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# Implicit Difference Schemes for Evolution Functional Differential Equations

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**Abstract.** We give a theorem on an error estimate of approximate solutions for functional difference equations of the Volterra type with unknown function of several variables. We apply this general result in the investigations of the stability of quasilinear implicit difference schemes generated by first order partial differential functional equations and by parabolic problems. A comparison technique is used with nonlinear estimates of the Perron type for given functions with respect to the functional variable. Equations with deviated variables and differential integral equations can be derived from a general model by specializing given operators.

**Keywords.** Functional differential equations, stability and convergence, interpolating operators, implicit difference schemes

**Mathematics Subject Classification (2000).** Primary 35R10, secondary 65M12

## 1. Introduction

We are interested in a numerical approximation of classical solutions to quasilinear functional differential equations or systems with initial boundary conditions. Difference schemes for evolution functional differential equations consist in replacing partial derivatives with difference operators. Moreover, because equations contain the functional variable, some interpolating operators are needed. This leads to functional difference equations which satisfy consistency conditions on classical solutions of original problems. The main task in these considerations is to find difference approximations of original problems which are stable. Comparison methods are used in the investigations of the stability of functional difference problems.

It is not our aim to show a full review of papers concerning explicit difference schemes for evolution functional differential equations. We shall mention only those which contain such reviews. They are [4, 15, 19] and the monograph [8].

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In recent years, a number of papers concerning implicit difference methods for functional partial differential equations have been published. Difference approximations of classical solutions to first order partial functional differential equations were investigated in [9,10]. Initial problems on the Haar pyramid and initial boundary value problems were considered. Implicit difference schemes for parabolic equations with initial boundary conditions of the Dirichlet type were studied in [5,12]. Monotone iterative methods and implicit difference schemes for computing approximate solutions to parabolic equations with time delays were analyzed in [13,20]. A numerical treatment of initial boundary value problems of the Neumann–Robin type can be found in [14].

A method of difference inequalities and theorems on recurrent inequalities are used in the investigations of the stability of implicit difference schemes. These considerations as a rule require a lot of calculations to reach the convergence result so the main property of the corresponding operators was not easy to be seen. The aim of the present paper is to show that results mentioned above as well as many others are consequences of a result on abstract difference functional equations with an unknown function of several variables.

We formulate our functional differential problems. For any metric spaces $X$ and $Y$ we denote by $C(X,Y)$ the class of all continuous functions from $X$ into $Y$. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let $M_{k\times n}$ be the class of all $k\times n$ matrices with real elements. For $x\in\mathbb{R}^n$, $U\in M_{k\times n}$ where $x=(x_1,\dots,x_n)$, $u=\big[\,u_{ij}\,\big]_{i=1,\dots,k,\,j=1,\dots,n}$ we write

$$\|x\|=\sum_{i=1}^{n}|x_i|,\quad \|U\|_{k\times n;\infty}=\max\Big\{\sum_{j=1}^{n}|u_{ij}|\,:\,1\le i\le n\Big\}.$$

Suppose that $a>0$, $b=(b_1,\dots,b_n)\in\mathbb{R}^n$, $b_i>0$ for $1\le i\le n$, and $d_0\in\mathbb{R}_+$, $d=(d_1,\dots,d_n)\in\mathbb{R}^n_+$, $\mathbb{R}_+=[0,+\infty)$, are given. Let $c=b+d$ and

$$E=[0,a]\times(-b,b),\quad D=[-d_0,0]\times[-d,d],\quad E_0=[-d_0,0]\times[-c,c]$$
$$\partial_0E=[0,a]\times\big([-c,c]\setminus(-b,b)\big),\quad \Omega=E\cup E_0\cup\partial_0E.$$

For a function $z:\Omega\to\mathbb{R}^k$ and for a point $(t,x)\in\bar{E}$ where $\bar{E}$ is the closure of $E$, we define a function $z_{(t,x)}:D\to\mathbb{R}^k$ by $z_{(t,x)}(\tau,y)=z(t+\tau,x+y)$, $(\tau,s)\in D$. Then $z_{(t,x)}$ is the restriction of $z$ to the set $[t-d_0,t]\times[x-d,x+d]$ and this restriction is shifted to the set $D$. Write $\Xi=E\times C(D,\mathbb{R}^k)$ and suppose that

$$f:\Xi\to M_{k\times n},\quad f=\big[\,f_{ij}\,\big]_{i=1,\dots,k,\,j=1,\dots,n},\qquad g:\Xi\to\mathbb{R}^k,\quad g=(g_1,\dots,g_k)$$
$$\varphi:E_0\cup\partial_0E\to\mathbb{R}^k,\ \varphi=(\varphi_1,\dots,\varphi_k)$$

are given functions. Let $z = (z_1, \ldots, z_k)$ be an unknown function of the variables $(t, x)$. We consider the system of functional differential equations

$$\partial_t z_i(t,x) = \sum_{j=1}^{n} f_{ij}(t,x,z_{(t,x)})\partial_{x_j} z_i(t,x) + g_i(t,x,z_{(t,x)}), \quad i = 1,\ldots,k, \tag{1}$$

with the initial boundary condition

$$z(t,x) = \varphi(t,x) \quad \text{on } E_0 \cup \partial_0 E. \tag{2}$$

Sufficient conditions for the existence and uniqueness of classical or generalized solutions of first order partial functional functional problems can be found in [1,6,8].

Now we formulate initial boundary value problems for parabolic functional differential equations. Suppose that

$$G : \Xi \to \mathbb{R}, \;\; \mathbf{G} : \Xi \to \mathbb{R}^n, \; \mathbf{G} = (G_1, \ldots, G_n)$$
$$\mathbf{F} : \Xi \to M_{n\times n}, \; \mathbf{F} = \left[\, F_{ij} \,\right]_{i,j=1,\ldots,n}, \quad \phi : E_0 \cup \partial_0 E \to \mathbb{R}$$

are given functions. Let $z$ be a real unknown function of the variables $(t, x)$. We consider the functional differential equation

$$\begin{aligned}\partial_t z(t,x) = &\sum_{i,j=1}^{n} F_{ij}(t,x,z_{(t,x)})\partial_{x_i x_j} z(t,x) \\ &+ \sum_{i=1}^{n} G_i(t,x,z_{(t,x)})\partial_{x_i} z(t,x) + G(t,x,z_{(t,x)}),\end{aligned} \tag{3}$$

with the initial boundary condition

$$z(t,x) = \phi(t,x) \quad \text{on } E_0 \cup \partial_0 E. \tag{4}$$

Sufficient conditions for the existence and uniqueness of classical or generalized solutions to parabolic functional differential functional problems can be found in [2,3,7,11,16].

Let us denote by $CL(D,\mathbb{R})$ the class of all linear and continuous operators defined on $C(D,\mathbb{R})$ and taking values in $\mathbb{R}$. Write $\Sigma = E \times C(D,\mathbb{R}) \times \mathbb{R}^n$ and suppose that $F : \Sigma \to \mathbb{R}$ and $\phi : E_0 \cup \partial_0 E$ are given functions. Let $z$ be an unknown function of the variables $(t, x)$. We consider the functional differential equation

$$\partial_t z(t,x) = F(t,x,z_{(t,x)}, \partial_x z(t,x)) \tag{5}$$

with the initial boundary condition (4) where $\partial_x z = (\partial_{x_1}, \ldots, \partial_{x_n} z)$. Existence results and a theory of difference methods for (4), (5) are based on the following method of quasilinearization. Suppose that the function $F$ of that variables $(t, x, w, q)$, $q = (q_1, \ldots, q_n)$, is continuous and:

(i) the partial derivatives $\partial_x F = (\partial_{x_1} F, \dots, \partial_{x_n} F)$ and $\partial_q F = (\partial_{q_1}, \dots, \partial_{q_n} F)$ exist on $\Sigma$ and $\partial_x F,\ \partial_q F \in C(\Sigma, \mathbb{R}^n)$;

(ii) there exists the Fréchet derivative $\partial_w F(P)$ and $\partial_w F(P) \in CL(D, \mathbb{R})$ for $P = (t, x, w, q) \in \Sigma$.

Suppose that $\phi \in C(E_0 \cup \partial_0 E, \mathbb{R})$ and there exists $\partial_x \phi = (\partial_{x_1}\phi, \dots, \partial_{x_n}\phi)$ and $\partial_x \phi \in C(E_0 \cup \partial_0 E, \mathbb{R}^n)$. Let $(z, u)$, $u = (u_1, \dots, u_n)$, be unknown functions of the variables $(t, x)$. First we introduce an additional unknown function $u = \partial_x z$ in (5). Then we consider the following linearization of (5) with respect to $u$:

$$\begin{aligned} \partial_t z(t,x) = F(t,x,z(t,x),u(t,x)) \\ + \sum_{i=1}^{n} \partial_{q_i} F(t,x,z_{(t,x)},u(t,x)) \big(\partial_{x_i} z(t,x) - u_i(t,x)\big). \end{aligned} \tag{6}$$

By virtue of (5) we get the functional differential equations for $u$:

$$\begin{aligned} \partial_t u(t,x) = \partial_x F(t,x,z_{(t,x)},u(t,x)) + \partial_w F(t,x,z_{(t,x)},u(t,x))\, u_{(t,x)} \\ + \partial_q F(t,x,z_{(t,x)},u(t,x)) \big[\partial_x u(t,x)\big]^T \end{aligned} \tag{7}$$

where $\partial_w F(P)\, u_{(t,x)} = (\partial_w F(P)\,(u_1)_{(t,x)}, \dots, \partial_w F(P)(u_n)_{(t,x)})$ and $\partial_t u = (\partial_t u_1, \dots, \partial_t u_n)$. We consider the following initial boundary condition for the equations (6), (7):

$$z(t,x) = \phi(t,x), \qquad u(t,x) = \partial_x \phi(t,x) \quad \text{on } E_0 \cup \partial_0 E. \tag{8}$$

Under natural assumptions on given functions the above problems have the following properties:

(i) if $(\tilde z, \tilde u) : \Omega \to \mathbb{R}^{1+n}$ is a solution of (6)–(8), then $\partial_x \tilde z = \tilde u$ and $\tilde z$ is a solution of (4), (5);

(ii) if $v : \Omega \to \mathbb{R}^n$ is a solution of (4), (5), then $(v, \partial_x v)$ satisfies (6)–(8).

The theory of implicit difference schemes for (4), (5) is based on the above method of quasilinearization. More exactly: difference methods for (6)–(8) are constructed and solutions of suitable difference functional problems approximate the solution $v$ of (5) and its partial derivatives $\partial_x v$, see [4, 10].

There are the following motivations for the construction of implicit difference schemes related to (1), (2) and (3), (4). Two types of assumptions are needed in theorems on the stability of explicit difference schemes generated by (1), (2) and (3), (4). The first type of conditions concerns the regularity of given functions, and they are the same for explicit and for implicit difference methods. It is required that $f$, $g$ and $\mathbf{F}$, $\mathbf{G}$, $G$ are continuous and that they satisfy nonlinear estimates of the Perron type with respect to the functional variable. The second type of conditions concern the mesh. It is required that explicit difference methods generated by (1), (2) satisfy the condition

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} |f_{ij}(t,x,w)| \ \geq 0 \quad \text{on } \Xi, \ \text{ for } i = 1, \dots, k, \tag{9}$$

where $h_0$ and $(h_1, \dots, h_n)$ are steps of the mesh with respect to $t$ and $(x_1, \dots, x_n)$, respectively. The above assumption is known as the Courant–Friedrichs–Levy condition for (1), (2), see [4, 8].

The following condition is needed in the analysis of the stability of explicit difference schemes for (3), (4):

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} f_{ii}(t,x,w) + h_0 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{1}{h_i h_j} \big| f_{ij}(t,x,w) \big| \geq 0 \quad \text{on } \Xi, \tag{10}$$

see [19]. Note that assumptions (9) and (10) require some relations between $h_0$ and $(h_1, \dots, h_n)$. It is important that conditions (9) and (10) are omitted in theorems on the stability of implicit difference schemes.

The motivations for the construction of implicit difference schemes for quasilinear problem (6)–(8) are the same. Numerical examples given in [4, 5, 9, 10, 12] show that implicit difference methods are natural tools for numerical solution of evolution functional differential equations.

We show that all known results on implicit difference methods for evolution functional differential equations can be obtained as particular cases of this general and simple theorem. We use a comparison technique with nonlinear estimates of the Perron type for given functions with respect to the functional variable.

The paper is divided into two parts. In the first part (Section 2) we propose a new method of the investigation of implicit difference schemes corresponding to initial boundary value problems for quasilinear evolution functional differential equations or systems. We formulate a general implicit difference functional problem with an unknown function of several variables. We give sufficient conditions for the existence and uniqueness of a solution of initial boundary value problems and we prove a theorem on error estimates of approximate solutions. The error is estimated by a solution of an initial problem for a nonlinear difference equation with an unknown function of one variable. In the second part of the paper we apply the above general results to quasilinear functional systems with first order partial derivatives (Section 3) and to quasilinear parabolic problems (Section 4). In Section 5 we construct implicit difference schemes for (6)–(8).

We use in the paper general ideas for finite difference equations which were introduced in [8, 17, 18].

## 2. Implicit difference functional equations

For any two sets $V$ and $W$ we denote by $\mathbf{F}(V,W)$ the class of all functions defined on $V$ and taking values in $W$. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. We define a mesh on $\Omega$ in the following way. Suppose that $(h_0,h')$, $h'=(h_1,\dots,h_n)$, $h_i>0$ for $0\le i\le n$, stand for steps of the mesh. For $h=(h_0,h')$ and $(r,m)\in\mathbb{Z}^{1+n}$ where $m=(m_1,\dots,m_n)$, we define nodal points as follows: $t^{(r)}=rh_0$, $x^{(m)}=(x_1^{(m_1)},\dots,x_n^{(m_n)})=(m_1h_1,\dots,m_nh_n)$. Let us denote by $H$ the set of all $h$ such that there are $K_0\in\mathbb{Z}$ and $K=(K_1,\dots,K_n)\in\mathbb{Z}^n$ satisfying the conditions: $K_0h_0=d_0$ and $(K_1h_1,\dots,K_nh_n)=d$. Let $N_0\in\mathbb{N}$ and $N=(N_1,\dots,N_n)\in\mathbb{N}$ be defined by the relations:

$$N_0h_0\le a<(N_0+1)h_0,\quad N_ih_i<b_i\le(N_i+1)h_i\quad\text{for }\ i=1,\dots,n,$$

and we assume that $(N_i+1)h_i=b_i$ if $d_i=0$. Write

$$R_h^{1+n}=\big\{\big(t^{(r)},x^{(m)}\big):\,(r,m)\in\mathbb{Z}^{1+n}\big\}$$

and

$$D_h=D\cap\mathbb{R}_h^{1+n},\ E_h=E\cap\mathbb{R}_h^{1+n},\ E_{0.h}=E_0\cap\mathbb{R}_h^{1+n}$$
$$\partial_0E_h=\partial_0E\cap\mathbb{R}_h^{1+n},\ \ \Omega_h=E_h\cup E_{0.h}\cup\partial_0E_h.$$

Set

$$E_h'=\big\{\big(t^{(r)},x^{(m)}\big)\in E_h:\,0\le r\le N_0-1\big\}$$

and

$$\Omega_{h.r}=\Omega_h\cap\big(\big[-d_0,t^{(r)}\big]\times\mathbb{R}^n\big),\quad 1\le r\le N_0.$$

We consider implicit difference functional equations with unknown functions $(z_1,\dots,z_p)=z$ of the variables $(t^{(r)},x^{(m)})\in\Omega_h$. The norm in the space $R^p$ is denoted by $\|\cdot\|_\star$.

For $z\in\mathbb{F}(\Omega_h,\mathbb{R}^p)$, $w\in\mathbb{F}(D_h,\mathbb{R}^p)$ we write $z^{(r,m)}=z(t^{(r)},x^{(m)})$ on $\Omega_h$ and $w^{(r,m)}=w(t^{(r)},x^{(m)})$ on $D_h$. We will need a discrete version of the operator $(t,x)\to z_{(t,x)}$. If $z:\Omega_h\to\mathbb{R}^p$ and $(t^{(r)},x^{(m)})\in E_h$ then the function $z_{[r,m]}:D_h\to\mathbb{R}^p$ is defined by $z_{[r,m]}(\tau,y)=z(t^{(r)}+\tau,x^{(m)}+y)$, $(\tau,y)\in D_h$. For $w\in\mathbb{F}(D_h,\mathbb{R}^p)$ we put

$$\|w\|_{D_h}=\max\big\{\big\|w^{(r,m)}\big\|_\star:\,\big(t^{(r)},x^{(m)}\big)\in D_h\big\}.\tag{11}$$

Set $e_j=(0,\dots,0,1,\dots,0)\in\mathbb{R}^n$ with 1 standing on the $j$-th place, $1\le j\le n$. Write

$$\Lambda=\big\{\lambda=(\lambda_1,\dots,\lambda_n):\ \lambda_i\in\{-1,0,1\}\text{ for }1\le i\le n\text{ and }\|\lambda\|\le2\big\}$$
$$\Lambda'=\Lambda\setminus\{\,\theta\,\},\quad\theta=(0,\dots,0)\in\mathbb{R}^n,$$

and $\chi = 1 + 2n^2$. Note that $\chi$ is the number of elements of $\Lambda$. Let $\psi : \Lambda \to \{1, \dots, \chi\}$ be a function such that $\psi(\lambda) \neq \psi(\tilde\lambda)$ for $\lambda \neq \tilde\lambda$. We assume that $\prec$ is an order in $\Lambda$ defined in the following way: $\lambda \prec \tilde\lambda$ if $\psi(\lambda) < \psi(\tilde\lambda)$. Elements of the space $\mathbb{R}^\chi$ will be denoted by $\xi = \{\xi_\lambda\}_{\lambda\in\Lambda}$. Write

$$A_h = \big\{ x^{(m)} : m = (m_1, \dots, m_n) \in \Lambda \big\}.$$

For $\zeta : A_h \to \mathbb{R}$, $\eta : A_h \to \mathbb{R}^p$ we put $\zeta^{(m)} = \zeta(x^{(m)})$ and $\eta^{(m)} = \eta(x^{(m)})$ on $A_h$. If $z : \Omega_h \to \mathbb{R}^p$ and $(t^{(r)}, x^{(m)}) \in E_h$, then the function $z_{\langle r,m\rangle} : A_h \to \mathbb{R}^p$ is defined by $z_{\langle r,m\rangle}(y) = z(t^{(r)}, x^{(m)} + y)$, $y \in A_h$.

Suppose that

$$f_h : E'_h \times \mathbb{F}(D_h, \mathbb{R}^p) \to \mathbb{R}^p, \qquad f_h = \big(f_h^{(1)}, \dots, f_h^{(p)}\big)$$
$$G_h^{(i)} : E'_h \times \mathbb{F}(D_h, \mathbb{R}^p) \to \mathbb{R}^\chi, \quad G_h^{(i)} = \big\{G_{h.\lambda}^{(i)}\big\}_{\lambda\in\Lambda}, \quad i = 1, \dots, p,$$

are given functions. For $(t, x, w) \in E'_h \times \mathbb{F}(D_h, \mathbb{R}^p)$, $\zeta \in \mathbb{F}(A_h, \mathbb{R})$, $\eta \in \mathbb{F}(A_h, \mathbb{R}^p)$, $\eta = (\eta_1, \dots, \eta_p)$, we put

$$G_h^{(i)}(t, x, w) \circ \zeta = \sum_{\lambda\in\Lambda} G_{h.\lambda}^{(i)}(t, x, w)\, \zeta^{(\lambda)}, \quad i = 1, \dots, p,$$

and

$$G_h(t, x, w) \diamond \eta = \big(G_h^{(1)}(t, x, w) \circ \eta_1, \dots, G_h^{(p)}(t, x, w) \circ \eta_p\big).$$

Set $\Sigma_h = E'_h \times \mathbb{F}(D_h, \mathbb{R}^p) \times \mathbb{F}(A_h, \mathbb{R}^p)$. Let $F_h : \Sigma_h \to \mathbb{R}^p$, $F_h = (F_h^{(1)}, \dots, F_h^{(p)})$, be defined by

$$F_h(t, x, w, \eta) = f_h(t, x, w) + G_h(t, x, w) \diamond \eta. \tag{12}$$

For $(t^{(r)}, x^{(m)}, w, \eta) \in \Sigma_h$ we write

$$F_h[w, \eta]^{(r,m)} = F_h\big(t^{(r)}, x^{(m)}, w, \eta\big), \qquad f_h[w]^{(r,m)} = f_h\big(t^{(r)}, x^{(m)}, w\big)$$
$$G_h[w]^{(r,m)} = G_h\big(t^{(r)}, x^{(m)}, w\big), \qquad G_h^{(i)}[w]^{(r,m)} = G_h^{(i)}\big(t^{(r)}, x^{(m)}, w\big), \quad 1 \le i \le p.$$

Let $\delta_0$ be the difference operator defined by

$$\delta_0 z^{(r,m)} = \big(\delta_0 z_1^{(r,m)}, \dots, \delta_0 z_p^{(r,m)}\big) = \frac{1}{h_0}\big[\, z^{(r+1,m)} - z^{(r,m)}\,\big].$$

Given $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}^p$, we consider the functional difference equation

$$\delta_0 z^{(r,m)} = F_h\big[\, z_{[r,m]}, z_{\langle r+1,m\rangle}\,\big]^{(r,m)} \tag{13}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on } E_{0.h} \cup \partial_0 E_h. \tag{14}$$

Note that the vectors $z^{(r+1,m+\lambda)}$ where $\lambda \in \Lambda$ appear in $z_{\langle r+1,m\rangle}$. Then (13), (14) is an implicit functional difference problem.

There are the following motivations for investigations of problem (13), (14). Explicit difference equations for (1), (3) or (6), (7) have the form

$$\delta_0 z^{(r,m)} = \Phi_h\big(t^{(r)}, x^{(m)}, z\big) \tag{15}$$

where $\Phi_h : E'_h \times \mathbb{F}(\Omega_h, \mathbb{R}^p) \to \mathbb{R}^p$ is an operator of the Volterra type and $p = k$ for (1), $p = 1$ for (3) and $p = n + 1$ for (6), (7). Discretization of partial derivatives $\partial_x z_i = (\partial_{x_1} z_i, \dots, \partial_{x_n} z_i)$ and $\partial^2 z_i = [\partial_{x_\mu x_\nu} z_i]_{\mu,\nu=1,\dots,n}$, $i = 1, \dots, p$, leads to the following observation: the numbers $z_i^{(r,m+\lambda)}$ where $\lambda \in \Lambda$, $1 \le i \le p$, appear in definitions of difference operators corresponding to these derivatives. It follows that the right hand side of the $i$-th equation in (15) depends on the functional variable $(z_i)_{\langle r,m\rangle}$, $1 \le i \le p$. Since (1), (3) and (6), (7) contain the functional variable we conclude that $\Phi_h$ in (15) depends on $z_{[r,m]}$. It is clear that assumptions on $z_{[r,m]}$ and $z_{\langle r,m\rangle}$ are not the same in theorems on convergence of difference methods. Then it is convenient to consider the following explicit difference scheme for (1) and (3):

$$\delta_0 z^{(r,m)} = F_h\big(t^{(r)}, x^{(m)}, z_{[r,m]}, z_{\langle r,m\rangle}\big) \tag{16}$$

where $F_h : \Sigma_h \to \mathbb{R}$. The initial boundary condition (14) is associated with (16). It is important that two functional variables: $z_{[r,m]}$ and $z_{\langle ,m\rangle}$ appear in (16).

Systems (1) and (6), (7) and equation (3) are linear with respect to partial derivatives. It follows that explicit difference schemes for (1), (3) and (6), (7) are linear with respect to $\delta z_i = (\delta_1 z_i, \dots, \delta_n z_i)$ and $\delta^{(2)} z_i = [\delta_{\mu\nu} z_i]_{\mu,\nu=1,\dots,n}$, $i = 1, \dots, p$. Then they have the form (16) with $F_h$ defined by (12). The implicit difference methods corresponding to (16) have the form (13).

We give sufficient conditions for the existence and uniqueness of a solution to (13), (14).

**Assumption** $H[G_h]$**.** The functions $G_h^{(i)} : E'_h \times \mathbb{F}(D_h, \mathbb{R}^p) \to \mathbb{R}^\chi$, $1 \le i \le p$, satisfy the conditions:

$$G_{h.\lambda}^{(i)}(t, x, w) \ge 0 \quad \text{for } \lambda \in \Lambda' \quad \text{and} \quad \sum_{\lambda\in\Lambda} G_{h.\lambda}^{(i)}(t, x, w) = 0, \quad i = 1, \dots, p.$$

We beginwith a lemma on difference inequalities corresponding to (13), (14).

**Lemma 2.1.** *Suppose that Assumption $H[G_h]$ is satisfied and $h \in H$, $z_h : \Omega_h \to \mathbb{R}^p$, $z_h = (z_{h.1}, \dots, z_{h.p})$.*

(I) *If $z_h$ satisfies the difference inequality*

$$z_h^{(r+1,m)} \le h_0 G_h[(z_h)_{[r,m]}]^{(r,m)} \diamond (z_h)_{\langle r+1,m\rangle}, \quad \big(t^{(r)}, x^{(m)}\big) \in E'_h,$$

*and* $z_h^{(r,m)} \leq \theta_{[p]}$ *on* $E_{0.h} \cup E_h$ *where* $\theta_{[p]} = (0, \dots, 0) \in \mathbb{R}^p$*, then*

$$z_h^{(r,m)} \leq \theta_{[p]} \quad \text{on } E_h. \tag{17}$$

(II) *If* $z_h$ *satisfies the difference inequality*

$$z_h^{(r+1,m)} \geq h_0 G_h[(z_h)_{[r,m]}]^{(r,m)} \diamond (z_h)_{\langle r+1,m\rangle}, \quad \big(t^{(r)}, x^{(m)}\big) \in E'_h,$$

*and* $z_h^{(r,m)} \geq \theta_{[p]}$ *on* $E_{0.h} \cup \partial_0 E_h$*, then* $z_h^{(r,m)} \geq \theta_{[p]}$ *on* $E_h$.

*Proof.* Consider the case (I). Suppose that $0 \leq r \leq N_0 - 1$ is fixed and there exist $\tilde{m} \in \mathbb{Z}^n$, $-N \leq \tilde{m} \leq N$, and $j$, $1 \leq j \leq p$, such that $z_{h.j}^{(r+1,\tilde{m})} = M$ where $M = \max\{z_{h.j}^{(r+1,m)},\ x^{(m)} \in [-c, c]\}$, and

$$z_{h.j}^{(r+1,\tilde{m})} > 0. \tag{18}$$

Then $(t^{(r)}, x^{(\tilde{m})}) \in E'_h$. It follows from Assumption $H[G_h]$ that

$$\begin{aligned} z_{h.j}^{(r+1,\tilde{m})} &\leq h_0 \sum_{\lambda \in \Lambda'} G_{h.\lambda}^{(j)}[\,(z_h)_{[r,\tilde{m}]}\,]^{(r,\tilde{m})}\, z_{h.j}^{(r+1,\tilde{m}+\lambda)} + h_0 M G_{h.\theta}^{(j)}[\,(z_h)_{[r,\tilde{m}]}\,]^{(r+1,\tilde{m})} \\ &\leq M h_0 \sum_{\lambda \in \Lambda} G_{h.\lambda}^{(j)}[\,(z_h)_{[r,\tilde{m}]}\,]^{(r+1,\tilde{m})} = 0 \end{aligned}$$

which contradicts (18). Then the proof of (17) is completed. The case (II) can be treated in a similar way. This proves the lemma. □

**Theorem 2.2.** *If Assumption* $H[G_h]$ *is satisfied and* $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}^p$*,* $h \in H$*, then there exists exactly one solution* $z_h : \Omega_h \to \mathbb{R}^p$ *to* (13), (14).

*Proof.* Suppose that $0 \leq r \leq N_0 - 1$ is fixed and that $z_h$ is known on the set $\Omega_{h.r}$. Consider the linear system

$$\begin{aligned} z^{(r+1,m)} = z_h^{(r,m)} &+ h_0 f_h[\,(z_h)_{r,m]}\,]^{(r,m)} \\ &+ h_0 G_h[(z_h)_{[r,m]}]^{(r,m)} \diamond z_{\langle r+1,m\rangle}, \end{aligned} \qquad -N \leq m \leq N, \tag{19}$$

and

$$z^{(r+1,m)} = \varphi_h^{(r+1,m)} \quad \text{for } \big(t^{(r+1)}, x^{(m)}\big) \in \partial_0 E_h \tag{20}$$

with unknowns $z^{(r+1,m)}$. It follows from Lemma 2.1 that the homogeneous system corresponding to (19), (20) has exactly one zero solution. Then system (19), (20) has exactly one solution and $z_h$ is defined on the set $\Omega_{h.r+1}$. Since $z_h$ is given on $E_{0.h}$ then the proof is completed by induction with respect to $r$, $0 \leq r \leq N_0$. □

We will consider approximate solutions to (13), (14). Let $X_h \subset \mathbb{F}(D_h, \mathbb{R}^p)$ and $Y_h \subset \mathbb{F}(A_h, \mathbb{R}^p)$ be fixed subsets. Suppose that the functions $v_h : \Omega_h \to \mathbb{R}^p$ and $\alpha_0,\ \gamma : H \to \mathbb{R}_+$ satisfy the conditions:

$$\big\|\, \delta_0 v_h^{(r,m)} - F_h\big[\, (v_h)_{[r,m]}, (v_h)_{\langle r+1,m\rangle}\,\big]^{(r,m)} \,\big\|_\star \le \gamma(h) \quad \text{on } E'_h \tag{21}$$

$$\big\|\, \varphi_h^{(r,m)} - v_h^{(r,m)} \,\big\|_\star \le \alpha_0(h) \quad \text{on } E_{0.h} \cup \partial_0 E_h \tag{22}$$

$$\lim_{h\to 0} \gamma(h) = 0, \quad \lim_{h\to 0} \alpha_0(h) = 0 \tag{23}$$

and

$$\big((v_h)_{[r,m]}, (v_h)_{\langle r,m\rangle}\big) \in X_h \times Y_h \quad \text{for } \big(t^{(r)}, x^{(m)}\big) \in E_h. \tag{24}$$

The function $v_h$ satisfying the above relations is considered as an approximate solution to (13), (14). It is important in our considerations that we look for approximate solutions to (13), (14) such that condition (24) is satisfied with a fixed subspace $X_h \times Y_h \subset \mathbb{F}(D_h, \mathbb{R}^p) \times \mathbb{F}(A_h, \mathbb{R}^p)$. Remark 2.4 contains additional comments on (24).

We give a theorem on the estimate of the difference between the exact and approximate solutions to (13), (14).

**Assumption** $H[\sigma]$**.** The function $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions:

1) $\sigma$ is continuous and it is nondecreasing with respect to the both variables;
2) $\sigma(t,0) = 0$ for $t \in [0,a]$ and the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(\,t, \omega(t)), \quad \omega(0) = 0$$

is $\tilde{\omega}(t) = 0$ for $t \in [0,a]$.

We formulate a general result on error estimates of approximate solutions to (13), (14).

**Theorem 2.3.** *Suppose that*

1) $h \in H$*, Assumption* $H[G_h]$ *is satisfied and* $z_h : \Omega_h \to \mathbb{R}^p$ *is the solution to* (13), (14)*;*
2) $v_h : \Omega_h \to \mathbb{R}^p$ *and there are* $\alpha_0,\ \gamma : H \to \mathbb{R}_+$ *such that conditions* (21)–(24) *are satisfied;*
3) *there exists* $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that Assumption* $H[\sigma]$ *is satisfied and for each* $(\tilde{w}, \eta) \in X_h \times Y_h$*,* $w \in \mathbb{F}(D_h, \mathbb{R}^p)$ *we have*

$$\big\| F_h[w,\eta]^{(r,m)} - F_h[\tilde{w},\eta]^{(r,m)} \big\|_\star \le \sigma\big(t^{(r)}, \|w - \tilde{w}\|_{D_h}\big)$$

*where* $(t^{(r)}, x^{(m)}) \in E'_h$.

*Then there is* $\alpha : H \to \mathbb{R}_+$ *such that*

$$\big\|(z_h - v_h)^{(r,m)}\big\|_\star \le \alpha(h) \ \text{ on } E_h \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0. \tag{25}$$

*Proof.* Write

$$\Gamma_h^{(r,m)} = \delta v_h^{(r,m)} - F_h[(v_h)_{r,m]}, (v_h)_{\langle r+1,m\rangle}]^{(r,m)}, \quad \Gamma_h^{(r,m)} = (\Gamma_{h.1}^{(r,m)}, \ldots, \Gamma_{h.p}^{(r,m)}).$$

We conclude from (12) that, for $i = 1, \ldots, p$,

$$\begin{aligned}
&\big(z_{h.i} - v_{h.i}\big)^{(r+1,m)}\big[1 - h_0\, G_{h,\theta}^{(i)}[(z_h)_{[r,m]}]^{(r,m)}\big] \\
&= \big(z_{h.i} - v_{h.i}\big)^{(r,m)} \\
&\quad + h_0\Big\{F_h^{(i)}[(z_h)_{[r,m]}, (v_h)_{\langle r+1,m\rangle}]^{(r,m)} - F_h^{(i)}[(v_h)_{[r,m]}, (v_h)_{\langle r+1,m\rangle}]^{(r,m)}\Big\} \qquad (26)\\
&\quad + h_0 \sum_{\lambda\in\Lambda'} G_{h.\lambda}^{(i)}[(z_h)_{[r,m]}]^{(r,m)}\big(z_{h.i} - v_{h.i}\big)^{(r+1,m+\lambda)} - h_0\Gamma_{h.i}^{(r,m)}.
\end{aligned}$$

Write $\varepsilon_h^{(r)} = \max\big\{\|(z_h - v_h)^{(i,m)}\|_\star :\ (t^{(i)}, x^{(m)}) \in \Omega_{h.r}\big\}$, $0 \le r \le N_0$. It follows from Assumptions $G[G_h]$, $H[\sigma]$ and (21), (22), (26) that

$$\varepsilon_h^{(0)} \le \alpha_0(h), \quad \varepsilon_h^{(r+1)} \le \varepsilon_h^{(r)} + h_0\sigma\big(t^{(r)}, \varepsilon_h^{(r)}\big) + h_0\gamma(h), \quad 0 \le r \le N_0 - 1. \qquad (27)$$

Let us denote by $\omega(\,\cdot\,, h)$ the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t)) + \gamma(h), \quad \omega(0) = \alpha_0(h). \qquad (28)$$

It follows that $\omega(\,\cdot\,, h)$ is defined on $[0, a]$ and $\lim_{h\to 0}\omega(t, h) = 0$ uniformly on $[0, a]$. We conclude from (27) that $\varepsilon_h^{(r)} \le \omega(t^{(r)}, h)$ for $0 \le r \le N_0$. Then condition (25) is satisfied with $\alpha(h) = \omega(a, h)$. This completes the proof. □

**Remark 2.4.** Let us consider the following condition:

3') there exists $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that Assumption $H[\sigma]$ is satisfied and for each $w$, $\tilde w \in \mathbb{F}(D_h, \mathbb{R}^p)$, $\eta \in \mathbb{F}(A_h, \mathbb{R}^p)$, we have

$$\big\|F_h[w, \eta]^{(r,m)} - F_h[\tilde w, \eta]^{(r,m)}\big\|_\star \le \sigma\big(t^{(r)}, \|w - \tilde w\|_{D_h}\big)$$

where $(t^{(r)}, x^{(m)}) \in E_h'$.

It is clear that Theorem 2.3 remains true if assumption 3) is replaced by 3'). There are differential functional problems such that the corresponding operators $F_h$ satisfy 3'). We show that assumption 3) is important in our considerations. The operators $F_h$ generated by (1), (3) or (6), (7) satisfy condition 3) and they do not satisfy 3').

Now we formulate a particular case of Theorem 2.3. We assume that the function $\sigma(t, \,\cdot\,)$ is linear.

**Remark 2.5.** Suppose that all the assumptions of Theorem 2.3 holds true and $\sigma(t,p) = Lp$ on $[0,a] \times \mathbb{R}_+$ where $L \in \mathbb{R}_+$. Then we have assumed that the operator $F_h$ satisfies the Lipschitz condition with respect to the functional variable $w$ for each fixed $(\tilde{w},\eta) \in X_h \times Y_h$. Then

$$\|(z_h - v_h)^{(r,m)}\|_\star \le \tilde{\alpha}(h) \text{ on } E_h$$

where

$$\tilde{\alpha}(h) = \alpha_0(h)e^{La} + \gamma(h)\frac{e^{La}-1}{L} \quad \text{if } L > 0 \tag{29}$$

$$\tilde{\alpha}(h) = \alpha_0(h) + a\gamma(h) \quad \text{if } L = 0. \tag{30}$$

The above estimates are obtained by solving problem (28) with $\sigma(t,p) = Lp$.

## 3. Implicit difference schemes for hyperbolic functional differential systems

In this part of the paper we put $\mathbb{R}^p = \mathbb{R}^k$. For $\zeta \in \mathbb{R}^k$, $\zeta = (\zeta_1,\dots,\zeta_k)$, we define the norm

$$\|\zeta\|_\star = \|\zeta\|_\infty = \max\{|\zeta_i| : 1 \le i \le k\}.$$

For $w \in C(D,\mathbb{R}^k)$ we put

$$\|w\|_D = \max\{\|w(t,x)\|_\infty : (t,x) \in D\}.$$

The norm of $w \in \mathbb{F}(D_h,\mathbb{R}^k)$ is defined by (11) with the above given $\|\cdot\|_\star$.

We formulate a difference method for (1), (2). Let $T_h : \mathbb{F}(D_h,\mathbb{R}^k) \to C(D,\mathbb{R}^k)$ be an interpolating operator. We consider the system of functional difference equations

$$\begin{aligned}\delta_0 z_i^{(r,m)} = &\sum_{j=1}^n f_{ij}\big(t^{(r)},x^{(m)},T_h z_{[r,m]}\big)\delta_j z_i^{(r+1,m)} \\ &+ g_i\big(t^{(r)},x^{(m)},T_h z_{[r,m]}\big),\end{aligned} \qquad i = 1,\dots,k, \tag{31}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on } E_{0.h} \cup \partial_0 E_h \tag{32}$$

where $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}^k$ is a given function. The difference operators $(\delta_1,\dots,\delta_n)$ are defined in the following way. Suppose that $(t^{(r)},x^{(m)}) \in E'_h$ and that the function $z = (z_1,\dots,z_k)$ is known on the set $\Omega_{h.r}$. We put

$$\text{if } f_{ij}\big(t^{(r)},x^{(m)},T_h z_{[r,m]}\big) \ge 0, \quad \text{then } \delta_j z_i^{(r+1,m)} = \frac{1}{h_j}\big[z_i^{(r+1,m+e_j)} - z_i^{(r+1,m)}\big]$$

$$\text{if } f_{ij}\big(t^{(r)},x^{(m)},T_h z_{[r,m]}\big) < 0, \quad \text{then } \delta_j z_i^{(r+1,m)} = \frac{1}{h_j}\big[z_i^{(r+1,m)} - z_i^{(r+1,m-e_j)}\big],$$

and we take $i = 1, \ldots, k$, $j = 1, \ldots, n$ in the above definitions. We claim that we have obtained a difference problem which is a particular case of (13), (14). Consider the operator $F_h : \Sigma_h \to \mathbb{R}^k$, $F_h = (F_h^{(1)}, \ldots, F_h^{(k)})$, defined in the following way. Suppose that $(t^{(r)}, x^{(m)}, w, \eta) \in \Sigma_h$. Write

$$\begin{aligned} J_{i.+}^{(r,m)}[w] &= \big\{ j \in \{1, \ldots, n\} : \; f_{ij}\big(t^{(r)}, x^{(m)}, T_h w\big) \geq 0 \big\} \\ J_{i.-}^{(r,m)} &= \{1, \ldots, n\} \setminus J_{i.+}^{(r,m)}[w] \end{aligned}$$

and

$$F_h^{(i)}\big(t^{(r)}, x^{(m)}, w, \eta\big) = \begin{aligned} &\sum_{j=1}^{n} f_{ij}\big(t^{(r)}, x^{(m)}, T_h w\big)\, \delta_j \eta_i^{(\theta)} \\ &+ g_i\big(t^{(r)}, x^{(m)}, T_h w\big)\,, \end{aligned} \qquad i = 1, \ldots, k\,.$$

The expressions $(\delta_1 \eta_i^{(\theta)}, \ldots, \delta_n \eta_i^{(\theta)})$, $1 \leq i \leq k$, are defined in the following way:

$$\begin{aligned} {}_j\eta_i^{(\theta)} &= \frac{1}{h_j}\big[\eta_i^{(e_j)} - \eta_i^{(\theta)}\big] \quad \text{ for } j \in J_{i.+}^{(r,m)}[w] \\ \delta_j \eta_i^{(\theta)} &= \frac{1}{h_j}\big[\eta_i^{(\theta)} - \eta_i^{(-e_j)}\big] \quad \text{ for } j \in J_{i.-}^{(r,m)}[w], \end{aligned}$$

and we put $i = 1, \ldots, k$, $j = 1, \ldots, n$ in the above formulas. It is clear that system (31) is equivalent to (13) with the above defined $F_h$ and $p = k$.

**Lemma 3.1.** *Suppose that* $f : \Xi \to M_{k \times n}$, $g : \Xi \to \mathbb{R}^k$, $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}^k$ *and* $T_h : \mathbb{F}(D_h, \mathbb{R}^k) \to C(D, \mathbb{R}^k)$, $h \in H$. *Then there exists exactly one solution* $z_h : \Omega_h \to \mathbb{R}^k$ *of system* (31) *with initial boundary condition* (32).

*Proof.* We apply Theorem 2.2. Let us define

$$\begin{aligned} f_h &: E_h' \times \mathbb{F}(D_h, \mathbb{R}^k) \to \mathbb{R}^k, \quad f_h = \big(f_h^{(1)}, \ldots, f_h^{(k)}\big) \\ G_h^{(i)} &: E_h' \times \mathbb{F}(D_h, \mathbb{R}^k) \to \mathbb{R}^\chi, \quad i = 1, \ldots, k, \end{aligned}$$

in the following way. Suppose that $(t^{(r)}, x^{(m)}, w) \in E_h' \times \mathbb{F}(D_h, \mathbb{R}^k)$. Write

$$\begin{aligned} \Lambda_{i.+}^{(r,m)}[w] &= \big\{ \lambda \in \Lambda : \text{ there is } j \in J_{i.+}^{(r,m)}[w] \text{ such that } \lambda = e_j \big\} \\ \Lambda_{i.-}^{(r,m)}[w] &= \big\{ \lambda \in \Lambda : \text{ there is } j \in J_{i.-}^{(r,m)}[w] \text{ such that } \lambda = -e_j \big\}, \end{aligned}$$

where $i = 1, \ldots, k$. Set

$$\begin{aligned} f_h[w]^{(r,m)} &= g\big(t^{(r)}, x^{(m)}, T_h w\big) \\ G_{h.\theta}^{(i)}[w]^{(r,m)} &= -\sum_{j=1}^{n} \frac{1}{h_j} \big| f_{ij}\big(t^{(r)}, x^{(m)}, T_h w\big) \big| \end{aligned}$$

and

$$G^{(i)}_{h.e_j}[w]^{(r,m)} = \frac{1}{h_j} f_{ij}\big(t^{(r)}, x^{(m)}, T_h w\big) \text{ for } j \in J^{(r,m)}_{i.+}[w]$$
$$G^{(i)}_{h.-e_j}[w]^{(r,m)} = -\frac{1}{h_j} f_{ij}\big(t^{(r)}, x^{(m)}, T_h w\big) \text{ for } j \in J^{(r,m)}_{i.-}[w]$$
$$G^{(i)}_{h.\lambda}[w]^{(r,m)} = 0 \quad \text{for } \lambda \in \Lambda \setminus \big[\Lambda^{(r,m)}_{i.+}[w] \cup \Lambda^{(r,m)}_{i.-}[w] \cup \{\theta\}\big].$$

We take $i = 1, \ldots, k$ in the above definitions. Then Assumption $H[G_h]$ is satisfied and $F_h$ is given by (12). Our theorem follows from Theorem 2.2. □

**Assumption** $H[T_h]$**.** The operator $T_h : \mathbb{F}(D_h, \mathbb{R}^k) \to C(D, \mathbb{R}^k)$ satisfies the conditions:

1) for any $w, \tilde{w} \in \mathbb{F}(D_h, \mathbb{R}^k)$ we have $\|T_h w - T_h \tilde{w}\|_D \le \|w - \tilde{w}\|_{D_h}$;
2) if $w : D \to \mathbb{R}^k$ is of class $C^1$, then there is $\tilde{\gamma} : H \to \mathbb{R}_+$ such that $\|w - T_h w_h\|_D \le \tilde{\gamma}(h)$ and $\lim_{h\to 0} \tilde{\gamma}(h) = 0$ where $w_h$ is the restriction $w$ to the set $D_h$.

**Remark 3.2.** The above condition 1) states that $T_h$ satisfies the Lipschitz condition with the constant $L = 1$. The meaning of condition 2) is that $T_h w_h$ is an approximation of $w$ and the error of the approximation is estimated by $\tilde{\gamma}(h)$.

An example of the operator $T_h$ satisfying Assumption $H[T_h]$ can be found in [8, Chapter 5].

**Assumption** $H_\star[\sigma]$**.** The function $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions:

1) $\sigma$ is continuous and it is nondecreasing with respect to both variables;
2) $\sigma(t, 0) = 0$ for $t \in [0, a]$, and for each $\tilde{c} \ge 1$ the maximal solution of the Cauchy problem
$$\omega'(t) = \tilde{c}\,\sigma(t, \omega(t)), \quad \omega(0) = 0,$$
is $\tilde{\omega}(t) = 0$ for $t \in [0, a]$.

**Assumption** $H[f, g]$**.** The functions $f : \Xi \to M_{k\times n}$, $g : \Xi \to \mathbb{R}^k$ are continuous and there is $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that Assumption $H_\star[\sigma]$ is satisfied and, on $\Xi$,

$$\|f(t, x, w) - f(t, x, \tilde{w})\|_{k\times n;\infty} \le \sigma(t, \|w - \tilde{w}\|_D)$$
$$\|g(t, x, w) - g(t, x, \tilde{w})\|_\infty \le \sigma(t, \|w - \tilde{w}\|_D)\,.$$

**Theorem 3.3.** *Suppose that Assumptions $H[T_h]$ and $H[f, g]$ are satisfied and:*

1) *$\varphi : E_0 \cup E \to \mathbb{R}^k$ is of class $C^1$ and $v : \Omega \to \mathbb{R}^k$ is a solution to* (1), (2) *and $v$ is of class $C^1$;*

2) *$h \in H$ and $z_h : \Omega_h \to \mathbb{R}^k$ is a solution of equation* (31) *with the initial boundary condition* (32) *and there is $\alpha_0 : H \to \mathbb{R}_+$ such that*

$$\left\|\varphi_h^{(r,m)} - \varphi^{(r,m)}\right\|_\infty \le \alpha_0(h) \quad on \ E_{0.h} \cup \partial_0 E_h \quad and \quad \lim_{h\to 0} \alpha_0(h) = 0.$$

*Then there is $\alpha : H \to \mathbb{R}_+$ such that*

$$\|(z_h - v_h)^{(r,m)}\|_\infty \le \alpha(h) \quad on \ E_h \quad and \quad \lim_{h\to 0} \alpha(h) = 0, \tag{33}$$

*where $v_h$ is the restriction of $v$ to the set $\Omega_h$.*

*Proof.* We apply Theorem 2.3 to prove (33). Write $X_h = \mathbb{F}(D_h, \mathbb{R}^k)$. Let $\tilde{c} \in \mathbb{R}_+$ be defined by the relation:

$$\|\partial_x v(t,x)\|_{k\times n;\infty} \le \tilde{c} \quad \text{for} \ (t,x) \in E. \tag{34}$$

Let us denote by $Y_h$ the class of all $\eta \in \mathbb{F}(A_h, \mathbb{R}^k)$, $\eta = (\eta_1, \dots, \eta_n)$, such that

$$\left|\frac{1}{h_j}\big(\eta_i^{(e_j)} - \eta_i^{(\theta)}\big)\right| \le \tilde{c} \left|\frac{1}{h_j}\big(\eta_i^{(\theta)} - \eta_i^{(-\eta_j)}\big)\right| \le \tilde{c}, \quad i = 1,\dots,k, \ j = 1,\dots,n.$$

Then $\big((v_h)_{[r,m]}, (v_h)_{\langle r,m\rangle}\big) \in X_h \times Y_h$ for $(t^{(r)}, x^{(m)}) \in E_h$. It follows from Assumption $H[T_h]$ and from (34) that condition (21)–(23) are satisfied. For $w, \tilde{w} \in \mathbb{F}(D_h, \mathbb{R}^k)$ and $\eta \in Y_h$ we have

$$\left\|F_h[w,\eta]^{(r,m)} - F_h[\tilde{w},\eta]^{(r,m)}\right\|_\infty \le (1+\tilde{c})\,\sigma\big(t^{(r)}, \|w - \tilde{w}\|_{D_h}\big),$$

where $(t^{(r)}, x^{(m)}) \in E'_h$, Then all the assumptions of Theorem 2.3 are satisfied and the assertion (33) follows. □

**Remark 3.4.** Suppose that the assumptions of Theorem 3.3 are satisfied and $\sigma(t,p) = \tilde{L}p$ on $[0,a] \times \mathbb{R}_+$ where $\tilde{L} \in \mathbb{R}_+$. Then there is $L \in \mathbb{R}_+$ such that $\|(z_h - v_h)^{(r,m)}\|_\infty \le \tilde{\alpha}(h)$ on $E_h$ where $\tilde{\alpha}$ is given by (29), (30).

## 4. Implicit difference schemes for parabolic problems

In this part of the paper we apply the results presented in Section 2 for $\mathbb{R}^p = \mathbb{R}$. We construct a class of difference schemes for (3), (4). Given $\phi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$, $T_h : \mathbb{F}(D_h, \mathbb{R}) \to C(D, \mathbb{R}$, we consider the functional difference equation

$$\begin{aligned}\delta_0 z^{(r,m)} = &\sum_{i,j=1}^{n} F_{ij}\big(t^{(r)}, x^{(m)}, T_h z_{[r,m]}\big)\delta_{ij} z^{(r+1,m)} \\ &+ \sum_{i=1}^{n} G_i\big(t^{(r)}, x^{(m)}, T_h z_{[r,m]}\big)\delta_i z^{(r+1,m)}) + G\big(t^{(r)}, x^{(m)}, T_h z_{[r,m]}\big)\end{aligned} \tag{35}$$

with the initial boundary condition

$$z^{(r,m)} = \phi_h^{(r,m)} \quad \text{on } E_{0.h} \cup \partial_0 E_h. \tag{36}$$

The difference operators $(\delta_1, \ldots, \delta_n)$ are given by

$$\delta_i z^{(r+1,m)} = \frac{1}{2h_i}\big[z^{(r+1,m+e_i)} - z^{(r+1,m-e_i)}\big], \quad i = 1, \ldots, n.$$

Write

$$\left.\begin{aligned} \delta_i^+ z^{(r+1,m)} &= \frac{1}{h_i}\big[z^{(r+1,m+e_i)} - z^{(r+1,m)}\big] \\ \delta_i^- z^{(r+1,m)} &= \frac{1}{h_i}\big[z^{(r+1,m)} - z^{(r+1,m-e_i)}\big], \end{aligned}\right. \qquad i = 1, \ldots, n.$$

In the same way we define the expressions $\delta_i^+ \eta^{(\theta)}$ and $\delta_i^- \eta^{(\theta)}$ for $1 \le i \le n$ where $\eta : A_h \to \mathbb{R}$. We apply the difference operators $\delta^{(2)} = [\,\delta_{ij}\,]_{i,j=1,\ldots,n}$ defined in the following way. Put

$$\delta_{ii} z^{(r+1,m)} = \delta_i^+ \delta_i^- z^{(r+1,m)} \quad \text{for } i = 1, \ldots, n.$$

The difference expressions $\delta_{ij} z^{(r+1,m)}$ for $1 \le i, j \le n$, $i \ne j$, are given in the following way:

$$\text{if } F_{i,j}(t^{(r)}, x^{(m)}, T_h z_{[r,m]}) \ge 0, \text{ then } \delta_{ij} z^{(r+1,m)} = \frac{1}{2}\big[\,\delta_i^+ \delta_j^+ z^{(r+1,m)} + \delta_i^- \delta_j^- z^{(r+1,m)}\big]$$

$$\text{if } F_{i,j}(t^{(r)}, x^{(m)}, T_h z_{[r,m]}) < 0, \text{ then } \delta_{ij} z^{(r+1,m)} = \frac{1}{2}\big[\,\delta_i^+ \delta_j^- z^{(r+1,m)} + \delta_i^- \delta_j^+ z^{(r+1,m)}\big].$$

We claim that difference functional equation (35) is a particular case of (13) for $k = 1$. Consider the operator $F_h : \Sigma_h \to \mathbb{R}$ defined in the following way. Suppose that $(t^{(r)}, x^{(m)}, w, \eta) \in \Sigma_h$. Write

$$\begin{aligned} S_+^{(r,m)}[w] &= \big\{(i,j) : 1 \le i,\ j \le n,\ i \ne j,\ F_{ij}(t^{(r)}, x^{(m)}, T_h w) \ge 0\big\} \\ S_-^{(r,m)}[w] &= \big\{(i,j) : 1 \le i,\ j \le n,\ i \ne j,\ F_{ij}(t^{(r)}, x^{(m)}, T_h w) < 0\big\}, \end{aligned}$$

and

$$\begin{aligned} F_h(t^{(r)}, x^{(m)}, w, \eta) = \sum_{i,j=1}^{n} & F_{ij}\big(t^{(r)}, x^{(m)}, T_h w\big)\delta_{ij}\eta^{(\theta)} \\ & + \sum_{i=1}^{n} G_i\big(t^{(r)}, x^{(m)}, T_h w\big)\delta_i \eta^{(\theta)} + g\big(t^{(r)}, x^{(m)}, T_h w\big)\,. \end{aligned}$$

The expressions $(\delta_1 \eta^{(\theta)}, \ldots, \delta_n \theta^{(\theta)})$ is given by

$$\delta_i \eta^{(\theta)} = \frac{1}{2h_i}\big[\eta^{(e_i)} - \eta^{(-e_i)}\big] \quad \text{for } i = 1, \ldots, n.$$

The difference operators $\left[\delta_{ij}\eta^{(\theta)}\right]_{i,j=1,\dots,n}$ are defined in the following way:

$$\delta_{ii}\eta^{(\theta)} = \delta_i^+\delta_i^-\eta^{(\theta)} \quad \text{for } \ i = 1,\dots,n.$$

and

$$\delta_{ij}\eta^{(\theta)} = \frac{1}{2}\left[\delta_i^+\delta_j^+\eta^{(\theta)} + \delta_i^-\delta_j^-\eta^{(\theta)}\right] \quad \text{for } \ (i,j) \in S_+^{(r,m)}[w]$$
$$\delta_{ij}\eta^{(\theta)} = \frac{1}{2}\left[\delta_i^+\delta_j^-\eta^{(\theta)} + \delta_i^-\delta_j^+\eta^{(\theta)}\right] \quad \text{for } \ (i,j) \in S_-^{(r,m)}[w].$$

It is clear that equation (35) is equivalent to (13) with the above given $F_h$ and $p = 1$.

**Lemma 4.1.** *Suppose that* $h \in H$ *and* $\mathbf{F} : \Xi \to M_{n\times n}$, $\mathbf{G} : \Xi \to \mathbb{R}^n$, $G : \Xi \to \mathbb{R}$, $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$, $T_h : \mathbb{F}(D_h, \mathbb{R}) \to C(D, \mathbb{R})$. *Then there exists exactly one solution* $z_h : \Omega_h \to \mathbb{R}$ *of equation* (35) *with initial boundary condition* (36).

*Proof.* We apply Theorem 2.2. We define $f_h : E_h' \times \mathbb{F}(D_h, \mathbb{R}) \to \mathbb{R}$, $G_h : E_h' \times \mathbb{F}(D_h, \mathbb{R}) \to \mathbb{R}^\kappa$, $G_h = \{ G_{h.\lambda} \}_{\lambda\in\Lambda}$, in the following way. Suppose that $(t^{(r)}, x^{(m)}, w) \in E_h' \times \mathbb{F}(D_h, \mathbb{R})$. Write

$$\Lambda_0^{(r,m)}[w] = \{\lambda \in \Lambda : \text{there is } i,\ 1 \le i \le n, \text{ such that } \lambda = e_i \text{ or } \ \lambda = -e_i \}$$

$$\Lambda_I^{(r,m)}[w] = \left\{ \lambda \in \Lambda : \begin{array}{l} \text{there is } (i,j) \in S_+^{(r,m)}[w] \text{ such that} \\ \lambda = e_i + e_j \text{ or } \lambda = -e_i - e_j \end{array} \right\}$$

$$\Lambda_{II}^{(r,m)}[w] = \left\{ \lambda \in \Lambda : \begin{array}{l} \text{there is } (i,j) \in S_-^{(r,m)}[w] \text{ such that} \\ \lambda = e_i - e_j \text{ or } \lambda = -e_i + e_j \end{array} \right\}$$

$$\tilde{\Lambda}^{(r,m)}[w] = \Lambda \setminus \{ \Lambda_0^{(r,m)}[w] \cup \Lambda_I^{(r,m)}[w] \cup \Lambda_{II}^{(r,m)}[w] \cup \{ \theta \} \}$$

and

$$f_h[w]^{(r,m)} = G\big(t^{(r)}, x^{(m)} T_h w\big)$$

$$G_{h.\theta}[w]^{(r,m)} = -2\sum_{i=1}^{n} \frac{1}{h_i^2} F_{ii}\big(t^{(r)}, x^{(m)}, T_h w\big) + \sum_{\substack{i,j=1\\ i\ne j}}^{n} \frac{1}{h_i h_j}\big|F_{ij}\big(t^{(r)}, x^{(m)}, T_w\big)\big|$$

$$G_{h.e_i}[w]^{(r,m)} = \frac{1}{h_i^2} F_{ii}\big(t^{(r)}, x^{(m)}, T_h w\big) - \sum_{\substack{j=1\\ j\ne i}}^{n} \frac{1}{h_i h_j}\big|F_{ij}\big(t^{(r)}, x^{(m)}, T_h w\big)\big| + \frac{1}{2h_i} G_i\big(t^{(r)}, x^{(m)}, w\big)$$

$$G_{h.-e_i}[w]^{(r,m)} = \frac{1}{h_i^2} F_{ii}\big(t^{(r)}, x^{(m)}, T_h w\big) - \sum_{\substack{j=1\\ j\ne i}}^{n} \frac{1}{h_i h_j}\big|F_{ij}\big(t^{(r)}, x^{(m)}, T_h w\big)\big| - \frac{1}{2h_i} G_i\big(t^{(r)}, x^{(m)}, T_h w\big),$$

$$
\begin{aligned}
G_{h.e_i+e_j}[w]^{(r,m)} &= G_{h.-e_i-e_j}[w]^{(r,m)} \\
&= \frac{1}{2h_ih_j} F_{ij}\big(t^{(r)}, x^{(m)}, T_h w\big), \qquad (i,j) \in S_+^{(r,m)}[w] \\
G_{h.e_i-e_j}[w]^{(r,m)} &= G_{h.-e_i+e_j}[w]^{(r,m)} \\
&= -\frac{1}{2h_ih_j} F_{ij}\big(t^{(r)}, x^{(m)}, T_h w\big), \quad (i,j) \in S_-^{(r,m)}[w] \\
G_{h.\lambda}[w]^{(r,m)} &= 0 \quad \text{for } \lambda \in \tilde{\Lambda}[w]^{(r,m)},
\end{aligned}
$$

and we put $i, j = 1, \dots, n$ in the above formulas. Then $F_h$ satisfies (12) and Assumption $H[G_h]$ holds true. Then our theorem follows from Theorem 2.2. □

**Assumption** $H[\mathbf{F}, \mathbf{G}, G]$. The functions $\mathbf{F} : \Xi \to M_{n\times n}$, $\mathbf{G} : \Xi \to \mathbb{R}^n$, $G : \Xi \to \mathbb{R}$ are continuous and there is $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that Assumption $H_\star[\sigma]$ is satisfied and the terms

$$\|\mathbf{F}(t,x,w) - \mathbf{F}(t,x,\tilde{w})\|_{n\times n;\infty}, \ \|\mathbf{G}(t,x,w) - \mathbf{G}(t,x,\tilde{w})\|, \ |G(t,x,w) - G(t,x,\tilde{w})|$$

are bounded from above by $\sigma(t, \|w - \tilde{w}\|_D)$.

**Theorem 4.2.** *Suppose that Assumptions $H[T_h]$ and $H[\mathbf{F}, \mathbf{G}\, G]$ are satisfied and*

1) *$\varphi : E_0 \cup \partial_0 E \to \mathbb{R}$ is of class $C^2$ and $v : \Omega \to \mathbb{R}$ is a solution of (3), (4) and $v$ is of class $C^2$,*
2) *$h \in H$, there is $c_0 > 0$ such that $h_i h_j^{-1} \le c_0$ for $i, j = 1, \dots, n$ and $z_h : \Omega_h \to \mathbb{R}$ is a solution of (35), (36) and there is $\alpha_0 : H \to \mathbb{R}_+$ such that*

$$|\phi^{(r,m)} - \phi_h^{(r,m)}| \le \alpha_0(h) \text{ on } E_{0.h} \cup \partial_0 E_h \quad \text{and} \quad \lim_{h\to 0} \alpha_0(h) = 0.$$

*Then there is $\alpha : h \to \mathbb{R}_+$ such that*

$$\big|(z_h - v_h)^{(r,m)}\big| \le \alpha(h) \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0, \tag{37}$$

*where $v_h$ is the restriction of $v$ to the set $\Omega_h$.*

*Proof.* We apply Theorem 2.3 to prove (37). Let $\tilde{c} \in \mathbb{R}_+$ be defined by the relations

$$\|\partial_x v(t,x)\| \le \tilde{c}, \quad \|\partial_{xx} v(t,x)\|_{n\times n;\infty} \le \tilde{c} \quad \text{for } (t,x) \in E. \tag{38}$$

Set $X_h = \mathbb{F}(D_h, \mathbb{R})$. Let us denote by $Y_h$ the class of all $\eta \in \mathbb{F}(A_h, \mathbb{R})$ satisfying the conditions:

$$
\left.
\begin{aligned}
\frac{1}{2}\big|\delta_i^+ \eta^{(\theta)} + \delta_i^- \eta^{(\theta)}\big| &\le \tilde{c} \\
\frac{1}{2}\big|\delta_i^+\delta_j^+ \eta^{(\theta)} + \delta_i^-\delta_j^- \eta^{(\theta)}\big| &\le \tilde{c} \\
\frac{1}{2}\big|\delta_i^+\delta_j^- \eta^{(\theta)} + \delta_i^-\delta_j^+ \eta^{(\theta)}\big| &\le \tilde{c},
\end{aligned}
\right\} \quad i, j = 1, \dots, n.
$$

Then $((v_h)_{[r,m]},(v_h)_{\langle r,m\rangle}) \in X_h \times Y_h$ for $(t^{(r)},x^{(m)}) \in E_h$. It follows from Assumption $H[T_h]$ and from (38) that conditions (21)–(23) are satisfied. There is $\bar{c}>0$ such that for $w,\tilde{w} \in \mathbb{F}(D_h,\mathbb{R})$ and $\eta \in Y_h$ we have

$$\big|F_h[w,\eta]^{(r,m)} - F_h[\tilde{w},\eta]^{(r,m)}\big| \le (1+\bar{c})\,\sigma\big(t^{(r)},\|w-\tilde{w}\|_{D_h}\big).$$

where $(t^{(r)},x^{(m)}) \in E'_h$. Then the assumptions of Theorem 2.3 are satisfied and the assertion (37) follows. □

**Remark 4.3.** If the assumptions of Theorem 4.2 are satisfied and $\sigma(t,p)=\tilde{L}p$ on $[0,a]\times\mathbb{R}_+$ where $\tilde{L}\in\mathbb{R}_+$, then there is $L\in\mathbb{R}_+$ such that $|(z_h-v_h)^{(r,m)}| \le \tilde{\alpha}(h)$ on $E_h$ where $\tilde{\alpha}$ is given by (29), (30).

## 5. Generalized Euler method for nonlinear functional differential equations

In this part of the paper we put $\mathbb{R}^p=\mathbb{R}^{1+n)}$. For $\zeta=(x_0,x)\in\mathbb{R}^{1+n}$, $x\in\mathbb{R}^n$, we define the norm $\|\zeta\|_\star = |x_0|+\|x\|$. The norm in the space $CL(D,\mathbb{R})$ generated by the maximum norm in $C(D,\mathbb{R})$ will be dented by $\|\cdot\|_C$.

We construct implicit difference schemes for (6), (7). Let $(z,u)$, $u=(u_1,\ldots,u_n)$, be unknown functions of the variables $(t^{(r)},x^{(m)})\in\Omega_h$. Given $\phi_h : E_{0.h}\cup\partial_0E_h\to\mathbb{R}$, $\psi_h : E_{0.h}\cup\partial_0E_h\to\mathbb{R}^n$, $T_h : \mathbb{F}(D_h,\mathbb{R})\to C(D,\mathbb{R})$. Write

$$P^{(r,m)}[z,u] = \big(\, t^{(r)}, x^{(m)}, T_h z_{[r,m]}, u^{(r,m)}\,\big)$$

and $T_h u_{[r,m]} = \big(\, T_h(u_1)_{[r,m]},\ldots,T_h(u_h)_{[r,m]}\,\big)$. We consider the functional difference equations

$$\delta_0 z^{(r,m)} = F(P^{(r,m)}[z,u]) + \sum_{j=1}^n \partial_{q_j}F(P^{(r,m)}[z,u])\,\big(\delta_j z^{(r,m)} - u_j^{(r,m)}\big) \tag{39}$$

$$\begin{aligned}\delta_0 u^{(r,m)} &= \partial_x F(P^{(r,m)}[z,u]) + \partial_w F(P^{(r,m)}[z,u])\,T_h u_{[r,m]}\\ &\quad + \partial_q F(P^{(r,m)}[z,u])\big[\delta u^{(r+1,m)}\big]^T\end{aligned} \tag{40}$$

with initial boundary conditions

$$z^{(r,m)} = \phi_h^{(r,m)},\quad u^{(r,m)} = \psi_h^{(r,m)} \quad \text{on } E_{0.h}\cup\partial_0E_h, \tag{41}$$

where $\delta u^{(r+1,m)} = \big[\,\delta_j u_i^{(r+1,m)}\,\big]_{i,j=1,\ldots,n}$. The difference operators $(\delta_1,\ldots,\delta_n)$ is defined in the following way. Suppose that the functions $(z,u)$ are known on the set $\Omega_{h.r}$ and $(t^{(r)},x^{(m)})\in E'_h$. We put

$$\begin{aligned}&\text{if } \partial_{q_j}F(P^{(r,m)}[z,u])\ge 0,\\ &\quad\text{then } \delta_j z^{(r+1,m)} = \frac{1}{h_j}\big(z^{(r+1,m+e_j)} - z^{(r+1,m)}\big)\\ &\quad\text{and } \delta_j u_i^{(r+1,m)} = \frac{1}{h_j}\big(u_i^{(r+1,m+e_j)} - u_i^{(r+1,m)}\big),\quad 1\le i\le n.\end{aligned} \tag{42}$$

Moreover we put

$$\begin{aligned} &\text{if } \partial_{q_j} F(P^{(r,m)}[z,u]) < 0, \\ &\qquad \text{then } \delta_j z^{(r+1,m)} = \frac{1}{h_j}\big(z^{(r+1,m)} - z^{(r+1,m-e_j)}\big) \\ &\qquad \text{and } \delta_j u_i^{(r+1,m)} = \frac{1}{h_j}\big(u_i^{(r+1,m)} - u_i^{(r+1,m-e_j)}\big), \quad 1 \le i \le n, \end{aligned} \tag{43}$$

and we take $j = 1,\ldots,n$ in (42), (43). The difference problem consisting of system (39), (40) and initial boundary conditions (41) is called a generalized Euler method for (4), (5). We claim that (39), (40) is a particular case of (13).

Write $k = 1+n$ and $\Sigma_h = E'_h \times \mathbb{F}(D_h, \mathbb{R}^{1+n}) \times \mathbb{F}(A_h, \mathbb{R}^{1+n})$. Consider the operator $F_h : \Sigma_h \to \mathbb{R}^{1+n}$, $F_h = (F_h^{(0)}, F_h^{(1)}, \ldots, F_h^{(n)})$, defined in the following way. Suppose that $(t^{(r)}, x^{(m)}, w, \eta) \in \Sigma_h$ and $w = (w_0, w')$, $w' = (w_1.\ldots, w_n)$, $\eta = (\eta_0, \eta')$, $\eta' = (\eta_1, \ldots, \eta_n)$. Write

$$Q[w]^{(r,m)} = \big(t^{(r)}, x^{(m)}, T_h w_0, w'((0,\theta)\big)$$

and

$$\begin{aligned} &J_+^{(r,m)}[w] = \big\{ j \in \{1,\ldots,n\} : \partial_{q_j} F(Q[w]^{(r,m)}) \ge 0 \big\} \\ &J_-^{(r,m)}[w] = \{1,\ldots,n\} \setminus J_+^{(r,m)}[w]. \end{aligned}$$

Set

$$F_h^{(0)}(t^{(r)}, x^{(m)}, w, \eta) = F(Q[w]^{(r,m)}) + \sum_{j=1}^{n} \partial_{q_j} F(Q[w]^{(r,m)}) \big(\delta_j \eta_0^{(\theta)} - w_j^{(0,\theta)}\big)$$

and

$$\begin{aligned} F_h^{(i)}(t^{(r)}, x^{(m)}, w, \eta) = {} & \partial_{x_i} F(Q[w]^{(r,m)}) + \partial_w F(Q[w]^{(r,m)})\, T_h w_i \\ & + \sum_{j=1}^{n} \partial_{q_j} F(Q[w]^{(r,m)})\, \delta_j \eta_i^{(\theta)}, \quad i = 1,\ldots,n. \end{aligned}$$

The expressions $\delta \eta_i^{(\theta)} = (\delta_1 \eta_i^{(\theta)}, \ldots, \delta_n \eta_i^{(\theta)})$, $i = 0,1,\ldots,n$, are defined in the following way:

$$\begin{aligned} \delta_j \eta_i^{(\theta)} &= \frac{1}{h_j}\big[\eta_i^{(e_j)} - \eta_i^{(\theta)}\,\big] \quad \text{for } j \in J_+^{(r,m)}[w] \\ \delta_j \eta_i^{(\theta)} &= \frac{1}{h_j}\big[\eta_i^{(\theta)} - \eta_i^{(-e_j)}\,\big] \quad \text{for } j \in J_-^{(r,m)}[w]. \end{aligned}$$

We put $i = 0,1,\ldots,n$, $j = 1,\ldots,n$ in the above definitions. It is clear that system (39), (40) is equivalent to (13) with the above defined $F_h$.

**Assumption** $H_\star[F]$**.** The function $F : \Sigma \to \mathbb{R}$ is continuous and:

1) the partial derivatives $\partial_x F$, $\partial_q F$ exist on $\Sigma$ and $\partial_x F$, $\partial_q F \in C(\Sigma, \mathbb{R}^n)$;
2) there exists the Fréchet derivative $\partial_w F(P)$ and $\partial_w F(P) \in CL(D, \mathbb{R})$ for $P \in \Sigma$.

**Lemma 5.1.** *Suppose that Assumption* $H_\star[F]$ *is satisfied and* $\phi_h : E_{0.h} \cup \partial_0 E \to \mathbb{R}$, $\psi_h : E_{0.h} \cup \partial_0 E \to \mathbb{R}^n$, $h \in H$ *and* $T_h : \mathbb{F}(D_h, \mathbb{R}) \to C(D, \mathbb{R})$*. Then there exists exactly one solution* $(z_h, u_h) : \Omega_h \to \mathbb{R}^{1+n}$*, to problem* (39)–(41)*.*

*Proof.* We apply Theorem 2.2. Define

$$f_h : E'_h \times \mathbb{F}(D_h, \mathbb{R}^{1+n}) \to \mathbb{R}^{1+n}, \quad f_h = \big( f_h^{(0)}, f_h^{(1)}, \dots, f_h^{(n)} \big)$$
$$G_h^{(i)} : E'_h \times \mathbb{F}(D_h, \mathbb{R}^{1+n}) \to \mathbb{R}^{1+n}, \quad G_h^{(i)} = \big\{ G_{h.\lambda}^{(i)} \big\}_{\lambda \in \Lambda}, \quad i = 0, 1, \dots, n,$$

in the following way. Suppose that $(t^{(r)}, x^{(m)}, w) \in E'_h \times \mathbb{F}(D_h, \mathbb{R}^{1+n})$. Write

$$S_+^{(r,m)}[w] = \big\{ \lambda \in \Lambda : \text{there is } \; j \in J_+^{(r,m)}[w] \; \text{ such that } \lambda = e_j \big\}$$
$$S_-^{(r,m)}[w] = \big\{ \lambda \in \Lambda : \text{there is } \; j \in J_-^{(r,m)}[w] \; \text{ such that } \lambda = -e_j \big\},$$

and

$$G_{h.\theta}^{(i)}[w]^{(r,m)} = -\sum_{j=1}^{n} \frac{1}{h_j} \left| \partial_{q_j} F(Q[w]^{(r,m)}) \right|$$
$$G_{h.e_j}^{(i)}[w]^{(r,m)} = \frac{1}{h_j} \partial_{q_j} F(Q[w]^{(r,m)}) \quad \text{for } j \in J_+^{(r,m)}[w],$$
$$G_{h.-e_j}^{(i)}[w]^{(r,m)} = -\frac{1}{h_j} \partial_{q_j} F(Q[w]^{(r,m)}) \quad \text{for } j \in J_-^{(r,m)}[w],$$
$$G_{h.\lambda}^{(i)}[w]^{(r,m)} = 0 \quad \text{for } \lambda \in \Lambda \setminus \big[ S_+^{(r,m)}[w] \cup S_-^{(r,m)}[w] \cup \{ \theta \} \big].$$

We take $i = 0, 1, \dots, n$ in the above definitions. Set

$$f_h^{(0)}[w]^{(r,m)} = F(Q[w]^{(r,m)}) - \sum_{j=1}^{n} \partial_{q_j} F(Q[w]^{(r,m)}) \, w_j^{(0,\theta)}$$
$$f_h^{(i)}[w]^{(r,m)} = \partial_{x_i} F(Q^{(r,m)}) + \partial_w F(Q[w]^{(r,m)}) \, T_h w_i, \quad i = 1, \dots, n.$$

Then Assumption $H[G_h]$ is satisfied and $F_h$ is given by (12). Our theorem follows from Theorem 2.2. □

**Assumption** $H_\star[\sigma]$**.** The function $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions:

1) $\sigma$ is continuous and it is nondecreasing with respect to the both variables;

2) $\sigma(t,0)=0$ for $t\in[0,a]$ and for each $C\in\mathbb{R}_+$, $\tilde{c}\geq 1$ the maximal solution of the Cauchy problem

$$\omega'(t)=C\,\omega(t)+\tilde{c}\,\sigma(t,\omega(t)),\quad \omega(0)=0,$$

is $\tilde{\omega}(t)=0$ for $t\in[0,a]$.

**Assumption** $H[F]$**.** The function $F:\Sigma\to\mathbb{R}$ satisfies Assumption $H_\star[F]$ and:

1) there is $L\in\mathbb{R}_+$ such that, for $P=(t,x,w,q)\in\Sigma$,

$$\|\partial_x F(P)\|,\quad \|\partial_q F(P)\|,\quad \|\partial_w F(P)\|_C\leq L\,;$$

2) there exists $\sigma:[0,a]\times\mathbb{R}_+\to\mathbb{R}_+$ such that Assumption $H_\star[\sigma]$ is satisfied and the terms

$$\|\partial_x F(t,x,w,q)-\partial_x F(t,x,\tilde{w},\tilde{q})\|,\quad \|\partial_q F(t,x,w,q)-\partial_q F(t,x,\tilde{w},\tilde{q})\|$$
$$\|\partial_w F(t,x,w,q)-\partial_w F(t,x,\tilde{w},\tilde{q})\|_C$$

are bounded from above by $\sigma(t,\|w-\tilde{w}\|_D+\|q-\tilde{q}\|)$ on $\Sigma$.

**Theorem 5.2.** *Suppose that Assumption $H[F]$ is satisfied and:*

1) $\varphi:E_0\cup\partial_0 E\to\mathbb{R}$ *is of class* $C^2$ *and* $v:\Omega\to\mathbb{R}$ *is a solution of* (4), (5) *and* $v$ *is of class* $C^2$;
2) $\phi_h:E_{0.h}\cup\partial_0 E_h\to\mathbb{R}$, $\psi_h:E_{0.h}\cup\partial_0 E_h\to\mathbb{R}^n$, $h\in H$ *and* $(z_h,u_h):\Omega_h\to\mathbb{R}^{1+n}$ *is a solution of* (39)–(41)*;*
3) $T_h:\mathbb{F}(D_h,\mathbb{R})\to C(D,\mathbb{R})$ *and Assumption* $H[T_h]$ *is satisfied with* $k=1$,
4) *there is* $\alpha_0:H\to\mathbb{R}_+$ *such that*

$$\left|\phi^{(r,m)}-\phi_h^{(r,m)}\right|+\left\|\partial_x\phi^{(r,m)}-\psi_h^{(r,m)}\right\|\leq\alpha_0(h)\quad\textit{on}\ \ E_{0.h}\cup\partial_0 E_h$$

*and* $lim_{h\to)}\alpha_0(h)=0$.

*Then there is* $\alpha:H\to\mathbb{R}_+$ *such that*

$$|(v_h-z_h)^{(r,m)}|+\|(\,\partial_x v_h-u_h\,)^{(r,m)}\|\leq\alpha(h)\quad\textit{on}\ \ E_h$$

*and* $lim_{h\to 0}\alpha(h)=0$ *where* $v_h$ *and* $\partial_x v_h$ *are the restrictions of* $v$ *and* $\partial_x v$ *th the set* $\Omega_h$.

*Proof.* We use Theorem 2.3. It follows that the functions $(v,\,\partial_x v):\Omega\to\mathbb{R}^{1+n}$ satisfy (6)–(8). Let $\bar{c},\,\bar{C}\in\mathbb{R}_+$ be define by the relations $\|\partial_x v(t,x)\|\leq\bar{c}$, $\|\partial_{xx}v(t,x)\|_{n\times n;\infty}\leq\bar{C}$ on $\Omega$.

Let $X_h\subset\mathbb{F}(D_h,\mathbb{R}^{1+n})$ be the class of all functions $w=(w_0,w')$, $w'=(w_1,\ldots,w_n)$, such that $\|w'(t^{(r)},x^{(m)})\|\leq\bar{c}$ for $(t^{(r)},x^{(m)})\in D_h$. Let $Y_h\subset$

$\mathbb{F}(A_h, \mathbb{R}^{1+n})$ denote the class of functions $\eta : A_h \to \mathbb{R}^{1+n}$, $\eta = (\eta_0, \eta_1, \dots, \eta_n)$, satisfying the conditions:

$$\left| \frac{1}{h_j} \big[\eta_i^{(e_j)} - \eta_i^{(\theta)}\big] \right|, \; \left| \frac{1}{h_j} \big[\eta_i^{(\theta)} - \eta_i^{(e_j)}\big] \right| \le \max\{ \bar{c}, \bar{C} \}$$

where $i = 0, 1, \dots, n$, $j = 1, \dots, n$. Write $V_h = (v_h, \partial_x v_h)$. Then we have

$$\big((V_h)_{[r,m]},\ (V_h)_{\langle r,m\rangle}\big) \in X_h \times Y_h \quad \text{for} \ \big(t^{(r)}, x^{(m)}\big) \in E_h.$$

It follows from Assumption $H[F]$ that there are $C \in \mathbb{R}_+$, $\tilde{c} \ge 1$ such that for $w \in \mathbb{F}(D_h, \mathbb{R}^{1+n})$, $(\tilde{w}, \eta) \in X_h \times Y_h$ we have

$$\big\| F_h[w,\eta]^{(r,m)} - F_h[\tilde{w},\eta]^{(r,m)} \big\|_\star \le C \|w - \tilde{w}\|_{D_h} + \tilde{c}\, \sigma\big(t^{(r)}, \|w - \tilde{w}\|_{D_h}\big)$$

where $\|\cdot\|_{D_h}$ is defined by (11) and $(t^{(r)}, x^{(m)}) \in E_h'$. It is easily seen that that conditions (21)–(23) holds true. Then all the assumptions of Theorem 2.3 are satisfied and our assertion follows. □

**Remark 5.3.** Suppose that the assumptions of Theorem 5.2 are satisfied and $\sigma(t,p) = \tilde{L}p$ on $[0,a] \times \mathbb{R}_+$ where $\tilde{L} \in \mathbb{R}_+$. Then there is $L \in \mathbb{R}_+$ such that

$$\big|v^{(r,m)} - z_h^{(r,m)}\big| + \big\|\partial_x v^{(r,m)} - u_h^{(r,m)}\big\| \le \tilde{\alpha}(h) \quad \text{on} \ E_h$$

where $\tilde{\alpha}$ is given by (29), (30).

**Remark 5.4.** It is easily seen that the results on parabolic functional differential equations and on nonlinear first order partial functional differential problems can be extended on weakly coupled functional differential systems.

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