Formal Solutions of
Second Order Evolution Equations

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Abstract. We study the initial value problem for a second order evolution equation $\partial_t u = F(x, u, \nabla_x u, \nabla_x^2 u)$, $u|_{t=0} = u_0$, where $F(x, u, p, q)$ is a polynomial function in variables $u \in \mathbb{R}, p \in \mathbb{R}^d, q \in \mathbb{R}^{d^2}$ with coefficients analytic on a domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$ and u_0 is analytic on Ω. We construct a formal power series solution $\widehat{u}(t, x) = \sum_{n=0}^{\infty} \varphi_n(x) t^n$ of the equation and prove that it satisfies Gevrey type estimates $|\varphi_n(x)| \leq C^{n+1} n!$ for $x \in K \Subset \Omega$ and $n \in \mathbb{N}_0$, where C does not depend on n. The proof is based on some combinatorial identities and estimates which may be of independent interest.

Keywords. Nonlinear evolution equations, formal solutions, Gevrey estimates

Mathematics Subject Classification (2000). Primary 35C10, 35K15, secondary 05A10, 35K55

1. Introduction

We study the initial value problem for a second order evolution equation

$$
\begin{cases}
\partial_t u = F(x, u, \nabla_x u, \nabla_x^2 u) \\
u|_{t=0} = u_0,
\end{cases}
\tag{1}
$$

where $F(x, u, p, q)$ is a polynomial function in variables $u \in \mathbb{R}, p \in \mathbb{R}^d, q \in \mathbb{R}^{d^2}$ with coefficients analytic on a domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$. The initial data u_0 is supposed to be analytic on Ω and we are interested in solving (1) in the class of formal power series in variable t,

$$
\widehat{u}(t, x) = \sum_{n=0}^{\infty} \varphi_n(x) t^n.
\tag{2}
$$

In order to study the growth properties of formal solutions we use the following definition.

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Definition. Let $\Omega \subseteq \mathbb{R}^d$ and $s \geq 1$. A formal power series (2) is said to belong to the *Gevrey class $G^s(\Omega)$-in time* if for any compact set $K \Subset \Omega$ one can find $L < \infty$ such that

$$\sup_{n \in \mathbb{N}_0} \sup_{x \in K} \frac{|\varphi_n(x)|}{L^n (n!)^{s-1}} < \infty. \tag{3}$$

Equivalently, the class $G^s(\Omega)$-in time consists of power series (2) for which the Borel transform of order s, $B^s \widehat{u}(\tau, x) = \sum_{n=0}^{\infty} \frac{\varphi_n(x)}{\Gamma(1+(s-1)n)} \tau^n$ is convergent for small $|\tau|$ locally uniformly in Ω. In particular, for $s = 1$ we get the convergence of (2) for small $|t|$.

Let us mention here that formal power series solutions to nonlinear partial differential equations were studied by H. Chen and Z. Luo [1], H. Chen, Z. Luo and H. Tahara [2], H. Chen, Z. Luo and C. Zhang [3], H. Chen and Z. Zhang [4], R. Gérard and H. Tahara [6], and by S. Ōuchi [9–11]. In particular, S. Ōuchi [9] obtained, by means of majorant functions, Gevrey estimates of formal solutions for a quite general class of nonlinear PDEs. From his results one should be able to infer an estimation for formal solutions of (1). However in the main theorems of Ōuchi's paper (Theorems 1.7, 1.8, 2.4, 2.6 and 2.7 in [9]) he imposed very technical assumptions on the equation which are difficult to check in concrete cases. Also the estimations of the Gevrey index given in that theorems are difficult to compute.

In our paper F is an arbitrary polynomial and the Gevrey index is given explicitly, so the result is more *user friendly*. Also our proof is direct and more elementary. In fact, we construct a formal power series solution (2) to (1) and prove that it belongs to the formal Gevrey class G^2 provided that the initial data u_0 is analytic. Our main result is a Maillet type theorem and reads as follows.

Main Theorem. *Let Ω be a domain in $\mathbb{R}^d$, $F(x, u, p, q)$ be a polynomial function in variables $u \in \mathbb{R}, p \in \mathbb{R}^d, q \in \mathbb{R}^{d^2}$ with coefficients analytic on Ω and let u_0 be an analytic function on Ω. Then there exists a unique formal power series solution (2) of (1) and it belongs to $G^2(\Omega)$-in time.*

The Main Theorem is an extension of [8, Theorem 1] and [7, Theorem 1], where the semilinear heat equation $\partial_t u = \Delta u + f(u)$ and the Burgers type equation $\partial_t u = \Delta u + \frac{\partial}{\partial x_1} f(u)$ were treated. However it applies to a much wider class of equations including the nonlinear Schrödinger equation $i \partial_t u = \Delta u + f(u, \nabla u)$ and the fast diffusion equation $\partial_t u = \Delta(u^m)$. Note that by an analytic change of coordinates the initial hyperplane $\{t = 0\}$ can be replaced by an arbitrary analytic hypersurface S in $\mathbb{R}^{d+1}$. The Main Theorem can be also regarded as an extension of the well known Cauchy-Kowalevski theorem (see [5, Chapter 1.D]) to the case of initial data on a characteristic hypersurface. Finally, let us mention that our combinatorial method of the proof of the Main Theorem can be

extended to higher order equations of the type $\partial_t^l u = F(x, u, \nabla u, \ldots, \nabla^k u)$ with $l < k$.

The formal power series solution (2) of (1) is constructed in Section 2. In Section 3 we prove some lemmas of combinatorial type which are useful in the proof of the Main Theorem, but may be of independent interest, too. Finally, the proof of the Main Theorem is done in Section 4.

2. Formal solutions

In this section we construct the formal power series solution of (1). To this end note that the IVP (1) can be written in the form

$$
\begin{cases}
\partial_t u = \displaystyle\sum_{\ell = (l^0, l^1, l^2) \in \Lambda} a^\ell u^{l^0} (\nabla_x u)^{l^1} (\nabla_x^2 u)^{l^2} \\
u\big|_{t=0} = u_0,
\end{cases}
\tag{4}
$$

where the sum is over a finite subset $\Lambda \subset \mathbb{N}_0 \times \mathbb{N}_0^d \times \mathbb{N}_0^{d^2}$, $a^\ell \in \mathcal{A}(\Omega)$ and $u_0 \in \mathcal{A}(\Omega)$. Now let us look for a solution of (4) in the form of a formal power series (2). Then

$$
\partial_t \widehat{u}(t, x) = \sum_{n=0}^{\infty} (n+1) \varphi_{n+1}(x) t^n,
$$

and for $\ell = (l^0, l^1, l^2) \in \Lambda$,

$$
\left(\widehat{u}(t,x) \right)^{l^0} \left(\nabla_x \widehat{u}(t,x) \right)^{l^1} \left(\nabla_x^2 \widehat{u}(t,x) \right)^{l^2}
$$

$$
= \left(\sum_{n=0}^{\infty} \varphi_n(x) t^n \right)^{l^0} \prod_{i=1}^{d} \left(\sum_{n=0}^{\infty} \partial_i \varphi_n(x) t^n \right)^{l_i^1} \prod_{i,j=1}^{d} \left(\sum_{n=0}^{\infty} \partial_{ij}^2 \varphi_n(x) t^n \right)^{l_{i,j}^2}
$$

$$
= \sum_{n=0}^{\infty} \left(\sum_{\kappa \in \mathcal{K}_0^\ell(n)} \varphi_{\kappa_1^0} \cdots \varphi_{\kappa_{l^0}^0} \cdot \partial_1 \varphi_{\kappa_1^{1;1}} \cdots \partial_1 \varphi_{\kappa_{l_1^1}^{1;1}} \cdots \partial_d \varphi_{\kappa_1^{1;d}} \cdots \partial_d \varphi_{\kappa_{l_d^1}^{1;d}} \right.
$$

$$
\left. \cdot \, \partial_{11}^2 \varphi_{\kappa_1^{2;1,1}} \cdots \partial_{dd}^2 \varphi_{\kappa_{l_{d,d}^2}^{2;d,d}} \right) t^n,
$$

where the internal sum is over the set

$$
\mathcal{K}_0^\ell(n) = \left\{ \kappa = (\kappa^0, \kappa^1, \kappa^2) \,\middle|\,
\begin{array}{l}
\kappa^0 \in \mathbb{N}_0^{l^0}; \\[4pt]
\kappa^1 = (\kappa^{1;1}, \ldots, \kappa^{1;d}); \quad \kappa^{1;i} \in \mathbb{N}_0^{l_i^1}; \quad i = 1, \ldots, d; \\[4pt]
\kappa^2 = (\kappa^{2;1,1}, \ldots, \kappa^{2;d,d}); \; \kappa^{2;i,j} \in \mathbb{N}_0^{l_{i,j}^2}; \; i, j = 1, \ldots, d; \\[4pt]
n = |\kappa^0| + |\kappa^1| + |\kappa^2|
\end{array}
\right\}.
$$

So the formal solution of (4) is given by (2), with $\varphi_0 = u_0$ and φ_n given by the recurrence relations

$$\varphi_{n+1} = \frac{1}{n+1} \sum_{\ell \in \Lambda} a^\ell \cdot \sum_{\kappa \in \mathcal{K}_0^\ell(n)} (\varphi)_{\kappa^0} \cdot (\nabla_x \varphi)_{\kappa^1} \cdot (\nabla_x^2 \varphi)_{\kappa^2}, \quad n \in \mathbb{N}_0, \qquad (5)$$

where $(\varphi)_{\kappa^0} = \prod_{\nu=1}^{l^0} \varphi_{\kappa_\nu^0}$, $(\nabla_x \varphi)_{\kappa^1} = \prod_{i=1}^{d} \prod_{\nu=1}^{l_i^1} \partial_i \varphi_{\kappa_\nu^{1;i}}$, $(\nabla_x^2 \varphi)_{\kappa^2} = \prod_{i,j=1}^{d} \prod_{\nu=1}^{l_{i,j}^2} \partial_{i,j}^2 \varphi_{\kappa_\nu^{2;i,j}}$.

In particular,

$$\varphi_1 = \sum_{\ell=(l^0,l^1,l^2)\in\Lambda} a^\ell u_0^{l^0} (\nabla_x u_0)^{l^1} (\nabla_x^2 u_0)^{l^2} = F(x, u_0, \nabla_x u_0, \nabla_x^2 u_0).$$

3. Combinatorial lemmas

Here we prove combinatorial lemmas useful in the proof of the Main Theorem.

Lemma 1. *Let $d, j \in \mathbb{N}, \alpha \in \mathbb{N}_0^d, \kappa \in \mathbb{N}_0^j$. Then*

$$\sum \frac{(|\beta^1| + \kappa_1)!}{\beta^1! \kappa_1!} \cdots \frac{(|\beta^j| + \kappa_j)!}{\beta^j! \kappa_j!} = \frac{(|\alpha| + |\kappa| + j - 1)!}{\alpha!(|\kappa| + j - 1)!}, \qquad (6)$$

where the sum is over $\beta^1, \ldots, \beta^j \in \mathbb{N}_0^d$ with $\beta^1 + \cdots + \beta^j = \alpha$.

Proof. For $x \in \mathbb{R}^d$ with $\sigma(x) := x_1 + \cdots + x_d < 1$ put $f(x) = \left(\frac{1}{1-\sigma(x)}\right)^{|\kappa|+j}$. Then

$$\partial_x^\alpha f(x)|_{x=0} = \frac{(|\kappa| + j) \cdots (|\kappa| + j + |\alpha| - 1)}{(1 - \sigma(x))^{|\kappa|+j+|\alpha|}}\bigg|_{x=0} = \frac{(|\alpha| + |\kappa| + j - 1)!}{(|\kappa| + j - 1)!}. \qquad (7)$$

On the other hand using the Leibniz formula we get

$$\partial^\alpha f(x)|_{x=0}$$
$$= \partial^\alpha \left(\left(\frac{1}{1-\sigma(x)}\right)^{\kappa_1+1} \cdots \left(\frac{1}{1-\sigma(x)}\right)^{\kappa_j+1} \right)\bigg|_{x=0}$$
$$= \sum_{\beta^1+\cdots+\beta^j=\alpha} \binom{\alpha}{\beta^1 \cdots \beta^j} \left(\partial^{\beta^1} \left(\frac{1}{1-\sigma(x)}\right)^{\kappa_1+1} \cdots \partial^{\beta^j} \left(\frac{1}{1-\sigma(x)}\right)^{\kappa_j+1} \right)\bigg|_{x=0} \qquad (8)$$
$$= \sum_{\beta^1+\cdots+\beta^j=\alpha} \frac{\alpha!}{\beta^1! \cdots \beta^j!} \frac{(\kappa_1+1)\cdots(\kappa_1+|\beta^1|)}{(1-\sigma(x))^{\kappa_1+|\beta^1|+1}} \cdots \frac{(\kappa_j+1)\cdots(\kappa_j+|\beta^j|)}{(1-\sigma(x))^{\kappa_j+|\beta^j|+1}}\bigg|_{x=0}$$
$$= \alpha! \cdot \sum_{\beta^1+\cdots+\beta^j=\alpha} \frac{(|\beta^1|+\kappa_1)!}{\beta^1! \kappa_1!} \cdots \frac{(|\beta^j|+\kappa_j)!}{\beta^j! \kappa_j!}.$$

Comparing right hand sides of (7) and (8) we get (6).   □

Corollary 1. *Let* $d, j \in \mathbb{N}, \alpha \in \mathbb{N}_0^d, \kappa \in \mathbb{N}_0^j.$ *Then*

$$\sum \frac{|\beta^0|!}{\beta^0!} \frac{(|\beta^1| + \kappa_1)!}{\beta^1! \kappa_1!} \cdots \frac{(|\beta^j| + \kappa_j)!}{\beta^j! \kappa_j!} = \frac{(|\alpha| + |\kappa| + j)!}{\alpha!(|\kappa| + j)!}, \qquad (9)$$

where the sum is over $\beta^0, \beta^1, \ldots, \beta^j \in \mathbb{N}_0^d$ *with* $\beta^0 + \beta^1 + \cdots + \beta^j = \alpha.$

Lemma 2. *Let* $\gamma \in \mathbb{N}, \ \ell = (l^0, l^1, l^2) \in \Lambda$ *and* $\eta = (\eta^0, \eta^1, \eta^2) \in \Lambda$ *with* $0 \le \eta \le \ell.$ *For* $n \in \mathbb{N}_0$ *set*

$$A_\eta^\ell(n) = \sum_{i \in \mathcal{K}_1^\eta(n)} \frac{\left(\genfrac{}{}{0pt}{}{n}{i^0, i^1, i^2}\right)^{\gamma-1}}{\left(\genfrac{}{}{0pt}{}{\gamma n + |l^1| + 2|l^2|}{\gamma i^0, \mathbf{0}_{l^0 - \eta^0}, \gamma i^1 + \mathbf{1}_{\eta^1}, \mathbf{1}_{\ell^1 - \eta^1}, \gamma i^2 + \mathbf{2}_{\eta^2}, \mathbf{2}_{\ell^2 - \eta^2}}\right)}, \qquad (10)$$

where the sum is over the set

$$\mathcal{K}_1^\eta(n) = \left\{ i = (i^0, i^1, i^2) \,\middle|\, \begin{array}{l} i^0 \in \mathbb{N}^{\eta^0}; \\[4pt] i^1 = (i^{1;1}, \ldots, i^{1;d}); \quad i^{1;\nu} \in \mathbb{N}^{\eta_\nu^1}; \quad \nu = 1, \ldots, d; \\[4pt] i^2 = (i^{2;1,1}, \ldots, i^{2;d,d}); \ i^{2;\nu,\mu} \in \mathbb{N}^{l_{\nu,\mu}^2}; \ \nu, \mu = 1, \ldots, d; \\[4pt] n = |i^0| + |i^1| + |i^2|; \end{array} \right\}$$

and $\mathbf{0}_{\eta^0} = (0, \ldots, 0) \in \mathbb{N}_0^{\eta^0}, \ \mathbf{1}_{\eta^1} = (1, \ldots, 1) \in \mathbb{N}_0^{|\eta^1|}, \ \mathbf{2}_{\eta^2} = (2, \ldots, 2) \in \mathbb{N}_0^{|\eta^2|}.$
Then there exists a constant $L = L(\ell, \eta) < \infty$ *such that*

$$A_\eta^\ell(n) \le \frac{L}{(n+1)^{|l^1| + 2|l^2| - 2}} \quad \text{for } n \in \mathbb{N}_0. \qquad (11)$$

Proof. First of all note that for $n < \eta^0 + |\eta^1| + |\eta^2|$ the set $\mathcal{K}_1^\eta(n)$ is empty and so $A_\eta^\ell(n) = 0$. Next, for $n \ge \eta^0 + |\eta^1| + |\eta^2|$ take a term in the sum (10),

$$\frac{\left(\genfrac{}{}{0pt}{}{n}{i^0, i^1, i^2}\right)^{\gamma-1}}{\left(\genfrac{}{}{0pt}{}{\gamma n + |l^1| + 2|l^2|}{\gamma i^0, \mathbf{0}_{l^0 - \eta^0}, \gamma i^1 + \mathbf{1}_{\eta^1}, \mathbf{1}_{\ell^1 - \eta^1}, \gamma i^2 + \mathbf{2}_{\eta^2}, \mathbf{2}_{\ell^2 - \eta^2}}\right)}$$

$$= \left(\frac{n!}{i^0! i^1! i^2!}\right)^{\gamma-1} \cdot \frac{(\gamma i^0)! (\gamma i^1 + \mathbf{1}_{\eta^1})! \cdot 1^{|l^1 - \eta^1|} (\gamma i^2 + \mathbf{2}_{\eta^2})! \cdot 2^{|l^2 - \eta^2|}}{(\gamma n + |l^1| + 2|l^2|)!}$$

$$\le 2^{|l^2 - \eta^2|} \left(\frac{n!}{i^0! i^1! i^2!}\right)^{\gamma-1} \cdot \frac{(\gamma i^0 + \mathbf{2}_{\eta^0})! (\gamma i^1 + \mathbf{2}_{\eta^1})! (\gamma i^2 + \mathbf{2}_{\eta^2})!}{(\gamma n + |l^1| + 2|l^2|)!}.$$

Since the last expression is symmetric with respect to the group of permutations of $(\eta^0 + |\eta^1| + |\eta^2|)$ elements we can assume that $i_1^0 \ge \cdots \ge i_{\eta^0}^0 \ge i_1^{1;1} \ge \cdots \ge i_{\eta_d^1}^{1;d} \ge i_1^{2;1,1} \ge \cdots \ge i_{\eta_{d,d}^2}^{2;d,d}$. Set $j_1 = i_1^0, j' = (i_2^0, \ldots, i_{\eta_{d,d}^2}^{2;d,d})$ and note that a term

in the sum is bounded by

$$\frac{2^{|l^2-\eta^2|}(\gamma j' + \mathbf{2}_{(\eta^0-1,\eta^1,\eta^2)})!}{(j'!)^{\gamma-1}} \cdot \left(\frac{n!}{j_1!}\right)^{\gamma-1} \cdot \frac{(\gamma j_1 + 2)!}{(\gamma n + |l^1| + 2|l^2|)!}$$

$$= \frac{2^{|l^2-\eta^2|}(\gamma j' + \mathbf{2}_{(\eta^0-1,\eta^1,\eta^2)})!}{(j'!)^{\gamma-1}} \cdot \frac{\left((j_1+1)(j_1+2)\cdots n\right)^{\gamma-1}}{(\gamma j_1 + 3)(\gamma j_1 + 4)\cdots(\gamma n + |l^1| + 2|l^2|)}.$$

Clearly the numerator of the last factor is bounded by $(n+1)^{(\gamma-1)(n-j_1)}$. Next, since $n - |j'| = j_1 \geq \frac{n}{\eta^0+|\eta^1|+|\eta^2|}$ we get

$$\gamma j_1 + k \geq \frac{\gamma n}{\eta^0 + |\eta^1| + |\eta^2|} + k \geq \min\left(\frac{\gamma}{\eta^0 + |\eta^1| + |\eta^2|}, k\right) \cdot (n+1)$$

for $k = 3, 4, \ldots, \gamma n + |l^1| + 2|l^2| - \gamma j_1$ and $n \geq 0$. So the nominator of the last factor is not less then

$$\frac{c_0}{(n+1)^{\gamma n + |l^1| + 2|l^2| - \gamma j_1 - 2}} \quad \text{with some } c_0 > 0.$$

Hence a term in the sum (10) is bounded by

$$\frac{L}{(n+1)^{|l^1|+2|l^2|+n-j_1-2}} \leq \frac{L}{(n+1)^{|l^1|+2|l^2|+\eta^0+|\eta^1|+|\eta^2|-3}}$$

with some $L < \infty$ (since $n - j_1 = |j'| \geq \eta^0 + |\eta^1| + |\eta^2| - 1$). Finally since the sum (10) contains no more then $(n+1)^{\eta^0+|\eta^1|+|\eta^2|-1}$ terms we get (11).   □

Lemma 3. *For $\ell \in \Lambda$ and $n \in \mathbb{N}_0$ set*

$$A^\ell(n) = \sum_{\kappa \in \mathcal{K}_0^\ell(n)} \frac{\binom{n}{\kappa^0, \kappa^1, \kappa^2}^{\gamma-1}}{\binom{\gamma n + |l^1| + 2|l^2|}{\gamma\kappa^0, \gamma\kappa^1+\mathbf{1}_{l^1}, \gamma\kappa^2+\mathbf{2}_{l^2}}}.$$

Then there exists a constant $L(\ell) < \infty$ such that

$$A^\ell(n) \leq \frac{L(\ell)}{(n+1)^{|l^1|+2|l^2|-2}} \quad \text{for } n \in \mathbb{N}_0.$$

Proof. Note that

$$A^\ell(n) = \sum \binom{l^0}{\eta^0}\binom{l^1}{\eta^1}\binom{l^2}{\eta^2} A_\eta^\ell(n),$$

where the sum is over $\eta = (\eta^0, \eta^1, \eta^2)$ with $\eta^0 \in \mathbb{N}_0, \eta^1 \in \mathbb{N}_0^d, \eta^2 \in \mathbb{N}_0^{d^2}$ and $0 \leq \eta^0 \leq l^0$, $0 \leq \eta^1 \leq l^1$, $0 \leq \eta^2 \leq l^2$. Hence Lemma 3 is a consequence of Lemma 2.   □

4. Proof of the Main Theorem

The existence and uniqueness of a formal solution to (1) was established in Section 2. Furthermore, the functions φ_n of the formal solution (2) satisfy recurrence relations (5). Hence we only need to show that formal solution belongs to $G^2(\Omega)$-in time. To this end set $\gamma = \max_{\ell \in \Lambda} \gamma^\ell$, where $\gamma^\ell = l^0 + 2|l^1| + 3|l^2|$. Note that if $\gamma \le 2$, then $|l^2| = 0$ for all $\ell \in \Lambda$. In that case the equation satisfies the assumption of the Cauchy-Kowalevski theorem (see [5, Theorem 1.25]), and hence the solution of (1) is analytic in time at $\{t = 0\}$. So we can assume that $\gamma \ge 3$.

First off all we shall prove inductively that for any compact set $K \Subset \Omega$ one can find $1 \le C < \infty$ such that for any $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$,

$$\sup_{x \in K} |\partial^\alpha \varphi_n(x)| \le b_n C^{|\alpha| + \gamma n + 1} \frac{(|\alpha| + \gamma n)!}{n!^{\gamma - 1}}, \tag{12}$$

with $b_0 = 1$ and

$$
b_{n+1} = \frac{1}{n+1} \frac{(n+1)!^{\gamma-1}}{n!^{\gamma-1}} \left[\sum_{\ell \in \Lambda} \frac{(\gamma n + |l^1| + 2|l^2|)!}{(\gamma n + \gamma)!} \sum_{\kappa \in \mathcal{K}_0^\ell(n)} (b)_{\kappa^0} \cdot (b)_{\kappa^1} \cdot (b)_{\kappa^2} \right.
$$
$$
\left. \times \binom{n}{\kappa^0, \kappa^1, \kappa^2}^{\gamma-1} \cdot \binom{\gamma n + |l^1| + 2|l^2|}{\gamma \kappa^0, \gamma \kappa^1 + \mathbf{1}_{l^1}, \gamma \kappa^2 + \mathbf{2}_{l^2}}^{-1} \right]. \tag{13}
$$

Fix $K \Subset \Omega$. Clearly, since $u_0 = \varphi_0 \in \mathcal{A}(\Omega)$, (12) holds for $n = 0$ with $b_0 = 1$ and some $1 \le C < \infty$. Now assume that (12) holds some for $n \in \mathbb{N}_0$. Since $a^\ell \in \mathcal{A}(\Omega)$ for $\ell \in \Lambda$, for any $x \in K$, $\ell \in \Lambda$ and $\beta^0 \in \mathbb{N}_0^d$ increasing eventually C we have $|\partial^{\beta^0} a^\ell(x)| \le C^{|\beta^0|+1}|\beta^0|!$. So by (5), the Leibniz rule, the inductive assumption and Corollary 1 we estimate for $\alpha \in \mathbb{N}_0^d$ and $x \in K$,

$$|\partial^\alpha \varphi_{n+1}(x)|$$
$$\le \frac{1}{n+1} \sum_{\ell \in \Lambda} \sum_{\kappa \in \mathcal{K}_0^\ell(n)} \left| \partial^\alpha \big(a^\ell(x) \cdot (\varphi(x))_{\kappa^0} \cdot (\nabla_x \varphi(x))_{\kappa^1} \cdot (\nabla_x^2 \varphi(x))_{\kappa^2} \big) \right|$$
$$\le \frac{1}{n+1} \sum_{\ell \in \Lambda} \sum_{\kappa \in \mathcal{K}_0^\ell(n)} \sum_{\substack{\beta^0, \beta^1, \ldots, \beta^{l^0+|l^1|+|l^2|} \in \mathbb{N}_0^d \\ \beta^0 + \beta^1 + \cdots + \beta^{l^0+|l^1|+|l^2|} = \alpha}} \binom{\alpha}{\beta} |\partial^{\beta^0} a^\ell(x)|$$
$$\times \left| \partial^{\beta^1} \varphi_{\kappa_1^0}(x) \right| \cdots \left| \partial^{\beta^{l^0}} \varphi_{\kappa_{l^0}^0}(x) \right|$$
$$\times \left| \partial^{\beta^{l^0+1}} \partial_1 \varphi_{\kappa_1^{1;1}}(x) \right| \cdots \left| \partial^{\beta^{l^0+l_1^1}} \partial_1 \varphi_{\kappa_{l_1^1}^{1;1}}(x) \right| \cdots \left| \partial^{\beta^{l^0+|l^1|}} \partial_d \varphi_{\kappa_{l_d^1}^{1;d}}(x) \right|$$
$$\times \left| \partial^{\beta^{l^0+|l^1|+1}} \partial_{11}^2 \varphi_{\kappa_1^{2;1,1}}(x) \right| \cdots \left| \partial^{\beta^{l^0+|l^1|+|l^2|}} \partial_{dd}^2 \varphi_{\kappa_{l_{d,d}^2}^{2;d,d}}(x) \right|$$

$$\leq \frac{1}{n+1} \sum_{\ell \in \Lambda} \sum_{\kappa \in \mathcal{K}_0^\ell(n)} \sum_{\substack{\beta^0,\ldots,\beta^{l^0+|l^1|+|l^2|} \in \mathbb{N}_0^d \\ \beta^0+\cdots+\beta^{l^0+|l^1|+|l^2|}=\alpha}} \binom{\alpha}{\beta} C^{|\beta^0|+1} |\beta^0|!$$

$$\times \; b_{\kappa_1^0} C^{|\beta^1|+\gamma\kappa_1^0+1} \frac{(|\beta^1|+\gamma\kappa_1^0)!}{(\kappa_1^0!)^{\gamma-1}} \cdots b_{\kappa_{l^0}^0} C^{|\beta^{l^0}|+\gamma\kappa_{l^0}^0+1} \frac{(|\beta^{l^0}|+\gamma\kappa_{l^0}^0)!}{(\kappa_{l^0}^0!)^{\gamma-1}}$$

$$\times \; b_{\kappa_1^{1;1}} C^{|\beta^{l^0+1}|+1+\gamma\kappa_1^{1;1}+1} \frac{(|\beta^{l^0+1}|+1+\gamma\kappa_1^{1;1})!}{\left(\kappa_1^{1;1}!\right)^{\gamma-1}} \cdots b_{\kappa_{l_1^1}^{1;1}} C^{|\beta^{l^0+|l_1^1|}|+1+\gamma\kappa_{l_1^1}^{1;1}+1}$$

$$\times \; \frac{\left(|\beta^{l^0+|l_1^1|}|+1+\gamma\kappa_{l_1^1}^{1;1}\right)!}{\left(\kappa_{l_1^1}^{1;1}!\right)^{\gamma-1}} \cdots b_{\kappa_{l_d^1}^{1;d}} C^{|\beta^{l^0+|l^1|}|+1+\gamma\kappa_{l_d^1}^{1;d}+1} \frac{\left(|\beta^{l^0+|l^1|}|+1+\gamma\kappa_{l_d^1}^{1;d}\right)!}{\left(\kappa_{l_d^1}^{1;d}!\right)^{\gamma-1}}$$

$$\times \; b_{\kappa_1^{2;1,1}} C^{|\beta^{l^0+|l^1|+1}|+2+\gamma\kappa_1^{2;1,1}+1} \frac{\left(|\beta^{l^0+|l^1|+1}|+2+\gamma\kappa_1^{2;1,1}\right)!}{\left(\kappa_1^{2;1,1}!\right)^{\gamma-1}} \cdots$$

$$\times \; b_{\kappa_{l_{d,d}^2}^{2;d,d}} C^{|\beta^{l^0+|l^1|+|l^2|}|+2+\gamma\kappa_{l_{d,d}^2}^{2;d,d}+1} \frac{\left(|\beta^{l^0+|l^1|+|l^2|}|+2+\gamma\kappa_{l_{d,d}^2}^{2;d,d}\right)!}{\left(\kappa_{l_{d,d}^2}^{2;d,d}!\right)^{\gamma-1}}$$

$$= \frac{1}{n+1} \sum_{\ell \in \Lambda} C^{|\alpha|+\gamma n+1+l^0+2|l^1|+3|l^2|} \sum_{\kappa \in \mathcal{K}_0^\ell(n)} \frac{(b)_{\kappa^0}(b)_{\kappa^1}(b)_{\kappa^2}}{(\kappa^0!)^{\gamma-1}(\kappa^1!)^{\gamma-1}(\kappa^2!)^{\gamma-1}}$$

$$\times \; \alpha! \sum_{\substack{\beta^0,\ldots,\beta^{l^0+|l^1|+|l^2|} \in \mathbb{N}_0^d \\ \beta^0+\cdots+\beta^{l^0+|l^1|+|l^2|}=\alpha}} \frac{|\beta^0|!}{\beta^0!} \cdot \frac{(|\beta^1|+\gamma\kappa_1^0)!\cdots(|\beta^{l^0}|+\gamma\kappa_{l^0}^0)!}{\beta^1!\cdots\beta^{l^0}!}$$

$$\times \; \frac{\left(\beta^{l^0+1}+\gamma\kappa_1^{1;1}+1\right)!\cdots\left(\beta^{l^0+|l^1|}+\gamma\kappa_{l_d^1}^{1;d}+1\right)!}{\beta^{l^0+1}!\cdots\beta^{l^0+|l^1|}!}$$

$$\times \; \frac{\left(\beta^{l^0+|l^1|+1}+\gamma\kappa_1^{2;1,1}+2\right)!\cdots\left(\beta^{l^0+|l^1|+|l^2|}+\gamma\kappa_{l_{d,d}^2}^{2;d,d}+2\right)!}{\beta^{l^0+|l^1|+1}!\cdots\beta^{l^0+|l^1|+|l^2|}!}$$

$$\overset{(9)}{=} \frac{1}{n+1} \sum_{\ell \in \Lambda} C^{|\alpha|+\gamma n+l^0+2|l^1|+3|l^2|+1} \frac{(|\alpha|+\gamma n+|l^1|+2|l^2|+l^0+|l^1|+|l^2|)!}{(\gamma n+|l^1|+2|l^2|+l^0+|l^1|+|l^2|)!}$$

$$\times \; \sum_{\kappa \in \mathcal{K}_0^\ell(n)} (b)_{\kappa^0}(b)_{\kappa^1}(b)_{\kappa^2} \frac{(\gamma\kappa^0)!(\gamma\kappa^1+\mathbf{1}_{l^1})!(\gamma\kappa^2+\mathbf{2}_{l^2})!}{(\kappa^0!)^{\gamma-1}(\kappa^1!)^{\gamma-1}(\kappa^2!)^{\gamma-1}}$$

$$= \frac{1}{n+1} \sum_{\ell \in \Lambda} C^{|\alpha|+\gamma n+\gamma^\ell+1} \frac{(|\alpha|+\gamma n+\gamma^\ell)!}{(\gamma n+\gamma^\ell)!} \cdot \frac{(\gamma n+|l^1|+2|l^2|)!}{n!^{\gamma-1}}$$

$$\times \; \sum_{\kappa \in \mathcal{K}_0^\ell(n)} (b)_{\kappa^0}(b)_{\kappa^1}(b)_{\kappa^2} \cdot \frac{\binom{n}{\kappa^0,\kappa^1,\kappa^2}^{\gamma-1}}{\binom{\gamma n+|l^1|+2|l^2|}{\gamma\kappa^0,\gamma\kappa^1+\mathbf{1}_{l^1},\gamma\kappa^2+\mathbf{2}_{l^2}}}$$

$$\leq b_{n+1} \cdot C^{|\alpha|+\gamma(n+1)+1} \frac{(|\alpha|+\gamma(n+1))!}{(n+1)!^{\gamma-1}},$$

where b_{n+1} is given by (13). To finish the proof assume that $b_i \leq L^i$ for $i \leq n$ with $L \geq L(\ell)$ for $\ell \in \Lambda$, where $L(\ell)$ is the constant in Lemma 3. Then by Lemma 3,

$$
\begin{aligned}
b_{n+1} &\leq \frac{1}{n+1}\frac{(n+1)!^{\gamma-1}}{n!^{\gamma-1}} \sum_{\ell \in \Lambda} \frac{(\gamma n + |l^1| + 2|l^2|)!}{(\gamma n + \gamma)!} \cdot L^n \cdot \frac{1}{(n+1)^{|l^1|+2|l^2|-2}} \\
&\leq L^n \sum_{\ell \in \Lambda} (n+1)^{\gamma-|l^1|-2|l^2|} \cdot \frac{1}{(\gamma n + |l^1| + 2|l^2| + 1)\cdots(\gamma n + \gamma)} \\
&\leq L^{n+1}
\end{aligned}
$$

if $L \geq |\Lambda|$. Hence applying (12) with $m = 0$ and $(\gamma n)! \leq \gamma^{\gamma n} \cdot n!^{\gamma}$ we get,

$$
\sup_{x \in K} |\varphi_n(x)| \leq L^n C^{\gamma n + 1}\frac{(\gamma n)!}{n!^{\gamma-1}} \leq C(LC^{\gamma}\gamma^{\gamma})^n \cdot n! \qquad \text{for} \quad n \in \mathbb{N}_0
$$

which implies (3) with $s = 2$. $\qquad\qquad\square$

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