

Regularity Criteria in Terms of the Pressure for the Navier-Stokes Equations in the Critical Morrey-Campanato Space

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Abstract. In this paper, we establish a Serrin-type regularity criterion in terms of the pressure for Leray weak solutions to the Navier–Stokes equation in \mathbb{R}^3 . It is proved that the solution is regular if the associate pressure satisfies

$$p \in L^{\frac{2}{2-r}}((0, T); \dot{\mathcal{M}}_{2, \frac{3}{r}}(\mathbb{R}^3)) \quad \text{or} \quad \nabla p \in L^{\frac{2}{3-r}}((0, T); \dot{\mathcal{M}}_{2, \frac{3}{r}}(\mathbb{R}^3))$$

for $0 < r < 1$, where $\dot{\mathcal{M}}_{2, \frac{3}{r}}(\mathbb{R}^3)$ is the critical Morrey–Campanato space. Regularity criteria for the 3D MHD equations are also given.

Keywords. Navier–Stokes equations, Morrey–Campanato space, weak solution, regularity criterion

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1. Introduction

Consider the Navier–Stokes equations in \mathbb{R}^3 :

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, & (x, t) \in \mathbb{R}^3 \times (0, T) \\ \operatorname{div} u = 0, & (x, t) \in \mathbb{R}^3 \times (0, T) \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^3, \end{cases} \quad (1)$$

where $u = u(x, t) \in \mathbb{R}^3$: the velocity field, and $p = p(x, t) \in \mathbb{R}$: the scalar pressure are unknowns. For simplicity, we assume that the external force has a scalar potential and is included into the pressure gradient.

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It is well known that for $u_0 \in L^2(\mathbb{R}^3)$ problem (1) possesses at least one weak solution, see [10]. Its uniqueness is an open problem. Moreover, if $u_0 \in W^{1,2}(\mathbb{R}^3)$, then at least on a short time interval, $u \in L^2(0, T^*; W^{2,2}(\mathbb{R}^3)) \cap L^\infty(0, T^*; W^{1,2}(\mathbb{R}^3))$. The fundamental open question is whether T^* can be arbitrarily large or whether there is finite time blow up, i.e., whether there is $T^* < \infty$ such that $\limsup_{\tau \rightarrow T^*} \|\nabla u(\tau)\|_{L^2} = +\infty$. In the general situation, to ensure the smoothness, we may consider certain minimal smoothness which excludes the possibility of the blow up., i.e., which ensures $u \in C^\infty((0, \infty) \times \mathbb{R}^3)$. The first such a criterion is usually referred as the Prodi-Serrin condition (see [13, 15] and for $s = 3$ [3]) saying that if the weak solution u additionally belongs to $L^t(0, T; L^s(\mathbb{R}^3))$, $\frac{2}{t} + \frac{3}{s} = 1$, $s \in [3, \infty]$, then the solution is as regular as the data allow and unique in the class of all weak solutions satisfying the energy inequality. Further criteria, including several components of velocity field or the gradient of velocity, pressure or other quantities can be found, e.g., in [1, 4, 12, 17–19, 21, 25].

In L^p framework, a final Serrin-type regularity criterion in terms of the pressure was obtained by Berselli and Galdi [2] (a much simpler proof was given by the Zhou [21] recently). In terms of the gradient of pressure, Zhou also established a final version Serrin-type regularity criterion in [21]. For arbitrary dimensional case, we refer to [16, 22]. It is shown that if the pressure $p \in L^\alpha((0, T); L^\gamma(\mathbb{R}^3))$ with $\frac{2}{\alpha} + \frac{3}{\gamma} < 2$, $1 < \alpha \leq \infty$, $\frac{3}{2} < \gamma < \infty$ or $\nabla p \in L^\alpha((0, T); L^q(\mathbb{R}^3))$ with $\frac{2}{\alpha} + \frac{3}{q} \leq 3$, then the corresponding weak solution actually is strong. Very recently, Fan, Jiang and Ni [4] extended regularity criteria in terms of the pressure to a multiplier space (see Definition 2.1) and a homogeneous Morrey–Campanato space (see Definition 2.2).

The purpose of this work is to establish a Serrin-type regularity criterion in terms of the pressure or gradient of the pressure in the the critical Morrey–Campanato space by an equivalence between a multiplier space and a homogeneous Morrey–Campanato space.

We will prove

Theorem 1.1. *Let $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for some $q \geq 4$, and $\nabla \cdot u_0 = 0$ in the sense of distributions. Assume that $u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ is a weak solution of the Navier–Stokes equations (1) with an associated pressure p . If the pressure p satisfies*

$$p \in L^{\frac{2}{2-r}}((0, T), \dot{\mathcal{M}}_{2, \frac{3}{r}}(\mathbb{R}^3)) \quad \text{with } r \in (0, 1),$$

then $u(t, x)$ is a strong solution in $(0, T]$.

and the following regularity theorem in terms of the gradient of pressure:

Theorem 1.2. *Let $u_0 \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for some $q \geq 4$, and $\nabla \cdot u_0 = 0$ in the sense of distributions. Suppose that $u(x, t)$ is a Leray-Hopf solution of (1). If*

$$\nabla p \in L^{\frac{2}{3-r}}((0, T), \dot{\mathcal{M}}_{2, \frac{3}{r}}(\mathbb{R}^3)) \quad \text{for } 0 < r < 1,$$

then $u(t, x)$ is a regular solution in the sense that $u \in C^\infty([0, T] \times \mathbb{R}^3)$.

Remark 1.3. By a strong solution we mean a weak solution of the Navier–Stokes equation such that $u \in L^\infty((0, T); H^1) \cap L^2((0, T); H^2)$. It is well-known that strong solutions are regular (we say classical) and unique in the class of weak solutions.

2. Morrey–Campanato spaces

In this section, we will recall the definition and some properties of the space we are going to use. This kind of spaces play an important role in studying the regularity of solutions to partial differential equations (see [9] and references therein).

First, we give the definition of \dot{X}_r , which was used in [4, 27].

Definition 2.1. For $0 \leq r < \frac{3}{2}$, the space \dot{X}_r is defined as the space of $f(x) \in L^2_{loc}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \leq 1} \|fg\|_{L^2} < \infty,$$

where we denote by $\dot{H}^r(\mathbb{R}^3)$ the completion of the space $\mathcal{D}(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\dot{H}^r} = \|(-\Delta)^{\frac{r}{2}} u\|_{L^2}$.

Now we recall the definition of the Morrey–Campanato spaces:

Definition 2.2. For $1 < p \leq q \leq \infty$, the Morrey–Campanato space $\dot{\mathcal{M}}_{p,q}$ is defined by

$$\dot{\mathcal{M}}_{p,q} = \left\{ f \in L^p_{loc}(\mathbb{R}^3) : \|f\|_{\dot{\mathcal{M}}_{p,q}} = \sup_{x \in \mathbb{R}^3} \sup_{R > 0} R^{\frac{3}{q} - \frac{3}{p}} \|f(y) 1_{B(x,R)}(y)\|_{L^p(dy)} < \infty \right\}.$$

One can find that $\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3)$ is a Banach space under the norm $\|\cdot\|_{\dot{\mathcal{M}}_{p,q}}$. Furthermore, it is easy to check the following:

$$\|f(\lambda \cdot)\|_{\dot{\mathcal{M}}_{p,q}} = \frac{1}{\lambda^{\frac{3}{q}}} \|f(\cdot)\|_{\dot{\mathcal{M}}_{p,q}}, \quad \lambda > 0.$$

Hence, for (u, p) solves the Navier–Stokes equations, we have

$$\|u_\lambda\|_{L^{\frac{2}{1-r}}((0,T); \dot{\mathcal{M}}_{2, \frac{3}{r}}(\mathbb{R}^3))} = \|u\|_{L^{\frac{2}{1-r}}((0,T); \dot{\mathcal{M}}_{2, \frac{3}{r}}(\mathbb{R}^3))},$$

and

$$\|p_\lambda\|_{L^{2-\frac{2}{r}}(0,T);\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)} = \|p\|_{L^{2-\frac{2}{r}}((0,T);\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3))},$$

with $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$ and $p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t)$ for $\lambda > 0$. Here, the point is that if (u, p) solves the MHD model, then so does (u_λ, p_λ) for all $\lambda > 0$. This is so called scaling dimension zero property.

Then, we have the following comparison between the Lorentz spaces and the Morrey–Campanato spaces: for $p \geq 2$,

$$L^{\frac{3}{r}}(\mathbb{R}^3) \subset L^{\frac{3}{r},\infty}(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{p,\frac{3}{r}}(\mathbb{R}^3).$$

The relation $L^{\frac{3}{r},\infty}(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{p,\frac{3}{r}}(\mathbb{R}^3)$ is shown in the following:

$$\begin{aligned} \|f\|_{\dot{\mathcal{M}}_{p,\frac{3}{r}}} &\leq \sup_E |E|^{\frac{r}{3}-\frac{1}{2}} \left(\int_E |f(y)|^p dy \right)^{\frac{1}{p}} && \left(f \in L^{\frac{3}{r},\infty}(\mathbb{R}^3) \right) \\ &= \left(\sup_E |E|^{\frac{pr}{3}-1} \int_E |f(y)|^p dy \right)^{\frac{1}{p}} \\ &\cong \left(\sup_{R>0} R \left| \{x \in \mathbb{R}^3 : |f(y)|^p > R\} \right|^{\frac{pr}{3}} \right)^{\frac{1}{p}} \\ &= \sup_{R>0} R \left| \{x \in \mathbb{R}^p : |f(y)| > R\} \right|^{\frac{r}{3}} \\ &\cong \|f\|_{L^{\frac{3}{r},\infty}}. \end{aligned}$$

The following lemma [9] gives an equivalence between $\dot{\mathcal{M}}_{2,\frac{3}{r}}$ and a multiplier space.

Lemma 2.3. *For $0 \leq r < \frac{3}{2}$, the space \dot{Z}_r is defined as the space of $f(x) \in L^2_{loc}(\mathbb{R}^3)$ such that*

$$\|f\|_{\dot{Z}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}} \leq 1} \|fg\|_{L^2} < \infty.$$

Then $f \in \dot{\mathcal{M}}_{2,\frac{3}{r}}$ if and only if $f \in \dot{Z}_r$ with equivalence of norms.

Additionally, for $2 < p \leq \frac{3}{r}$ and $0 \leq r < \frac{3}{2}$, we have the following inclusion relations ([9]):

$$\dot{\mathcal{M}}_{p,\frac{3}{r}}(\mathbb{R}^3) \subset \dot{X}_r(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3) = \dot{Z}_r(\mathbb{R}^3).$$

The relation $\dot{X}_r(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$ is shown as follows. Let $f \in \dot{X}_r(\mathbb{R}^3)$, $0 < R \leq 1$,

$x_0 \in \mathbb{R}^3$ and $\phi \in C_0^\infty(\mathbb{R}^3)$, $\phi \equiv 1$ on $B(\frac{x_0}{R}, 1)$. We have

$$\begin{aligned} R^{r-\frac{d}{2}} \left(\int_{|x-x_0| \leq R} |f(x)|^2 dx \right)^{\frac{1}{2}} &= R^r \left(\int_{|y-\frac{x_0}{R}| \leq 1} |f(Ry)|^2 dy \right)^{\frac{1}{2}} \\ &\leq R^r \left(\int_{y \in \mathbb{R}^3} |f(Ry)\phi(y)|^2 dy \right)^{\frac{1}{2}} \\ &\leq R^r \|f(R\cdot)\|_{\dot{X}_r} \|\phi\|_{H^r} \\ &\leq \|f\|_{\dot{X}_r} \|\phi\|_{H^r} \\ &\leq C \|f\|_{\dot{X}_r}. \end{aligned}$$

Another easy result is a direct application of Proposition 5.1 in [28]:

Lemma 2.4. *For $r \in (0, \frac{3}{2})$, the following embeddings hold:*

1. $\dot{B}_{p,\infty}^{\frac{3}{2}-r}(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$, provided $p < \frac{3}{r}$;
2. $\dot{B}_{\frac{3}{r},2}^0(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$.

Remark 2.5. Since $L^{\frac{3}{r}}(\mathbb{R}^3) \subset \dot{X}_r(\mathbb{R}^3) \subset \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$, Theorem 1.1 and 1.2 are improvements of Zhou's result [21] and those in [4, (1.9), (1.11) and (1.13) in Theorem 1.5]. Thanks to Lemma 2.4, we also extend regularity in homogeneous Besov spaces.

Remark 2.6. One can find that the space $\dot{\mathcal{M}}_{2,\frac{3}{r}}$ is a quite large one. Moreover, by using space $\dot{\mathcal{M}}_{2,\frac{3}{r}}$, one could find or exclude some type of singularity. We refer to [27] for detailed discussion.

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we recall the well-known pressure-velocity relation in \mathbb{R}^3 , given by

$$p = \sum_{i,j=1}^3 \mathcal{R}_i \mathcal{R}_j (u_i u_j),$$

where $(\mathcal{R}_i)_{i=1}^3$ are the Riesz transforms in \mathbb{R}^3 . Then the Calderón-Zygmund inequality implies

$$\|p\|_{L^\alpha} \leq \|u\|_{L^{2\alpha}}^2, \quad 1 < \alpha < \infty. \quad (2)$$

We also recall the following result due to Giga [5] (see also [8]) that will be used in the proof of Theorem 1.1 and 1.2. Here, BC denotes the class of bounded and continuous functions.

Lemma 3.1.

- (i) Suppose that $u_0 \in L^\alpha(\mathbb{R}^3)$, for $\alpha \geq 3$ and $\nabla \cdot u_0 = 0$. Then, there exists $T_0 > 0$ and a unique solution of (1) on $[0, T_0)$ such that

$$\begin{cases} u \in BC([0, T_0); L^\alpha(\mathbb{R}^3)) \cap L^r([0, T_0); L^s(\mathbb{R}^3)), \\ t^{\frac{1}{r}} u \in BC([0, T_0); L^s(\mathbb{R}^3)), \end{cases} \quad (3)$$

where $\frac{2}{r} + \frac{3}{s} = \frac{3}{\alpha}$, $s > 3$.

- (ii) Moreover, let $(0, T^*)$ be the maximal interval such that u solves (1) in $C((0, T^*); L^\alpha(\mathbb{R}^3))$, $\alpha > 3$. Then for any $t \in (0, T^*)$

$$\|u(t)\|_{L^\alpha} \geq \frac{C}{(T^* - t)^{\frac{\alpha-3}{2\alpha}}},$$

with the constant C independent of T^* and α .

- (iii) Let u be a strong solution satisfying

$$u \in L^\alpha((0, T); L^\beta(\mathbb{R}^3)) \quad \text{for} \quad \frac{2}{\alpha} + \frac{3}{\beta} = 1 \quad \text{and} \quad \beta > 3.$$

Then u belongs to $C^\infty(\mathbb{R}^3 \times (0, T))$.

The proof of Theorem 1.1 consists in first obtaining a continuation principle for strong solutions and then in applying it to weak solutions. By using the results of the previous Lemma 3.1, the weak solution u is smooth in some time interval $(0, T^*)$, $T^* \leq T$. In particular, $(u, p) \in C^\infty(\mathbb{R}^3 \times (0, T^*))$ and u is in the class (3). Thus, for any $T > 0$ we suppose that u is a smooth solution to (1) on $\mathbb{R}^3 \times (0, T)$ and will establish a priori bounds that will allow us to extend u for all time. Hence, it suffices to establish the following a priori estimate

$$\sup_{0 \leq t \leq T} \|u(t)\|_{L^4}^4 \leq \|a\|_{L^4}^4 \exp\left(C \int_0^t \|p\|_{\dot{M}_{2, \frac{3}{r}}}^{\frac{2}{2-r}} ds\right)$$

where C is independent of T .

Multiplying both sides of (1) by $4u|u|^2$ and integrating over $(0, t) \times \mathbb{R}^3$, after suitable integration by parts, we obtain (see e.g. [21])

$$\begin{aligned} & \|u(\cdot, t)\|_{L^4}^4 + 4 \int_0^t \|\nabla u |u|\|_{L^2}^2 ds + 2 \int_0^t \|\nabla |u|^2\|_{L^2}^2 ds \\ & \leq 4 \int_0^t \int_{\mathbb{R}^3} |p| |u| |\nabla |u|^2| dx ds + \|u_0\|_{L^4}^4, \end{aligned}$$

for $t \in (0, T)$. Let $w = |u|^2$. Then Cauchy's inequality implies that

$$\begin{aligned}
& \|u(\cdot, t)\|_{L^4}^4 + 4 \|\nabla u |u|\|_{L^{2,2}}^2 + 2 \|\nabla w\|_{L^{2,2}}^2 \\
& \leq 4 \int_0^t \int_{\mathbb{R}^d} |p| |w|^{\frac{1}{2}} |\nabla w| dx ds + \|u_0\|_{L^4}^4 \\
& \leq 2 \left[\int_0^t \int_{\mathbb{R}^d} |\nabla w|^2 dx ds + \int_0^t \int_{\mathbb{R}^d} |p|^2 |w| dx ds \right] + \|u_0\|_{L^4}^4.
\end{aligned} \tag{4}$$

Let us estimate the integral $I = \int_0^t \int_{\mathbb{R}^d} |p|^2 |w| dx ds$ on the right hand side of (4). By Hölder's and Young's inequalities, we have

$$\begin{aligned}
I & \leq \int_0^t \|pw\|_{L^2} \|p\|_{L^2} ds \\
& \leq \int_0^t \left(\|p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}} \|w\|_{\dot{B}_{2,1}^r} \right) \| |u|^2 \|_{L^2} ds \\
& \leq C \int_0^t \|p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}} \|w\|_{L^2}^{1-r} \|\nabla w\|_{L^2}^r \|w\|_{L^2} ds \\
& \leq C \int_0^t \|p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}} \|w\|_{L^2}^{2-r} \|\nabla w\|_{L^2}^r ds \\
& \leq C \int_0^t \left(\|p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}} \|w\|_{L^2}^{2-r} \right) \|\nabla w\|_{L^2}^r ds \\
& \leq \frac{1}{2} \int_0^t \|\nabla w\|_{L^2}^2 ds + C \int_0^t \|p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}}^{\frac{2}{2-r}} \|w\|_{L^2}^2 ds,
\end{aligned}$$

where we used the following inequality proved in [11]:

$$\|w\|_{\dot{B}_{2,1}^r} \leq C \|w\|_{L^2}^{1-r} \|\nabla w\|_{L^2}^r$$

and (2). Since $\|\nabla u |u|\|_{L^2}^2 = \frac{1}{4} \|\nabla |u|^2\|_{L^2}^2 = \frac{1}{4} \|\nabla w\|_{L^2}^2$, then by (4) and the above equality, we derive

$$\|u(\cdot, t)\|_{L^4}^4 \leq C \int_0^t \|p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}}^{\frac{2}{2-r}} \|w\|_{L^2}^2 ds + \|u_0\|_{L^4}^4. \tag{5}$$

Due to Gronwall's inequality, it follows from (5) that

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^4}^4 \leq \|a\|_{L^4}^4 \exp \left(C \int_0^T \|p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}}^{\frac{2}{2-r}} ds \right).$$

This completes the proof of Theorem 1.1. \square

4. Proof of Theorem 1.2

Taking $\nabla \operatorname{div}$ on both sides of the first equation in (1) for smooth (u, p) , one can obtain

$$-\Delta (\nabla p) = \sum_{i,j=1}^3 \partial_i \partial_j (\nabla(u_i u_j)).$$

Therefore the Calderon-Zygmund inequality

$$\|\nabla p\|_{L^q} \leq C \| |u| |\nabla u| \|_{L^q} \quad (6)$$

holds for any $1 < q < \infty$. This relation (6) between ∇p and derivatives of the velocity was derived firstly in [21] and will play a very important role in the following proof.

Multiplying both sides of the first equation of (1) by $4u|u|^2$, suitable integration by parts yields

$$\frac{d}{dt} \|u(\cdot, t)\|_{L^4}^4 + 4 \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 dx + 2 \int_{\mathbb{R}^3} |\nabla |u|^2|^2 dx \leq 2 \int_{\mathbb{R}^3} |\nabla p| |u|^3 dx \quad (7)$$

for $t \in (0, T)$. Let us estimate the integral $I = \int_{\mathbb{R}^3} |\nabla p| |u|^3 dx$ on the right hand side of (7):

$$\begin{aligned} I &\leq \| |\nabla p| \cdot |u| \|_{L^2} \| |u|^2 \|_{L^2} \\ &= \| |\nabla p| \cdot |u| \|_{L^2} \|u\|_{L^4}^2 \\ &\leq \|\nabla p\|_{L^2}^{\frac{1}{2}} \| |\nabla p| \cdot |u|^2 \|_{L^2}^{\frac{1}{2}} \|u\|_{L^4}^2 \\ &\leq C \| |u| \cdot |\nabla u| \|_{L^2}^{\frac{1}{2}} \|\nabla p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}}^{\frac{1}{2}} \| |u|^2 \|_{B_{2,1}^{\frac{1}{2}}}^{\frac{1}{2}} \|u\|_{L^4}^2 \\ &\leq C \| |u| \cdot |\nabla u| \|_{L^2}^{\frac{1}{2}} \|\nabla p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}}^{\frac{1}{2}} \| |u|^2 \|_{L^2}^{\frac{1-r}{2}} \| \nabla |u|^2 \|_{L^2}^{\frac{r}{2}} \|u\|_{L^4}^2 \\ &\leq C \| |u| \cdot |\nabla u| \|_{L^2}^{\frac{1+r}{2}} \|\nabla p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}}^{\frac{1}{2}} \|u\|_{L^4}^{3-r} \\ &\leq C (\| |u| \cdot |\nabla u| \|_{L^2}^2)^{\frac{1+r}{4}} \left(\|\nabla p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}}^{\frac{2}{3-r}} \|u\|_{L^4}^4 \right)^{\frac{3-r}{4}} \\ &\leq \| |u| \cdot |\nabla u| \|_{L^2}^2 + C \|\nabla p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}}^{\frac{2}{3-r}} \|u\|_{L^4}^4, \end{aligned}$$

where we used (6) for $q = 2$ and $\|u\|_{\dot{B}_{2,1}^r} \leq C \|u\|_{L^2}^{1-r} \|\nabla u\|_{L^2}^r$. Then by (7) and the above inequality, we derive

$$\frac{d}{dt} \|u(\cdot, t)\|_{L^4}^4 + 2 \int_{\mathbb{R}^3} |\nabla u|^2 |u|^2 dx + 2 \int_{\mathbb{R}^3} |\nabla |u|^2|^2 dx \leq C \|\nabla p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}}^{\frac{2}{3-r}} \|u\|_{L^4}^4. \quad (8)$$

Due to Gronwall's inequality, it follows from (8) that

$$\sup_{0 \leq t \leq T} \|u(t)\|_{L^4}^4 \leq \|u_0\|_{L^4}^4 \exp \left(C \int_0^T \|\nabla p\|_{\dot{\mathcal{M}}_{2, \frac{3}{r}}}^{\frac{2}{3-r}} ds \right).$$

This completes the proof of Theorem 1.2. \square

5. Regularity criteria for the MHD equations

In this paper, we will also consider the following 3D incompressible viscous MHD equations:

$$\begin{cases} \partial_t u - \mu \Delta u + u \cdot \nabla u + \nabla p + \frac{1}{2} \nabla |b|^2 - b \cdot \nabla b = 0 \\ \partial_t b - \nu \Delta b + u \cdot \nabla b - b \cdot \nabla u = 0 \\ \nabla \cdot u = \nabla \cdot b = 0 \\ u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x), \end{cases} \quad (9)$$

where $u = u(x, t) \in \mathbb{R}^3$ is the velocity field, $b \in \mathbb{R}^3$ is the magnetic field, $p = p(x, t)$ is the scalar pressure, $\mu > 0$ is the kinematic viscosity and $\nu > 0$ is the resistivity, while u_0 and b_0 are given initial velocity and initial magnetic field with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ in the sense of distribution. For simplicity, we assume that the external force has a scalar potential and is included into the pressure gradient. In what follows, we assume $\mu = \nu = 1$ for convenience.

It is well-known [14] that the problem (9) is local well-posed for any given initial datum $u_0, b_0 \in H^s(\mathbb{R}^3)$, $s \geq 3$. But whether this unique local solution can exist globally is an outstanding challenge problem. Some fundamental Serrin-type regularity criteria in term of the velocity only were shown in [7, 20] independently. In [24] direction of the vorticity field $\omega = \nabla \times u$ was discussed (see also [7]).

In particular, in [23] regularity is guaranteed under Serrin-type conditions both for the pressure and the magnetic field. It is reasonable in the following sense: p in the first equation of (9) can guarantee the regularity of u , but the smoothness of b can be kept under some condition of itself due to the second equation. Recently, Zhou and Fan [26] did a breakthrough and established regularity criteria in L^p space only in terms of the pressure.

Here, we can extend regularity criteria for the MHD equations to the critical Morrey–Campanato space.

Theorem 5.1. *Let $T > 0$ and $(u_0, b_0) \in H^s(\mathbb{R}^3)$ with $s \geq 3$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. If the pressure p associated with the corresponding smooth solution (u, b) satisfies one of the following conditions:*

$$p \in L^{\frac{2}{2-r}}(0, T; \dot{\mathcal{M}}_{2, \frac{3}{r}}(\mathbb{R}^3)) \quad \text{or} \quad \nabla p \in L^{\frac{2}{3-r}}(0, T; \dot{\mathcal{M}}_{2, \frac{3}{r}}(\mathbb{R}^3))$$

with $0 < r < 1$, then (u, b) can be extended smoothly beyond $t = T$.

There are two main ingredients in the proof. First, we rewrite (9) as (firstly given in [6])

$$\begin{cases} \partial_t w^+ + w^- \cdot \nabla w^+ = \Delta w^+ - \nabla p \\ \partial_t w^- + w^+ \cdot \nabla w^- = \Delta w^- - \nabla p \\ \nabla \cdot w^+ = \nabla \cdot w^- = 0, \end{cases} \quad (10)$$

with $w^\pm := u \pm b$. It is sufficient to establish regularity for $w^\pm := u \pm b$ instead of for u and b . Then, we can combine the arguments in [26] and that in Section 4. For concision, we omit the detailed proof.

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