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Singular Perturbations of Curved Boundaries in Three Dimensions. The Spectrum of the Neumann Laplacian

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Abstract. We calculate the main asymptotic terms for eigenvalues, both simple and multiple, and eigenfunctions of the Neumann Laplacian in a three-dimensional domain $\Omega(h)$ perturbed by a small (with diameter $O(h)$) Lipschitz cavern $\overline{\omega_h}$ in a smooth boundary $\partial\Omega = \partial\Omega(0)$. The case of the hole $\overline{\omega_h}$ inside the domain but very close to the boundary $\partial\Omega$ is under consideration as well. It is proven that the main correction term in the asymptotics of eigenvalues does not depend on the curvature of $\partial\Omega$ while terms in the asymptotics of eigenfunctions do. The influence of the shape of the cavern to the eigenvalue asymptotics relies mainly upon a certain matrix integral characteristics like the tensor of virtual masses. Asymptotically exact estimates of the remainders are derived in weighted norms.

Keywords. Asymptotic analysis, singular perturbations, spectral problem, asymptotics of eigenfunctions and eigenvalues

Mathematics Subject Classification (2000). Primary 35C20, 35J25, 35B40, secondary 35J20, 46E35, 49Q10, 74P15

1. Introduction

1.1. Preamble. In the seventies and eighties of the last century two asymptotic methods, namely the method of matched [9] and compound [20] expansions, were successfully developed to construct asymptotic expansions of solutions to elliptic boundary value problems in domains with singularly perturbed

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boundaries as well as intrinsic functionals calculated for these solutions. In this context the singular perturbation of the boundary means the creation of a small hole (opening) inside the domain, smoothing corner and conical points or edges on the boundary and so on. Among the above-mentionned functionals one finds the energy functional [15,21,24,25], eigenvalues [10,18,19,28,29], the capacity [17] and others. The theory of elliptic problems in singularly perturbed domains is presented in [16,20] in much generality: systems of partial differential equations, elliptic in the Agmon-Douglis-Nirenberg sense, multi-dimensional domains, two-scaled coefficients, miscellaneous perturbation types, and, besides, the procedures to construct and justify asymptotics of solutions, a qualitative analysis of the problems is performed, that is "almost inverse" operators (paramatrices) are constructed, asymptotically sharp estimates in weighted norms are derived and formulas for the index are obtained.

The asymptotic analysis of the Neumann Laplacian in a three-dimensional domain with a small cavern (Figure 1 with the spatial domain and its two-dimensional dummy) follows the general scheme in [16,20] because a point on a smooth surface can be readily regarded as the top of the cone $\mathbb{R}^3_+$, i.e., the half-space. However, the most interesting and important question cannot be answered by the general procedure which only gives a structure of the asymptotic ansätze, identifies problems to be solved, proves the existence of solutions and provides the principal asymptotic forms. At the same time, the procedure leaves open the appearance of logarithmic terms in the decomposition of the auxiliary solutions, the detection of shape and integral characteristics of the perturbed domain that appear in the asymptotic expansions, and annulling of certain asymptotic terms. These particularities are to be specified by a direct calculation which, quite often becomes a very complicated task.

In this paper we compute the main asymptotic terms of eigenvalues, simple and multiple, and eigenfunctions of the Neumann problem for the Laplace operator in a three-dimensional domain $\Omega(h)$ with the small cavity $\overline{\omega_h}$ (Figure 1); notice that the case of a small hole at the distance $O(h)$ from the boundary $\partial\Omega$ (cf. the two-dimensional image in Figure 2) is also under consideration.

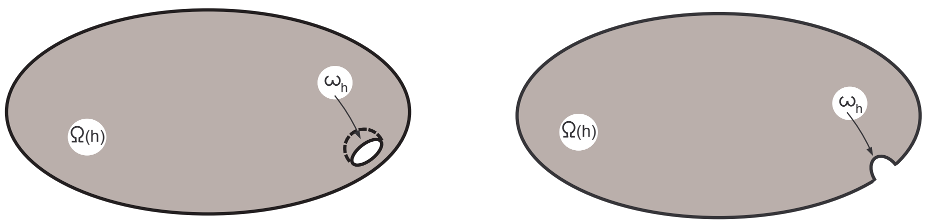

Figure 1: The domains $\Omega(h)$ and ω_h

The unperturbed boundary $\partial\Omega = \partial\Omega(0)$ must be smooth but both $\partial\omega_h$ and

$\partial\Omega(h)$ can be Lipschitz. We prove that the main correction term $O(h^3)$ in the asymptotics for eigenvalues is independent of the curvature of the surface $\partial\Omega$ at the point $\mathcal{O}$ to which the cavity $\overline{\omega_h}$ shrinks as $h \to +0$. Nevertheless, some asymptotic terms in the decomposition of eigenfunctions depend directly on the curvatures. The shape of the cavity influences the eigenvalue correction term by a special integral characteristics, like the virtual mass tensor [30]. The major difficulty in the treatment of perturbations of curved boundaries performed in this paper resides in the use of an appropriate system of curvilinear coordinates to derive the asymptotic expansions.

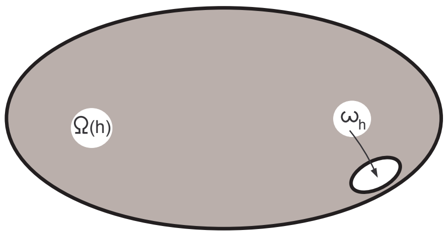

Figure 2: Small hole at the distance $O(h)$ from the boundary

Similar results on the boundary perturbations of spectral problems for the Laplace operator in two variables were recently obtained in [26, 27]. We also mention publications on the perturbation of eigenvalues by smooth perturbations of the boundary [2,7,8,32], by a small hole inside a domain [10,18,19,28,29], [20, Chapter 9], or by changing the type of boundary conditions in a small part of $\partial\Omega$ [4–6]. We especially emphasize that the case of a small cavern in the *flat* boundary is not interesting. Indeed, by the mirror reflection of the domain and the even extension of the eigenfunctions (Figure 3), one arrives at a domain with the interior being a small hole, and such class of perturbation problems has been investigated more than 25 years ago (see citations above).

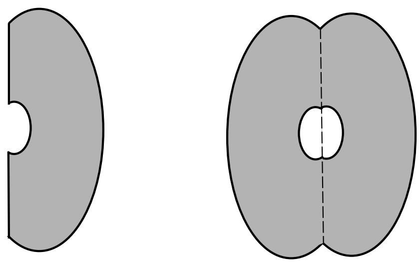

Figure 3: Mirror reflection of a domain with flat boundary

1.2. Problem formulation. Let $\Omega \subset \mathbb{R}^3$ be a domain with a smooth boundary Γ. We assume that the origin $\mathcal{O}$ of the Cartesian coordinates $x = (x_1, x_2, x_3)$ belongs to Γ. Since Γ is smooth, we can find a neighbourhood $\mathcal{U}$ of the point $\mathcal{O}$

such that there exists a conformal application which maps $\mathcal{U}$ onto a neighbourhood of the origin in $\mathbb{R}^3$, and thus there exists an orthonormal curvilinear coordinate system (n, s, ν) in $\mathcal{U}$ (see Figure 4), where (s, ν) are the parameters associated with the local surface parameterization of the origin $\mathcal{O}$, and n stands for the oriented distance to Γ, with $n > 0$ in $\Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}$.

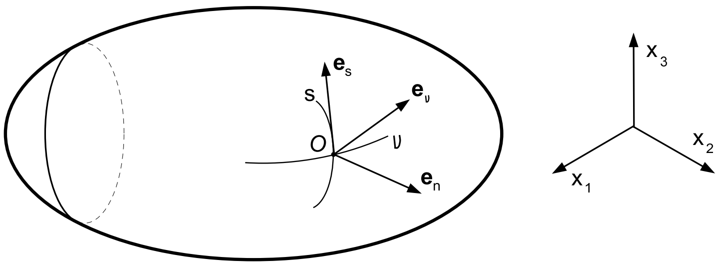

Figure 4: Orthonormal curvilinear coordinate system (n, s, ν) in $\mathcal{U}$

We denote $(\mathbf{e_n}, \mathbf{e_s}, \mathbf{e}_\nu)$ the basis corresponding to the curvilinear coordinate system (n, s, ν). By $\omega \subset \mathbb{R}^3_- = (-\infty, 0) \times \mathbb{R}^2$ (see Figure 5), we understand an open set (not necessarily connected) with the compact closure $\overline{\omega} = \omega \cup \partial\omega$ and such that $\partial\omega$ is Lipschitz. The boundary $\partial\Xi$ of the infinite domain $\Xi = \mathbb{R}^3_- \setminus \overline{\omega}$ is also assumed to be Lipschitz.

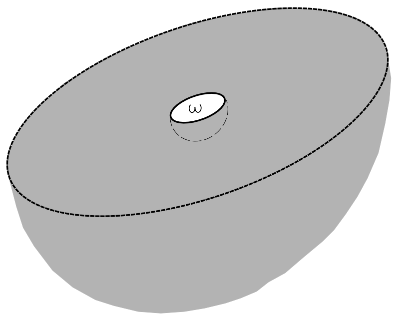

Figure 5: The domain ω

Introduce a family of domains depending on the small parameter $h > 0$ (see Figure 1),

$$\omega_h = \{(n, s, \nu) \mid \xi = (\xi_1, \xi_2, \xi_3) := (h^{-1}n, h^{-1}s, h^{-1}\nu) \in \omega\} \tag{1.1}$$

$$\Omega(h) = \Omega \setminus \overline{\omega_h}. \tag{1.2}$$

Let us consider the spectral Neumann problem

$$-\Delta_x u^h(x) = \lambda^h u^h(x), \quad x \in \Omega(h) \tag{1.3}$$

$$\partial_{n^h} u^h(x) = 0, \qquad x \in \Gamma(h) := \partial\Omega(h), \tag{1.4}$$

with the Laplace operator Δ_x, and where $\partial_{n^h} = n^h \cdot \nabla_x$ denotes the normal derivative along the outer normal n^h. Problem (1.3)–(1.4) admits the sequence of eigenvalues

$$0 = \lambda_0^h < \lambda_1^h \leq \lambda_2^h \leq \cdots \leq \lambda_m^h \leq \cdots \to +\infty, \tag{1.5}$$

where the multiplicity is explicitely indicated. The corresponding eigenfunctions $u_0^h, u_1^h, u_2^h, \ldots, u_m^h, \ldots$ are subject to the orthogonality and normalization conditions

$$(u_p^h, u_m^h)_{\Omega(h)} = \delta_{p,m}, \quad p, m \in \mathbb{N}_0, \tag{1.6}$$

where $(\cdot,\cdot)_D$ is the natural scalar product in the Lebesgue space $L_2(D)$, and $\delta_{p,m}$ the Kronecker symbol.

Our aim is to derive asymptotic formulas for the solution of the spectral problem (1.3)–(1.4) as $h \to 0$. We will intermediately conclude that for a fixed index m and with $h \to 0$, the entry λ_m^h of (1.5) converges to the element λ_m^0 in the sequence

$$0 = \lambda_0^0 < \lambda_1^0 \leq \lambda_2^0 \leq \cdots \leq \lambda_m^0 \leq \cdots \to +\infty \tag{1.7}$$

of eigenvalues for the limit spectral Neumann problem

$$-\Delta_x v^0(x) = \lambda^0 v^0(x), \quad x \in \Omega \tag{1.8}$$

$$\partial_n v^0(x) = 0, \qquad x \in \Gamma. \tag{1.9}$$

Therefore we will use an eigenfunction v^0 as our first approximation of u^h. The eigenfunctions of (1.8)–(1.9) are smooth in $\overline{\Omega}$ and admit the orthogonality and normalization conditions

$$(v_p^0, v_m^0)_\Omega = \delta_{p,m}, \quad p, m \in \mathbb{N}_0. \tag{1.10}$$

1.3. Preliminary description of the asymptotic procedure. We use the following asymptotic ansätze for λ_m^h and u_m^h.

$$\lambda_m^h = \lambda_m^0 + h^3 \lambda_m' + \cdots \tag{1.11}$$

$$u_m^h(x) = v_m^0(x) + h\chi(x) w_m^1(\xi) + h^2 \chi(x) w_m^2(\xi) + h^3 v_m^3(x) + \cdots \tag{1.12}$$

Here v_m^0 and v_m^3 are terms of regular type, and w_m^1, w_m^2 are terms of the boundary layer type, which depend on the *rapid variables* $\xi = (\xi_1, \xi_2, \xi_3)$. Finally $\chi \in C^\infty(\overline{\Omega})$ is a cut-off function equal to one in a fixed neighbourhood, independent of h, of the point $\mathcal{O}$, and null outside of a bigger neighbourhood $\mathcal{U}$. We emphasize that the coefficients of h^1 and h^2 vanish in (1.11) and the same happens for regular terms in (1.12). This simplification of the asymptotic ansätze is not predicted by the general procedure in [20] but is a result of our further calculations, and we now accept it as granted and verify this assumption in the sequel.

Inserting v_m^0 and λ_m^0 into the singularly perturbed problem (1.3)–(1.4) brings a discrepancy into the boundary condition on the surface $\partial\Omega(h) \cap \partial\omega_h$ of the cavern $\overline{\omega_h}$. This discrepancy cannot be compensated by a function depending on the variables n, s, ν smoothly and, using the stretched curvilinear coordinates ξ from (1.1), we come across the boundary layer phenomenon so that the first correction term becomes of the boundary layer type and must be found out while solving the Neumann problem in the infinite domain Ξ (Figure 5). The corresponding solution decays at infinity as a linear combination of derivatives of the fundamental solution for the Laplacian,

$$h\left(c_1\partial_{\xi_1} + c_2\partial_{\xi_2}\right)\frac{1}{4\pi|\xi|}, \tag{1.13}$$

and after the multiplication with an appropriate cut-off function the main asymptotic term (1.13) of the boundary layer produces lower order discrepancies in the differential equation (1.3) and the Neumann conditions (1.4) on $\partial\Omega(h) \setminus \partial\omega_h$. The expression (1.13) can be rewritten in the original coordinates n, s, ν and becomes

$$h^3\left(c_1\partial_s + c_2\partial_\nu\right)\left(4\pi\left(n^2 + s^2 + \nu^2\right)^{\frac{1}{2}}\right)^{-1}. \tag{1.14}$$

We emphasize that there appears an additional small factor and that the function (1.14) is not singular at a distance from the point $\mathcal{O}$ where the discrepancies are mainly located due to the cut-off function. The latter allows to compensate for them by means of the lower-order term of regular type (in the variable x) while the compatibility condition in the problem for this function gives the main asymptotic correction of the eigenvalue λ_j^0.

The above is a very simplified description of the asymptotic procedure to construct the compound expansion of the solution to the spectral problem (1.3)–(1.4). Much complication arises from the fact that coefficients of differential operators written in the curvilinear coordinates are no longer constant. The latter crucially influences both, the procedure to construct asymptotics and the derivation of estimates for the asymptotic remainders. For example, the discrepancies of the expression (1.13) appear in the problem in Ξ for the next term of the boundary layer type as well as in the problem for the above-mentionned next element of regular type. The correct statement of these problems is made by means of *the procedure to rearrange discrepancies* [20] which is silently used many times in our paper.

The most complicated task is to examine the behaviour of regular and boundary layer solutions for $x \to \mathcal{O}$ and $\xi \to \infty$, respectively. The general structure is predicted by the Kondratiev theory [11] (see, e.g., monographs [12, 23]) but exact formulas for the decompositions of the solutions need scrupulous and cumbersome calculations.

1.4. The asymptotic ansätze and the structure of the paper. In the paper, the method of compound asymptotic expansions [20] is applied to identify different terms of ansätze (1.11)–(1.12). In Section 2.1 and 2.2 the first and second boundary layers w_m^1 and w_m^2 in (1.11), respectively, are found out. Both the functions w_m^1 and w_m^2 enjoy, unlike in dimension two, the canonic property of boundary layers, i.e., they decay for $|\xi| \to \infty$, with order $|\xi|^{-2}$ and $|\xi|^{-1}$, respectively. The correction function of regular type v_m^3 in (1.12) is determined in section 2.3. From this correction we deduce λ_m' of ansatz (1.11), given by (2.46) in the case of a simple eigenvalue λ_m^0 and by (2.53) in the case of multiple eigenvalues in section 2.4.

The justification of asymptotics is based on the weighted Poincaré inequality (Lemma 3.1). We then reduce the problem to an abstract equation in a convenient Hilbert space and use the lemma on "almost eigenvalues and eigenfunctions" (Lemma 3.4) which allows to give estimates for the remainders in ansätze (1.11)–(1.12), for simple or multiple eigenvalues. The justification of the asymptotics consists of many steps: We need to estimate a remainder which is a combination of the terms appearing in ansätze (1.11)–(1.12). The remainder is then divided into several terms which, when combined in an appropriate fashion, provide an estimate of order $h^{\frac{7}{2}}$. The estimates of different terms rely mainly on the analysis of the behaviour of the boundary layers as $x \to \mathcal{O}$ and $|\xi| \to \infty$.

Finally, in Theorem 3.6 we derive the estimates for the remainders corresponding to ansätze (1.11)–(1.12), i.e., for the eigenvalues and the eigenfunctions, respectively. In the proof of Theorem 3.6, we use Lemma 3.4 to obtain the existence of a certain number of eigenvalues close to the eigenvalue λ_m^0 with the multiplicity $\varkappa_m$ in the sense of the desired estimate, and the main task of the proof is then to show that these eigenvalues exactly coincide with the eigenvalues corresponding to a small perturbation of the eigenvalue λ_m^0 with the multiplicity $\varkappa_m$.

2. Constructing the asymptotics

2.1. First term of the boundary layer type. Let P be a point in a neighbourhood $\mathcal{U}$ of $\mathcal{O}$, and P_Γ its projection onto Γ. Then we have $P = n\mathbf{e_n} + P_\Gamma(s,\nu)$. Thus, the components of the metric tensor are given by (see [3, pp. 83])

$$\begin{aligned} g_{nn} &= |\partial_n P|^2 = |\mathbf{e_n}|^2 = 1 \\ g_{ss} &= |\partial_s P|^2 = |n\partial_s \mathbf{e_n} + \partial_s P_\Gamma(s,\nu)|^2 \\ &= |n\varkappa_s(s,\nu)\mathbf{e_s} + n\tau_s(s,\nu)\mathbf{e}_\nu + \mathbf{e_s}|^2 \\ &= (1+n\varkappa_s(s,\nu))^2 + (n\tau_s(s,\nu))^2 \\ g_{\nu\nu} &= |\partial_\nu P|^2 = |\nu\partial_\nu \mathbf{e_n} + \partial_\nu P_\Gamma(s,\nu)|^2 \\ &= |\nu\varkappa_\nu(s,\nu)\mathbf{e}_\nu + n\tau_\nu(s,\nu)\mathbf{e_s} + \mathbf{e}_\nu|^2 \\ &= (1+n\varkappa_\nu(s,\nu))^2 + (n\tau_\nu(s,\nu))^2, \end{aligned}$$

where $\varkappa_s$ and $\varkappa_\nu$ stand for the two curvatures corresponding to the curves $\nu = const$ and $s = const$ containing the surface point (s, ν), respectively, while τ_s and τ_ν are the torsions of these curves, respectively. Since the coordinates system corresponding to (n, s, ν) is orthogonal, we have $g_{ns} = g_{n\nu} = g_{s\nu} = 0$. We can always assume, shrinking the neighbourhood $\mathcal{U}$, that $1 + n\varkappa_s > 0$ and $1 + n\varkappa_\nu > 0$ in $\mathcal{U}$. The Jacobian is thus equal to

$$J(n, s, \nu) = \left[(1 + n\varkappa_s)^2 + (n\tau_s)^2\right]^{\frac{1}{2}} \left[(1 + n\varkappa_s) + (n\tau_\nu)^2\right]^{\frac{1}{2}}$$

The Laplace operator Δ_x in the curvilinear coordinates (n, s, ν) admits the representation

$$\begin{aligned} \Delta_x &= J^{-1}\left[\partial_n(J\partial_n) + \partial_s\left(\frac{J}{g_{ss}}\partial_s\right) + \partial_\nu\left(\frac{J}{g_{\nu\nu}}\partial_\nu\right)\right] \\ &= \partial_n^2 + g_{ss}^{-1}\partial_s^2 + g_{\nu\nu}^{-1}\partial_\nu^2 + J^{-1}\partial_n J\partial_n \\ &\quad + J^{-1}\left(\left[\frac{\partial_s J}{g_{ss}} - \frac{J\partial_s g_{ss}}{g_{ss}^2}\right]\partial_s + \left[\frac{\partial_\nu J}{g_{\nu\nu}} - \frac{J\partial_\nu g_{\nu\nu}}{g_{\nu\nu}^2}\right]\partial_\nu\right) \end{aligned} \tag{2.1}$$

Under the transformation to the rapid variable $\xi = (\xi_1, \xi_2, \xi_3)$ introduced in (1.1), the elements depending on the torsion in $J(n, s, \nu)$ are of order h^2 and thus the Laplace operator is independent of the torsions at orders h^{-2} and h^{-1}, i.e.,

$$\Delta_x = h^{-2}\Delta_\xi + h^{-1}\left(\varkappa_s(\mathcal{O})(\partial_{\xi_1} - 2\xi_1\partial_{\xi_2}^2) + \varkappa_\nu(\mathcal{O})(\partial_{\xi_1} - 2\xi_1\partial_{\xi_3}^2)\right) + \cdots . \tag{2.2}$$

In the coordinates (n, s, ν) the gradient takes the form

$$\nabla_x = \left({g_{nn}}^{-\frac{1}{2}}\partial_n, {g_{ss}}^{-\frac{1}{2}}\partial_s, {g_{\nu\nu}}^{-\frac{1}{2}}\partial_\nu\right) = \left(\partial_n, (1 + n\varkappa_s)^{-1}\partial_s, (1 + n\varkappa_\nu)^{-1}\partial_\nu\right).$$

The decomposition of the unit normal vector n^h to $\Omega(h)$ in the basis $(\mathbf{e_n}, \mathbf{e_s}, \mathbf{e}_\nu)$ is as follows

$$n^h = d^{-\frac{1}{2}}\left[N_1 J\mathbf{e_n} + N_2(1 + n\varkappa_\nu)\mathbf{e_s} + N_3(1 + n\varkappa_s)\mathbf{e}_\nu\right] \tag{2.3}$$

with $d = [N_1 J]^2 + [N_2(1 + n\varkappa_\nu)]^2 + [N_3(1 + n\varkappa_s)]^2$ and $N = (N_1, N_2, N_3)$ is the outward unit normal vector on the boundary $\partial\Xi \subset \mathbb{R}^3$. Therefore, denoting by ∂_N the directional derivative along N, we obtain in the rapid coordinates the formula

$$\begin{aligned} \partial_{n^h} &= \nabla_x \cdot n^h \\ &= d^{-\frac{1}{2}}\left(N_1 J\partial_n + N_2\frac{1 + n\varkappa_\nu}{1 + n\varkappa_s}\partial_s + N_3\frac{1 + n\varkappa_s}{1 + n\varkappa_\nu}\partial_\nu\right) \\ &= h^{-1}\partial_N + \xi_1\left(N_2^2\varkappa_s(\mathcal{O}) + N_3^2\varkappa_\nu(\mathcal{O})\right)\partial_N \\ &\quad - 2\xi_1\left(N_2\varkappa_s(\mathcal{O})\partial_{\xi_2} + N_3\varkappa_\nu(\mathcal{O})\partial_{\xi_3}\right) + \cdots . \end{aligned} \tag{2.4}$$

In view of the homogeneous Neumann condition (1.9), the function v^0 in the Ch-neighbourhood of the point $\mathcal{O}$ has the expansion

$$\begin{aligned} v^0(x) &= v^0(\mathcal{O}) + s\partial_s v^0(\mathcal{O}) + \nu\partial_\nu v^0(\mathcal{O}) \\ &\quad + \frac{1}{2}\big(n^2\partial_n^2 v^0(\mathcal{O}) + s^2\partial_s^2 v^0(\mathcal{O}) + \nu^2\partial_\nu^2 v^0(\mathcal{O}) + 2s\nu\partial_{s\nu}^2 v^0(\mathcal{O})\big) \\ &\quad + O\big((n^2+s^2+\nu^2)^{\frac{3}{2}}\big) \\ &= v^0(\mathcal{O}) + h(\xi_2\partial_s v^0(\mathcal{O}) + \xi_3\partial_\nu v^0(\mathcal{O})) \\ &\quad + \frac{1}{2}h^2\left(\xi_1^2\partial_n^2 v^0(\mathcal{O}) + \xi_2^2\partial_s^2 v^0(\mathcal{O}) + \xi_3^2\partial_\nu^2 v^0(\mathcal{O}) + 2\xi_2\xi_3\partial_{\xi_2\xi_3}^2 v^0(\mathcal{O})\right) \\ &\quad + O(h^3). \end{aligned}$$

Under the coordinate dilation by factor h^{-1} and setting $h = 0$, the domain $\Omega(h)$ becomes $\Xi = \mathbb{R}^3 \setminus \overline{\omega}$, thus the boundary layer w^1 is defined in Ξ. After replacing u^h and Δ, ∂_{n^h} by their respective expansions (1.12) and (2.2), (2.4) one may collect terms of order h^{-1} in the equation, and of order h^0 in the boundary conditions. Then, setting formally $h = 0$, we arrive at the problem

$$-\Delta_\xi w^1(\xi) = 0, \qquad \xi \in \Xi \tag{2.5}$$

$$\partial_N w^1(\xi) = -N_2(\xi)\partial_s v^0(\mathcal{O}) - N_3(\xi)\partial_\nu v^0(\mathcal{O}), \quad \xi \in \partial\Xi. \tag{2.6}$$

For $j,k = 1,2,3$ we have the evident formulas

$$\int_{\partial\Xi\cap\partial\omega} N_k(\xi)\,ds_\xi = 0, \qquad \int_{\partial\Xi\cap\partial\omega} \xi_j N_k(\xi)\,ds_\xi = -\delta_{j,k}\,\mathrm{mes}_3(\omega). \tag{2.7}$$

The first formula in (2.7) shows that the right-hand side of the boundary condition in (2.6) has null integral over the surface $\partial\Xi$; note that $N_2 = N_3 = 0$ on the plane surface $\partial\Xi \setminus \partial\omega$ of the boundary, and, therefore, the right-hand side is compactly supported. Thus, there exists a unique generalized solution $w^1 \in H^1_{loc}(\overline{\Xi})$ of problem (2.5)–(2.6), decaying at infinity. The solution is represented in the form

$$w^1(\xi) = \partial_s v^0(\mathcal{O})W_2(\xi) + \partial_\nu v^0(\mathcal{O})W_3(\xi), \tag{2.8}$$

where W_2 and W_3 are canonical solutions of the Neumann problem

$$-\Delta_\xi W_k(\xi) = 0, \qquad \xi \in \Xi \tag{2.9}$$

$$\partial_n W_k(\xi) = -N_k(\xi), \quad \xi \in \partial\Xi. \tag{2.10}$$

They admit the representation

$$W_k(\xi) = -\sum_{j=2}^{3} \frac{m_{kj}}{2\pi}\frac{\xi_j}{\rho^3} + O(|\xi|^{-3}), \quad |\xi| \geq R, \tag{2.11}$$

where the coefficients m_{kj} have been introduced in Note G on virtual mass tensor in the classical monograph [30].

Remark 2.1. The approach we discussed in Section 1, with even extension of a harmonic function over the boundary with the homogeneous Neumann condition (see Figures 3 and 6), is applicable to the function W.

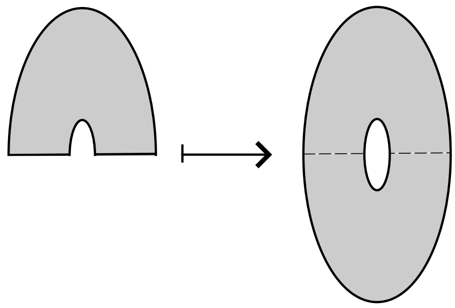

Figure 6: Even extension of the domain Ξ

As a result, problem (2.9)–(2.10) can be transformed to the exterior Neumann problem in the domain $\Xi^{00} = \{\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : (-|\xi_1|, \xi_2, \xi_3) \notin \overline{\omega}\}$. In this way, the extended functions W_2, W_3 become solutions to exactly the same problems as introduced in monograph [30, p. 239] for the description of the virtual mass tensor. Hence, the first term on the right-hand side of (2.11) is half the corresponding term of the virtual mass matrix (see [30, Note G]).

In the spherical coordinate system (ρ, θ, ϕ) we have $(\xi_1, \xi_2, \xi_3) = (\rho\cos\phi, \rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi)$ and

$$\begin{aligned} W_2(\xi) &= -\frac{m_{22}}{2\pi}\rho^{-2}\cos\theta\sin\phi - \frac{m_{23}}{2\pi}\rho^{-2}\sin\theta\sin\phi + O(\rho^{-3}) \\ W_3(\xi) &= -\frac{m_{33}}{2\pi}\rho^{-2}\sin\theta\sin\phi - \frac{m_{32}}{2\pi}\rho^{-2}\sin\theta\sin\phi + O(\rho^{-3}). \end{aligned}$$

In order to observe general properties of m_{kj}, we apply Green's formula on the set $\Xi_R = \{\xi \in \Xi : \rho < R\}$ with the functions W_k and $Y_k = \xi_k + W_k$, $k = 2, 3$,

$$\begin{aligned} \int_{\partial\Xi} Y_2 \partial_N W_2 \, ds_\xi &= \int_{\{\xi\in\mathbb{R}^3_- :\, \rho=R\}} (W_2\partial_\rho Y_2 - Y_2\partial_\rho W_2)\, ds_\xi \\ &= -\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \left(\frac{3m_{22}}{2\pi}R^{-2}\cos^2\theta\sin^2\phi\right) R^2\sin\phi\, d\phi d\theta \\ &\quad -\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \left(\frac{3m_{23}}{2\pi}R^{-2}\cos\theta\sin\theta\sin^2\phi\right) R^2\sin\phi\, d\phi d\theta + O(R^{-1}) \\ &= -\frac{3m_{22}}{2\pi}\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \cos^2\theta\sin^3\phi\, d\phi d\theta + O(R^{-1}) \\ &= -m_{22} + O(R^{-1}). \end{aligned}$$

On the other hand, applying Green's formula in ω and changing the direction of the normal, we have

$$\begin{aligned}\int_{\partial\Xi} Y_j \partial_N W_k \, ds_\xi &= \int_{\partial\Xi} W_j \partial_N W_k \, ds_\xi - \int_{\partial\Xi} \xi_j N_k \, ds_\xi \\ &= \int_{\Xi} \nabla_\xi W_k \cdot \nabla_\xi W_j \, d\xi + \delta_{kj} \operatorname{mes}_3(\omega).\end{aligned} \tag{2.12}$$

Therefore, as $R \to \infty$, and in a similar way for m_{33} and $m_{23} = m_{32}$ we get

$$m_{kj} = -\int_{\Xi} \nabla_\xi W_k \cdot \nabla_\xi W_j \, d\xi - \delta_{kj} \operatorname{mes}_3(\omega), \quad k, j = 1, 2. \tag{2.13}$$

In other words, the 2×2-matrix

$$\mathbf{m}(\Xi) = \begin{pmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{pmatrix} \tag{2.14}$$

is symmetric and negative definite as it is the sum of two Gram matrices.

Example 2.2. For a semi-ball of radius R, $\mathbf{m}(\Xi)$ is a multiple of the identity matrix with the coefficient $-\pi R^3$.

Example 2.3. If $\overline{\omega}$ is a plain crack then $\operatorname{mes}_3(\overline{\omega}) = 0$ and the matrix (2.14) becomes singular. For example, if $\overline{\omega}$ belongs to the plane $\{\xi_2 = 0\}$, then $m_{33} = m_{23} = 0$ while obviously $W_3 = 0$. However, for a curved or broken crack (cf. Figure 7b) the solutions W_2 and W_3 are linear independent and $\mathbf{m}(\Xi)$ is non-degenerate although $\operatorname{mes}_3(\overline{\omega}) = 0$.

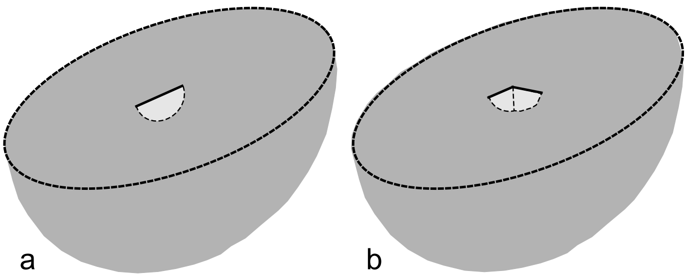

Figure 7: Straight a) and broken b) cracks

2.2. Second term of the boundary layer type. The right-hand sides in the problem

$$-\Delta_\xi w^2(\xi) = F^2(\xi), \quad \xi \in \Xi \tag{2.15}$$

$$\partial_N w^2(\xi) = G^2(\xi), \quad \xi \in \partial\Xi. \tag{2.16}$$

are to be determined using (1.12), (2.2) and (2.4), and collecting terms of order h^0 in the equation, and of order h^1 in the boundary conditions. As a result, we arrive at the following functions written in rapid variables

$$F^2(\xi) = \left[\varkappa_s(\mathcal{O})(\partial_{\xi_1} - 2\xi_1\partial^2_{\xi_2}) + \varkappa_\nu(\mathcal{O})(\partial_{\xi_1} - 2\xi_1\partial^2_{\xi_3})\right] w^1(\xi) \tag{2.17}$$

$$\begin{aligned}
G^2(\xi) &= G^2_1(\xi) + G^2_2(\xi) + G^2_3(\xi) + G^2_4(\xi) + G^2_5(\xi)\\
G^2_1(\xi) &:= -N_1\xi_1\partial^2_n v^0(\mathcal{O}) - N_2\xi_2\partial^2_s v^0(\mathcal{O}) - N_3\xi_3\partial^2_\nu v^0(\mathcal{O})\\
&\quad - (N_2\xi_3 + N_3\xi_2)\partial^2_{s\nu} v^0(\mathcal{O}),\\
G^2_2(\xi) &:= -\xi_1\left(N_2^2\varkappa_s(\mathcal{O}) + N_3^2\varkappa_\nu(\mathcal{O})\right)\left(N_2\partial_s v^0(\mathcal{O}) + N_3\partial_\nu v^0(\mathcal{O})\right), \qquad (2.18)\\
G^2_3(\xi) &:= +2N_2\xi_1\varkappa_s(\mathcal{O})\partial_s v^0(\mathcal{O}) + 2N_3\xi_1\varkappa_\nu(\mathcal{O})\partial_\nu v^0(\mathcal{O}),\\
G^2_4(\xi) &:= -\xi_1\left(N_2^2\varkappa_s(\mathcal{O}) + N_3^2\varkappa_\nu(\mathcal{O})\right)\partial_N w^1(\xi),\\
G^2_5(\xi) &:= +2N_2\xi_1\varkappa_s(\mathcal{O})\partial_{\xi_2} w^1(\xi) + 2N_3\xi_1\varkappa_\nu(\mathcal{O})\partial_{\xi_3} w^1(\xi).
\end{aligned}$$

We immediately notice that $G^2_2(\xi) + G^2_4(\xi) = 0$ according to the boundary conditions (2.6). In view of formulas (2.17) and (2.11), the following expansion holds true:

$$\begin{aligned}
F^2(\xi) &= \left[\varkappa_s(\mathcal{O})(\partial_{\xi_1} - 2\xi\partial^2_{\xi_2}) + \varkappa_\nu(\mathcal{O})(\partial_{\xi_1} - 2\xi\partial^2_{\xi_3})\right]\\
&\quad \times \left[\partial_s v^0(\mathcal{O}) W_2(\xi) + \partial_\nu v^0(\mathcal{O}) W_3(\xi)\right]\\
&= \varkappa_s(\mathcal{O})\partial_s v^0(\mathcal{O})\frac{m_2}{\pi}\left(15\frac{\xi_1\xi_2}{\rho^5} - 30\frac{\xi_1\xi_2^3}{\rho^7}\right)\\
&\quad + \varkappa_\nu(\mathcal{O})\partial_\nu v^0(\mathcal{O})\frac{m_3}{\pi}\left(15\frac{\xi_1\xi_3}{\rho^5} - 30\frac{\xi_1\xi_3^3}{\rho^7}\right)\\
&\quad + \varkappa_s(\mathcal{O})\partial_\nu v^0(\mathcal{O})\frac{m_3}{\pi}\left(-27\frac{\xi_1\xi_3}{\rho^5} + 30\frac{\xi_1^3\xi_3 + \xi_1\xi_3^3}{\rho^7}\right)\\
&\quad + \varkappa_\nu(\mathcal{O})\partial_s v^0(\mathcal{O})\frac{m_2}{\pi}\left(-27\frac{\xi_1\xi_2}{\rho^5} + 30\frac{\xi_1^3\xi_2 + \xi_1\xi_2^3}{\rho^7}\right) + O(\rho^{-2}),
\end{aligned} \tag{2.19}$$

as $\rho \to \infty$. The function

$$\begin{aligned}
U^2(\xi) &= \left[\varkappa_s(\mathcal{O})(\partial_{\xi_1} - 2\xi\partial^2_{\xi_2}) + \varkappa_\nu(\mathcal{O})(\partial_{\xi_1} - 2\xi\partial^2_{\xi_3})\right]\\
&\quad \times \left[\partial_s v^0(\mathcal{O}) W_2(\xi) + \partial_\nu v^0(\mathcal{O}) W_3(\xi)\right]\\
&= \varkappa_s(\mathcal{O})\partial_s v^0(\mathcal{O})\frac{m_2}{\pi}\left(15\frac{\xi_1\xi_2}{6\rho^3} - 30\frac{\xi_1^3\xi_2 + \xi_1\xi_3^2\xi_2 + \xi_1\xi_2^3}{20\rho^5}\right)\\
&\quad + \varkappa_\nu(\mathcal{O})\partial_\nu v^0(\mathcal{O})\frac{m_3}{\pi}\left(15\frac{\xi_1\xi_3}{6\rho^3} - 30\frac{\xi_1^3\xi_3 + \xi_1\xi_2^2\xi_3 + \xi_1\xi_3^3}{20\rho^5}\right)\\
&\quad + \varkappa_s(\mathcal{O})\partial_\nu v^0(\mathcal{O})\frac{m_3}{\pi}\left(-27\frac{\xi_1\xi_3}{6\rho^3} - 30\frac{3\xi_1^3\xi_3 + 2\xi_1\xi_2^2\xi_3 + 3\xi_1\xi_3^3}{20\rho^5}\right)\\
&\quad + \varkappa_\nu(\mathcal{O})\partial_s v^0(\mathcal{O})\frac{m_2}{\pi}\left(-27\frac{\xi_1\xi_2}{6\rho^3} - 30\frac{3\xi_1^3\xi_2 + 2\xi_1\xi_3^2\xi_2 + 3\xi_1\xi_2^3}{20\rho^5}\right)
\end{aligned}$$

has the homogeneity order -1 (the same as for the fundamental solutions) and compensates for the leading term of $F^2(\xi)$. Therefore, the expansion of $w^2(\xi)$ at infinity can be written as follows :

$$w^2(\xi) = a\rho^{-1} + U^2(\xi) + O(\rho^{-2}). \tag{2.20}$$

Remark 2.4. The formula $z(\xi) = z_0(\xi) + O(\rho^{-p})$ used in (2.11) and (2.19), (2.20) means that

$$z(\xi) = z_0(\xi) + \widetilde{z}(\xi), \quad |\nabla_\xi^q \widetilde{z}(\xi)| \le c_q \rho^{-p-q}, \quad q = 0, 1, \dots, \rho = |\xi| \ge R_0, \tag{2.21}$$

where $\nabla_\xi^q \widetilde{z}$ is the collection of all order q derivatives of the function $\widetilde{z}$, and the radius R_0 is selected such that $\overline{\omega} \subset \{\xi : \rho < R_0\}$. For a solution w^1 of problem (2.5)–(2.6) the estimate of form (2.21) for the remainder $\widetilde{w}^1$ is straightforward, since the remainder verifies the Laplace equation in the set $\{\xi \in \mathbb{R}^2 : \rho > R_0\}$. For such an equation, e.g., the Fourier method can be used in order to provide a solution representation in the form of a convergent series, with harmonic functions decaying at infinity. The pointwise estimates of the remainder in the representation (2.20) are justified again by the general theory (see [14] and, e.g., [23, Chapter 3]).

To evaluate the coefficient a, we compute the following integrals on the semi-sphere of radius R taking the expansion (2.20) into account :

$$\begin{aligned}\int_{\Xi_R} F^2(\xi)\,d\xi + \int_{\partial\omega\cap\partial\Xi} G^2(\xi)\,ds_\xi &= -\int_{\partial\Xi_R} \partial_N w^2(\xi)\,ds_\xi + \int_{\partial\omega\cap\partial\Xi} G^2(\xi)\,ds_\xi \\ &= -\int_{\{\xi\in\mathbb{R}^3_- : \rho=R\}} \partial_\rho w^2(\xi)\,ds_\xi,\end{aligned}$$

where we have used the fact that $G^2(\xi) = 0$ on $\partial\Xi_R \setminus \partial\omega$ due to the evident relations $\xi_1 = 0$ and $N_2 = N_3 = 0$ on $\partial\Xi_R \setminus \partial\omega$. In view of expansion (2.20) we obtain

$$\partial_\rho w^2(\xi) = -a\rho^{-2} + \partial_\rho U^2(\xi) + O(\rho^{-3}) = -a\rho^{-2} - \rho^{-1}U^2(\xi) + O(\rho^{-3})$$

and thus

$$\begin{aligned}-\int_{\{\xi\in\mathbb{R}^3_- : \rho=R\}} \partial_\rho w^2(\xi) ds_\xi &= a\int_{\{\xi\in\mathbb{R}^3_- : \rho=R\}} \rho^{-2}\,ds_\xi \\ &\quad - \int_{\{\xi\in\mathbb{R}^3_- : \rho=R\}} \partial_\rho U^2(\xi)\,ds_\xi + O(R^{-1}). \\ &= 2\pi a + O(R^{-1}).\end{aligned} \tag{2.22}$$

Note that all terms in $U^2(\xi)$ are odd in either ξ_2, or ξ_3, thus it is also true for $\partial_\rho U^2(\xi)$ so that $\int_{\{\xi\in\mathbb{R}^3_- : \rho=R\}} \partial_\rho U^2(\xi) = 0$. Using (2.18) we now study the integral

$\int_{\partial\omega\cap\partial\Xi} G^2(\xi)\, ds_\xi$. If we denote by ω^+ the domain obtained by adding to ω its mirror image with respect to the plane $\xi_1 = 0$ we, in view of (2.7), can write

$$\begin{aligned}\int_{\partial\omega\cap\partial\Xi} G_1^2(\xi)\, ds_\xi &= \frac{1}{2}\int_{\partial\omega^+} G_1^2(\xi)\, ds_\xi \\ &= -\frac{1}{2}\sum_{k=1}^{3} \partial_k^2 v^0(\mathcal{O}) \int_{\partial\omega^+} N_k\xi_k\, ds_\xi \\ &\quad - \frac{1}{2}\partial_{s\nu}^2 v^0(\mathcal{O}) \int_{\partial\omega^+} (N_2\xi_3 + N_3\xi_2)\, ds_\xi \\ &= -\frac{1}{2}\lambda^0 v^0(\mathcal{O})\, \mathrm{mes}_3(\omega^+) \\ &= -\lambda^0 v^0(\mathcal{O})\, \mathrm{mes}_3(\omega).\end{aligned} \tag{2.23}$$

According to (2.7), we also have $\int_{\partial\omega\cap\partial\Xi} G_3^2(\xi)\, ds_\xi = 0$. Now we process the integral $\int_{\Xi_R} F^2(\xi)\, d\xi$. Owing to (2.17), we first compute

$$\int_{\Xi_R} \partial_{\xi_1}(\xi)\, ds_\xi = \int_{\partial\omega\cap\partial\Xi} N_1(\xi) w^1(\xi) ds_\xi + \int_{\{\xi\in\mathbb{R}^3_-:\rho=R\}} \rho^{-1}\xi_1 w^1(\xi)\, ds_\xi. \tag{2.24}$$

The last integral on the right-hand side of (2.24) is of order R^{-1}. Indeed, the main terms of $w^1(\xi)$ are of order R^{-2}, however, according to (2.11), they are odd functions in either the variable ξ_2, or ξ_3. Therefore, the terms $O(1)$ vanish in the last integral on the right-hand side of (2.24) due to the full symmetry of the semi-sphere $\{\xi \in \mathbb{R}^3_- : \rho = R\}$. The first integral on the right-hand side of (2.24) is equal to

$$\begin{aligned}\int_{\partial\omega\cap\partial\Xi} N_1(\xi) w^1(\xi)\, ds_\xi &= \int_{\partial\omega\cap\partial\Xi} w^1(\xi)\partial_N\xi_1\, ds_\xi \\ &= \int_{\partial\omega\cap\partial\Xi} \xi_1\partial_N w^1(\xi)\, ds_\xi \\ &\quad + \int_{\{\xi\in\mathbb{R}^3_-:\,\rho=R\}} \big(\xi\partial_\rho w^1(\xi) - w^1(\xi)\partial_\rho\xi_1\big)\, ds_\xi.\end{aligned}$$

The integral $\int_{\{\xi\in\mathbb{R}^3_-:\rho=R\}} \big(\xi_1\partial_\rho w^1(\xi) - w^1(\xi)\partial_\rho\xi_1\big)\, ds_\xi$ is also of order R^{-1} by the same argument as above, since $\partial_\rho w^1(\xi)$ has the same symmetry in ξ_2 and ξ_3 as $w^1(\xi)$. We also have $\int_{\partial\omega\cap\partial\Xi} \xi_1\partial_N w^1(\xi)\, ds_\xi = 0$ due to the boundary conditions (2.10) and the second equality in (2.7).

We compute now

$$\begin{aligned}-2\varkappa_s(\mathcal{O})\int_{\Xi_R} \xi_1\partial_{\xi_2}^2 w^1(\xi)\, d\xi &= -2\varkappa_s(\mathcal{O})\int_{\partial\omega\cap\partial\Xi} \xi_1 N_2(\xi)\partial_{\xi_2} w^1(\xi)\, ds_\xi \\ &\quad - 2\varkappa_s(\mathcal{O})\int_{\{\xi\in\mathbb{R}^3_-:\rho=R\}} \rho^{-1}\xi_1\xi_2\partial_{\xi_2} w^1(\xi)\, ds_\xi.\end{aligned} \tag{2.25}$$

The latter integral is of order R^{-1}, hence the leading asymptotic term of order ρ^{-2} coming from the expression $\xi_2\partial_{\xi_2}$ is still odd with respect to the variable ξ_2 or ξ_3, therefore it is annihilated by integration. The first integrand on the right-hand side in (2.25) is the opposite of the first term in $G_5^2(\xi)$, and, hence, they cancel each other. Finally, recalling that $G_2^2(\xi) + G_4^2(\xi) = 0$, collecting the aforementionned integrals and taking (2.23) into account, we pass to the limit $R \to \infty$ and get the equality $a = -\frac{1}{2\pi}\lambda^0 v^0(\mathcal{O})\text{mes}_3(\omega)$. Note that the coefficient a does not depend on the curvatures $\varkappa_s(\mathcal{O})$ or $\varkappa_\nu(\mathcal{O})$, although the original expressions (2.18) and (2.19) do.

2.3. The correction term of regular type. We start by writing the boundary layers in the following condensed form

$$w^q(\xi) = t^q(\xi) + O(\rho^{q-4}), \quad \text{as } \rho \to \infty,\ q = 1, 2, \tag{2.26}$$

where t^1 and t^2 denote the sum of functions of the homogeneity orders -2 and -1 in (2.8), (2.11) and (2.20), respectively. In other words, $t^1(\xi) = h^2 t^1(n, s, \nu)$ and $t^2(\xi) = ht^2(n, s, \nu)$. Outside a small neighbourhood of the point $\mathcal{O}$ we have,

$$hw^1(\xi)+h^2w^2(\xi) = h^3(t^1(n, s, \nu)+t^2(n, s, \nu))+O(h^4) = h^3T(x)+O(h^4). \tag{2.27}$$

In view of the multiplier h^3, the expression for T should be present in the following problem for the function v^3 of regular type in the asymptotic ansatz (1.12)

$$-\Delta_x v^3(x) = \lambda^0 v^3(x) + \lambda' v^0(x) + f^3(x), \quad x \in \Omega \tag{2.28}$$

$$\partial_n v^3(x) = g^3(x), \quad x \in \Gamma. \tag{2.29}$$

The first two terms on the right-hand side of (2.28) are obtained if we replace the eigenvalues and eigenfunctions in (1.3) by the ansätze (1.11)–(1.12) and collect terms of order h^3 written in the slow variables x. The right-hand side g^3 of the boundary condition (2.29) is the discrepancy which results from the multiplication of the boundary layers with the cut-off function χ. If we assume that in the vicinity of the boundary the cut-off function χ depends only on the tangential variables s and ν, and it is independent of the normal variable n, then $g^3 = 0$, since the boundary conditions (2.6), (2.16) on $\partial\Xi\setminus\partial\omega$ are homogeneous. It is clear that such a requirement can be readily satisfied, and thus we further assume $g^3 = 0$. The correction f^3 in (2.28) is given by

$$f^3(x) = \lambda^0\chi(x)T(x) + \Delta_x(\chi(x)T(x)). \tag{2.30}$$

We will verify that the function f^3, smooth outside a neighbourhood of the origin $\mathcal{O}$, is of the growth $O(|x|^{-2})$ as $x \to \mathcal{O}$ which means that f^3 belongs to $H^{-1}(\Omega)$, since a function of order $|x|^{-\frac{5}{2}+\delta}$ is in $H^{-1}(\Omega)$ for all $\delta > 0$. This ensures that f^3 is admissible for the right-hand side of equation (2.28). The observation is obvious for the first term of f^3, since $t^1(n,s) = O(|x|^{-2})$ and $t^2(n,s) = O(|x|^{-1})$. Let us consider the second term $\Delta_x(\chi(x)T(x))$. According to (2.1), the representation of the Laplacian in curvilinear coordinates can be rewritten in the form

$$\Delta_x = L^0(\partial_n, \partial_s, \partial_\nu) + L^1(n, \partial_n, \partial_s, \partial_\nu) + L^2(n, s, \nu, \partial_n, \partial_s, \partial_\nu), \qquad (2.31)$$

with the ingredients

$$L^0(\partial_n, \partial_s, \partial_\nu) = (\partial_n^2 + \partial_s^2 + \partial_\nu^2) \qquad (2.32)$$

$$L^1(n, \partial_n, \partial_s, \partial_\nu) = \varkappa_s(\mathcal{O})(\partial_n - 2n\partial_s^2) + \varkappa_\nu(\mathcal{O})(\partial_n - 2n\partial_\nu^2) \qquad (2.33)$$

$$L^2(n, s, \nu, \partial_n, \partial_s, \partial_\nu) = a_{11}\partial_n^2 + a_{22}\partial_s^2 + a_{33}\partial_\nu^2 + a_1\partial_n + a_2\partial_s + a_3\partial_\nu, \qquad (2.34)$$

while the functions a_{jj} and a_j are smooth in a neighbourhood of $\mathcal{O}$, in variable n and s, and in addition they have the property

$$a_{jj}(0,0) = 0, \quad \partial_k a_{jj}(0,0) = 0, \quad a_j(0,0) = 0, \quad j = 1,2,3. \qquad (2.35)$$

Therefore, we can write

$$\Delta_x T = L^0 t^1 + (L^0 t^2 + L^1 t^1) + L^1 t^2 + L^2(t^1 + t^2). \qquad (2.36)$$

We readily check that $L^0 t^1 = 0$ and $L^0 t^2 + L^1 t^1 = 0$ due to the definition of w^1 and w^2, see (2.5) and (2.15). Function t^2 is of order $|x|^{-1}$ thus $L^1 t^2$ is of order $|x|^{-2}$, and $L^2(t^1 + t^2)$ is also of order $|x|^{-2}$ due to (2.35). Thus, we have concluded that $g^3 = 0$ and $f^3 \in H^{-1}(\Omega)$.

According to the Fredholm alternative, and under the assumption that λ^0 is a simple eigenvalue, the problem (2.28)–(2.29) with the described right-hand sides admits a solution v^3 in the Sobolev space $H^1(\Omega)$ if and only if the following orthogonality condition is satisfied by the right-hand side of (2.28)–(2.29) :

$$\lambda'(v^0, v^0)_\Omega + (f^3, v^0)_\Omega + (g^3, v^0)_{\partial\Omega} = 0. \qquad (2.37)$$

Owing to the normalization condition and since $g^3 = 0$, relation (2.37) becomes $\lambda' = -(f^3, v^0)_\Omega$. Integral of the product $f^3 v^0$ is convergent, which means that

$$(f^3, v^0)_\Omega = \lim_{\delta \to +0} \int_{\Omega_\delta} (\lambda^0 \chi T + \Delta_x(\chi T)) v^0 \, dx, \qquad (2.38)$$

where $\Omega_\delta = \Omega \setminus \{x : n^2 + s^2 + \nu^2 \le \delta^2\}$. The surface patch $S_\delta = \partial\Omega_\delta \setminus \partial\Omega$ turns out to be a semi-sphere in the curvilinear coordinate system. We imitate the spherical coordinate system in the curvilinear coordinates by setting $n = r\sin\theta\cos\varphi$, $s = r\sin\theta\sin\varphi$ and $\nu = r\cos\theta$ while denoting (r, φ, θ) the spherical coordinate system, with $r = h\rho \geq 0$, $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\theta \in (0, \pi)$. Using Green's formula for the smooth functions T and v^0 in the domain Ω_δ yields

$$\int_{\Omega_\delta} f^3 v^0 \, dx = \int_{S_\delta} \left(v^0 \partial_N T - T \partial_N v^0 \right) ds_x. \tag{2.39}$$

Let us observe that $ds_x = d(n,s)^{\frac{1}{2}} J(n,s)\, r^2 \sin\theta \, d\theta d\varphi$ on S_δ, and according to formulas (2.3) the derivative ∂_{N_S} along the normal to the patch S_δ satisfies the relation $\partial_{N_S} T = d^{\frac{1}{2}} (N_n \partial_n T + N_s \partial_s T + N_\nu \partial_\nu T)$, where

$$N_n = J \sin\theta \cos\varphi, \;\; N_s = (1 + n\varkappa_\nu) \sin\theta \sin\varphi, \;\; N_\nu = \cos\theta (1 + n\varkappa_s) \tag{2.40}$$
$$d = J^2 \sin^2\theta \cos^2\varphi + (1 + n\varkappa_\nu)^2 \sin^2\theta \sin^2\varphi + (1 + n\varkappa_s) \cos^2\theta.$$

We can split the integral (2.39) into several pieces

$$\int_{\Omega_\delta} f^3 v^0 \, dx = I_1 + I_2 + I_3 + I_4 + o(1) \tag{2.41}$$

with

$$I_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi v^0(\mathcal{O}) \partial_{N_S} T \delta^2 \sin\theta \, d\theta d\phi$$
$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi v^0(\mathcal{O}) \partial_{N_S} T n \left(\varkappa_s(\mathcal{O}) (1 + \sin^2\theta \cos^2\varphi + \cos^2\theta) \right.$$
$$\left. + \varkappa_\nu(\mathcal{O})(1 + \sin^2\theta) \right) \delta^2 \sin\theta \, d\theta d\phi$$
$$I_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi \left(\partial_s v^0(\mathcal{O}) s \partial_{N_S} T + \partial_\nu v^0(\mathcal{O}) \nu \partial_{N_S} T \right) \delta^2 \sin\theta \, d\theta d\phi$$
$$I_4 = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi T \partial_{N_S} v^0 d^{\frac{1}{2}} J \, \delta^2 \sin\theta \, d\theta d\phi \; .$$

In view of formulas (2.40), we get the following expansion for $\partial_{N_S} T$:

$$\partial_{N_S} T = \partial_r T + n \big[\partial_n T \left(\varkappa_\nu(\mathcal{O}) \cos^2\theta + \varkappa_s(\mathcal{O}) (\sin^2\theta \sin^2\varphi) \right) \sin\theta \cos\varphi$$
$$+ \partial_s T \left(\varkappa_\nu(\mathcal{O}) \cos^2\theta + \varkappa_s(\mathcal{O}) (\sin^2\theta \sin^2\varphi - 1) \right) \sin\theta \sin\varphi$$
$$+ \partial_\nu T \left(-\varkappa_\nu(\mathcal{O}) \cos^2\theta + \varkappa_s(\mathcal{O}) (\sin^2\theta \sin^2\varphi) \right) \cos\theta \big] + o(\delta).$$

The asymptotic expansions of integrands in I_1 and I_2, already derived, lead to

$$I_1 + I_2 = v^0(\mathcal{O}) \delta^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi \partial_r T \sin\theta \, d\theta d\phi$$
$$+ v^0(\mathcal{O}) \delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi n \big[n \partial_n T \left(\varkappa_\nu(\mathcal{O}) \cos^2\theta + \varkappa_s(\mathcal{O}) (\sin^2\theta \sin^2\varphi) \right)$$
$$+ s \partial_s T \left(\varkappa_\nu(\mathcal{O}) \cos^2\theta + \varkappa_s(\mathcal{O}) (\sin^2\theta \sin^2\varphi - 1) \right)$$
$$+ \nu \partial_\nu T \left(-\varkappa_\nu(\mathcal{O}) \cos^2\theta + \varkappa_s(\mathcal{O}) (\sin^2\theta \sin^2\varphi) \right) \big] \, d\theta d\phi + o(1).$$

After simplification of the expression in brackets we get

$$
\begin{aligned}
I_1 + I_2 &= v^0(\mathcal{O})\delta^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \partial_r T \sin\theta \, d\theta d\phi \\
&\quad + v^0(\mathcal{O})\varkappa_s(\mathcal{O})\delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} n(2n\partial_n T + s\partial_s T + 2\nu\partial_\nu T)\, d\theta d\phi \\
&\quad + v^0(\mathcal{O})\varkappa_\nu(\mathcal{O})\delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} n(2n\partial_n T + 2s\partial_s T + \nu\partial_\nu T)\, d\theta d\phi + o(1) \\
&= v^0(\mathcal{O})\delta^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \partial_r T \sin\theta \, d\theta d\phi + o(1).
\end{aligned}
$$

In the calculation above, we have taken into account the fact that the expressions $2n\partial_n T + s\partial_s T + 2\nu\partial_\nu T$ and $2n\partial_n T + 2s\partial_s T + \nu\partial_\nu T$ are odd in either s, or ν, therefore, the corresponding integrals over the patch S_δ vanish.

For integrals I_3 and I_4, we have

$$
\begin{aligned}
I_3 + I_4 &= \partial_s v^0(\mathcal{O})\delta^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} (s\partial_{N_S} T - T\partial_{N_S} s) \sin\theta \, d\theta d\phi \\
&\quad + \partial_\nu v^0(\mathcal{O})\delta^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} (\nu\partial_{N_S} T - T\partial_{N_S} \nu) \sin\theta \, d\theta d\phi + o(1) \\
&= \partial_s v^0(\mathcal{O})\delta^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} (s\partial_r t^1 - t^1\partial_r s)|_{r=\delta} \sin\theta \, d\theta d\phi \\
&\quad + \partial_\nu v^0(\mathcal{O})\delta^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} (\nu\partial_r t^1 - t^1\partial_r \nu)|_{r=\delta} \sin\theta \, d\theta d\phi + o(1).
\end{aligned}
$$

Gathering all the integrals in (2.41), we obtain

$$
\int_{\Omega_\delta} f^3 v^0 \, dx = v^0(\mathcal{O})\delta^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \partial_r T|_{r=\delta} \sin\theta \, d\theta d\phi \tag{2.42}
$$

$$
+ \partial_s v^0(\mathcal{O})\delta^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} (s\partial_r t^1 - t^1\partial_r s)|_{r=\delta} \sin\theta \, d\theta d\phi \tag{2.43}
$$

$$
+ \partial_\nu v^0(\mathcal{O})\delta^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} (\nu\partial_r t^1 - t^1\partial_r \nu)|_{r=\delta} \sin\theta \, d\theta d\phi + o(1). \tag{2.44}
$$

The first integral in (2.42) is equal to

$$
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \partial_r T|_{r=\delta} \sin\theta \, d\theta d\phi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \partial_r t^1|_{r=\delta} \sin\theta \, d\theta d\phi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \partial_r t^2|_{r=\delta} \sin\theta \, d\theta d\phi,
$$

and according to (2.7) we get $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi \partial_r t^1|_{r=\delta} \sin\theta \, d\theta d\phi = 0$. In view of (2.22) we also obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi \partial_r t^2|_{r=\delta} \sin\theta \, d\theta d\phi = -2\pi \frac{a}{\delta^2}.$$

The two integrals in (2.43) and (2.44) are calculated with the help of (2.12) and (2.13), and we obtain in a similar way that

$$\begin{aligned} &\partial_s v^0(\mathcal{O})\delta^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi (s\partial_r t^1 - t^1 \partial_r s)|_{r=\delta} \sin\theta \, d\theta d\phi \\ &+ \partial_\nu v^0(\mathcal{O})\delta^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi (\nu\partial_r t^1 - t^1 \partial_r \nu)|_{r=\delta} \sin\theta \, d\theta d\phi \\ &= \nabla_{s,\nu} v^0(\mathcal{O}) \mathbf{m}(\Xi) \nabla_{s,\nu} v^0(\mathcal{O}), \end{aligned}$$

where $\mathbf{m}(\Xi)$ is the virtual mass matrix of the cavity ω in the half-space which depends on the shape of Ξ and is given by

$$\mathbf{m}(\Xi) = \begin{pmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{pmatrix}. \tag{2.45}$$

Furthermore, $\nabla_{s,\nu} v^0(\mathcal{O}) = (\partial_s v^0(\mathcal{O}), \partial_\nu v^0(\mathcal{O}))^T$. The previous results show that $(f^3, v^0)_\Omega = \nabla v^0(\mathcal{O}) \mathbf{m}(\Xi) \nabla v^0(\mathcal{O}) - 2\pi a$, and finally the perturbation term in the asymptotic ansatz (1.11) of the simple eigenvalue λ_m^0 takes the form

$$\lambda'_m = (\nabla_{s,\nu} v_m^0(\mathcal{O}))^T \mathbf{m}(\Xi) \nabla_{s,\nu} v_m^0(\mathcal{O}) + \lambda_m^0 |v_m^0(\mathcal{O})|^2 \operatorname{mes}_3(\omega). \tag{2.46}$$

Remark 2.5. The max-min principle (see, e.g., [1]) reads:

$$\lambda_j^h = \max_{\mathcal{E}_j^h \subset H^1(\Omega(h))} \inf_{u^h \in \mathcal{E}_j^h \setminus \{0\}} \frac{\|\nabla_x u^h; L^2(\Omega(h))\|^2}{\|u^h; L^2(\Omega(h))\|^2} \tag{2.47}$$

$$\lambda_j^0 = \max_{\mathcal{E}_j^0 \subset H^1(\Omega)} \inf_{v \in \mathcal{E}_j^0 \setminus \{0\}} \frac{\|\nabla_x v; L^2(\Omega)\|^2}{\|v; L^2(\Omega)\|^2}, \tag{2.48}$$

where $\mathcal{E}_j^h$ and $\mathcal{E}_j^0$ stand for any subspaces of codimension $j-1$, i.e.,

$$\dim(H^1(\Omega(h)) \ominus \mathcal{E}_j^h) = j-1, \;\; \dim(H^1(\Omega) \ominus \mathcal{E}_j^0) = j-1.$$

For the cavity ω_h of a general shape, there is no obvious relation between $H^1(\Omega(h))$ and $H^1(\Omega)$ so that (2.48) and (2.47) do not allow to directly establish a link between λ_j^h and λ_j^0. Notice that in case $\operatorname{mes}_3 \omega > 0$, (2.46) can be made both, negative or positive. Indeed, assume that the eigenfunction v_m changes sign on the boundary Γ and put the coordinate origin $\mathcal{O}$ at a point where v_m vanishes. Then the last term in (2.46) becomes null and $\lambda'_m \le 0$ due to the above-mentionned properties of the matrix $\mathbf{m}(\Xi)$. On the contrary, if the point $\mathcal{O}$ constitutes an extremum of the function $\Gamma \ni x \mapsto v_m(x)$, then $\nabla_{s,\nu} v_m(\mathcal{O}) = 0$ and $\lambda'_m > 0$ provided $v_m(\mathcal{O}) \neq 0$ and $\mathrm{mes}_3\, \omega > 0$.

In the limiting case of a crack $\overline{\omega}$, i.e., a domain flattens into a two-dimensional surface (see Figure 7), one easily observes that $H^1(\Omega) \subset H^1(\Omega(h))$ since a function in $H^1(\Omega(h))$ can have a nontrivial jump over $\overline{\omega_h}$ but $v \in H^1(\Omega)$ cannot. As a consequence of (2.48), (2.47), we conclude the general relationship $\lambda_j^h \leq \lambda_j^0$. This formula is in agreement with (2.46) for the correction term in (1.11) because $\mathrm{mes}_3\, \omega = 0$ for a crack and, therefore,

$$\lambda'_m = (\nabla_{s,\nu} v_m^0(\mathcal{O}))^T \mathbf{m}(\Xi) \nabla_{s,\nu} v_m^0(\mathcal{O}) \leq 0$$

since the matrix $\mathbf{m}(\Xi)$ in the case of a crack is negative or negative definite (see Example 2.3).

2.4. Multiple eigenvalues. Assume now, that λ_m^0 is an eigenvalue of the multiplicity $\varkappa_m > 1$, i.e.,

$$\lambda_{m-1}^0 < \lambda_m^0 = \cdots = \lambda_{m+\varkappa_m-1}^0 < \lambda_{m+\varkappa_m}^0. \tag{2.49}$$

In such a case ansätze (1.11) and (1.12) are valid for $p = m, \dots, m+\varkappa_m-1$, however, the principal terms in the expansions of the eigenfunctions $u_m^h, \dots, u_{m+\varkappa_m-1}^h$ of problem (1.3)–(1.4) are predicted in the form of linear combinations

$$v^{p0} = a_1^p v_m^0 + \cdots + a_{\varkappa_m}^p v_{m+\varkappa_m-1}^0 \tag{2.50}$$

of eigenfunctions of problem (1.8)–(1.9) corresponding to the eigenvalue λ_m^0, and subject to the orthogonality and normalization conditions (1.10). The coefficients of the columns $a^p = (a_1^p, \dots, a_{\varkappa_m}^p)$ in (2.50) are to be determined. If the columns $a^m, \dots, a^{m+\varkappa_m-1}$ are unit vectors and

$$a^p \cdot a^q = \delta_{p,q}, \quad p, q = m, \dots, m + \varkappa_m - 1, \tag{2.51}$$

then the linear combinations (2.50) with $p = m, \dots, m + \varkappa_m - 1$, are simply a new orthonormal basis in the eigenspace of the eigenvalue λ_m.

The construction of boundary layers is performed in the same way as in the previous section. When solving problem (2.28)–(2.29) for the regular term v^{p3}, there appear $\varkappa_m$ compatibility conditions

$$\lambda^{p\prime}(v^{p0}, v_{m+k}^0)_\Omega + (f^{p3}, v_{m+k}^0)_\Omega = 0, \quad k = 0, \dots, \varkappa_m - 1, \tag{2.52}$$

which can be written in the form of the linear system of $\varkappa_m$ algebraic equations

$$\mathbf{M} a^p = \lambda^{p\prime} a^p \tag{2.53}$$

with the matrix $\mathbf{M} = (\mathbf{M}_{ik})_{j,k=0}^{\varkappa_m-1}$ of the size $\varkappa_m \times \varkappa_m$,

$$\mathbf{M}_{jk} = \big(\nabla_{s,\nu} v^0_{m+k}(\mathcal{O})\big)^T \mathbf{m}(\Xi) \nabla_{s,\nu} v^0_{m+j}(\mathcal{O}) + \lambda^0_m v^0_{m+k}(\mathcal{O}) v^0_{m+j}(\mathcal{O}) \operatorname{mes}_2(\omega). \quad (2.54)$$

Formula (2.54) is derived in exactly the same way as it is for formula (2.46). The matrix $\mathbf{M}$ is symmetric, and its real eigenvalues $\lambda^{m\prime}, \dots, \lambda^{m+\varkappa_m-1\prime}$ correspond to eigenvectors $a^m, \dots, a^{m+\varkappa_m-1}$, which satisfy conditions (2.51). Actually, just these attributes of the matrix $\mathbf{M}$ with elements (2.54) are included in ansätze (1.11) and (1.12), (2.50) for eigenvalues λ^h_p and eigenfunctions u^h_p of problem (1.3)–(1.4) for $p = m, \dots, m + \varkappa_m - 1$ in the case (2.49).

3. Justification of asymptotics

3.1. The weighted Poincaré inequality. Let $H^1(\Omega(h))_\perp$ denote a subspace of the Sobolev space $H^1(\Omega(h))$ which contains functions of zero mean over the set $\Omega(h)$.

Lemma 3.1. *The following inequality is valid*

$$\|u; L_2(\Omega(h))\| \le c\|r_h^{-1}u; L_2(\Omega(h))\| \le C\|\nabla_x u; L_2(\Omega(h))\|, \qquad (3.1)$$

where $r_h = r + h$ and $r(x) = \operatorname{dist}(x, \mathcal{O}) = |x|$, and the constants c and C are independent of the parameter $h \in (0, h_0]$ and function $u \in H^1(\Omega(h))_\perp$.

Proof. We use the representation $u(x) = u_*(x) + b_*$, where the constant b_* is chosen such that

$$\int_{\Omega_*} u_*(x)\,dx = 0, \quad b_* = -(\operatorname{mes}(\Omega_*))^{-1} \int_{\Omega_*} u(x)\,dx. \qquad (3.2)$$

In (3.2), the domain $\Omega_* \subset \Omega$ satisfies $\Omega_* \neq \varnothing$ and $\Omega_* \cap \omega_h = \varnothing$ for $h \in (0, h_0]$. Let us construct an extension $\hat{u}_*$ of u_* in the class H^1, from the set $\Omega_{Rh} := \Omega \setminus \mathbb{B}_{Rh}$ onto Ω, in such a way that the following estimate is valid

$$\|\nabla_x \hat{u}_*; L_2(\Omega)\| \le c\|\nabla_x u_*; L_2(\Omega_{Rh})\| = c\|\nabla_x u; L_2(\Omega_{Rh})\| \le c\|\nabla_x u_*; L_2(\Omega(h))\|. \quad (3.3)$$

Here $\mathbb{B}_{Rh}$ is the ball of radius Rh and center $\mathcal{O}$, with R a constant chosen such that $w_h \subset \mathbb{B}_{Rh}$.

The reason for such procedure is that a direct extension form $\Omega(h)$ onto Ω may not exist in the class H^1, for example in the case of a crack (cf. Remark 2.5). Stretching coordinates $x \mapsto \eta = h^{-1}x$ transforms the set $\Sigma_{Rh} = \{x \in \Omega : Rh > r > R\frac{h}{2}\}$ into the three-dimensional half-annulus $\Upsilon(h)$ with fixed radii and gently sloped ends, due to the smoothness of the boundary $\partial\Omega$. In stretched coordinates, we write $U_*(\eta) = u_*(x)$. Then, we proceed to the decomposition

$$U_*(\eta) = U_\perp(\eta) + b_\perp \qquad (3.4)$$

where the constant $b_\perp$ is chosen such that

$$\int_{\Upsilon(h)} U_\perp(\eta)\,d\eta = 0, \quad b_\perp = (\mathrm{mes}(\Upsilon(h))^{-1} \int_{\Upsilon(h)} U_*(\eta)\,d\eta. \tag{3.5}$$

The extension ought to be made in the stretched variables. Due to the orthogonality condition in (3.5), the Poincaré inequality holds true for $U_\perp$ in $\Upsilon(h)$

$$\|U_\perp; L_2(\Upsilon(h))\| \le c\|\nabla_\eta U_\perp; L_2(\Upsilon(h))\| = c\|\nabla_\eta U_*; L_2(\Upsilon(h))\|$$

where the constant c does not depend on h because $\Upsilon(h)$ has gently sloped ends. Therefore, there exists an extension $\hat{U}_\perp$ of $U_\perp$ from $\Upsilon(h)$ onto $\hat{\Upsilon}(h) = \{\eta : x \in \Omega, r < Rh\}$, such that

$$\|\hat{U}_\perp; H^1(\hat{\Upsilon}(h))\| \le c\|U_\perp; H^1(\Upsilon(h))\| \le c\|\nabla_\eta U_\perp; L_2(\Upsilon(h))\|,$$

where c is independent of $h \in (0, h_0]$ and $U_\perp$.
Choosing $\Omega_* = \Omega \setminus \mathbb{B}_{Rh}$, the required extension $\hat{u}_*$ is thus defined as follows:

$$\hat{u}_*(x) = \begin{cases} u_*(x), & x \in \Omega \setminus \mathbb{B}_{Rh} \\ \hat{U}_\perp(\eta) + b_\perp, & x \in \Omega \cap \mathbb{B}_{Rh}. \end{cases} \tag{3.6}$$

Now we give estimates for the extension $\hat{u}_*$

$$\|\nabla_x \hat{u}_*; L_2(\Omega)\| = \|\nabla_x u_*; L_2(\Omega \setminus \mathbb{B}_{Rh})\| + \|\nabla_x \hat{U}_\perp; L_2(\Omega \cap \mathbb{B}_{Rh})\|,$$

and further, using the previous estimates, we obtain

$$\begin{aligned} \|\nabla_x \hat{U}_\perp; L_2(\Omega \cap \mathbb{B}_{Rh})\| &= h^{\frac{1}{2}}\|\nabla_\eta \hat{U}_\perp; L_2(\hat{\Upsilon}(h))\| \\ &\le h^{\frac{1}{2}}\|\hat{U}_\perp; H^1(\hat{\Upsilon}(h))\| \\ &\le ch^{\frac{1}{2}}\|\nabla_\eta U_\perp; L_2(\Upsilon(h))\| \\ &\le ch^{\frac{1}{2}}\|\nabla_\eta U_*; L_2(\Upsilon(h))\| \\ &= c\|\nabla_x u_*; L_2(\Sigma_{Rh})\|. \end{aligned}$$

Gathering the two previous estimates for $\nabla_x \hat{u}_*$ we obtain due to the definition of Σ_{Rh} that

$$\|\nabla_x \hat{u}_*; L_2(\Omega)\| \le c\|\nabla_x u_*; L_2(\Omega \setminus \mathbb{B}_{Rh/2})\| \le c\|\nabla_x u; L_2(\Omega(h))\|. \tag{3.7}$$

The last inequality is true if $\Omega \setminus \mathbb{B}_{Rh/2} \subset \Omega(h)$, which is certainly verified for an appropriate choice of R and h small enough. The constant c in the previous inequality is independent of h.

We show, using the Poincaré inequality, that

$$\|\hat{u}_*; L_2(\Omega)\| \le c\|\nabla_x \hat{u}_*; L_2(\Omega)\| \le c\|\nabla_x u; L_2(\Omega(h))\|. \tag{3.8}$$

Precisely, we use the following auxiliary assertion,

Lemma 3.2. *Let* $\Omega_1 \subset \Omega_2$ *be two smooth domains, with* $\mathrm{mes}_3(\Omega_1) \neq 0$, *then for any* $w \in H^1(\Omega_2)$ *we have*

$$\|w; L_2(\Omega_2)\| \leq c\left(\|\nabla_x w; L_2(\Omega_2)\| + \|w; L_2(\Omega_1)\|\right), \tag{3.9}$$

where the constant c *depends on* Ω_1 *and* Ω_2.

Proof. Assume that (3.9) is not true and take a sequence w_n such that $\|w_n; L_2(\Omega_2)\| = 1$ and the right-hand side of (3.9) tends to zero. From the boundedness of w and $\nabla_x w$ in $L_2(\Omega_2)$ we get the boundedness of w in $H^1(\Omega_2)$. Thus, up to a subsequence, w_n converges to some $\bar{w} \in H^1(\Omega_2)$ and since $\|\nabla_x w_2; L_2(\Omega_2)\| \to 0$ we get $\nabla_x \bar{w} = 0$ and $\bar{w}$ is constant. Since $\|w_2; L_2(\Omega_1)\| \to 0$, this constant is zero and thus $\bar{w} \equiv 0$. This implies $\|w_n; L_2(\Omega_2)\| \to 0$, in contradiction with $\|w_n; L_2(\Omega_2)\| = 1$. Thus, (3.9) holds true. □

Applying Lemma 3.2 to our situation, we get

$$\begin{aligned}\|\hat{u}_*; L_2(\Omega)\| &\leq c(\|\nabla_x \hat{u}_*; L_2(\Omega)\| + \|\hat{u}_*; L_2(\Omega_*)\|)\\ &\leq c(\|\nabla_x \hat{u}_*; L_2(\Omega)\| + \|\nabla_x \hat{u}_*; L_2(\Omega_*)\|)\\ &\leq c\|\nabla_x \hat{u}_*; L_2(\Omega)\|,\end{aligned}$$

where we have also used the Poincaré inequality in Ω_*, since $\hat{u}_*$ coincides with u_* and has zero mean value on this set. Then, with (3.7) and the previous inequality, we obtain the desired estimate (3.8).

Next we invoke the one-dimensional Hardy inequality

$$\int_0^1 |z(r)|^2\, dr \leq 4 \int_0^1 r^2 |\partial_r z(r)|^2\, dr, \quad z \in \mathcal{C}_c^1([0,1)), \tag{3.10}$$

which, after the integration in the angular variables θ and ϕ, leads to

$$\|r^{-1}\hat{u}_*; L_2(\Omega)\| \leq \|\nabla_x \hat{u}_*; L_2(\Omega)\| \leq c\|\nabla_x u; L_2(\Omega(h))\|. \tag{3.11}$$

For the constant $b_\perp$ in decomposition (3.4) we now obtain

$$|b_\perp| = \left|(\mathrm{mes}(\Upsilon(h)))^{-1}\int_{\Upsilon(h)} U_*(\eta)\, d\eta\right| \leq c\|U_*; L_2(\Upsilon(h))\| \leq ch^{-\frac{1}{2}}\|r^{-1}\hat{u}_*; L_2(\Sigma_{Rh})\|$$

since $\|U_*; L_2(\Upsilon(h))\| = \|\hat{U}_*; L_2(\Upsilon(h))\| = h^{-\frac{3}{2}}\|\hat{u}_*; L_2(\Sigma_{Rh})\|$.

Further, the image $\Sigma_\omega(h)$ of the set $\Omega(h) \cap \mathbb{B}_{Rh}$ under stretching of coordinates, possesses a gently sloped boundary. Hence, applying Lemma 3.2 we obtain

$$\|U_*; L_2(\Sigma_\omega(h))\| \leq c(\|\nabla_\eta U_*; L_2(\Sigma_\omega(h))\| + \|U_*; L_2(\Upsilon(h))\|).$$

Recall that $r_h = r + h > h$. In this way we have

$$
\begin{aligned}
\|r_h^{-1}u_*; L_2(\Omega(h) \cap \mathbb{B}_{Rh})\| &\le h^{-1}\|u_*; L_2(\Omega(h) \cap \mathbb{B}_{Rh})\| \\
&= h^{\frac{1}{2}}\|U_*; L_2(\Sigma_\omega(h))\| \\
&\le ch^{\frac{1}{2}}(\|\nabla_\eta U_*; L_2(\Sigma_\omega(h))\| + \|U_*; L_2(\Upsilon(h))\|) \\
&\le ch^{\frac{1}{2}}(\|\nabla_\eta U_*; L_2(\Sigma_\omega(h))\| + \|U_\perp; L_2(\Upsilon(h))\| + |b_\perp|).
\end{aligned}
$$

Using the Poincaré inequality for $U_\perp$ in $\Upsilon(h)$ and the estimate for $b_\perp$, we get from the previous inequality

$$
\begin{aligned}
\|r_h^{-1}u_*; L_2(\Omega(h) \cap \mathbb{B}_{Rh})\| \le c\,(h\|\nabla_x u_*; L_2(\Omega(h) \cap \mathbb{B}_{Rh})\| \\
+ \|r^{-1}\hat{u}_*; L_2(\Sigma_{Rh})\|\big)\,.
\end{aligned}
\tag{3.12}
$$

We can now, applying (3.11) and (3.12), write

$$
\begin{aligned}
\|r_h^{-1}u_*; L_2(\Omega(h))\| &= \|r_h^{-1}u_*; L_2(\Omega \setminus \mathbb{B}_{Rh})\| + \|r_h^{-1}u_*; L_2(\Omega(h) \cap \mathbb{B}_{Rh})\| \\
&\le c\|r_h^{-1}\hat{u}_*; L_2(\Omega)\| \\
&\quad + c\left(h\|\nabla_x u_*; L_2(\Omega(h) \cap \mathbb{B}_{Rh})\| + \|r^{-1}\hat{u}_*; L_2(\Sigma_{Rh})\|\right) \\
&\le c\|\nabla_x u_*; L_2(\Omega(h))\|.
\end{aligned}
$$

We give an estimate for the constant $b_* = \left(\mathrm{mes}(\Omega(h))\right)^{-1}\int_{\Omega(h)}(u(x) - u_*(x))\,dx = \int_{\Omega(h)} u_*(x)\,dx$ as follows:

$$
|b_*| \le c\|u_*; L_2(\Omega(h))\| \le c\|r^{-1}u_*; L_2(\Omega(h))\| \le c\|\nabla_x u; L_2(\Omega(h))\|.
$$

Finally we have

$$
\begin{aligned}
\|r_h^{-1}u; L_2(\Omega(h))\| &\le c(\|r_h^{-1}u_*; L_2(\Omega(h))\| + \|r_h^{-1}b_*; L_2(\Omega(h))\|) \\
&\le c\|\nabla_x u; L_2(\Omega(h))\|,
\end{aligned}
$$

which proves Lemma 3.1. □

In the sequel we write $|||u; \Omega(h)||| = \|r_h^{-1}u; L_2(\Omega(h))\|$. In the proof of Lemma 3.1, an extension $\hat{u} := \hat{u}_* + b_*$ of the function $u \in H^1(\Omega(h))_\perp$ onto the domain Ω is constructed such that

$$
|||u; \Omega(h)\,||| + \|\nabla_x \hat{u}; L_2(\Omega)\| \le c\|\nabla_x u; L_2(\Omega(h))\|. \tag{3.13}
$$

Assume that $m \ge 1$ and $\hat{u}_m^h$ is the extension described above of the eigenfunction u_m^h; then, in view of (1.6) and the integral identity [13], namely

$$
(\nabla_x u_m^h, \nabla_x z)_{\Omega(h)} = \lambda_m^h (u_m^h, z)_{\Omega(h)}, \quad z \in H^1(\Omega(h))_\perp, \tag{3.14}
$$

which serves for the problem (1.3)–(1.4), the following relation is valid:

$$\|\hat{u}_m^h; H^1(\Omega)\|^2 \le c\|\nabla_x u_m^h; L_2(\Omega(h))\|^2 = c\lambda_m^h. \tag{3.15}$$

The max-min principle (see, e.g., [31]), where the test functions can be taken from the space $\mathcal{C}_c^\infty(\Omega_*)$, show that for an arbitrary m there exist positive numbers h_m and c_m, such that

$$\lambda_m^h \le c_m \quad \text{for } h \in (0, h_m]. \tag{3.16}$$

therefore the norms $\|\hat{u}_m^h; H^1(\Omega)\|$ are uniformly bounded with respect to the parameter $h \in (0, h_m]$ for a fixed m, i.e., the pairs $\{\lambda_m^h, \hat{u}_m^h\}$ admit the weak limit $\{\lambda_m^0, \hat{u}_0^h\} \in \mathbb{R} \times H^1(\Omega)$ for $h \to +0$ and the strong limit in $\mathbb{R} \times L_2(\Omega)$.

In the integral identity (3.14) we choose a test function $z \in \mathcal{C}_c^\infty(\overline{\Omega} \setminus \mathcal{O})$ with null mean value. For sufficiently small h, $\hat{u}_m^h = u_m^h$ on the support of the function z, thus passing to the limit in (3.14) leads to the inequality

$$(\nabla_x \hat{v}_m^0, \nabla_x z)_\Omega = \hat{\lambda}_m^0 (\hat{v}_m^0, z)_\Omega. \tag{3.17}$$

Since $\mathcal{C}_c^\infty(\overline{\Omega} \setminus \mathcal{O})$ is dense in $H^1(\Omega)$ (elements of the Sobolev space $H^1(\Omega)$ have no traces at a single point), by a density argument, we can assume that in (3.17), the test function z belongs to $H^1(\Omega)_\perp$.

In view of (3.13)–(3.15), it follows that

$$\begin{aligned} \left| \int_\Omega \hat{u}_m^h dx - \int_{\Omega(h)} u_m^h \, dx \right| &\le \left| \int_{\Omega \cap \mathbb{B}_{Rh}} |\hat{u}_m^h| \, dx - \int_{\Omega(h) \cap \mathbb{B}_{Rh}} |u_m^h| \, dx \right| \\ &\le ch^{\frac{5}{2}} \left(|||\hat{u}_m^h; \Omega \; ||| + ||| \; u_m^h; \Omega(h)||| \right) \\ &\le ch^{\frac{5}{2}} \end{aligned}$$

and $$\left| \int_\Omega |\hat{u}_m^h|^2 \, dx - \int_{\Omega(h)} |u_m^h|^2 \, dx \right| \le ch^2.$$

Since $\|\hat{u}_m^h; L_2(\Omega)\| \to \|\hat{v}_m^0; L_2(\Omega)\|$ and $\|\hat{u}_m^h; L_2(\Omega)\| = 1$, the previous inequality provides $\hat{v}_m^0 \in H^1(\Omega)$ and $\|\hat{v}_m^0; L_2(\Omega)\| = 1$, i.e., in view of (3.17), $\hat{\lambda}_m^0$ is an eigenvalue and $\hat{v}_m^0$ is a normalized eigenfunction of problem (1.8)–(1.9).

Proposition 3.3. *Entries of sequences* (1.5) *are related to* (1.7) *by passing to the limit*

$$\lambda_m^h \to \lambda_m^0 \quad \text{as} \quad h \to +0. \tag{3.18}$$

The proof is completed at the end of this section. We only observe that it has been already shown that $\lambda_m^h \to \lambda_p^0$, thus it suffices to prove that $p = m$.

From Lemma 3.1 it follows that the left-hand side of identity (3.14) can be chosen as the scalar product $\langle u_m^h, z\rangle$ in the space $H^1(\Omega(h))_\perp$. We define the operator K^h in the space $H^1(\Omega(h))_\perp$ by the formula

$$\langle K^h u, z\rangle = (u, z)_{\Omega(h)}, \quad u, z \in H^1(\Omega(h))_\perp. \tag{3.19}$$

It is easy to check that K^h is symmetric, positive and compact, therefore, self-adjoint. For $m \geq 1$ we set $\mu_m^h = (\lambda_m^h)^{-1}$. The positive eigenvalues and the corresponding eigenfunction of problem (1.3)–(1.4) can be considered in an abstract framework, so we deal with the spectral equation in the Hilbert space $H = H^1(\Omega(h))_\perp$:

$$K^h u^h = \mu^h u^h. \tag{3.20}$$

The norm, defined by the scalar product $\langle\cdot,\cdot\rangle_H = \langle\cdot,\cdot\rangle$ is denoted by $\|\cdot\|_H$. The following statement [33] is known as lemma on *almost eigenvalues and eigenvectors.*

Lemma 3.4. *Let μ and $U \in H$ be such that $\|K^hU - \mu U\|_H = \alpha$ and $\|U\|_H = 1$. Then there exists an eigenvalue μ_m^h of the operator K^h, which satisfies the inequality*

$$|\mu - \mu_m^h| \leq \alpha.$$

Moreover, for any $\alpha_\bullet > \alpha$ the inequality $\|U - U_\bullet\|_H \leq \frac{2\alpha}{\alpha_\bullet}$ holds where $U_\bullet$ is a linear combination of eigenfunctions of the operator K^h, corresponding to the eigenvalues from the segment $[\mu - \alpha_\bullet, \mu + \alpha_\bullet]$ and $\|U_\bullet\|_H = 1$.

The asymptotic approximations μ and U of a solution to equation (3.19) are defined by the number $(\lambda_m^0 + h^3\lambda_m')^{-1}$ and by the function $\|V_m^h\|_H^{-1}V_m^h$, respectively, where $m \geq 1$ and λ_m' with V_m^h are, respectively, the correction given by (2.46) and the sum of the first four terms in the ansatz (1.12). In the case of multiple eigenvalue λ_p^0, we consider the specification provided at the end of section 2.4.

We estimate the quantity α from Lemma 3.4. Since $\|V_m^h\|_H \geq \|v_m^0\|_H - c_m h$ and $\lambda_m^0 + h^3\lambda_m' \geq \lambda_m^0 - c_m h^3$, for h sufficiently small it follows that

$$\begin{aligned}\alpha &= \|K^hU - \mu U\|_H \\ &= (\lambda_m^0 + h^3\lambda_m')^{-1}\|V_m^h\|_H^{-1}\|(\lambda_m^0 + h^3\lambda_m')(K^h - \mu)V_m^h\|_H \\ &= (\lambda_m^0 + h^3\lambda_m')^{-1}\|V_m^h\|_H^{-1}\sup\left|\langle(\lambda_m^0 + h^3\lambda_m')(K^h - \mu)V_m^h, z\rangle\right| \\ &\leq c_m \sup\left|(\lambda_m^0 + h^3\lambda_m')(V_m^h, z)_{\Omega(h)} - \langle V_m^h, z\rangle_{\Omega(h)}\right|,\end{aligned} \tag{3.21}$$

where the supremum is taken over the set $\{z \in H^1(\Omega(h))_\perp : \|z\|_H = 1\}$ and, hence, the L_2-norms of the test function z indicated in inequality (3.1), both standard and weighted, are bounded by a constant $\mathcal{N}$. Besides that, the standard proof of the trace theorem [13, p. 30] implies

$$h^{-\frac{1}{2}}\|z; L_2(\partial\omega_h \cap \Gamma(h))\| \leq c(|||z; \Omega(h)\ ||| + \|\nabla z; L_2(\Omega(h))\|) \leq c\mathcal{N}. \tag{3.22}$$

The expression in the sup in (3.21) can be processed as follows:

$$\begin{aligned}
I &= (\lambda_m^0 + h^3\lambda_m')(V_m^h, z)_{\Omega(h)} - \langle V_m^h, z\rangle_{\Omega(h)} \\
&= I^1 + h^3 I^2 - h^6 I^3 + I^4 - I^5 - h^3 I^6 \\
&:= (\nabla_x v_m^0, \nabla_x z)_{\Omega(h)} - \lambda_m^0 (v_m^0, z)_{\Omega(h)} \\
&\quad + h^3 \big((\nabla_x v_m^3, \nabla_x z)_{\Omega(h)} - (\lambda_m^0 v_m^3 + \lambda_m' v_m^0, z)_{\Omega(h)} \big) \\
&\quad - h^6 \lambda_m' (v_m^3, z)_{\Omega(h)} + \big(\nabla_x \chi(h w_m^1 + h^2 w_m^2), \nabla_x z\big)_{\Omega(h)} \\
&\quad - \lambda_m^0 \big(\chi(h w_m^1 + h^2 w_m^2), z\big)_{\Omega(h)} - h^3 \lambda_m' \big(\chi(h w_m^1 + h^2 w_m^2), z\big)_{\Omega(h)}.
\end{aligned} \tag{3.23}$$

The estimates of I^3 and I^6 are straightforward, that is

$$|I^3| \le c_m \|v_m^3; L^2(\Omega)\| \mathcal{N} \le c_m \mathcal{N} \tag{3.24}$$

$$\begin{aligned}
|I^6| &\le c_m \left| \int_{\Omega(h)} \chi r_h (h w_m^1 + h^2 w_m^2)(r_h^{-1} z)\, dx \right| \\
&\le c_m \, ||| \, z; \Omega(h) \, ||| \left(\int_{\Omega(h)} (\chi r_h (h w_m^1 + h^2 w_m^2))^2 \, dx \right)^{\frac{1}{2}} \\
&\le c_m \mathcal{N} h^{\frac{3}{2}} \bigg(\int_{\Xi \cap B_R} h^2 (1+\rho)^2 (h w_m^1 + h^2 w_m^2)^2 \, d\xi \\
&\qquad + \int_{\Xi \setminus B_R} \chi^2 h^2 (1+\rho)^2 (h \rho^{-2} + h^2 \rho^{-1})^2 \, d\xi \bigg)^{\frac{1}{2}} \\
&\le c_m \mathcal{N} h^{\frac{5}{2}}.
\end{aligned} \tag{3.25}$$

Here, expressions (2.8) and (2.20) of the boundary layers are taken into account. The remaining integrals require additional work. In view of relations (1.8)–(1.9) and (2.27)–(2.30) we have

$$I^1 = (\partial_{n^h} v_m^0, z)_{\partial\omega_h \cap \Gamma(h)} \tag{3.26}$$

$$\begin{aligned}
I^2 &= I_1^2 + I_2^2 + I_3^2 \\
&:= (\partial_{n^h} v_m^3, z)_{\partial\omega_h \cap \Gamma(h)} + (f^3, z)_{\Omega(h)} \\
&= (\partial_{n^h} v_m^3, z)_{\partial\omega_h \cap \Gamma(h)} + \big(\Delta_x \chi (t_m^1 + t_m^2), z\big)_{\Omega(h)} + \lambda_m^0 \big(\chi(t_m^1 + t_m^2), z\big)_{\Omega(h)}.
\end{aligned}$$

To get the estimate for I_1^2, we, first of all, need to prove the following inequality:

$$\|r^{-\frac{1}{2}} z; L_2(\Gamma(h))\| + \|r^{-1} z; L_2(\Omega(h))\| \le c \|z; H^1(\Omega(h))\|. \tag{3.27}$$

By (3.22) and (3.1), we may write the inequality

$$\|r_h^{-\frac{1}{2}} z; L_2(\Gamma(h))\| + \|r_h^{-1} z; L_2(\Omega(h))\| \le c \|z; H^1(\Omega(h))\|. \tag{3.28}$$

Thus, we only need to verify that (3.27) is true in a h-neighbourhood of $\mathcal{O}$. Using the dilation by h^{-1}, we are left to verify inequality

$$\|\rho^{-\frac{1}{2}}Z; L_2(\partial\Xi_R)\| + \|\rho^{-1}Z; L_2(\Xi_R)\| \le c\|Z; H^1(\Xi_R)\|, \tag{3.29}$$

in the parameter-independent case, where $\Xi_R := \Xi \cap \mathbb{B}_R$, $\mathbb{B}_R$ is the ball of radius R centered at $\mathcal{O} = \{\rho = 0\}$ and $R > 0$ is chosen so that $\Xi_R \supset \overline{\omega}$. Three situations may then occur:

(i) If $\mathcal{O}$ lies outside $\overline{\Xi_R}$, then $\rho > c > 0$ and (3.29) is trivially satisfied.

(ii) If $\mathcal{O}$ is inside $\overline{\Xi_R}$, then $\rho > c > 0$ on $\partial\Xi_R$ and thus the first norm on the left-hand side of (3.29) is bounded by $c\|Z; H^1(\Xi_R)\|$ due to the standard trace inequality. The estimation of $\|\rho^{-1}Z; L_2(\Xi_R)\|$ in (3.29) derives from Hardy's inequality (3.10).

(iii) If $\mathcal{O}$ is on $\partial\Xi_R$, then we need to rectify the boundary $\partial\Xi$.

Note that the boundary $\partial\Xi$ is Lipschitz. Without loss of generality, let us assume that there exists a neighbourhood $\mathcal{V}$ of $\mathcal{O}$ such that $\partial\Xi_R \cap \mathcal{V}$ is the graph of a Lipschitz function ψ. We rectify the boundary $\partial\Xi_R \cap \mathcal{V}$ using the transformation

$$T : (\xi_1, \xi_2, \xi_3) \mapsto (\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3) = (\xi_1, \xi_2, \xi_3 - \psi(\xi_1, \xi_2)).$$

The image of $\partial\Xi_R \cap \mathcal{V}$ by T is a piece of plane. Let $(\tilde{\rho}, \tilde{\theta}, \tilde{\phi})$ be the spherical coordinate system associated with $(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3)$. Using the Lipschitz property of ψ, one readily checks that there exist constants $c_1 > 0$ and $c_2 > 0$, dependent on ψ, such that $c_1\rho < \tilde{\rho} < c_2\rho$. Using Hardy's inequality (3.10) and the equivalence of ρ and $\tilde{\rho}$, we have

$$\begin{aligned}\|\rho^{-1}Z; L_2(\Xi_R \cap \mathcal{V})\| &\le c\|\tilde{\rho}^{-1}\tilde{Z}; L_2(T(\Xi_R \cap \mathcal{V}))\| \\ &\le c\|\tilde{Z}; H^1(T(\Xi_R \cap \mathcal{V}))\| \\ &\le c\|Z; H^1(\Xi_R \cap \mathcal{V})\|.\end{aligned}$$

For the trace inequality, we separate the radial and angular variables and use the two-dimensional trace inequality in the angular variables:

$$\begin{aligned}\|\rho^{-\frac{1}{2}}Z; L_2(\partial\Xi_R \cap \mathcal{V})\| &\le c\|\tilde{\rho}^{-\frac{1}{2}}\tilde{Z}; L_2(T(\partial\Xi_R \cap \mathcal{V}))\| \\ &= c\int_0^{\tilde{R}}\int_0^{2\pi} \tilde{\rho}^{-1}|\tilde{Z}|^2 \rho\, d\tilde{\theta} d\tilde{\rho} \\ &= c\int_0^{\tilde{R}}\int_0^{\pi}\int_0^{2\pi} \left(|\tilde{Z}|^2 + |\partial_{\tilde{\theta}}\tilde{Z}|^2 + |\partial_{\tilde{\phi}}\tilde{Z}|^2\right) d\tilde{\theta} d\tilde{\phi} d\tilde{\rho}\end{aligned}$$

for some $\tilde{R} > 0$. Then we may use Friedrich's inequality to obtain

$$\int_0^{\tilde{R}}\int_0^{\pi}\int_0^{2\pi} \left(|\tilde{Z}|^2 + |\partial_{\tilde{\theta}}\tilde{Z}|^2 + |\partial_{\tilde{\phi}}\tilde{Z}|^2\right) d\tilde{\theta}d\tilde{\phi}d\tilde{\rho} \le c\|\tilde{Z}; H^1(T(\Xi_R \cap \mathcal{V}))\| \\ \le c\|Z; H^1(\Xi_R \cap \mathcal{V})\|.$$

Therefore, we have proved (3.29) and in view of the previous comments, (3.27) follows. Using (3.27), we get the estimate for I_1^2

$$\begin{aligned} |I_1^2| &\le c_m \|r^{\frac{1}{2}}\partial_{n^h} v_m^3; L_2(\partial\omega_h \cap \Gamma(h))\| \, \|r^{-\frac{1}{2}} z; L_2(\partial\omega_h \cap \Gamma(h))\| \\ &\le c_m \mathcal{N} h^{\frac{1}{2}} \|r^{\frac{1}{2}}\nabla v_m^3; H^1(\Omega(h))\| \\ &\le c_m h^{\frac{1}{2}}, \end{aligned} \tag{3.30}$$

where we have also used the estimates $|\nabla_x^p v_m^3(x)| \le c_p r^{-p}$, $p = 1, 2, \ldots$, for the solution of (2.28)–(2.29) which follow from the theory of elliptic boundary problems in domains with corners or conical points (see, e.g., [23]) and from the analysis (2.36) of the right-hand side of equation (2.28).

By Remark 2.4 and (2.26), the following estimates are valid for $\rho \ge R_0$

$$|\tilde{w}_m^1(\xi)| = |w_m^1(\xi) - t_m^1(\xi)| \le c\rho^{-3} \tag{3.31}$$
$$|\tilde{w}_m^2(\xi)| = |w_m^2(\xi) - t_m^2(\xi)| \le c\rho^{-2}, \tag{3.32}$$

which means that

$$\begin{aligned} |I^5 - h^3 I_3^2| &\le |||z, \Omega(h)\,||| \left(\int_{\Omega(h)} \left(r\chi(x)(h\widetilde{w}_m^1 + h^2\widetilde{w}_m^2)\right)^2 dx\right)^{\frac{1}{2}} \\ &\le \mathcal{N}\left(\int_{\Xi} h^2\rho^2\chi(h\xi)\left(h\widetilde{w}_m^1 + h^2\widetilde{w}_m^2\right)^2 h^3\, d\xi\right)^{\frac{1}{2}} \\ &\le \mathcal{N} h^{\frac{7}{2}} \left(\int_R^{h^{-1}d} \rho^{-4}\rho^2\, d\rho\right)^{\frac{1}{2}} \\ &\le \mathcal{N} h^{\frac{7}{2}}, \end{aligned} \tag{3.33}$$

where d is the diameter of the support of χ. We denote

$$I^4 = I_1^4 + I_2^4 := (\nabla_x(hw_m^1 + h^2 w_m^2), \nabla_x\chi z)_{\Omega(h)} - ([\Delta_x, \chi](hw_m^1 + h^2 w_m^2), z)_{\Omega(h)}$$
$$I_2^2 = I_4^2 + I_5^2 := (\chi\Delta_x(t_m^1 + t_m^2), z)_{\Omega(h)} + ([\Delta_x, \chi](t_m^1 + t_m^2), z)_{\Omega(h)}\,.$$

Here $[\Delta_x, \chi] = 2\nabla_x\chi\cdot\nabla_x + (\Delta_x\chi)$ is the commutator of the Laplace operator with the cut-off function χ. The supports of the coefficients of first order differential operator $[\Delta_x, \chi]$ are contained in the set $\operatorname{supp}|\nabla_x\chi|$ which is located at the distance d_χ from the origin. Thus, taking into account relation, Remark 2.4 and (2.26), we find

$$\begin{aligned} |I_2^4 - h^2 I_5^2| &= ([\Delta_x, \chi](h\widetilde{w}_m^1 + h^2\widetilde{w}_m^2), z)_{\Omega(h)} \\ &\le c_m \Bigg(\int_{d_\chi}^{d} \Big(h^2\rho^{-6} + h^4\rho^{-4} \Big) \Big|_{\rho = h^{-1}r} r\, dr \Bigg)^{\frac{1}{2}} \| z; L_2(\Omega(h)) \| \\ &\le c_m h^4 \mathcal{N}. \end{aligned} \tag{3.34}$$

Moreover,

$$\begin{aligned} I_1^4 + h^3 I_4^2 &= I_3^4 + I_4^4 \\ &:= -\big(\Delta_x(h\widetilde{w}_m^1 + h^2\widetilde{w}_m^2), \chi z\big)_{\Omega(h)} + \big(\partial_{n^h}(hw_m^1 + h^2 w_m^2), z\big)_{\partial\omega_h \cap \Omega(h)}. \end{aligned} \tag{3.35}$$

Remark 3.5. The presence of corners on the boundary of domain Ξ may result in the singularities of derivatives of the boundary layers, therefore the inclusions $\chi\Delta_x\widetilde{w}_m^q \in L_2(\Omega(h))$ and $\chi\partial_{n^h} w_m^q \in L_2(\Gamma(h))$, in general are not valid. However, the terms in (3.35) may be well defined in the sense of duality obtained by the extension of scalar products $(\cdot,\cdot)_{\Omega(h)}$ and $(\cdot,\cdot)_{\Gamma(h)}$ in the Lebesgue spaces to the appropriate weighted Kondratiev classes (see [11] and, e.g., [23, Chapter 2]). Additional weighted factors are local, i.e., the factors are written in fast variables. That is why the norms of test functions z can be bounded as before by the constant $\mathcal{N}$.

By definition, the function $\widetilde{w}_m^1$ remains harmonic, and according to (2.15)–(2.16) and (2.33), $\widetilde{w}_m^2$ verifies the equation

$$-\Delta_\xi \widetilde{w}_m^2(\xi) = L^1(\xi_1, \nabla_\xi)\widetilde{w}_m^1(\xi), \quad \xi \in \Xi. \tag{3.36}$$

Therefore,

$$\Delta_x(h\widetilde{w}_m^1 + h^2\widetilde{w}_m^2) = h^2 L^1 \widetilde{w}_m^2 + L^2(h\widetilde{w}_m^1 + h^2\widetilde{w}_m^2). \tag{3.37}$$

In (3.37) the operators L^q are written in the slow variables and the function $\widetilde{w}^q$ in fast variables (in contrast to (3.36)) where $\Delta_\xi = h^2 L^0(\partial_n, \partial_s, \partial_\nu)$ and $L^1(\xi_1, \nabla_\xi) = hL^1(n, \partial_n, \partial_s, \partial_\nu)$. Owing to (3.37), (2.26) and applying Remark 3.5, we have $L^2(h\widetilde{w}_m^1) = hO(\rho^{-3}), L^1(\widetilde{w}_m^2) = h^{-1}O(\rho^{-3})$ and $L^2(h^2\widetilde{w}_m^2) = h^2 O(\rho^{-2})$.

Thus, it follows that

$$\begin{aligned}|I_3^4| &\le |||z,\Omega(h)\,|||\left(\int_{\Omega(h)}\left(r\chi(x)\Delta_x(h\widetilde{w}_m^1+h^2\widetilde{w}_m^2)\right)^2 dx\right)^{\frac{1}{2}}\\ &\le \mathcal{N}\left(\int_{\Xi} h^2\rho^2\chi(h\xi)^2\left(\Delta_x(h\widetilde{w}_m^1+h^2\widetilde{w}_m^2)\right)^2 h^3\, d\xi\right)^{\frac{1}{2}}\\ &\le \mathcal{N}h^{\frac{5}{2}}\left(\int_{\Xi\setminus B_R}\rho^2\chi(h\xi)^2\left(h\rho^{-3}+h^2\rho^{-2}+h\rho^{-3}\right)^2 d\xi\right)^{\frac{1}{2}}\\ &\le \mathcal{N}h^{\frac{7}{2}}\left(\int_{d_0}^{h^{-1}d}\rho^{-4}\,\rho^2\, d\rho\right)^{\frac{1}{2}}\\ &\le \mathcal{N}h^{\frac{7}{2}}.\end{aligned} \tag{3.38}$$

For the two last terms it suffices to process the difference of integrals from (3.26) and (3.35): $I^1+I_4^4=-\left(\partial_{n^h}(hw_m^1+h^2w_m^2+v_m^0),z\right)_{\partial\omega_h\cap\Gamma(h)}$. Note that, due to the very construction of w_m^1 and w_m^2 we have $\partial_{n^h}(hw_m^1+h^2w_m^2+v_m^0)=O(h^2)$, see (2.15)–(2.18) for instance. Thus, we get the estimate

$$|I^1+I_4^4|\le c_m\|z;L_2(\partial\omega_h\cap\Gamma(h))\|h^2(\mathrm{mes}_2(\partial\omega_h))^{\frac{1}{2}}\le c_m h^{\frac{7}{2}}\mathcal{N}, \tag{3.39}$$

where mes_2 denotes the two-dimensional Hausdorff measure. Collecting estimates (3.24)–(3.25), (3.30), (3.33)–(3.34), (3.38)–(3.39) of the terms in (3.23), we arrive at the following estimate of α in (3.21):

$$\alpha\le c_m h^{\frac{7}{2}}. \tag{3.40}$$

We are ready now to verify the theorem on the asymptotics, which implies the main result of the paper.

Theorem 3.6. *For any positive eigenvalue λ_m^0 of multiplicity $\varkappa_m$ in problem* (1.8)–(1.9)*, see* (2.49)*, there exist numbers $\mathbf{c}_m>0$ and $h_m>0$ such that for $h\in(0,h_m]$ the eigenvalues $\lambda_m^h,\dots,\lambda_{m+\varkappa_m-1}^h$ of problem* (1.3)–(1.4) *and except for all other eigenvalues in sequence* (1.5) *satisfy the following inequalities*

$$|\lambda_q^h-\lambda_m^0-h^3\lambda^{q\prime}|\le\mathbf{c}_m h^{\frac{7}{2}},\quad q=m,\dots,m+\varkappa_m-1. \tag{3.41}$$

Moreover, there is a constant $\mathbf{C}_m$ and columns $a^{hm},\dots,a^{hm+\varkappa_m-1}$ which define an unitary matrix of the size $\varkappa_m\times\varkappa_m$ such that

$$\left\|v^{q0}+\chi(hw^{q1}+h^2w^{q2})+h^3v^{q3}-\sum_{p=m}^{m+\varkappa_m-1}a_p^{hq}u_p^h;H^1(\Omega(h))\right\|\le\mathbf{C}_m h \tag{3.42}$$

with $q = m, \ldots, m + \varkappa_m - 1$*. Here* v^{q0} *denotes the linear combination* (2.50) *of eigenfunctions in problem* (1.8)–(1.9)*, constructed in the end of Section* 2.4*, and* w^{q1}*,* w^{a2} *and* v^{q3} *are given functions which are determined for fixed* v^{q0} *in the way described in Section* 2*, finally* $\lambda^{q\prime}$ *is an eigenvalue of the matrix* $\mathbf{M}$ *with entries* (2.54)*. In the case of a simple eigenvalue* λ_m^0 *(i.e.,* $\varkappa_m = 1$*), we have* $v^{m0} = v_m^0$ *the corresponding eigenfunction, and* $\lambda^{m\prime} = \lambda_m^\prime$ *is given by* (2.46)*.*

Proof. Given eigenvectors $a^m, \ldots, a^{m+\varkappa_m - 1}$ of the matrix $\mathbf{M}$, we construct linear combinations (2.50) and the associated appropriate terms in asymptotic ansatz (1.12). As a result, approximation solutions $\big\{(\lambda_q^0 + h^3\lambda^{q\prime})^{-1}, U^q\big\}$ for $q = m, \ldots, m + \varkappa_m - 1$ are obtained for the abstract spectral problem (3.19).

Let $\lambda^{q\prime}$ be an eigenvalue of the matrix $\mathbf{M}$ of multiplicity κ_q, i.e.,

$$\lambda^{q-1\prime} < \lambda^{q\prime} = \cdots = \lambda^{q+\kappa_q-1\prime} < \lambda^{q+\kappa_q\prime}. \tag{3.43}$$

We choose the factor c_* in the value $\alpha_* = c_* h^3$ in Lemma 3.4 so small that the segment

$$\big[(\lambda_m^0 + h^3\lambda^{q\prime})^{-1} - c_* h^3, (\lambda_m^0 + h^3\lambda^{q\prime})^{-1} + c_* h^3\big] \tag{3.44}$$

does not contain the approximation eigenvalues $(\lambda_m^0 + h^3\lambda^{p\prime})^{-1}$ when $p \notin \{q, q + \kappa_q - 1\}$. Then Lemma 3.4 and (3.40) delivers the eigenvalues $\mu_{i(q)}^h, \ldots, \mu_{i(q+\kappa_q-1)}^h$ of the operator K^h such that

$$\big|\mu_{i(p)}^h - (\lambda_m^0 + h^3\lambda^{p\prime})^{-1}\big| \le \alpha \le \mathbf{c}_m h^{\frac{7}{2}}, \quad p = q, \ldots, q + \kappa_q - 1\,. \tag{3.45}$$

We here emphasize that, at the time being, we cannot infer that these eigenvalues are different. At the same moment, the second part of Lemma 3.4 gives the normed columns $b^{hp} = (b_{k_{mq}}^{hp}, \ldots, b_{k_{mq}+N_{mq}-1}^{hp})$ verifying the inequalities

$$\Bigg\| U^p - \sum_{k=k_{mq}}^{k_{mq}+N_{mq}-1} b_k^{hp} u_k^h; H^1(\Omega(h)) \Bigg\| \le c\frac{\alpha}{\alpha_*} \le c h^{\frac{1}{2}}. \tag{3.46}$$

Here $\{\mu_{k_{mq}}^h, \ldots, \mu_{k_{mq}+N_{mq}-1}^h\}$ implies the list of all eigenvalues of the operator K^h in segment (3.44). Note that the numbers k_{mq} and N_{mq} can depend on the parameter h but this fact is not reflected in the notation. Since

$$\begin{aligned} \|h\chi w^1; H^1(\Omega(h))\| &\le c h^{\frac{3}{2}} \\ \|h^2\chi w^2; H^1(\Omega(h))\| &\le c h^{\frac{5}{2}} \\ \|h^2 v^2; H^1(\Omega(h))\| &\le c h^3, \end{aligned} \tag{3.47}$$

the normalization condition (1.10) for the eigenfunctions of problem (1.8)–(1.9) and similar conditions for eigenvectors of the matrix $\mathbf{M}$ ensure that

$$\big|(U^p, U^t)_{L_2(\Omega(h))} - \delta_{p,t}\big| \le c h^{\frac{3}{2}}, \quad p, t = q, \ldots, q + \kappa_q + 1. \tag{3.48}$$

In a similar way, inequalities (3.46) and the orthogonality and normalization conditions (1.6) for eigenfunctions u_k^h of problem (1.3)–(1.4) lead to the relation

$$\left|(U^p, U^t)_{L_2(\Omega(h))} - \sum_{k=k_{mq}}^{k_{mq}+N_{mq}-1} b_k^{hp} b_k^{ht}\right| \le ch^{\frac{1}{2}}. \tag{3.49}$$

Formulas (3.48) and (3.49) are true simultaneously if and only if

$$N_{mq} \ge \kappa_q\,, \tag{3.50}$$

otherwise we arrive at a contradiction where at least one of the coefficients b_k^{hp} has to be close to zero and to one simultaneously. To actually prove that the equality occurs in (3.50), we first of all, notice that, for a sufficiently small $h > 0$, the relations of type (3.50) are valid for all eigenvalues $\lambda_1^0, \dots, \lambda_m^0$ of problem (1.8)–(1.9) and all eigenvalues $\lambda^{q'}$ of the associated matrices $\mathbf{M}$.

We have verified above Proposition 3.3 that each eigenvalue λ_p^h and the corresponding eigenfunction u_p^h of singularly perturbed problem (1.3)–(1.4) converge to an eigenvalue and an eigenfunction of the limit problem (1.8)–(1.9), respectively. This observation ensures that the number of entries of the eigenvalue sequence (1.5), which live on the interval $(0, \lambda_m^0)$, does not exceed $m + \varkappa_m - 1$ for a small $h > 0$. Summing up the inequalities (3.50) over all $\lambda_1^0, \dots, \lambda_m^0$ and $\lambda^{q'}$, we conclude that the equalities $N_{mq} = \kappa_q$ are necessary. Moreover, we now are able to confirm that the eigenvalues $\mu_{i(q)}^h, \dots, \mu_{i(q+\kappa_q-1)}^h$ can be chosen different one from another. Indeed, take $\alpha_* = C_* h^{\frac{7}{2}}$ in Lemma 3.4 and fix C_* so large that the inequality (3.46) with the new bound $\frac{c}{C_*}$ still guarantees that the segment

$$\Lambda_q(h) = \left[(\lambda_m^0 + h^3\lambda^{q'})^{-1} - C_* h^{\frac{7}{2}}, (\lambda_m^0 + h^3\lambda^{q'})^{-1} + C_* h^{\frac{7}{2}}\right] \tag{3.51}$$

contains exactly κ_q eigenvalues of the operator K^h. It suffices to mention two facts. First, for a small $h > 0$, the intervals $\Lambda_q(h)$ and $\Lambda_p(h)$ with $\lambda^{q'} \ne \lambda^{p'}$ do not intersect. Second, any eigenvalue $\mu_k^h = (\lambda_k^h)^{-1}$ in the interval (3.51) meets the inequality (3.41). □

Remark 3.7. Estimates (3.47) show that the bound in (3.42) is larger than the norms of the functions w^{q1}, w^{q2} and v^{q3} included into the approximation solution and, therefore, estimate (3.42) remains valid for the function v^{q0} alone, without three correcting terms. This is the usual situation in the asymptotic analysis of singular spectral problems: One needs to construct additional asymptotic terms of eigenfunctions in order to prove that the correcting term in the asymptotics of an eigenvalue is found properly. In theory, one can employ the general procedure [20] and construct higher order asymptotic terms of eigenvalues and eigenfunctions. We keep the boundary layer and regular corrections in the estimate (3.42) because they form a so-called asymptotic *conglomerate* which is replicated in the asymptotic series (see [20, 22]; actually the notion of asymptotic conglomerates was introduced in [22]).

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References

[1] Birman, M. Sh. and Solomjak, M. Z., *Spectral Theory of Selfadjoint Operators in Hilbert Space* (in Russian). Leningrad: Leningrad. Univ. 1980; Engl. transl.: Math. Appl. (Soviet Ser.). Dordrecht: D. Reidel 1987.

[2] Bucur, D. and Buttazzo, G., *Variational Methods in Shape Optimization Problems.* Progress Nonlin. Diff. Equ. Appl. 65. Boston (MA): Birkhäuser 2005.

[3] Dodson, C. T. J. and Poston, T., *Tensor Geometry.* Graduate Texts Math. 130 (2nd ed.). Berlin: Springer 1991.

[4] Gadyl'shin, R. R., Asymptotics of the eigenvalue of a singularly perturbed selfadjoint elliptic problem with a small parameter in the boundary conditions (in Russian). *Differentsial'nye Uravneniya* 22 (1986)(4), 640 – 652; Engl. transl.: *Differential Equations* 22 (1986)(4), 474 – 483.

[5] Gadyl'shin, R. R., The method of matched asymptotic expansions in a problem on the acoustic Helmholtz resonator (in Russian). *Prikl. Mat. Mekh.* 56 (1992)(3), 412 – 418; Engl. transl.: *J. Appl. Math. Mech.* 56 (1992)(3), 340 – 345.

[6] Gadyl'shin, R. R., Splitting of a multiple eigenvalue of the Dirichlet problem for the Laplace operator under singular perturbation of the boundary condition (in Russian). *Mat. Zametki* 52 (1992)(4), 42 – 55; Engl. transl.: *Math. Notes* 52 (1992)(3–4), 1020 – 1029.

[7] Haug, E. J. and Rousselet, B., Design sensitivity analysis of eigenvalue variations. In: *Optimization of Distributed Parameter Structures II* (Iowa City, Iowa, 1980; eds.: E. J. Haug et al.). NATO Adv. Study Inst. Ser. E: Appl. Sci. 50. The Hague: Nijhoff 1981, pp. 1371 – 1396.

[8] Henrot, A., *Extremum Problems for Eigenvalues of Elliptic Operators.* Frontiers in Mathematics. Basel: Birkhäuser 2006.

[9] Il'in, A. M., *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems.* Transl. Math. Monographs 102. Providence (RI): Amer. Math. Soc. 1992.

[10] Kamotskiĭ, I. V. and Nazarov, S. A., Spectral problems in singularly perturbed domains and self-adjoint extensions of differential operators (in Russian). In: *Trudy S.-Peterburg Mat. Obshch. 6* (ed.: N. N. Uraltseva). Novosibirsk: Nauchn. Kniga 1998, pp. 151 – 212; Engl. transl.: *Proceedings St. Petersburg Math. Soc.* VI. Amer. Math. Soc. Transl. Ser. 2, 199. Providence (RI): Amer. Math. Soc. 2000, pp. 127 – 181.

[11] Kondratiev, V. A., Boundary problems for elliptic equations in domains with conical or angular points (in Russian). *Trudy Moskov. Mat. Obshch.* (1967), 209 – 292; Engl. transl.: *Trans. Moscow Math. Soc.* 16 (1967), 227 – 313.

[12] Kozlov, V. A., Maz'ya, V. G. and Rossmann, J., *Elliptic Boundary Value Problems in Domains with Point Singularities.* Math. Surveys Monographs 52. Providence (RI): Amer. Math. Soc. 1997.

[13] Ladyzhenskaya, O. A., *The Boundary Value Problems of Mathematical Physics.* Appl. Math. Sci. 49. New York: Springer 1985.

[14] Maz'ja, V. G. and Plamenevskiĭ, B. A., Estimates in L_p and in Hölder classes, and the Miranda-Agmon maximum principle for the solutions of elliptic boundary value problems in domains with singular points on the boundary. *Math. Nachr.* 81 (1978), 25 – 82.

[15] Maz'ya, V. G. and Nazarov, S. A., Asymptotic behavior of energy integrals under small perturbations of the boundary near corner and conic points (in Russian). *Trudy Moskov. Mat. Obshch.* 50 (1987), 79 – 129; Engl. transl.: *Trans. Moscow Math. Soc.* 1988, 77 – 127.

[16] Maz'ya, V. G., Nazarov, S. A. and Plamenevskiĭ, B. A., On the asymptotic behavior of solutions of elliptic boundary value problems with irregular perturbations of the domain (in Russian). *Probl. Mat. Anal.* 8 (1981), 72 – 153.

[17] Maz'ya, V. G., Nazarov, S. A. and Plamenevskiĭ, B. A., Asymptotic behavior of the solution of the Dirichlet problem in a domain with a thin bridge (in Russian). *Funktsional. Anal. i Prilozhen.* 16 (1982)(2), 39 – 46.

[18] Maz'ya, V. G., Nazarov, S. A. and Plamenevskiĭ, B. A., Singularities of solutions of the Dirichlet problem in the exterior of a thin cone (in Russian). *Mat. Sb.* 122 (1983)(4), 435 – 457 Engl. transl.: *Math. USSR-Sb.* 50 (1985)(2), 415 – 437.

[19] Maz'ya, V. G., Nazarov, S. A. and Plamenevskiĭ, B. A., Asymptotic expansions of eigenvalues of boundary value problems for the Laplace operator in domains with small openings (in Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 48 (1984)(2), 347 – 371; Engl. transl.: *Math. USSR-Izv.* 24 (1985)(2), 321 – 346.

[20] Maz'ya, V. G., Nazarov, S. A. and Plamenevskiĭ, B. A., *Asymptotic Behavior of Solutions of Elliptic Boundary Value Problems under Singular Perturbations of the Domain* (in Russian). Tbilisi: Tbilis. Gos. Univ., Inst. Prikl. Mat. 1981; German transl.: Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten, I. Berlin: Akademie-Verlag 1991.

[21] Nazarov, S. A., Asymptotic conditions at a point, self-adjoint extensions of operators and the method of matched asymptotic expansions (in Russian). In: *Trudy S.-Peterburg Mat. Obshch.* 5 (ed.: N. N. Uraltseva). Novosibirsk: Nauchn. Kniga 1998, pp. 112 – 183; Engl. transl.: *Proceedings St. Petersburg Math. Soc.* V. Amer. Math. Soc. Transl. Ser. 2, 193. Providence (RI): Amer. Math. Soc. 1999, pp. 77 – 125.

[22] Nazarov, S. A., *Asymptotic Theory of Thin Plates and Rods 1. Dimension Reduction and Integral Estimates* (in Russian). Novosibirsk: Nauchn. Kniga 2001.

[23] Nazarov, S. A. and Plamenevsky, B. A., *Elliptic Problems in Domains with Piecewise Smooth Boundaries.* De Gruyter Expos. Math. 13. Berlin: de Gruyter 1994.

[24] Nazarov, S. A. and Sokołowski, J., Asymptotic analysis of shape functionals. *J. Math. Pures Appl.* (9) 82 (2003)(2), 125 – 196.

[25] Nazarov, S. A. and Sokołowski, J., Self-adjoint extensions for the Neumann Laplacian and applications. *Acta Math. Sin.* (Engl. Ser.) 22 (2006), 879 – 906.

[26] Nazarov, S. A. and Sokołowski, J., Spectral problems in shape optimization. Singular boundary perturbations. *Asymptotic Anal.* 56 (2008)(3–4), 159 – 196.

[27] Nazarov, S. A. and Sokołowski, J., Shape sensitivity analysis of eigenvalues revisited. *Control Cybernet.* 37 (2008)(4), 999 – 1012.

[28] Ozawa, S., An asymptotic formula for the eigenvalues of the Laplacian in a three-dimensional domain with a small hole. *J. Fac. Sci. Univ. Tokyo, Sect. IA Math.* 30 (1983)(2), 243 – 257.

[29] Ozawa, S., Asymptotic property of an eigenfunction of the Laplacian under singular variation of domains – the Neumann condition. *Osaka J. Math.* 20 (1985)(4), 639 – 655.

[30] Pólya, G. and Szegö, G., *Isoperimetric Inequalities in Mathematical Physics.* Ann. Math. Studies 27. Princeton (NJ): Princeton Univ. Press 1951.

[31] Sanchez-Hubert, J. and Sánchez-Palencia, E., *Vibration and Coupling of Continuous Systems. Asymptotic methods.* Berlin: Springer 1989.

[32] Sokołowski, J. and Zolésio, J.-P., *Introduction to Shape Optimization. Shape Sensitivity Analysis.* Springer Ser. Comput. Math. 16. Berlin: Springer 1992.

[33] Višik, M. I. and Ljusternik, L. A., Regular degeneration and boundary layer for linear differential equations with small parameter (in Russian). *Uspehi Mat. Nauk (N.S.)* 12 (1957)(5), 3 – 122; Engl. transl.: *Amer. Math. Soc. Transl. (2)* 20 (1962), 239 – 364.

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