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Some Non-Local Boundary-Value Problems and their Relationship to Problems for Loaded Equations

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Abstract. In several mathematical models of physical or technical processes there are non-local boundary-value problems in terms of partial differential equations with integral conditions. In this article we consider hyperbolic differential equations of second order in the rectangle with some integral conditions and their relationship to boundary-value problems for some certain type of loaded equations.

Keywords. Non-local boundary-value problems, problems with integral conditions, loaded differential equations

Mathematics Subject Classification (2000). Primary 35L20, secondary 45K05, 35Q80

1. Introduction

In actual natural sciences there steadily arise new problems differing from the classical ones. Therefore there is a need for further development of the theory of partial differential equations.

J. R. Cannon [1] studied the one-dimensional heat equation

$$u_t(x,t) = u_{xx}(x,t), \quad x > 0,\ t > 0,$$

with Dirichlet-conditions on some part of the boundary and integral conditions.

Given continuous mappings E, x, φ on $[0, \infty)$ satisfying the initial condition

$$E(0) = \int_0^{x(0)} \varphi(x)\,\mathrm{d}x$$

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the problem is in finding the temperature distribution u such that the equations

$$\int_0^{x(t)} u(x,t)\,\mathrm{d}x = E(t), \quad x(t) > 0,\ t > 0$$
$$u(x,0) = \varphi(x), \quad x \geq 0$$

hold. This problem describes the temperature distribution in a homogeneous semi-infinite conductor. Assume the total heat energy of a certain part of the conductor to be given. Furthermore let the initial temperature and the behaviour of the temperature distribution on the boundary of the conductor be known. Then the unique existence of a solution can be shown.

The Cannon-problem belongs to the so-called non-local problems. In this class of problems given boundary-values are partially or completely replaced by additional conditions on the functions connecting the values at the boundary and the inner of the domain, e.g.,

$$u(0,\cdot) + u(l,\cdot) + u(1,\cdot) = 0, \quad l \in (0,1).$$

Definition 1.1. Let I be an index set, $\Omega \subset \mathbb{R}^n$ and $\varnothing \neq \bigcup_{i\in I} \omega_i = \omega \subset \overline{\Omega}$. Let $F(M,\mathbb{R})$ denote the set of all real-valued functions on a set M. Suppose $u, f \in F(\Omega,\mathbb{R})$, $\varphi_i \in F(\omega_i,\mathbb{R})$, $i \in I$. Let $L : D(L) \subset F(\Omega,\mathbb{R}) \longrightarrow F(\Omega,\mathbb{R})$ be a differential operator and for $i \in I$

$$B_i : D(B_i) \subset F(\Omega,\mathbb{R}) \longrightarrow F(\omega_i,\mathbb{R}).$$

The problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ B_i u = \varphi_i & \text{on } \omega_i,\ i \in I \end{cases}$$

is called *non-local*, if there are $i \in I$, $z \in \Omega$, $v \in F(\Omega,\mathbb{R})$ such that

$$\partial^\alpha u(z) = \partial^\alpha v(z), \ \ \alpha \in \mathbb{N}_0^n, \qquad \text{and} \qquad (B_i u)(z) \neq (B_i v)(z).$$

That is, the problem is non-local if the image of some $z \in \Omega$ under the map $B_i u$ is not uniquely determined by the value of the derivatives of all orders of u at z.

In the class of non-local boundary-value problems those problems with integral conditions play an important role. One can consider such problems as a necessary step towards the generalisation of classical problems of mathematical physics.

In recent publications such problems are studied for various types of equations. The respective investigations started with elliptic and parabolic differential equations. Hyperbolic equations are more difficult to study. Findings in this area of research one can find in the work of L. S. Pulkina, e.g., in [6].

In this article a linear hyperbolic equation of second order is studied. We consider a rectangle as underlying domain, the two boundary-value conditions are thereby replaced by integral conditions.

2. Problem with integral conditions

We consider differential equations on the domain $\Omega = (0,a) \times (0,b) \subset \mathbb{R}^2$. Let therefore $A, B, C \in \mathbf{C}(\overline{\Omega})$, $f \in \mathbf{C}(\Omega)$ be given and let φ, ψ be integrable functions on $[0,a]$ and $[0,b]$ respectively such that

$$\int_0^a \varphi(x)\,\mathrm{d}x = \int_0^b \psi(y)\,\mathrm{d}y. \tag{1}$$

The investigated differential equation has the form

$$u_{xy}(x,y) + A(x,y)u_x(x,y) + B(x,y)u_y(x,y) + C(x,y)u(x,y) = f(x,y), \tag{2}$$

$(x,y) \in \Omega$, and the integral conditions are given as follows:

$$\int_0^a u(x,y)\,\mathrm{d}x = \psi(y), \quad y \in (0,b) \tag{3}$$

$$\int_0^b u(x,y)\,\mathrm{d}y = \varphi(x), \quad x \in (0,a). \tag{4}$$

Let

$$U := \left\{ u \in F(\Omega, \mathbb{R}) \mid u \in \mathbf{C}^1(\Omega),\ u_{xy} \in \mathbf{C}(\Omega) \right\}$$

be the set of all continuously differentiable functions on Ω such that u_{xy} is continuous on Ω.

Problem A: For given $A, B, C, f, \varphi, \psi$ as above, find a function $u \in U$, such that for all $(x,y) \in \Omega$ the equation (2) holds and the integral conditions (3) and (4) are satisfied.

Remark 2.1. The classical problem of Goursat is given by

$$u_{xy}(x,y) + A(x,y)u_x(x,y) + B(x,y)u_y(x,y) + C(x,y)u(x,y) = f(x,y)$$

$$u(0,y) = \psi(y), \quad y \in (0,b)$$
$$u(x,0) = \varphi(x), \quad x \in (0,a).$$

The first boundary-value condition can be written in the form of (3), if one understands ψ as a mean-value of u in $(0,a)$, i.e., $\psi(y) = \frac{1}{a}\int_0^a u(x,y)\,\mathrm{d}x$. Analogously one gets condition (4). That is why this problem is often called integral-analogue to the Goursat-problem.

In the light of physics one can interpret conditions like (3) and (4) as an averaging process made by a sensor in order to measure for example the temperature in a certain area.

Remark 2.2. Mathematical modelling and integral conditions arise in many areas, e.g.,

- in particle diffusion processes in turbulent plasma, in processes of heat diffusion, in water conducting processes in porous media and in modelling of some technical processes ([4]),
- in the description of population in mathematical biology ([5]),
- and in demographic problems.

Such problems often arise in mathematical physics if one investigates heat diffusion or diffusion processes where one does not know all physical quantities, i.e., some relevant quantities of the experiment are impossible to measure. On the other hand it is possible to get additional information for the processes.

3. Relationships of non-local boundary-value problems to boundary-value problems with loaded equations

3.1. Loaded equations. The investigation of such integral problems with standard methods goes along with big difficulties due to the non-local integral conditions. That is why those problems need the development of new methods in proving the solvability and in finding solution strategies. An analysis of problems with non-local boundary-value conditions reveals that there is an important relationship to so-called loaded equations ([3]).

Let $\Omega \subset \mathbb{R}^n$, $M \subset \overline{\Omega}$ locally of dimension less than n. We consider the differential equation $Lu = f$ on the domain Ω with L a differential operator.

Definition 3.1. The equation $Lu = f$ is called *loaded*, if f depends on $\partial^\alpha u|_M$ for some $\alpha \in \mathbb{N}_0^n$.

An example for an ordinary loaded differential equation is $y' = f(x, y, y(0))$. The interest in loaded equations has grown in the last years. Obviously this is related to two main facts. Loaded equations arise in mathematical research itself and descriptions of thermodynamical or diffusion problems with loaded equations are more accurate than the classical ones.

We will show that non-local problems for differential equations can be investigated by studying local problems with specially chosen loaded equations of the same type and order.

3.2. Formal motivation. For a motivation of how we establish a connection between the non-local differential equation in Problem A and a Goursat-problem with classical conditions but a loaded integro-differential equation of hyperbolic type, we give some formal calculations and arguments. Assume there exists a solution $u \in U$ of Problem A and consider the (formal) substitution

$$v(x,y) := xyu(x,y) + x\int_y^b u(x,\eta)\,\mathrm{d}\eta + y\int_x^a u(\xi,y)\,\mathrm{d}\xi + \int_0^y\int_0^x u(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \quad (5)$$

Differentiation w.r.t. the x-variable and afterwards w.r.t. the y-variable equation (5) becomes $v_{xy}(x,y) = xyu_{xy}(x,y)$, or equivalently

$$u_{xy}(x,y) = \frac{v_{xy}(x,y)}{xy}. \tag{6}$$

Integrating equation (6) we obtain

$$u(x,y) = \int_y^b \int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta + C_1(x) + C_2(y). \tag{7}$$

Due to (3) and (4) we get the following system of equations for $C_1(x)$ and $C_2(y)$:

$$\begin{cases} \displaystyle\int_0^a \int_y^b \int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}x + \int_0^a C_1(x)\, \mathrm{d}x + aC_2(y) = \psi(y) \\ \displaystyle\int_0^b \int_y^b \int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}y + \int_0^b C_2(y)\, \mathrm{d}y + bC_1(x) = \varphi(x). \end{cases}$$

From the first equation we obtain

$$C_2(y) = \frac{1}{a}\left(\psi(y) - \int_0^a \int_y^b \int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}x - \int_0^a C_1(x)\, \mathrm{d}x\right).$$

Using this expression in the second equation we get

$$\begin{aligned} &\int_0^b \int_y^b \int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}y \\ &+\frac{1}{a}\int_0^b \left(\psi(y) - \int_0^a \int_y^b \int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}x - \int_0^a C_1(x)\, \mathrm{d}x\right) \mathrm{d}y + bC_1(x) = \varphi(x). \end{aligned}$$

With this we have

$$\begin{aligned} C_1(x) = {}& \frac{\varphi(x)}{b} - \frac{1}{b}\int_0^b \int_y^b \int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}y - \frac{1}{ab}\int_0^b \psi(y)\, \mathrm{d}y \\ & + \frac{1}{ab}\int_0^b \int_0^a \int_y^b \int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}x\, \mathrm{d}y + \frac{1}{a}\int_0^a C_1(x)\, \mathrm{d}x. \end{aligned}$$

Hence a particular solution of the problem (6)–(3)–(4) is given by

$$\begin{aligned} u(x,y) = {}& \int_y^b \int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta + \frac{\varphi(x)}{b} - \frac{1}{b}\int_0^b \int_y^b \int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}y \\ & - \frac{1}{ab}\int_0^b \psi(y)\, \mathrm{d}y + \frac{1}{ab}\int_0^b \int_0^a \int_y^b \int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}x\, \mathrm{d}y \\ & + \frac{\psi(y)}{a} - \frac{1}{a}\int_0^a \int_y^b \int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}x. \end{aligned} \tag{8}$$

By changing the order of integration in some expressions of the terms on the right-hand side of (8) and making use of equations (5), (3) and (4) we obtain a simplified expression for u without using partial derivatives of v:

$$\begin{aligned} u(x,y) = {} & \frac{v(x,y)}{xy} + \frac{1}{a}\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta + \frac{1}{b}\int_x^a \frac{\varphi(\xi)}{\xi}\,\mathrm{d}\xi - \frac{1}{ab}\int_0^b \psi(\eta)\,\mathrm{d}\eta \\ & - \frac{1}{x}\int_y^b \frac{v(x,\eta)}{\eta^2}\,\mathrm{d}\eta - \frac{1}{y}\int_x^a \frac{v(\xi,y)}{\xi^2}\,\mathrm{d}\xi + \int_x^a\int_y^b \frac{v(\xi,\eta)}{\xi^2\eta^2}\,\mathrm{d}\eta\,\mathrm{d}\xi \\ =: {} & S_{\varphi,\psi}(v). \end{aligned} \tag{9}$$

Equation (9) serves us as the definition of the operator $S_{\varphi,\psi}$, which we apply later to elements of $V := \{v \in F(\overline{\Omega}) \mid v \in \mathbf{C}^1(\overline{\Omega}), v_{xy} \in \mathbf{C}(\Omega)\}$.

3.3. Passing to the problem with classical conditions. Next, we derive a differential equation for v by plugging $S_{\varphi,\psi}(v) = u$ in equation (2). Differentiation of (9) w.r.t. x and y yields:

$$u_x(x,y) = \frac{v_x(x,y)}{xy} - \frac{1}{b}\frac{\varphi(x)}{x} - \frac{1}{x}\int_y^b \frac{v_x(x,\eta)}{\eta^2}\,\mathrm{d}\eta, \tag{10}$$

$$u_y(x,y) = \frac{v_y(x,y)}{xy} - \frac{1}{a}\frac{\psi(y)}{y} - \frac{1}{y}\int_x^a \frac{v_y(\xi,y)}{\xi^2}\,\mathrm{d}\xi, \tag{11}$$

together with the already known equation (6), $u_{xy}(x,y) = \frac{v_{xy}(x,y)}{xy}$. Using these expressions in equation (2) we get

$$\begin{aligned} \frac{v_{xy}(x,y)}{xy} & + A(x,y)\left(\frac{v_x(x,y)}{xy} - \frac{1}{b}\frac{\varphi(x)}{x} - \frac{1}{x}\int_y^b \frac{v_x(x,\eta)}{\eta^2}\,\mathrm{d}\eta\right) \\ & + B(x,y)\left(\frac{v_y(x,y)}{xy} - \frac{1}{a}\frac{\psi(y)}{y} - \frac{1}{y}\int_x^a \frac{v_y(\xi,y)}{\xi^2}\,\mathrm{d}\xi\right) \\ & + C(x,y)\left(\frac{v(x,y)}{xy} + \frac{1}{a}\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta + \frac{1}{b}\int_x^a \frac{\varphi(\xi)}{\xi}\,\mathrm{d}\xi\right. \\ & - \frac{1}{ab}\int_0^b \psi(\eta)\,\mathrm{d}\eta \frac{1}{x}\int_y^b \frac{v(x,\eta)}{\eta^2}\,\mathrm{d}\eta - \frac{1}{y}\int_x^a \frac{v(\xi,y)}{\xi^2}\,\mathrm{d}\xi \\ & \left. + \int_x^a\int_y^b \frac{v(\xi,\eta)}{\xi^2\eta^2}\,\mathrm{d}\eta\,\mathrm{d}\xi\right) = f(x,y). \end{aligned} \tag{12}$$

This leads to a differential equation for v which we consider now:

$$v_{xy}(x,y) + A(x,y)v_x(x,y) + B(x,y)v_y(x,y) + C(x,y)v(x,y) = F(x,y;v), \tag{13}$$

where F can be expressed as $F(x,y;v) = F_1(x,y) + F_2(x,y;v)$, the function

$$F_1(x,y) = xyf(x,y) + \frac{A(x,y)}{b} y\varphi(x) + \frac{B(x,y)}{a} x\psi(y) + \frac{C(x,y)}{ab} xy \left(\int_0^b \psi(\eta)\,\mathrm{d}\eta - b\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta - a\int_x^a \frac{\varphi(\xi)}{\xi}\,\mathrm{d}\xi \right)$$

is known and the expression

$$\begin{aligned} F_2(x,y;v) = {} & A(x,y)y\int_y^b \frac{v_x(x,\eta)}{\eta^2}\,\mathrm{d}\eta + B(x,y)x\int_x^a \frac{v_y(\xi,y)}{\xi^2}\,\mathrm{d}\xi \\ & - C(x,y)xy\int_x^a\int_y^b \frac{v(\xi,\eta)}{\xi^2\eta^2}\,\mathrm{d}\eta\,\mathrm{d}\xi + C(x,y)x\int_x^a \frac{v(\xi,y)}{\xi^2}\,\mathrm{d}\xi \\ & + C(x,y)y\int_y^b \frac{v(x,\eta)}{\eta^2}\,\mathrm{d}\eta, \end{aligned}$$

contains the unknown function v. Consequently, equation (13) is loaded. As boundary conditions to equation (13) let

$$v(x,0) = x\varphi(x) =: \varphi_1(x) \tag{14}$$
$$v(0,y) = y\psi(y) =: \varphi_2(y), \tag{15}$$

which can be easily formally motivated as consequences of (3) and (4). Note that $\varphi_1(0) = \varphi_2(0) = 0$. The functions A, B, C, f are assumed to be of equal quality as in Problem A. Additionally, let φ, ψ be differentiable on $(0,a]$ and $(0,b]$, respectively. Furthermore assume $\varphi_1 \in \mathbf{C}([0,a])$ and $\varphi_2 \in \mathbf{C}([0,b])$.

Problem B: For given $A, B, C, f, \varphi_1, \varphi_2$ as above, find a function $v \in V$, such that for all $(x,y) \in \Omega$ the loaded equation (13) holds and the classical conditions (14) and (15) are satisfied.

Proposition 3.2. *Let $v \in V$ be a solution of Problem B and $u := S_{\varphi,\psi}(v)$, where the operator $S_{\varphi,\psi}$ is defined in equation* (9)*. Then u satisfies the integral conditions* (3) *and* (4)*.*

Proof. First we proof the convergence of the integrals involved in the respective expressions. Consider the integral $\int_0^a u(x,y)\,\mathrm{d}x$. The function u can be written as follows:

$$\begin{aligned} u(x,y) = {} & \int_y^b\int_x^a \frac{v(\xi,\eta) - v(x,\eta) - v(\xi,y) + v(x,y)}{\xi^2\eta^2}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ & - \frac{1}{a}\int_y^b \frac{v_\eta(x,\eta)}{\eta}\,\mathrm{d}\eta - \frac{1}{b}\int_x^a \frac{v_\xi(\xi,y)}{\xi}\,\mathrm{d}\xi - \frac{v(x,y)}{ab} + \frac{v(x,b)}{ab} \\ & + \frac{v(a,y)}{ab} + \frac{1}{a}\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta + \frac{1}{b}\int_x^a \frac{\varphi(\xi)}{\xi}\,\mathrm{d}\xi - \frac{1}{ab}\int_0^b \psi(\eta)\,\mathrm{d}\eta. \end{aligned} \tag{16}$$

From (16) we easily get

$$\int_0^a u(x,y)\,\mathrm{d}x = \int_0^a \Bigg(\int_y^b\int_x^a \frac{v(\xi,\eta)-v(x,\eta)-v(\xi,y)+v(x,y)}{\xi^2\eta^2}\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$- \frac{1}{a}\int_y^b \frac{v_\eta(x,\eta)}{\eta}\,\mathrm{d}\eta - \frac{1}{b}\int_x^a \frac{v_\xi(\xi,y)}{\xi}\,\mathrm{d}\xi - \frac{v(x,y)}{ab} + \frac{v(x,b)}{ab}$$
$$+ \frac{v(a,y)}{ab} + \frac{1}{a}\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta \Bigg)\,\mathrm{d}x + \int_0^a \left(\frac{1}{b}\int_x^a \frac{\varphi(\xi)}{\xi}\,\mathrm{d}\xi\right)\mathrm{d}x$$
$$- \int_0^a \left(\frac{1}{ab}\int_0^b \psi(\eta)\,\mathrm{d}\eta\right)\mathrm{d}x.$$

The last two integrals can be calculated explicitly:

$$\int_0^a \left(\frac{1}{b}\int_x^a \frac{\varphi(\xi)}{\xi}\,\mathrm{d}\xi\right)\mathrm{d}x = \frac{1}{b}\int_0^a \left(\int_0^\xi \frac{\varphi(\xi)}{\xi}\,\mathrm{d}x\right)\mathrm{d}\xi = \frac{1}{b}\int_0^a \varphi(\xi)\,\mathrm{d}\xi$$
$$\int_0^a \left(\frac{1}{ab}\int_0^b \psi(\eta)\,\mathrm{d}\eta\right)\mathrm{d}x = \frac{1}{b}\int_0^b \psi(\eta)\,\mathrm{d}\eta.$$

Because of (1) we get

$$\int_0^a u(x,y)\,\mathrm{d}x = \int_0^a \Bigg(\int_y^b\int_x^a \frac{v(\xi,\eta)-v(x,\eta)-v(\xi,y)+v(x,y)}{\xi^2\eta^2}\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$- \frac{1}{a}\int_y^b \frac{v_\eta(x,\eta)}{\eta}\,\mathrm{d}\eta - \frac{1}{b}\int_x^a \frac{v_\xi(\xi,y)}{\xi}\,\mathrm{d}\xi - \frac{v(x,y)}{ab} + \frac{v(x,b)}{ab}$$
$$+ \frac{v(a,y)}{ab} + \frac{1}{a}\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta \Bigg)\,\mathrm{d}x.$$

Denote the integrand of the right-hand side of the above equation by $T(x,y)$. We estimate the modulus of $T(x,y)$. Obviously we have

$$|T(x,y)| \leq \int_y^b\int_x^a \frac{|v(\xi,\eta)-v(x,\eta)-v(\xi,y)+v(x,y)|}{\xi^2\eta^2}\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+ \frac{1}{a}\int_y^b \frac{|v_\eta(x,\eta)|}{\eta}\,\mathrm{d}\eta + \frac{1}{b}\int_x^a \frac{|v_\xi(\xi,y)|}{\xi}\,\mathrm{d}\xi + \frac{|v(x,y)|}{ab} + \frac{|v(x,b)|}{ab}$$
$$+ \frac{|v(a,y)|}{ab} + \frac{1}{a}\left|\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta\right|.$$

Due to $v \in \mathbf{C}^1(\overline{\Omega})$ the function v and its derivatives are bounded by some constant M. Therefore we get

$$|T(x,y)| \leq \int_y^b\int_x^a \frac{|v(\xi,\eta)-v(x,\eta)|+|v(\xi,y)-v(x,y)|}{\xi^2\eta^2}\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+ \frac{M}{a}\int_y^b \frac{1}{\eta}\,\mathrm{d}\eta + \frac{M}{b}\int_x^a \frac{1}{\xi}\,\mathrm{d}\xi + \frac{3M}{ab} + \frac{1}{a}\left|\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta\right|,$$

and the following Lipschitz condition holds: $|v(x,y) - v(\bar{x},y)| \le M\,|x-\bar{x}|$. That is why we have

$$|T(x,y)| \le M\left(2\int_y^b\int_x^a \frac{\xi-x}{\xi^2\eta^2}\,\mathrm{d}\xi\,\mathrm{d}\eta + \frac{1}{a}\int_y^b \frac{1}{\eta}\,\mathrm{d}\eta + \frac{1}{b}\int_x^a \frac{1}{\xi}\,\mathrm{d}\xi + \frac{3}{ab}\right) + \frac{1}{a}\left|\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta\right|,$$

and

$$\int_y^b\int_x^a \frac{\xi-x}{\xi^2\eta^2}\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_y^b \frac{1}{\eta^2}\,\mathrm{d}\eta\int_x^a \frac{\xi-x}{\xi^2}\,\mathrm{d}\xi \le \left(-\frac{1}{b}+\frac{1}{y}\right)\int_x^a \frac{1}{\xi}\,\mathrm{d}\xi \le \frac{1}{y}\,(\ln a - \ln x).$$

Hence we get

$$\begin{aligned}|T(x,y)| \le{} & M\left[\frac{2}{y}\,(\ln a - \ln x) + \frac{1}{a}\,(\ln b - \ln y) + \frac{1}{b}\,(\ln a - \ln x) + \frac{3}{ab}\right] \\ & + \frac{1}{a}\left|\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta\right|.\end{aligned}$$

The integral

$$\begin{aligned}&\int_0^a\left(M\left[-\left(\frac{2}{y}+\frac{1}{b}\right)\ln x + \left(\frac{2}{y}+\frac{1}{b}\right)\ln a + \frac{1}{a}(\ln b - \ln y) + \frac{3}{ab}\right]\right. \\ &\quad\left.+\frac{1}{a}\left|\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta\right|\right)\,\mathrm{d}x \\ &= M\left(\frac{2a}{y} + \frac{a+3}{b} + \ln b - \ln y\right) + \left|\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta\right|\end{aligned}$$

converges and because of the majorant criterion so does $\int_0^a u(x,y)\,\mathrm{d}x$. The convergence of $\int_0^b u(x,y)\,\mathrm{d}y$ can be proved analogously.

In order to show the equalities $\int_0^b u(x,y)\,\mathrm{d}y = \varphi(x)$ and $\int_0^a u(x,y)\,\mathrm{d}x = \psi(y)$ we use equation (8). We have

$$\begin{aligned}\int_0^a u(x,y)\,\mathrm{d}x = \int_0^a & \left(\left(\int_y^b\int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta - \frac{1}{b}\int_0^b\int_y^b\int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}y\right.\right. \\ & - \frac{1}{ab}\int_0^b \psi(y)\,\mathrm{d}y + \frac{1}{ab}\int_0^b\int_0^a\int_y^b\int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x\,\mathrm{d}y \\ & \left. + \frac{\psi(y)}{a} + \frac{\varphi(x)}{b} - \frac{1}{a}\int_0^a\int_y^b\int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x\right)\,\mathrm{d}x.\end{aligned}$$

Therefore we get

$$\int_0^a u(x,y)\,\mathrm{d}x = \int_0^a\int_y^b\int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x - \frac{1}{b}\int_0^a\int_0^b\int_y^b\int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}y\,\mathrm{d}x$$
$$-\frac{1}{b}\int_0^b \psi(y)\,\mathrm{d}y + \frac{1}{b}\int_0^b\int_0^a\int_y^b\int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x\,\mathrm{d}y + \psi(y)$$
$$+\frac{1}{b}\int_0^a \varphi(x)\,\mathrm{d}x - -\int_0^a\int_y^b\int_x^a \frac{v_{\xi\eta}(\xi,\eta)}{\xi\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x,$$

this yields $\int_0^a u(x,y)\,\mathrm{d}x = \psi(y)$. Analogously we get $\int_0^b u(x,y)\,\mathrm{d}y = \varphi(x)$. □

Proposition 3.3. *Let $v \in V$ be a solution of Problem B and $u := S_{\varphi,\psi}(v)$, where the operator $S_{\varphi,\psi}$ is defined in equation* (9)*. Then u satisfies the differential equation* (2) *on $\Omega = (0,a) \times (0,b)$.*

Proof. By partial differentiation of equation (9) we first recover equations (10), (11) and (6):

$$\begin{aligned}
u(x,y) &= \frac{v(x,y)}{xy} + \frac{1}{a}\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta + \frac{1}{b}\int_x^a \frac{\varphi(\xi)}{\xi}\,\mathrm{d}\xi - \frac{1}{ab}\int_0^b \psi(\eta)\,\mathrm{d}\eta \\
&\quad - \frac{1}{x}\int_y^b \frac{v(x,\eta)}{\eta^2}\,\mathrm{d}\eta - \frac{1}{y}\int_x^a \frac{v(\xi,y)}{\xi^2}\,\mathrm{d}\xi + \int_x^a\int_y^b \frac{v(\xi,\eta)}{\xi^2\eta^2}\,\mathrm{d}\eta\,\mathrm{d}\xi \\
u_x(x,y) &= \frac{v_x(x,y)}{xy} - \frac{1}{b}\frac{\varphi(x)}{x} - \frac{1}{x}\int_y^b \frac{v_x(x,\eta)}{\eta^2}\,\mathrm{d}\eta \\
u_y(x,y) &= \frac{v_y(x,y)}{xy} - \frac{1}{a}\frac{\psi(y)}{y} - \frac{1}{y}\int_x^a \frac{v_y(\xi,y)}{\xi^2}\,\mathrm{d}\xi \\
u_{xy}(x,y) &= \frac{v_{xy}(x,y)}{xy}.
\end{aligned}$$

By assumption, equation (13) holds for $(x,y) \in \Omega$. Reordering the terms, we obtain (cf. equation (12))

$$\begin{aligned}
v_{xy}(x,y) &+ A(x,y)\left(v_x(x,y) - \frac{y}{b}\varphi(x) - y\int_y^b \frac{v_x(x,\eta)}{\eta^2}\,\mathrm{d}\eta\right) \\
&+ B(x,y)\left(v_y(x,y) - \frac{x}{a}\psi(y) - x\int_x^a \frac{v_y(\xi,y)}{\xi^2}\,\mathrm{d}\xi\right) \\
&+ C(x,y)\left(v(x,y) + \frac{xy}{a}\int_y^b \frac{\psi(\eta)}{\eta}\,\mathrm{d}\eta + \frac{xy}{b}\int_x^a \frac{\varphi(\xi)}{\xi}\,\mathrm{d}\xi\right. \\
&- \frac{xy}{ab}\int_0^b \psi(\eta)\,\mathrm{d}\eta - y\int_y^b \frac{v(x,\eta)}{\eta^2}\,\mathrm{d}\eta - x\int_x^a \frac{v(\xi,y)}{\xi^2}\,\mathrm{d}\xi \\
&\left.+xy\int_x^a\int_y^b \frac{v(\xi,\eta)}{\xi^2\eta^2}\,\mathrm{d}\eta\,\mathrm{d}\xi\right) = xyf(x,y),
\end{aligned}$$

from which we derive

$$xyu_{xy}(x,y) + Axyu_x(x,y) + Bxyu_y(x,y) + Cxyu(x,y) = xyf(x,y).$$

Here we omitted the dependence of the coefficients A, B, C on x and y for convenience. This proves that u solves equation (2) for all $(x,y) \in \Omega$. $\square$

Theorem 3.4. *Let $v \in V$ be a solution of Problem B and $u := S_{\varphi,\psi}(v)$, where the operator S is defined in equation* (9)*. Then u is a solution of Problem A.*

Proof. It is easy to see that $u \in \mathbf{C}^1(\Omega)$. Because of $u_{xy}(x,y) = \frac{v_{xy}(x,y)}{xy}$ the function u_{xy} is continuous on Ω and therefore $u \in U$. The rest follows by Propositions 3.2 and 3.3. $\square$

4. Conclusions

By relating problem A with problem B we shift the question of solvability of problem A to the question of solvability of problem B. In fact, existence and uniqueness of a solution of Problem B can be shown under certain nontrivial assumptions. This result will be published in a forthcoming paper [2].

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