,\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}) = \Lambda^{1,\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n})$.

Proof of Corollary 1.9. Suppose that (1.17) holds. Then, by [16, Corollary 1.7], we have

$$H^{\sigma} L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{1,\lambda_{qr}(\cdot)}_{\infty,r}(\overline{\mathbb{R}^n}). \tag{6.19}$$

Since Ω is bounded, there is an open ball B_1 such that $\Omega \subset B_1$ and (6.19) implies that

$$H^{\sigma} L_{p,q;b}(\mathbb{R}^n) \hookrightarrow \Lambda^{1,\lambda_{qr}(\cdot)}_{\infty,r}(\overline{B_1}). \tag{6.20}$$

On the other hand (1.17) is equivalent to

$$\sup_{x \in (0,\xi)} \frac{\left\| t^{-\frac{1}{r}} (\mu(t))^{-1} \right\|_{r;(x,\xi)}}{\left\| t^{-\frac{1}{r}} (\lambda_{qr}(t))^{-1} \right\|_{r;(x,1)}} \to 0 \quad \text{as } \xi \to 0_+ \tag{6.21}$$

(see the proof of Theorem 1.8 (i)). Thus, as in the proof of Corollary 1.7, we have

$$\Lambda^{1,\lambda_{qr}(\cdot)}_{\infty,r}(\overline{B_1}) \hookrightarrow\hookrightarrow \Lambda^{1,\mu(\cdot)}_{\infty,r}(\overline{B_1}). \tag{6.22}$$

Since $\Omega \subset B_1$, (1.20) follows from (6.20) and (6.22).

Conversely, if (1.20) holds, Remark 6.6 implies that (1.17) is satisfied. □

Remark 6.8. (i) In Theorem 1.6 (i) the implication

$$(1.11) \implies H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$$

remains true even if we extend the range of q and r to $1 \le q \le r \le +\infty$. (Indeed, this can be seen from the proof of Theorem 1.6 (i), where we use Theorem 1.4 and Remark 5.2 (i) instead of Theorem 1.4.)

Theorem 1.6 (i) continues to hold if we assume that $1 \le q \le r \le +\infty$, $q < +\infty$, and (5.5) is satisfied. (This follows from Remarks 5.2 (ii), (iii). Note that the condition $q < +\infty$ implies that the space $L_{p,q;b}(\mathbb{R}^n)$ has absolutely continuous norm - cf. [22, Lemma 3.2].)

(ii) Similarly, in Theorem 1.8 (i) the implication

$$(1.17) \implies H^\sigma L_{p,q;b}(\mathbb{R}^n) \hookrightarrow\hookrightarrow \Lambda^{k,\mu(\cdot)}_{\infty,r}(\overline{\Omega})$$

remains true if we extend the range of q and r to $1 < q \le r \le +\infty$.

Theorem 1.8 (i) continues to hold if we assume that $1 < q \le r \le +\infty$, $q < +\infty$, and (5.5) is satisfied. (This follows from Remarks 5.2 (ii), (iii).)

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