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On Compactness of Minimizing Sequences Subject to a Linear Differential Constraint

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Abstract. For $\Omega \subset \mathbb{R}^N$ open, we consider integral functionals of the form

$$F(u) := \int_\Omega f(x,u)\,dx,$$

defined on the subspace of L^p consisting of those vector fields u which satisfy the system $\mathcal{A}u = 0$ on Ω in the sense of distributions. Here, $\mathcal{A}$ may be any linear differential operator of first order with constant coefficients satisfying Murat's condition of constant rank. The main results provide sharp conditions for the compactness of minimizing sequences with respect to the strong topology in L^p. Although our results hold for bounded domains as well, our main focus is on domains with infinite measure, especially exterior domains.

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1. Introduction

We consider integral functionals of the form

$$F(u) := \int_\Omega f(x,u)\,dx, \quad u \in \mathcal{U}_\mathcal{A}, \tag{1.1}$$

with the class of admissible functions given by the set of $\mathcal{A}$-free functions in L^p, i.e.,

$$\mathcal{U}_\mathcal{A} := \left\{ u \in L^p(\Omega;\mathbb{R}^M) \mid \mathcal{A}u = 0 \text{ in } \Omega \right\}, \tag{1.2}$$

Here, $1 < p < \infty$, $\Omega \subset \mathbb{R}^N$ is open (and possibly unbounded), $\mathcal{A}$ is a linear first order differential operator as in Section 3, formally mapping $u : \Omega \to \mathbb{R}^M$

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onto $\mathcal{A}u : \Omega \to \mathbb{R}^L$, and the equation $\mathcal{A}u = 0$ is understood in the sense of distributions. Throughout, we assume that

$\mathcal{A}$ satisfies the condition of constant rank

introduced by Murat [17], as specified in (3.1) below. Such differential constraints arise naturally in a variety of physical models, and in particular, both curl and divergence are admissible. For further examples, we refer to [4, 7]. As to f, we assume that

(f:0) $f : \Omega \times \mathbb{R}^M \to \mathbb{R}$ is a Carathéodory function,

i.e., $f = f(x, \mu)$ is measurable in $x \in \Omega$ for every μ and continuous in $\mu \in \mathbb{R}^M$ for a.e. x, as well as the following p-growth and p-Lipschitz conditions:

(f:1) $|f(x, \mu)| \leq C(|\mu| + |h(x)|)^p,$

(f:2) $|f(x, \mu) - f(x, \eta)| \leq C(|\mu| + |\eta| + |h(x)|)^{p-1} |\mu - \eta|,$

for a.e. $x \in \Omega$ and every $\mu, \eta \in \mathbb{R}^M$, where $C > 0$ is a constant and $h \in L^p(\Omega)$.

The main purpose of this paper is the study of the so-called weak-strong convergence property of F, that is, we ask under which additional conditions on f and Ω we have that

$$F(u_n) \to F(u) \text{ and } u_n \rightharpoonup u \text{ weakly in } L^p \quad \Longrightarrow \quad u_n \to u \text{ strongly in } L^p \tag{1.3}$$

for any given sequence $(u_n) \subset \mathcal{U}_{\mathcal{A}}$. For unconstrained functionals, this question has been investigated by Visintin [21], and in the case of gradients on bounded domains, where $\mathcal{U}_{\mathcal{A}}$ is replaced by $\mathcal{U}' := \{u \in L^p(\Omega; \mathbb{R}^{N \times d}) \mid u = \nabla v \text{ for a } v \in W^{1,p}(\Omega; \mathbb{R}^d)\}$, it was studied by Evans and Gariepy [5], Zhang [22] and later by Sychev [19,20]. Results for more general[1] $\mathcal{A}$-free vector fields instead of gradients have not been obtained so far. Sychev's results provide optimal conditions for ruling out possible oscillations of $u_n = \nabla v_n$, but neither of the aforementioned articles attempts a comprehensive study of concentration effects. In fact, while in [5, 22] at least sufficient conditions for ruling out concentrations are given (in the case of [5] only partially, since concentrations near the boundary are not discussed), Sychev uses a slightly different definition for the weak-strong convergence property, namely

$$F(u_n) \to F(u) \text{ and } u_n \rightharpoonup u \text{ weakly in } L^p \quad \Longrightarrow \quad u_n \to u \text{ strongly in } L^1$$

for $(u_n) \subset \mathcal{U}'$. On a bounded domain, this variant allows one to ignore concentrations of u_n in L^p altogether. An alternative approach, still on bounded domains but taking concentrations into account, is possible with the methods developed in [8, 9] for gradients, which were extended to the $\mathcal{A}$-free case

[1]Note that $\mathcal{U}' = \mathcal{U}_{\mathrm{Curl}}$ on a bounded, simply connected domain.

in [6]. Our main results stated in the next section in particular provide optimal conditions for ruling out concentrations and similar effects occurring only on unbounded domains. Their proofs are collected in Section 5.

A second goal of this article and its main technical challenge is the extension of the decomposition result of [7] to unbounded domains. We employ this as an essential tool for studying the weak-strong convergence property, but it also is of independent interest. The decomposition lemma of [7] states that, up to a subsequence, any $\mathcal{A}$-free, bounded sequence in L^p on a bounded domain can be decomposed into the sum of two $\mathcal{A}$-free, bounded sequences, the first p-equiintegrable ("purely oscillating") and the second converging to zero in measure ("purely concentrating"). On general domains, we need to split into more parts, taking into account the additional obstacles for compactness other than oscillations and concentrations which may occur if the domain has infinite measure. This is carried out in Section 4, based on some preliminary observations collected in Section 3. As in [7], we heavily rely on a projection onto $\mathcal{A}$-free fields defined via the Fourier transform, now on the whole space instead of in the framework of periodic functions, whose main properties are derived with the help of suitable Fourier multiplier theorems.

2. Main results

Just as Morrey's by now classical notion of quasiconvexity is important for functionals depending on gradients, $\mathcal{A}$-quasiconvexity is relevant in our setting.

Definition 2.1. Let $x_0 \in \Omega$. Following [7], we say that $f(x_0, \cdot)$ is $\mathcal{A}$-*quasiconvex* at $\xi \in \mathbb{R}^M$ if

$$\int_Q \big[f(x_0, \xi + \varphi(y)) - f(x_0, \xi)\big]\, dy \geq 0 \quad \text{for every } \varphi \in \phi_{\mathcal{A}},$$

Here, $Q := (0,1)^N \subset \mathbb{R}^N$ and

$$\phi_{\mathcal{A}} := \left\{ \varphi \in C^\infty_\sharp(\mathbb{R}^N; \mathbb{R}^M) \;\middle|\; \mathcal{A}\varphi = 0 \text{ on } \mathbb{R}^N \text{ and } \int_Q \varphi\, dx = 0 \right\},$$

where $C^\infty_\sharp(\mathbb{R}^N; \mathbb{R}^M)$ denotes the set of all functions $f \in C^\infty(\mathbb{R}^N; \mathbb{R}^M)$ which are Q-periodic in the sense that $f(y) = f(y+z)$ for every $z \in \mathbb{Z}^N$ and every $y \in \mathbb{R}^N$.

Moreover, for $p > 1$ we say that $f(x_0, \cdot)$ is *strongly* p-$\mathcal{A}$-*quasiconvex* at $\xi \in \mathbb{R}^M$ if

$$\int_Q \big[f(x_0, \xi + \varphi(y)) - f(x_0, \xi)\big]\, dy \geq g\bigg(\int_Q |\varphi|\; dx, \int_Q |\varphi|^p\; dx\bigg) \quad \text{for every } \varphi \in \phi_{\mathcal{A}},$$

with a function $g : [0, \infty)^2 \to [0, \infty]$ which is increasing in its first variable, decreasing in the second, and satisfies $g(t, T) > 0$ for all $t > 0$, $T \geq 0$. (The monotonicity of g need not be strict, and g may depend on x_0 and ξ.)

Finally, we say that f is *(strongly p-) $\mathcal{A}$-quasiconvex*, if $f(x, \cdot)$ is (strongly p-) $\mathcal{A}$-quasiconvex at every $\xi \in \mathbb{R}^M$, for a.e. $x \in \Omega$.

Remark 2.2. If $f(x_0, \cdot)$ is uniformly strictly $\mathcal{A}$-quasiconvex in a sense analogous to the one used in [5] for the gradient case, i.e., if

$$\int_Q \big[f(x_0, \xi + \varphi(y)) - f(x_0, \xi) \big] \, dy \geq c \int_Q |\varphi|^p \, dx \quad \text{for every } \varphi \in \phi_{\mathcal{A}},$$

with some constant $c > 0$, then $f(x_0, \cdot)$ is strongly p-$\mathcal{A}$-quasiconvex at every $\xi \in \mathbb{R}^M$ with $g(t, T) := ct^p$, since $\int_Q |\varphi|^p \, dx \geq \big(\int_Q |\varphi| \, dx \big)^p$ by Hölder's inequality. Note that in particular, $f(x, \xi) := |\xi|^p$ is uniformly strictly $\mathcal{A}$-quasiconvex even in the unconstrained case $\mathcal{A} = 0$, and thus for any $\mathcal{A}$.

Strong p-$\mathcal{A}$-quasiconvexity can be characterized in the following way.

Proposition 2.3. *Let $N \geq 2$, let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be open and suppose that f satisfies* (f:0)–(f:2)*. Then for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^M$, $f(x, \cdot)$ is strongly p-$\mathcal{A}$-quasiconvex at ξ if and only if*

$$\begin{aligned} &f(x, \cdot) \text{ is } \mathcal{A}\text{-quasiconvex at } \xi \text{ and for every sequence } (\varphi_n) \in \Phi_{osc}, \\ &\int_Q f(x, \xi + \varphi_n(y)) \, dy \underset{n \to \infty}{\longrightarrow} f(x, \xi) \quad \Longrightarrow \quad \varphi_n \to 0 \text{ locally in measure}, \end{aligned} \tag{2.1}$$

where Φ_{osc} denotes the set of "purely oscillating" sequences weakly converging to zero in L^p, i.e.,

$$\Phi_{osc} := \left\{ (\varphi_n) \subset \Phi_{\mathcal{A}} \;\middle|\; \begin{array}{l} \varphi_n \rightharpoonup 0 \text{ weakly in } L^p(Q; \mathbb{R}^M) \text{ and} \\ \varphi_n \text{ is equiintegrable in } L^p(Q; \mathbb{R}^M) \end{array} \right\}.$$

Here, "equiintegrable in L^p" is meant in the sense of Definition 2.8 *below, and $\varphi_n \to 0$ locally in measure iff $|K \cap \{|\varphi_n| \geq \delta\}| \to 0$ as $n \to \infty$, for every $\delta > 0$ and every compact $K \subset \mathbb{R}^N$.*

Remark 2.4. Strong p-$\mathcal{A}$-quasiconvexity can also be rephrased in terms of Young measures as follows: $f(x, \cdot)$ is strongly p-$\mathcal{A}$-quasiconvex at $\xi \in \mathbb{R}^M$ if and only if

$$\begin{aligned} &f(x, \cdot) \text{ is } \mathcal{A}\text{-quasiconvex at } \xi \quad \text{and} \\ &\int_{\mathbb{R}^M} f(x, \xi + \mu) \, d\nu(\mu) = f(x, \xi) \quad \Longrightarrow \quad \nu \text{ is a Dirac mass at 0, for} \\ &\text{every homogeneous Young measure } \nu \text{ generated by a sequence in } \Phi_{osc}. \end{aligned} \tag{2.2}$$

The equivalence of (2.1) and (2.2) essentially is a consequence of the results concerning Young measures collected in Section 5. In particular, strong p-$\mathcal{A}$-quasiconvexity is the analogue of strict closed p-quasiconvexity as defined in [20]. Also note that if the sequence generating ν is not required to be equiintegrable in L^p, this still gives an equivalent definition, cf. Remark 5.6.

Remark 2.5. If in addition to (f:0) and (f:1), f is $\mathcal{A}$-quasiconvex, then the p-Lipschitz condition (f:2) automatically holds for certain examples of $\mathcal{A}$. In particular, this is the case for the curl and the divergence (of matrix-valued fields, applied row by row) since Curl-quasiconvexity and Div-quasiconvexity both imply rank-1-convexity. For more details see [4].

As observed in [7], $\mathcal{A}$-quasiconvexity is vital to ensure weak lower semicontinuity of F along $\mathcal{A}$-free sequences and, consequently, the existence of minimizers.

Theorem 2.6 (existence of minimizers for general domains)**.** *Let $N \geq 2$, let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be open, and suppose that f is $\mathcal{A}$-quasiconvex and satisfies* (f:0)–(f:2)*. Moreover, suppose that $I := \inf\{F(v) \mid v \in \mathcal{U}_\mathcal{A}\} > -\infty$ and that there exists a sequence $(u_n) \subset \mathcal{U}_\mathcal{A}$, bounded in L^p, such that $F(u_n) \to I$. Then there exists a $u^* \in \mathcal{U}_\mathcal{A}$ such that $F(u^*) = I$.*

Remark 2.7. Essentially, Theorem 2.6 is a standard application of the direct methods in the calculus of variations. In particular, it suffices to show that F is lower semicontinuous along sequences in $\mathcal{U}_\mathcal{A}$ which weakly converge in L^p. If $\Omega \subset \mathbb{R}^N$ is open and bounded and $f \geq 0$, this is due to [7, Theorem 3.7], and the result easily extends to unbounded domains as $F = \sup_{k \in \mathbb{N}} F_k$ with $F_k(u) := \int_{\Omega_k} f(x,u)\,dx$ defined on the bounded sets $\Omega_k := B_k(0) \cap \Omega$. This works even if (f:2) does not hold, and instead of $f \geq 0$, it actually suffices to have that $f^-(x,u_n)$ (the negative part of f) is weakly relatively compact in L^1 for a minimizing sequence u_n which is bounded in L^p. If, on the other hand, (f:2) holds, then we can use the fact that F is bounded from below to prove weak lower semicontinuity of F without any additional assumptions on the negative part of f as shown in Section 5. More details on role of lower bounds in this context for the gradient case can be found in [13].

In analogy to the case of functionals depending on gradients on bounded domains [5,19,20], strong $\mathcal{A}$-quasiconvexity turns out to be the right condition to rule out possible oscillations of minimizing sequences. Of course, oscillations are not the only obstacle for compactness, and we want to investigate others as well. We employ the following terms to describe some of them, in L^p and related spaces.

Definition 2.8. Let $\Omega \subset \mathbb{R}^N$ be an open set and let X be a normed space of measurable functions mapping Ω into $\mathbb{R}^M$ such that for every $u \in X$ and every

$E \subset \Omega$ measurable, the product $\chi_E u$ also belongs to X. Here and throughout the rest of this article, $\chi_E : \Omega \to \{0,1\}$ denotes the characteristic function of E, i.e., $\chi_E = 1$ on E and $\chi_E = 0$ elsewhere. Furthermore, let (u_n) be a sequence in X. We say that

u_n *does not concentrate in* X	if	$\sup_{n\in\mathbb{N}} \sup_{E\subset\Omega,\ \lvert E\rvert\leq\delta} \lVert\chi_E u_n\rVert_X \underset{\delta\to 0^+}{\longrightarrow} 0,$
u_n *is* $\mathbb{R}^N$*-tight in* X	if	$\sup_{n\in\mathbb{N}} \lVert\chi_{\Omega\setminus B_R(0)} u_n\rVert_X \underset{R\to\infty}{\longrightarrow} 0,$
u_n *is* Ω*-tight in* X	if	for every $\varepsilon > 0$, there is a compact set $K \subset \Omega$ such that $\sup_{n\in\mathbb{N}} \lVert\chi_{\Omega\setminus K} u_n\rVert_X \leq \varepsilon,$
u_n *does not spread out in* X	if	$\sup_{n\in\mathbb{N}} \lVert\chi_{\{\lvert u_n\rvert\leq\delta\}} u_n\rVert_X \underset{\delta\to 0^+}{\longrightarrow} 0,$
u_n *is equiintegrable in* X	if	u_n does not concentrate in X and u_n is $\mathbb{R}^N$-tight X.

Remark 2.9. Equiintegrability implies all four preceding properties, Ω-tight sequences are $\mathbb{R}^N$-tight and $\mathbb{R}^N$-tight sequences do not spread out. The converse of any one of the preceding statements does not hold in general. However, any $\mathbb{R}^N$-tight sequence which does not concentrate is equiintegrable. Some examples for sequences lacking one or more of these properties are given in Remark 2.10 below.

Next, we list conditions on f to rule out possible concentrations of minimizing sequences or a lack of tightness. They all amount to requiring that

$$\begin{gathered}\text{for every sequence } (\varphi_n) \in \Psi, \\ \int_\Omega f(x,\varphi_n(x))\,dx \underset{n\to\infty}{\longrightarrow} \int_\Omega f(x,0)\,dx \quad\Longrightarrow\quad \varphi_n \to 0 \text{ in } L^p,\end{gathered} \tag{2.3}$$

for certain classes of sequences

$$\Psi \subset \Phi := \left\{ (\varphi_n) \subset \mathcal{U}_\mathcal{A} \;\middle|\; \begin{array}{l} (\varphi_n) \text{ is bounded in } L^p \text{ and} \\ \varphi_n \to 0 \text{ locally in measure} \end{array} \right\}$$

with suitable additional properties, each of which is stronger than the convergence to zero locally in measure required so far. In particular, we are interested in the following subsets of Φ:

$$\begin{aligned}
\Phi_c &:= \{(\varphi_n) \in \Phi \mid \varphi_n \to 0 \text{ in } L^p + L^q \text{ for every } q \in (1,p)\} \\
\Phi_{ci} &:= \{(\varphi_n) \in \Phi_c \mid \varphi_n \text{ is } \Omega\text{-tight in } L^p\} \\
\Phi_{cb} &:= \{(\varphi_n) \in \Phi_c \mid \chi_E\varphi_n \to 0 \text{ in } L^p \text{ for every closed } E \subset \Omega\}, \\
\Phi_{c\infty} &:= \left\{ (\varphi_n) \in \Phi_c \;\middle|\; \begin{array}{l} \chi_B\varphi_n \text{ is } \Omega\text{-tight in } L^p \text{ and } \chi_B\varphi_n \to 0 \text{ in } L^p \\ \text{for every bounded, measurable } B \subset \Omega \end{array} \right\}
\end{aligned}$$

$$\Phi_{mov} := \left\{ (\varphi_n) \in \Phi \;\middle|\; \begin{array}{l} \chi_B \varphi_n \to 0 \text{ in } L^p \text{ for all bounded sets } B \subset \Omega \\ \varphi_n \text{ does not concentrate in } L^p \text{ and not spread out in } L^p \end{array} \right\}$$

$$\Phi_{spr} := \{(\varphi_n) \in \Phi \mid \varphi_n \to 0 \text{ in } L^p + L^r \text{ for every } r \in (p, \infty)\}$$

$$\Phi_{ext} := \{(\varphi_n) \in \Phi \mid \varphi_n \to 0 \text{ in } L^p_{\text{loc}}\}.$$

Here, $(L^p + L^q)(\Omega; \mathbb{R}^M) := \{u = v + w \in L^1_{\text{loc}}(\Omega; \mathbb{R}^M) \mid v \in L^p,\ w \in L^q\}$, which is a Banach space with respect to the norm

$$\|u\|_{L^p+L^q} := \inf\{\|v\|_{L^p} + \|w\|_{L^q} \mid v \in L^p \text{ and } w \in L^q \text{ such that } u = v + w\}.$$

Remark 2.10. For a bounded sequence in L^p, $u_n \to 0$ in $L^p + L^q$ for a $q < p$ if and only if $\chi_{\{|u_n| \leq T\}} u_n \to 0$ in L^p for every $T > 0$. In this sense, Φ_c consists of all "purely concentrating" sequences, such as, e.g.,

$$\varphi_n(x) := h_n^{\frac{N}{p}} u(h_n(x - x_n) + x_n), \quad \text{where } (x_n) \subset \Omega,\ h_n \to +\infty.$$

Here and in the examples below, u is a fixed, $\mathcal{A}$-free function in $L^p(\mathbb{R}^N; \mathbb{R}^M)$. The subsets Φ_{ci}, Φ_{cb} and $\Phi_{c\infty}$ of Φ_c distinguish concentrations in the interior, at the boundary and at infinity. Moreover, $u_n \to 0$ in $L^p + L^r$ for a $r > p$ if and only if $\chi_{\{|u_n| \geq t\}} u_n \to 0$ in L^p for every $t > 0$, whence Φ_{spr} consists of all "purely spreading" sequences, e.g.,

$$\varphi_n(x) := \delta_n^{\frac{N}{p}} u(\delta_n x) \text{ on } \mathbb{R}^N, \quad \text{where } \delta_n \to 0^+.$$

Examples for sequences in the set Φ_{mov} include bulks of mass moving off to infinity, e.g., $\varphi_n(x) := u(y_n + x)$, with a sequence $(y_n) \subset \Omega$ with $|y_n| \to \infty$, and possibly also certain sequences that do not spread out but still "vanish" in the sense of [15], e.g., $\varphi_n(n) := \chi_{[0, \frac{1}{n}] \times [0,n]}$ for $\mathcal{A} = 0$ and $\Omega = \mathbb{R}^2$.

Remark 2.11. The validity of (2.3) on Φ_c, Φ_{mov}, Φ_{spr} and Φ_{ext}, respectively, not only depends on f but in general also on Ω. In particular, (2.3) automatically holds for $\Psi = \Phi_{c\infty} \cup \Phi_{mov} \cup \Phi_{spr}$ if Ω is bounded, and $\Phi_{ext} = \Phi_{cb}$ in this case, consisting of sequences purely concentrating at the boundary of Ω.

Remark 2.12. As opposed to the definition of strong p-$\mathcal{A}$-quasiconvexity, (2.3) is not a pointwise property in the first variable of f. It remains an open question whether it is equivalent to a pointwise condition, at least for $\Psi = \Phi_{ci}$ under additional assumptions on f, in particular continuity in x.

Remark 2.13. It is not difficult to give sufficient conditions for (2.3) on the classes listed above. For instance, suppose that f satisfies

$$f(x, \mu) \geq V(\mu) - |h(x)|^p \quad \text{with an } h \in L^p \text{ and a } V \text{ satisfying (2.5)}, \tag{2.4}$$

for every $\mu \in \mathbb{R}^M$ and a.e. $x \in \Omega$, where

$$\begin{aligned} &V : \mathbb{R}^M \to \mathbb{R} \text{ is continuous, } V(0) = 0, \ |V(\mu)| \leq C\,|\mu|^p + C, \quad \text{and} \\ &\int_\Omega V(u)\,dx \geq c \int_\Omega |u|^p\,dx \quad \text{for all } u \in \mathcal{U}_\mathcal{A}, \text{ with a constant } c > 0. \end{aligned} \tag{2.5}$$

Then (2.3) holds for $\Psi = \Phi$ (and thus also for all of the subsets of Φ). In addition, any sequence $(u_n) \subset \mathcal{U}_\mathcal{A}$ such that $F(u_n)$ is bounded in $\mathbb{R}$ is bounded in L^p. Note that depending on $\mathcal{A}$, (f:3) can be significantly weaker than a coercivity condition on f given in a purely pointwise form such as $f(x, \mu) \geq c\,|\mu|^p - |h(x)|^p$.

Our main results are the following.

Theorem 2.14 (domains with compact boundary)**.** *Let $N \geq 2$, let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be open with compact boundary and let $u \in U_\mathcal{A}$. Moreover, suppose that f satisfies* (f:0)–(f:2)*, that $f(x, \cdot)$ is $\mathcal{A}$-quasiconvex at $u(x)$ for a.e. $x \in \Omega$, that* $\inf\{F(v) \mid v \in \mathcal{U}_\mathcal{A}\} > -\infty$*, and that*

$$(u_n) \subset U_\mathcal{A} \textit{ is a sequence s.t. } u_n \rightharpoonup u \textit{ in } L^p(\Omega; \mathbb{R}^M) \textit{ and } \limsup_{n\to\infty} F(u_n) \leq F(u).$$

If $f(x, \cdot)$ is strongly p-$\mathcal{A}$-quasiconvex at $u(x)$ for a.e. $x \in \Omega$ and (2.3) *holds for $\Psi = \Phi_c$, $\Psi = \Phi_{mov}$ and $\Psi = \Phi_{spr}$, then $u_n \to u$ strongly in L^p. More precisely, we have the following:*

(i) *If $f(x, \cdot)$ is strongly p-$\mathcal{A}$-quasiconvex at $u(x)$ for a.e. $x \in \Omega$, then $u_n \to u$ locally in measure.*

(ii) *If* (2.3) *holds for $\Psi = \Phi_c$, then u_n does not concentrate in L^p.*

(iii) *If* (2.3) *holds for $\Psi = \Phi_{mov}$, then $\chi_{\{s^{-1}<|u_n|<s\}} u_n$ is $\mathbb{R}^N$-tight in L^p for every fixed $s \geq 1$.*

(iv) *If* (2.3) *holds for $\Psi = \Phi_{spr}$, then u_n does not spread out in L^p.*

Using the classes Φ_{ci}, $\Phi_{c\infty}$ and Φ_{cb} instead of Φ_c, possible concentrations of u_n can be studied in even greater detail:

Corollary 2.15. *Under the assumptions of Theorem* 2.14*, the following is true:*

(ii.1) *If* (2.3) *holds for $\Psi = \Phi_{ci}$, then $\chi_K u_n$ does not concentrate in L^p, for every compact $K \subset \Omega$.*

(ii.2) *If* (2.3) *holds for $\Psi = \Phi_{c\infty}$, then $\chi_{\Omega_{\delta_n} \setminus B_{R_n}} u_n$ does not concentrate in L^p, for every pair of sequences $R_n \to \infty$ and $\delta_n \to 0^+$.*

(ii.3) *If* (2.3) *holds for $\Psi = \Phi_{cb}$, then $\chi_{\Omega \setminus \Omega_{\delta_n}} u_n$ does not concentrate in L^p, for every sequence $\delta_n \to 0^+$.*

Here, $B_R := \{x \in \mathbb{R}^N \mid |x| < R\}$ and $\Omega_\delta := \{x \in \Omega \mid \operatorname{dist}(x; \partial\Omega) > \delta\}$. In particular, Φ_c can be replaced by $\Phi_{ci} \cup \Phi_{c\infty} \cup \Phi_{cb}$ in Theorem 2.14 (ii)*.*

If Ω is an exterior domain and $f(x,\mu)$ has a limit as $|x| \to \infty$ which is uniform in μ in a suitable sense, Theorem 2.14 can be partially simplified by using a more tangible characterization of (2.3) for $\Psi = \Phi_{c\infty} \cup \Phi_{mov} \cup \Phi_{spr}$ (all the cases related to the behavior of f as $|x| \to \infty$):

Proposition 2.16. *Let $N \geq 2$, let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be the complement of a compact set. Moreover, suppose that f satisfies* (f:0)–(f:2) *and that there exists a function $f_\infty : \mathbb{R}^M \to \mathbb{R}$ such that*

$$\alpha(x) := \sup_{\mu \in \mathbb{R}^M} \frac{|f_\infty(\mu) - f(x,\mu)|}{|\mu|^p + |h(x)|^p} \underset{|x|\to\infty}{\longrightarrow} 0 \quad \textit{for a suitable } h \in L^p(\mathbb{R}^N), \tag{2.6}$$

possibly ignoring a set of measure zero (i.e., $\chi_{\mathbb{R}^N \setminus Z}(x)\alpha(x) \to 0$ as $|x| \to \infty$ for some $Z \subset \mathbb{R}^N$ with $|Z| = 0$). Then (2.3) *holds for $\Psi = \Phi_{c\infty} \cup \Phi_{mov} \cup \Phi_{spr}$ if and only if*

$$\begin{aligned} &\int_{\mathbb{R}^N} f_\infty(\varphi)\,dx \geq g(\|\varphi\|_{L^p}) \quad \textit{for every } \mathcal{A}\textit{-free } \varphi \in L^p(\mathbb{R}^N;\mathbb{R}^M), \\ &\textit{with a suitable } g : [0,\infty) \to \mathbb{R} \textit{ continuous such that } g > 0 \textit{ on } (0,\infty). \end{aligned} \tag{2.7}$$

If the boundary of Ω is not compact, we can still say the following.

Theorem 2.17 (general domains)**.** *Let $N \geq 2$, let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be open and let $u \in U_{\mathcal{A}}$. Moreover, suppose that f satisfies* (f:0)–(f:2)*, that $f(x,\cdot)$ is $\mathcal{A}$-quasiconvex at $u(x)$ for a.e. $x \in \Omega$ and that* $\inf\{F(v) \mid v \in \mathcal{U}_{\mathcal{A}}\} > -\infty$*. Then any bounded sequence $(u_n) \subset U_{\mathcal{A}}$ satisfying $u_n \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^M)$ and* $\limsup_{n\to\infty} F(u_n) \leq F(u)$ *has following properties.*

(i) *If $f(x,\cdot)$ is strongly p-$\mathcal{A}$-quasiconvex at $u(x)$ for a.e. $x \in \Omega$, then $u_n \to u$ locally in measure.*

(ii) *If* (2.3) *holds for $\Psi = \Phi_{ci}$, then $\chi_K u_n$ does not concentrate in L^p, for every compact $K \subset \Omega$.*

(iii) *If* (2.3) *holds for $\Psi = \Phi_{ext}$, then u_n is Ω-tight in L^p.*

In particular, if f and Ω are such that the assumptions of (i)–(iii) *are satisfied, then $u_n \to u$ strongly in L^p.*

By Remark 2.13, this immediately entails the following.

Corollary 2.18. *Let $N \geq 2$, let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be open. Moreover, suppose that f satisfies* (f:0)–(f:2) *as well as* (f:3) *and that f is strongly p-$\mathcal{A}$-quasiconvex. Then any minimizing sequence $(u_n) \subset \mathcal{U}_{\mathcal{A}}$ has a subsequence which strongly converges in L^p.*

Remark 2.19. In fact, all results stated above as well as those of Section 4 are also true for $N = 1$. However, this case requires some minor technical changes in the proofs which we omit for the sake of brevity. For more details see Remark 3.3 below.

Remark 2.20. The sufficient conditions listed in Theorem 2.14 (i)–(iv), Corollary 2.14 (ii.1)–(ii.3) and Theorem 2.17 (i)–(iii), respectively, are also necessary. For instance, if in the situation of Theorem 2.14, the assumption of (ii) does not hold, i.e., (2.3) is violated for $\Psi = \Phi_c$, then there exists a sequence $\varphi_n \in \Phi_c$ with $F(\varphi_n) \to F(0)$ and $\varphi_n \not\to 0$ in L^p (a bounded, $\mathcal{A}$-free, purely concentrating sequence). In particular, for any $u \in \mathcal{U}_{\mathcal{A}}$, $u_n := u + \varphi_n$ is a bounded, $\mathcal{A}$-free sequence in L^p which does concentrate, and $\lim_{n\to\infty} [F(u_n) - F(u)] = \lim_{n\to\infty} [F(\varphi_n) - F(0)] = 0$ as a consequence of Proposition 5.10, whence u_n is admissible for the theorem. Similar arguments also show that the conditions of Theorem 2.14 (iii), (iv), Corollary 2.14 (ii.1)–(ii.3) and and Theorem 2.17 (ii), (iii) are sharp. The necessity of strong $\mathcal{A}$-quasiconvexity for the local convergence in measure in part (i) of both theorems is equivalent to the converse of the second part of Proposition 5.4 discussed in Remark 5.5.

3. Preliminaries

Throughout this article, $\mathcal{A}$ denotes a homogeneous linear differential operator of first order, formally mapping $u : \Omega \to \mathbb{R}^M$ onto $\mathcal{A}u : \Omega \to \mathbb{R}^L$, defined by

$$\mathcal{A}u := \sum_{i=1}^{N} A_i \partial_{x_i} u,$$

with given matrix coefficients $A_i \in \mathbb{R}^{L\times M}$. Its formal adjoint is denoted by $\mathcal{A}^*$, which maps $v : \Omega \to \mathbb{R}^L$ to $\mathcal{A}^* v : \Omega \to \mathbb{R}^M$, where

$$\mathcal{A}^* v = -\sum_{i=1}^{N} A_i^T \partial_{x_i} v.$$

In particular, $\int_\Omega (\mathcal{A}u) \cdot \varphi \, dx = \int_\Omega u \cdot (\mathcal{A}^*\varphi) \, dx$ holds for all $u \in C^1(\Omega; \mathbb{R}^M)$ and all $\varphi \in C_c^1(\Omega; \mathbb{R}^L)$ due to integration by parts. A function $u \in L^1_{\text{loc}}(\Omega; \mathbb{R}^M)$ is called $\mathcal{A}$-*free* if $\mathcal{A}u = 0$ in Ω in the sense of distributions. The symbol of $\mathcal{A}$, i.e., the linear matrix-valued function

$$\mathbb{A} : \mathbb{R}^N \to \mathbb{R}^{L\times M}, \quad \mathbb{A}(\xi) := \sum_{i=1}^{N} A_i \xi_i, \quad \text{where } \xi = (\xi_1, \ldots, \xi_N),$$

is related to $\mathcal{A}$ via the Fourier transform $\mathcal{F}(\mathcal{A}u)(\xi) = \mathbb{A}(\xi)(\mathcal{F}u)(\xi)$. Here, $(\mathcal{F}u)(\xi) := \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi} u(x) \, dx$ for $u \in L^1(\mathbb{R}^N)$ and $\xi \in \mathbb{R}^N$, and the definition is extended to $u \in \mathcal{S}'$ as usual [18], where $\mathcal{S}$ denotes the Schwartz space of rapidly decaying functions of class C^∞ and $\mathcal{S}'$ is its dual. In the vector-valued case, $\mathcal{F}$

operates component-wise. We assume that $\mathcal{A}$ (and hence also $\mathcal{A}^*$) satisfies the condition of constant rank, that is,

$$\text{the rank of } \mathbb{A}(\xi) \in \mathbb{R}^{L\times M} \text{ is constant as a function of } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{3.1}$$

As a consequence, the orthogonal projection $\mathbb{P}(\xi) \in \mathbb{R}^{M\times M}$ onto the kernel of $\mathbb{A}(\xi)$ in $\mathbb{R}^M$ is continuous as a function of $\xi \in \mathbb{R}^N \setminus \{0\}$. We define $\mathbb{P}(0)$ to be the identity matrix. The Fourier multiplier $\mathcal{P} : \mathcal{S}^M \to (\mathcal{S}^M)'$ associated to $\mathbb{P}$ is given by

$$\mathcal{P}\varphi := \mathcal{F}^{-1}(\mathbb{P}\mathcal{F}\varphi), \quad \text{for } \varphi \in \mathcal{S}^M, \text{ with } (\mathbb{P}\mathcal{F}\varphi)(\xi) := \mathbb{P}(\xi)[\mathcal{F}\varphi(\xi)].$$

By definition, $\mathcal{P}$ is a projection onto the kernel of $\mathcal{A}$. Moreover, by the classical Hörmander-Mikhlin multiplier theorem, it extends to a continuous operator $\mathcal{P} : L^p(\mathbb{R}^N;\mathbb{R}^M) \to L^p(\mathbb{R}^N;\mathbb{R}^M)$ projecting $L^p(\mathbb{R}^N;\mathbb{R}^M)$ onto the kernel of $\mathcal{A}$. We also need this property in a broader class of weighted spaces of the form

$$L^p_w(\Omega;\mathbb{R}^M) := \Big\{ u : \Omega \to \mathbb{R}^M \text{ measurable} \,\Big|\, \|u\|_{L^p_w(\Omega;\mathbb{R}^M)} < \infty \Big\},$$

where $\|u\|^p_{L^p_w(\Omega;\mathbb{R}^M)} := \int_\Omega |u(x)|^p\, w(x)\, dx$, and the weight $w : \Omega \to (0,\infty)$ is a measurable function. Due to a result of [14], $\mathcal{P}$ extends to a continuous projection operator on $L^p_w(\Omega;\mathbb{R}^M)$ for various classes of weights. We only reproduce a special case which suffices for our purposes:

Lemma 3.1. *Let $1 < p < \infty$, let $w(x) := \min\{1, |x|^\beta\}$ with a constant $-N < \beta < N(p-1)$, and let $m : \mathbb{R}^N \to \mathbb{R}$ be a bounded function which is 0-homogeneous and of class C^N on $\mathbb{R}^N \setminus \{0\}$. Then the associated Fourier multiplier T given by $T(u) := \mathcal{F}^{-1}(m\mathcal{F}u)$ is a bounded linear operator mapping $L^p_w(\mathbb{R}^N)$ into itself.*

Proof. Since $m(x) = m(x/|x|)$, we have that $\left|D^k m(x)\right| \le C_1 |x|^{-k}$ for every $x \in \mathbb{R}^N \setminus \{0\}$ and every $k = 0,\dots,N$, with a constant $C_1 > 0$ only depending on m and N. As a consequence, for every $s \in (1,2]$ such that $sk \neq N$ for $k = 0,\dots,N$,

$$R^{sk-N} \int_{R<|x|<2R} \left|D^k m(x)\right|^s dx \le C_2 \quad \text{for every } R > 0 \text{ and every } k = 0,\dots,N,$$

with a constant C_2 only depending on N, s and C_1. This means that $m \in M(s,N)$ in the notation of [14], and with this property established, [14, Theorem 2] yields the assertion. □

In particular, this applies to the space L^p which corresponds to the case $w \equiv 1$ ($\beta = 0$). Besides weighted spaces, we also employ Orlitz-type spaces that allow for functions whose growth near singularities and decay at infinity (in an unbounded domain) is governed by different exponents. For us, spaces of the form $L^p + L^q$ and $L^p \cap L^q$ suffice, where the associated norms are given by

$$\begin{aligned} \|u\|_{L^p+L^q} &:= \inf \{ \|v\|_{L^p} + \|w\|_{L^q} \mid v \in L^p,\ w \in L^q,\ u = v + w \} \\ \|u\|_{L^p \cap L^q} &:= \|u\|_{L^p} + \|u\|_{L^q} . \end{aligned}$$

Note that if $p \geq q$ and $u \in L^p + L^q$, then $\chi_{\{|u| \leq 1\}} \in L^p$ and $\chi_{\{|u| \geq 1\}} \in L^q$ (the larger exponent p determines decay and the smaller exponent q determines growth), and we have continuous embeddings $L^p,\ L^q \subset L^p + L^q \subset L^{\tilde{p}} + L^{\tilde{q}}$ if $\tilde{p} \geq p \geq q \geq \tilde{q}$. Lemma 3.1 can be extended to $L^p + L^q$ as follows.

Lemma 3.2. *Let $1 < q < p < \infty$ and let $m : \mathbb{R}^N \to \mathbb{R}$ be a bounded function which is 0-homogeneous and of class C^N on $\mathbb{R}^N \setminus \{0\}$. Then the associated Fourier multiplier T given by $T(u) := \mathcal{F}^{-1}[m(\mathcal{F}u)]$ is a bounded linear operator mapping $L^q(\mathbb{R}^N;\mathbb{R}) + L^p(\mathbb{R}^N;\mathbb{R})$ into itself.*

Proof. For every $\varepsilon > 0$, there exists $v \in L^q$ and $w \in L^p$ with $v + w = u$ such that $\|v\|_{L^q} + \|w\|_{L^p} \leq \|u\|_{L^q+L^p} + \varepsilon$. Lemma 3.1 with $\beta = 0$ thus implies that

$$\|Tu\|_{L^q+L^p} \leq \|Tv\|_{L^q} + \|Tw\|_{L^p} \leq C \left(\|v\|_{L^q} + \|w\|_{L^p}\right) \leq C \left(\|u\|_{L^q+L^p} + \varepsilon\right)$$

for arbitrary ε with a constant C independent of u and ε. □

In the following, norms involving certain inverse derivatives will play a role, which we express by means of the operator $(-\Delta)^{-\frac{1}{2}}$, defined by

$$(-\Delta)^{-\frac{1}{2}} u := \mathcal{F}^{-1} |2\pi\xi|^{-1} \mathcal{F}u, \tag{3.2}$$

for any $u \in \mathcal{S}'$ such that the "pointwise" product of $|2\pi\xi|^{-1}$ with $(\mathcal{F}u)(\xi)$ is well defined in $\mathcal{S}'$. If $u \in L^1$, a more explicit definition of $(-\Delta)^{-\frac{1}{2}}$ can be given in terms of the corresponding Riesz potential, namely,

$$((-\Delta)^{-\frac{1}{2}} u)(x) = \frac{1}{\sigma} \int_{\mathbb{R}^N} |x - y|^{1-N} u(y)\, dy, \tag{3.3}$$

with a normalizing constant $\sigma = \sigma(N) > 0$, cf. [18, p. 117].

Remark 3.3. To be precise, (3.3) only holds if $N \geq 2$, which is the reason for this assumption in our main results as well as in any other statement of this note directly or indirectly exploiting (3.3) in form of Lemma 3.5 below. Of course, this is just a minor technical issue. The case $N = 1$ could easily be treated separately, for instance using the antiderivative instead of $(-\Delta)^{-\frac{1}{2}}$.

Extending [7, Lemma 2.14], which in turn is largely based on ideas of [17], the properties of the projection $\mathcal{P}$ in $L^p + L^q$ and in L^p_w can be summarized as follows.

Lemma 3.4. *Let $1 < q \le p < \infty$ and suppose that* (3.1) *holds. Then we have the following.*

(i) $\mathcal{P} : (L^q{+}L^p)(\mathbb{R}^N;\mathbb{R}^M) \to (L^q{+}L^p)(\mathbb{R}^N;\mathbb{R}^M)$ *is a linear, bounded operator.*

(ii) $\mathcal{P}v = v$ *for every $\mathcal{A}$-free $v \in (L^q + L^p)(\mathbb{R}^N;\mathbb{R}^M)$, and $\mathcal{A} \circ \mathcal{P} = 0$.*

(iii) *Let u_n be a bounded sequence in $(L^q + L^p)(\mathbb{R}^N;\mathbb{R}^M)$. If u_n does not concentrate in $L^q + L^p$, neither does $\mathcal{P}u_n$. Similarly, if u_n is $\mathbb{R}^N$-tight in $L^q + L^p$ then so is $\mathcal{P}u_n$, and if u_n does not spread out in $L^q + L^p$, then neither does $\mathcal{P}u_n$.*

(iv) $c\|(-\Delta)^{-\frac{1}{2}}\mathcal{A}u\|_{L^q+L^p} \le \|(I-\mathcal{P})u\|_{L^q+L^p} \le C\|(-\Delta)^{-\frac{1}{2}}\mathcal{A}u\|_{L^q+L^p}$ *for every $u \in (L^q + L^p)(\mathbb{R}^N;\mathbb{R}^M)$, with constants $c, C > 0$ independent of u.*

Moreover, all of the above stays true if $L^q + L^p$ is replaced with L^p_w, where w may be any positive weight function such that Lemma 3.1 *holds.*

Proof. We essentially proceed as in [7]. As a consequence of (3.1), the projection $\mathbb{P}(\xi)$ is a 0-homogeneous function of ξ of class C^∞ on $\mathbb{R}^N \setminus \{0\}$, whence Lemma 3.2 yields (i). The definition of $\mathcal{P}$ implies (ii): In view of (i), since C^∞_c is dense in $L^q + L^p$ and the set of $\mathcal{A}$-free functions is a closed subspace of $L^q + L^p$, it suffices to show that $\mathcal{P}v = v$ for every $\mathcal{A}$-free $v \in C^\infty_c$ and $\mathcal{A}\mathcal{P}v = 0$ for every $v \in C^\infty_c$. Abbreviating $\hat{v}(\xi) := \mathcal{F}(v)(\xi)$, this is the case if and only if $\mathbb{P}(\xi)\hat{v}(\xi) = \hat{v}(\xi)$ for every $\xi \neq 0$ provided $\mathbb{A}(\xi)\hat{v}(\xi) = 0$, and $\mathbb{A}(\xi)\mathbb{P}(\xi)\hat{v}(\xi) = 0$ for every $\xi \neq 0$. Both properties are clear since by definition, $\mathbb{P}(\xi)$ is a projection onto the kernel of $\mathbb{A}(\xi)$.

For the proof of (iii) consider a bounded sequence u_n in $L^q + L^p$. If u_n does not concentrate in $L^q + L^p$, we have that $\sup_{n\in\mathbb{N}} \left\|\chi_{\{|u_n|\le h\}}u_n - u_n\right\|_{L^q+L^p} \to 0$ as $h \to \infty$, and since $\mathcal{P}$ is continuous in $L^q + L^p$, we also get that

$$\sup_{n\in\mathbb{N}} \left\|\mathcal{P}(\chi_{\{|u_n|\le h\}}u_n) - \mathcal{P}u_n\right\|_{L^q+L^p} \underset{h\to\infty}{\longrightarrow} 0. \tag{3.4}$$

On the other hand, for fixed h, $\chi_{\{|u_n|\le h\}}u_n$ is bounded in L^∞ and thus also in L^s for any $s > p$. By continuity of $\mathcal{P}$ in L^s, this implies that $\mathcal{P}(\chi_{\{|u_n|\le h\}}u_n)$ is bounded in L^s. By Hölder's inequality we infer that $\mathcal{P}(\chi_{\{|u_n|\le h\}}u_n)$ does not concentrate in L^q since $s > q$, which also means that $\mathcal{P}(\chi_{\{|u_n|\le h\}}u_n)$ does not concentrate in $L^q + L^p$ for fixed h, since $p \ge q$. Together with (3.4), this implies that $\mathcal{P}u_n$ does not concentrate in $L^q + L^p$. If u_n does not spread out in $L^q + L^p$, an analogous argument gives that

$$\sup_{n\in\mathbb{N}} \left\|\mathcal{P}(\chi_{\{|u_n|\ge h\}}u_n) - \mathcal{P}u_n\right\|_{L^q+L^p} \underset{h\to 0^+}{\longrightarrow} 0$$

and that $\mathcal{P}(\chi_{\{|u_n|\geq h\}}u_n)$ does not spread out in L^q+L^p for fixed $h>0$ (since it is bounded in L^s with $1<s<q$), which implies that $\mathcal{P}u_n$ does not spread out in L^q+L^p. Last but not least, if u_n is $\mathbb{R}^N$-tight in L^q+L^p, we get that

$$\sup_{n\in\mathbb{N}}\left\|\mathcal{P}(\chi_{B_h(0)}u_n)-\mathcal{P}u_n\right\|_{L^q+L^p}\underset{h\to\infty}{\longrightarrow}0$$

and that $\mathcal{P}(\chi_{B_h(0)}u_n)$ is $\mathbb{R}^N$-tight in L^q+L^p for fixed h (since it is bounded in $L^q_{\tilde{w}}$ with $\tilde{w}(x):=\min\{1,|x|^{\tilde{\beta}}\}$, for any $0<\tilde{\beta}<N(q-1)$), whence $\mathcal{P}u_n$ is $\mathbb{R}^N$-tight in L^q+L^p.

To get (iv), first observe that

$$(I-\mathbb{P})(\xi)\mathcal{F}u(\xi)=\mathbb{Q}(\xi)\mathbb{A}(\xi)\mathcal{F}u(\xi)=2\pi\mathbb{Q}\left(\frac{\xi}{|\xi|}\right)\mathcal{F}((-\Delta)^{-\frac{1}{2}}\mathcal{A}u)(\xi), \tag{3.5}$$

where $\mathbb{Q}:\mathbb{R}^N\setminus\{0\}\to\mathbb{R}^{L\times L}$ is defined by

$$\begin{aligned}\mathbb{Q}(\xi)\mathbb{A}(\xi)\eta:=\eta\quad&\text{for any }\eta\in(\ker\mathbb{A}(\xi))^\perp\subset\mathbb{R}^M\\ \mathbb{Q}(\xi)\mu:=0\quad&\text{for any }\mu\in(\operatorname{range}\mathbb{A}(\xi))^\perp\subset\mathbb{R}^L.\end{aligned}$$

Note that $\mathbb{Q}$ is homogeneous of degree -1 as a function of ξ since $\mathbb{A}$ is homogeneous of degree 1, which justifies the second equality in (3.5). Moreover, as a consequence of (3.1), both $\operatorname{range}\mathbb{A}(\xi)$ and $(\operatorname{range}\mathbb{A}(\xi))^\perp$ have constant dimension and vary smoothly with $\xi\in\mathbb{R}^N\setminus\{0\}$, and $\mathbb{A}(\xi):(\ker\mathbb{A}(\xi))^\perp\to\operatorname{range}\mathbb{A}(\xi)$ is invertible with inverse smoothly depending on ξ, whence $\mathbb{Q}$ is of class C^∞. In particular, $\mathbb{Q}\big(\frac{\xi}{|\xi|}\big)$ gives rise to a Fourier multiplier in L^q+L^p by Lemma 3.2, whence (3.5) implies the second inequality in (iv). The first inequality follows in the same way, since $\mathbb{A}\big(\frac{\xi}{|\xi|}\big)\mathbb{Q}\big(\frac{\xi}{|\xi|}\big)\mathbb{A}\big(\frac{\xi}{|\xi|}\big)=\mathbb{A}\big(\frac{\xi}{|\xi|}\big)$ and $\mathbb{A}\big(\frac{\xi}{|\xi|}\big)$ also gives rise to a continuous Fourier multiplier in L^q+L^p.

Finally, note that all of the arguments above also work for L^p_w instead of L^q+L^p if we use Lemma 3.1 instead of Lemma 3.2 and suitably adapt the auxiliary spaces employed in the proof of (iii) and (iv). □

We will use Lemma 3.4 (iv) to handle domains other than the whole space, and for this purpose, the following compactness result is also crucial.

Lemma 3.5. *Let v_n be a bounded sequence in $L^p(\mathbb{R}^N)$ with some $1<p<\infty$. Moreover, suppose that there is a fixed compact set $K\subset\mathbb{R}^N$ containing the support of v_n for every n and that $\int_{\mathbb{R}^N}v_n\,dx=0$ for every n. Then $w_n:=(-\Delta)^{-\frac{1}{2}}v_n$ is bounded in $L^p(\mathbb{R}^N)$, and it has a subsequence which converges strongly in $L^p(\mathbb{R}^N)$.*

Lemma 3.5 is probably known, but since I was unable to find a suitable reference, a proof is given below.

Proof. Let B_r denote a ball with radius r centered at 0, containing K. Observe that for fixed $R > 0$, (3.3) yields

$$\int_{B_R} \left|(-\Delta)^{-\frac{1}{2}} v_n(x)\right|^p dx = \|\kappa * v_n\|^p_{L^p(B_R)} \leq \|\kappa\|^p_{L^1(B_{R+r})} \|v_n\|^p_{L^p(B_r)}, \tag{3.6}$$

where $*$ denotes the convolution and $\kappa(z) := \sigma^{-1} |z|^{1-N}$. Moreover, for every $R \geq 2r$, there is a constant $C = C(N, r) > 0$ such that

$$\sup_{y \in B_r} \left| |x-y|^{1-N} - |x|^{1-N} \right| \leq C |x|^{-N} \quad \text{for every } x \text{ with } |x| > R. \tag{3.7}$$

Since $\int_{B_r} v_n \, dx = 0$, (3.7) implies that

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R} \left|(-\Delta)^{-\frac{1}{2}} v_n\right|^p dx &= \int_{\mathbb{R}^N \setminus B_R} \left| \int_{B_r} (|x-y|^{1-N} - |x|^{1-N}) v_n(y) \, dy \right|^p dx \\ &\leq \|v_n\|^p_{L^p} \int_{\mathbb{R}^N \setminus B_R} C^p |x|^{-pN} \, dx \end{aligned} \tag{3.8}$$

for $R \geq 2r$. Note that $|x|^{-Np}$ is integrable on $\mathbb{R}^N \setminus B_{2r}$ for every $p > 1$. In particular, $(-\Delta)^{-\frac{1}{2}} v_n$ is bounded in $L^p(\mathbb{R}^N)$ by (3.6) and (3.8) combined. In addition, (3.8) implies that

$$\int_{\mathbb{R}^N \setminus B_R} \left|(-\Delta)^{-\frac{1}{2}} v_n\right|^p dx \underset{R \to \infty}{\longrightarrow} 0 \quad \text{uniformly in } n. \tag{3.9}$$

Moreover, as in (3.6) we get

$$\begin{aligned} &\int_{B_R} \left|(-\Delta)^{-\frac{1}{2}} v_n(x) - (-\Delta)^{-\frac{1}{2}} v_n(x+h)\right|^p dx \\ &\leq \|(\kappa(\cdot) - \kappa(\cdot + h))\|^p_{L^1(B_{R+r})} \|v_n\|^p_{L^p(B_r)} \underset{|h| \to 0}{\longrightarrow} 0 \quad \text{uniformly in } n, \end{aligned} \tag{3.10}$$

for any fixed $R > 0$, since κ is integrable on bounded sets and the shift is continuous in L^1. Together, (3.9) and (3.10) imply that $\{(-\Delta)^{-\frac{1}{2}} v_n \mid n \in \mathbb{N}\}$ is contained in a compact subset of $L^p(\mathbb{R}^N)$, by a standard criterion for relative compactness in L^p (e.g., [2]). □

4. Decomposition of $\mathcal{A}$-free sequences

We now derive a decomposition lemma in the tradition of [1, 7, 8, 11] and [12], here for a sequence of $\mathcal{A}$-free fields on the whole space. This result and suitable extensions to other unbounded domains will be our main tool for obtaining compactness of minimizing sequences.

Lemma 4.1. *Let $1 < p < \infty$ and let $\mathcal{A}$ be a linear differential operator of first order satisfying* (3.1). *Moreover, suppose that u_n is a bounded, $\mathcal{A}$-free sequence in $L^p(\mathbb{R}^N;\mathbb{R}^M)$ with $u_n \rightharpoonup u$ weakly in L^p. Then there exist a subsequence $u_{k(n)}$ of u_n and five bounded, $\mathcal{A}$-free sequences $w_n^0,\dots,w_n^4$ in $L^p(\mathbb{R}^N;\mathbb{R}^M)$ such that*

$$u_{k(n)} = u + w_n^0 + w_n^1 + w_n^2 + w_n^3 + w_n^4 \quad \text{for every } n \in \mathbb{N}$$

and the following properties hold:

(a) $w_n^0 \rightharpoonup 0$ *weakly in L^p, and w_n^0 is equiintegrable in L^p.*
(b) w_n^1 *is $\mathbb{R}^N$-tight in L^p, and $w_n^1 \to 0$ in $L^p + L^q$ for every $q \in (1,p)$.*
(c) $\chi_B w_n^2 \to 0$ *in L^p for any bounded, measurable set $B \subset \mathbb{R}^N$ and* $w_n^2 \to 0$ *in $L^p + L^q$ for every $q \in (1,p)$.*
(d) $\chi_B w_n^3 \to 0$ *in L^p for any bounded, measurable set $B \subset \mathbb{R}^N$,* w_n^3 *does not spread out in L^p and w_n^3 does not concentrate in L^p.*
(e) $w_n^4 \to 0$ *in $L^r + L^p$ for every $r \in (p,\infty)$.*

Remark 4.2. Using (a)–(e) to check Vitali's criteria for compactness in L^p, it is not difficult to see that if $u_{k(n)} = u + \tilde{w}_n^0 + \ldots + \tilde{w}_n^4$ is another decomposition with the same properties, then $w_n^j - \tilde{w}_n^j \to 0$ strongly in L^p. In this sense, the component sequences are uniquely determined.

For the proof of Lemma 4.1, we first need an auxiliary result which represents a decomposition lemma in L^p, summarizing Chacon's biting lemma and suitable variants for unbounded domains. It is based on four different kinds of truncations of L^p-functions.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^N$ be open and let $1 \le p < \infty$. Then every bounded sequence $(v_n) \subset L^p(\Omega;\mathbb{R}^M)$ has a subsequence $(v_{k(n)})$ such that*

$$\begin{aligned} &\chi_{\{|v_{k(n)}|\le n\}} v_{k(n)} \text{ does not concentrate in } L^p(\Omega;\mathbb{R}^M),\\ &\chi_{\{|v_{k(n)}|\ge \frac{1}{n}\}} v_{k(n)} \text{ does not spread out in } L^p(\Omega;\mathbb{R}^M),\\ &\chi_{B_n(0)} v_{k(n)} \text{ is } \mathbb{R}^N\text{-tight in } L^p(\Omega;\mathbb{R}^M) \text{ and}\\ &\chi_{K_n} v_{k(n)} \text{ is } \Omega\text{-tight in } L^p(\Omega;\mathbb{R}^M), \end{aligned} \tag{4.1}$$

where $K_n := \left\{x \in \Omega \;\middle|\; |x| \le n \text{ and } \operatorname{dist}(x;\partial\Omega) \ge \frac{1}{n}\right\}$.

Proof. This is essentially well known. For instance, the first three lines of (4.1) immediately follow from [12, Lemma 3.3 – Lemma 3.5], and the fourth line can be obtained analogously to the third. We omit the details. □

Proof of Lemma 4.1. W.l.o.g. we may assume that $u = 0$ (otherwise, since u is $\mathcal{A}$-free, we can decompose $\tilde{u}_n := u_n - u$ instead). For $j = 0,\dots,4$, let

$w_n^j := \mathcal{P} W_n^j \in L^p(\mathbb{R}^N;\mathbb{R}^M)$ with

$$\begin{aligned}
W_n^0 &:= \chi_{B_n(0)}\chi_{\{|u_{k(n)}|\leq n\}}u_{k(n)}\\
W_n^1 &:= \chi_{B_n(0)}\big(1-\chi_{\{|u_{k(n)}|\leq n\}}\big)u_{k(n)}\\
W_n^2 &:= \big(1-\chi_{B_n(0)}\big)\big(1-\chi_{\{|u_{k(n)}|\leq n\}}\big)u_{k(n)}\\
W_n^3 &:= \chi_{\{|u_{k(n)}|\geq \frac{1}{n}\}}\big(1-\chi_{B_n(0)}\big)\chi_{\{|u_{k(n)}|\leq n\}}u_{k(n)}\\
W_n^4 &:= \big(1-\chi_{\{|u_{k(n)}|\geq \frac{1}{n}\}}\big)\big(1-\chi_{B_n(0)}\big)\chi_{\{|u_{k(n)}|\leq n\}}u_{k(n)},
\end{aligned}$$

where the subsequence $u_{k(n)}$ is chosen according to Lemma 4.3 with $v_n := u_n$. By definition, $u_{k(n)} = \mathcal{P}u_{k(n)} = w_n^0 + \ldots + w_n^4$, each w_n^j is $\mathcal{A}$-free, and the sequences w_n^j are bounded in L^p by continuity of $\mathcal{P}$ in L^p. Moreover, due to the choice of $u_{k(n)}$ and the definition of W_n^j, the sequences W_n^j (in place of w_n^j) have the properties (a)–(e) listed in the assertion. The projected sequences w_n^j inherit these: $\mathbb{R}^N$-tightness, absence of concentration, absence of spreading and equiintegrability in L^p all survive the application of $\mathcal{P}$ due to Lemma 3.4 (iii). Convergence in $L^p + L^q$ or in $L^r + L^p$ with $1 < q < p$ and $p < r < \infty$ is also preserved, as a consequence of Lemma 3.4 (i), as is weak convergence to zero in L^p. Finally, note that for a bounded sequence v_n in $L^p(\mathbb{R}^N;\mathbb{R}^M)$, $\chi_B v_n \to 0$ in L^p for every bounded, open $B \subset \Omega$ if and only if $v_n \to 0$ in L^p_w with the weight $w(x) := \min\{1, |x|^{-\frac{1}{2}}\}$ (or any other bounded weight which is locally bounded away from zero and converges to zero as $|x| \to \infty$). Hence, the continuity of $\mathcal{P}$ in L^p_w also yields that $\chi_B w_n^{2,3} \to 0$ in L^p just as $W_n^{2,3}$. □

As it turns out, Lemma 4.1 can be extended to any domain but only with a somewhat coarser decomposition.

Lemma 4.4. *Let $N \geq 2$, let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be open and let $\mathcal{A}$ be a linear differential operator of first order satisfying* (3.1)*. Moreover, suppose that u_n is a bounded, $\mathcal{A}$-free sequence in $L^p(\Omega;\mathbb{R}^M)$ with $u_n \rightharpoonup u$ weakly in L^p. Then there exist a subsequence $u_{k(n)}$ of u_n and bounded, $\mathcal{A}$-free sequences $(v_n), (w_n) \subset L^p(\mathbb{R}^N;\mathbb{R}^M)$ and $(z_n) \subset L^p(\Omega;\mathbb{R}^M)$ such that*

$$u_{k(n)} = u + v_n + w_n + z_n \quad \text{in } \Omega \text{ for every } n \in \mathbb{N}$$

and the following properties hold:

(a) *$v_n \rightharpoonup 0$ weakly in $L^p(\mathbb{R}^N;\mathbb{R}^M)$ and v_n is equiintegrable in $L^p(\mathbb{R}^N;\mathbb{R}^M)$.*

(b) *$w_n \to 0$ in $(L^p + L^q)(\mathbb{R}^N;\mathbb{R}^M)$ for every $1 \leq q < p$, and w_n is Ω-tight in $L^p(\Omega;\mathbb{R}^M)$.*

(c) *$z_n \to 0$ in $L^p_{loc}(\Omega;\mathbb{R}^M)$.*

Proof. Observe that $\mathcal{A}u = 0$ in Ω. We choose a sequence of cut-off functions

$(\gamma_j) \subset C^1(\mathbb{R}^N;[0,1])$ such that

$$\begin{aligned} \{\gamma_j > 0\} &\subset \Big\{x \in \Omega \;\Big|\; |x| < j \text{ and } \operatorname{dist}(x;\partial\Omega) > \tfrac{1}{j}\Big\} \quad \text{and} \\ \{0 < \gamma_j < 1\} &\subset \Big\{x \in \Omega \;\Big|\; |x| > j-1 \text{ or } \operatorname{dist}(x;\partial\Omega) < \tfrac{2}{j}\Big\} \end{aligned} \tag{4.2}$$

For every fixed j, we have $\mathcal{A}(\gamma_j(u_n - u)) = \sum_{i=1}^N A_i\,(\partial_{x_i}\gamma_j)\,(u_n-u) \rightharpoonup 0$ weakly in $L^p(\mathbb{R}^N)$, as $n \to \infty$. Since $\operatorname{supp}\mathcal{A}(\gamma_j(u_n-u)) \subset \operatorname{supp}\nabla\gamma_j \subset \{0<\gamma_j<1\}$, whose closure is a compact set, and since $\int_{\mathbb{R}^N}\mathcal{A}(\gamma_j(u_n-u))\,dx = 0$ due to integration by parts, Lemma 3.5 is applicable to $\mathcal{A}(\gamma_j(u_n-u))$ and it yields that up to a subsequence,

$$\big\|(-\Delta)^{-\frac{1}{2}}\mathcal{A}(\gamma_j(u_n-u))\big\|_{L^p(\mathbb{R}^N;\mathbb{R}^L)} \underset{n\to\infty}{\longrightarrow} 0 \tag{4.3}$$

for fixed j. As a consequence of (4.3), we can select a subsequence $k(n)$ of n (fast enough) such that

$$\big\|(-\Delta)^{-\frac{1}{2}}\mathcal{A}(\gamma_n(u_m-u))\big\|_{L^p(\mathbb{R}^N;\mathbb{R}^L)} \le \frac{1}{n} \quad \text{for every } m \ge k(n). \tag{4.4}$$

Moreover, by Lemma 4.3 we can pass to another subsequence of $k(n)$ (not relabeled) such that

$$\gamma_n(u_{k(n)}-u) \text{ is } \Omega\text{-tight in } L^p(\Omega;\mathbb{R}^M). \tag{4.5}$$

Now define $\tilde{u}_n := \mathcal{P}(\gamma_n(u_n-u))$, which is a bounded sequence in $L^p(\mathbb{R}^N;\mathbb{R}^M)$ satisfying $\mathcal{A}\tilde{u}_n = 0$ on $\mathbb{R}^N$, and decompose $\tilde{u}_n = \tilde{w}_n^0 + \ldots + \tilde{w}_n^4$ according to Lemma 4.1 (again passing to a subsequence if necessary). We claim that the decomposition $u_n = u + v_n + w_n + z_n$ with

$$\begin{aligned} v_n &:= \tilde{w}_n^0 \\ w_n &:= \tilde{w}_n^1 \\ z_n &:= (I-\mathcal{P})[\gamma_n(u_n-u)] + (1-\gamma_n)(u_n-u) + \tilde{w}_n^2 + \tilde{w}_n^3 + \tilde{w}_n^4, \end{aligned}$$

then has the asserted properties. First note that v_n and w_n are bounded sequences in L^p and $\mathcal{A}$-free on Ω by definition, whence the same holds for $z_n = u_n - u - v_n - w_n$. Since v_n satisfies (a) by construction, it remains to show that (b) and (c) hold.

(c): Since $\gamma_n(u_{k(n)}-u)$ is supported in Ω and Ω-tight in $L^p(\Omega;\mathbb{R}^M)$, it is $\mathbb{R}^N$-tight in $L^p(\mathbb{R}^N;\mathbb{R}^M)$. Hence, $\tilde{u}_n$ is $\mathbb{R}^N$-tight in $L^p(\mathbb{R}^N;\mathbb{R}^M)$ by Lemma 3.4 (iii). Consequently, $R_n := \tilde{w}_n^2 + \tilde{w}_n^3 + \tilde{w}_n^4 = \tilde{u}_n - \tilde{w}_n^0 - \tilde{w}_n^1$ is $\mathbb{R}^N$-tight in $L^p(\mathbb{R}^N;\mathbb{R}^M)$, and by the properties of $\tilde{w}_n^j$, $j=2,3,4$, we also have $R_n \to 0$ in $L^p(B;\mathbb{R}^M)$ for every open, bounded $B \subset \mathbb{R}^N$. Combined, this implies that

$$R_n \to 0 \quad \text{in } L^p(\mathbb{R}^N;\mathbb{R}^M). \tag{4.6}$$

Moreover, by Lemma 3.4 (iv), (4.4) yields that

$$(I-\mathcal{P})[\gamma_n(u_{k(n)}-u)] \to 0 \quad \text{strongly in } L^p(\mathbb{R}^N,\mathbb{R}^M). \tag{4.7}$$

As a consequence of (4.6), (4.7) and the second line of (4.2), we now get that $z_n \to 0$ in $L^p(K;\mathbb{R}^M)$ for any compact $K \subset \Omega$.

(b): Combined, (4.7) and (4.5) imply that $\tilde{u}_n = \mathcal{P}[\gamma_n(u_{k(n)}-u)]$ is Ω-tight in $L^p(\Omega;\mathbb{R}^M)$. Hence, $w_n = \tilde{w}_n^1 = \tilde{u}_n - \tilde{w}_n^0 - R_n$ is Ω-tight in $L^p(\Omega;\mathbb{R}^M)$ as well, where we also used that $\tilde{w}_n^0$ and R_n are Ω-tight in $L^p(\Omega;\mathbb{R}^M)$, the former since it is equintegrable in L^p and the latter because of (4.6). In addition, we clearly have $w_n = \tilde{w}_n^1 \to 0$ in $(L^p+L^q)(\mathbb{R}^N;\mathbb{R}^M)$ for any $q \in (1,p)$ by definition of $\tilde{w}_n^1$. □

The result of Lemma 4.4 could be improved if the domain admits a continuous extension operator for $\mathcal{A}$-free vector fields in L^p from Ω to $\mathbb{R}^N$. However, to my knowledge, extension of $\mathcal{A}$-free fields has not yet been investigated even on bounded domains except in a few special cases such as gradient fields (e.g. [2]) and divergence-free fields [10]. In any case, for domains with compact boundary, the ideas already used in Lemma 4.4 suffice to obtain a refined decomposition without relying on general extension results. In comparison to Lemma 4.1, the decomposition now has an additional component w_n^5 which carries concentrations at the boundary.

Lemma 4.5. *Let $N \geq 2$, let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be open with compact boundary and let $\mathcal{A}$ be a linear differential operator of first order satisfying* (3.1). *Moreover, suppose that u_n is a bounded, $\mathcal{A}$-free sequence in $L^p(\mathbb{R}^N;\mathbb{R}^M)$ with $u_n \rightharpoonup u$ weakly in L^p. Then there exist a subsequence $u_{k(n)}$ of u_n and six bounded, $\mathcal{A}$-free sequences $w_n^0,\dots,w_n^5$ in $L^p(\Omega;\mathbb{R}^M)$ such that*

$$u_{k(n)} = u + w_n^0 + w_n^1 + w_n^2 + w_n^3 + w_n^4 + w_n^5 \quad \text{for every } n \in \mathbb{N}$$

and the following properties hold:

(a) *$w_n^0 \rightharpoonup 0$ weakly in L^p, and w_n^0 is equiintegrable in L^p.*
(b) *w_n^1 is Ω-tight in L^p and $w_n^1 \to 0$ in $L^p + L^q$ for every $q \in (1,p)$.*
(c) *$\chi_B w_n^2 \to 0$ in L^p for any bounded, measurable set $B \subset \mathbb{R}^N$ and $w_n^2 \to 0$ in $L^p + L^q$ for every $q \in (1,p)$.*
(d) *$\chi_B w_n^3 \to 0$ in L^p for any bounded, measurable set $B \subset \mathbb{R}^N$, w_n^3 does not spread out in L^p and w_n^3 does not concentrate in L^p.*
(e) *$w_n^4 \to 0$ in $L^r + L^p$ for every $r \in (p,\infty)$.*
(f) *$\chi_E w_n^5 \to 0$ in L^p for any closed set $E \subset \Omega$.*

Moreover, the component sequences $w_n^0,\dots,w_n^4 \in L^p(\Omega;\mathbb{R}^M)$ can be extended to bounded, $\mathcal{A}$-free sequences in $L^p(\mathbb{R}^N;\mathbb{R}^M)$ satisfying (a)–(e) *even on $\mathbb{R}^N$.*

Proof. Using Lemma 3.5 as in the proof of Lemma 4.4, we can find a sequence of cut-off functions $\gamma_n \in C^1(\mathbb{R}^N;[0,1])$ and an associated subsequence $k(n)$ of n such that

$$\begin{aligned} \{\gamma_n > 0\} \subset \big\{x \in \Omega \,\big|\, \operatorname{dist}(x;\partial\Omega) \geq \tfrac{1}{n}\big\} \quad \text{and} \\ \{0 < \gamma_n < 1\} \subset \big\{x \in \Omega \,\big|\, \operatorname{dist}(x;\partial\Omega) \leq \tfrac{2}{n}\big\} \end{aligned} \tag{4.8}$$

and

$$\big\|(-\Delta)^{-\frac{1}{2}}\mathcal{A}(\gamma_n(u_{k(n)}-u))\big\|_{L^p(\mathbb{R}^N;\mathbb{R}^L)} \leq \frac{1}{n} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.9}$$

Once again employing Lemma 4.3 to extract another subsequence of $k(n)$ (if necessary; not relabeled), we may also assume that

$$\chi_K\gamma_n(u_{k(n)}-u) \text{ is } \Omega\text{-tight in } L^p(\Omega;\mathbb{R}^M), \tag{4.10}$$

where K is a fixed compact set containing $\partial\Omega$ in its interior. (Thus (4.10) essentially means that $\gamma_n(u_{k(n)}-u)$ does not develop concentrations "at the boundary" of Ω.) Decomposing $\mathcal{P}[\gamma_n(u_{k(n)}-u)] =: \tilde{u}_n = \tilde{w}_n^0+\cdots+\tilde{w}_n^4$ according to Lemma 4.1, we define

$$\begin{aligned} &w_n^j := \tilde{w}_n^j \quad \text{for } j=0,1,2,3,4, \\ &w_n^5 := (I-\mathcal{P})[\gamma_n(u_{k(n)}-u)] + (1-\gamma_n)(u_{k(n)}-u). \end{aligned}$$

By construction, $u_{k(n)} = u + w_n^0 + \cdots + w_n^5$, $w_n^0,\dots,w_n^5$ are bounded, $\mathcal{A}$-free sequences in $L^p(\mathbb{R}^N;\mathbb{R}^M)$, and the properties (a), (c), (d) and (e) are satisfied. It remains to show (b) and (f).

(f): By Lemma 3.4 (iv), (4.9) yields that

$$(I-\mathcal{P})[\gamma_n(u_{k(n)}-u)] \to 0 \quad \text{strongly in } L^p(\mathbb{R}^N,\mathbb{R}^M). \tag{4.11}$$

Moreover, if E is a closed subset of Ω, the compact set $\partial\Omega$ has positive distance to E, whence $\chi_E(1-\gamma_n)(u_{k(n)}-u) = 0$ for every n large enough by (4.8). Together with (4.11), this implies that $\chi_E w_n^5 \to 0$ in L^p.

(b): By construction, w_n^1 is $\mathbb{R}^N$-tight and satisfies $w_n^1 \to 0$ in $L^p + L^q$ for any $q < p$. Hence, it suffices to show that

$$\chi_K w_n^1 = \chi_K\big[\tilde{u}_n - \tilde{w}_n^0 - \tilde{w}_n^2 - \tilde{w}_n^3 - \tilde{w}_n^4\big] \quad \text{is } \Omega\text{-tight in } L^p(\Omega;\mathbb{R}^M)$$

with a compact set $K \subset \mathbb{R}^N$ containing $\partial\Omega$ in its interior. Combined, (4.11) and (4.10) imply that $\chi_K\tilde{u}_n = \chi_K\mathcal{P}[\gamma_n(u_{k(n)}-u)]$ is Ω-tight in $L^p(\Omega;\mathbb{R}^M)$. Moreover, for $j \in \{0,3,4\}$, $\tilde{w}_n^j$ does not concentrate in L^p whence $\chi_K\tilde{w}_n^j$ is Ω-tight in L^p. Finally, $\chi_K\tilde{w}_n^2 \to 0$ in L^p and thus also is Ω-tight in L^p. □

5. Proof of the main results

The proofs are grouped into four subsections. The first subsumes various results for Young measures which are needed later. The second contains the proofs of Theorem 2.14 and Corollary 2.15 while the third is dedicated to showing Theorem 2.6 and Theorem 2.17. In the final subsection, we discuss some of the assumptions of the aforementioned theorems by proving Proposition 2.3 and Proposition 2.16.

5.1. Auxiliary results. Possible oscillations of minimizing sequences will be discussed with Young measures as the main tool.

Theorem 5.1 (fundamental theorem for Young measures [3,16])**.** *Let $\Omega \subset \mathbb{R}^N$ be measurable and let $u_n : \Omega \to \mathbb{R}^M$ be a sequence of measurable functions. Then there exists a subsequence $(u_{k(n)})$ and a family $\nu = (\nu_x)_{x\in\Omega}$ of nonnegative Radon measures on $\mathbb{R}^M$, weak*-measurable*[2] *in x, such that the following holds:*

(i) $\nu_x(\mathbb{R}^M) \leq 1$ *for a.e.* $x \in \Omega$.
(ii) *If* $\lim_{h\to\infty} \sup_{n\in\mathbb{N}} \left|\{|u_{k(n)}| \geq h\} \cap \Omega \cap B_R(0)\right| = 0$ *for every* $R > 0$*, then* $\nu_x(\mathbb{R}^M) = 1$ *for a.e.* $x \in \Omega$.
(iii) *For every Carathéodory function* $f : \Omega \times \mathbb{R}^M \to \mathbb{R}$ *such that* $f(\cdot, u_{k(n)})$ *is equiintegrable*[3] *in* $L^1(\Omega)$*, we have that*

$$\int_\Omega f(x, u_{k(n)}(x))\, dx \underset{n\to\infty}{\longrightarrow} \int_\Omega \int_{\mathbb{R}^M} f(x,\mu)\, d\nu_x(\mu) dx.$$

As a consequence of (iii), ν is uniquely determined by $(u_{k(n)})$ and it is called the Young measure generated by $u_{k(n)}$. Moreover, if $\nu_x = \nu_a$ for a.e. $x \in \Omega$ with a fixed $a \in \Omega$, then it is called a homogeneous Young measure. Another useful consequence of (iii) is the following.

Corollary 5.2. *Let $1 \leq p < \infty$ and let $(u_n) \subset L^p(\Omega; \mathbb{R}^M)$ be a bounded sequence which generates a Young measure $\nu = (\nu_x)$. Then $u_n \rightharpoonup u$ weakly in L^p with $u(x) = \langle \nu_x, id\rangle := \int_{\mathbb{R}^M} \mu\, d\nu_x(\mu)$ for a.e. $x \in \Omega$, and $u_n \to u$ locally in measure if and only if $\nu_x = \delta_{u(x)}$ for a.e. $x \in \Omega$. Here, δ_μ denotes the Dirac mass at the point $\mu \in \mathbb{R}^M$.*

Young measures generated by $\mathcal{A}$-free sequences on bounded domains have been characterized in [7]. Here, we only employ a version of an approximation result of [7] used to "localize" the Young measure, adapted to the whole space instead of bounded domains.

[2] i.e., $x \mapsto \int_{\mathbb{R}^M} f(\mu) d\nu_x(\mu)$ is measurable for every $f \in C_0(\mathbb{R}^M)$

[3] Note that equiintegrablility in L^1 in the sense of Definition 2.8 is equivalent to weak relative compactness in L^1 by the de la Vallé-Poussin criterion.

Proposition 5.3 (cf. [7, Proposition 3.8][4]). *Let $1 \leq p < \infty$ and let $\nu = (\nu_x)$ be a Young measure generated by a bounded sequence $(u_n) \subset L^p(\mathbb{R}^N;\mathbb{R}^M)$ which does not concentrate in L^p and satisfies $\|(-\Delta)^{-\frac{1}{2}}\mathcal{A}u_n\|_{L^p} \to 0$. Then for a.e. $a \in \mathbb{R}^N$, there exists a sequence $(w_n) \subset L^p_\sharp(\mathbb{R}^N;\mathbb{R}^M)$ with the following properties:*

(w_n) is bounded in $L^p(Q;\mathbb{R}^M)$ and does not concentrate in L^p;
(w_n) generates the homogeneous Young measure ν_a;
$\mathcal{A}w_n = 0$ in $\mathbb{R}^N$ and $\int_Q w_n\,dx = \langle \nu_a, id\rangle = \int_{\mathbb{R}^M} \mu\,d\nu_x(\mu)$ for all n.

Proof. Let $B \subset \mathbb{R}^N$ be an open ball containing a. For $p' := \frac{p}{p-1}$, we have

$$\begin{aligned}
&\|\mathcal{A}u_n\|_{W^{-1,p}(B;\mathbb{R}^M)}\\
&= \sup\left\{\int_{\mathbb{R}^N} u_n\cdot\mathcal{A}^*\eta\,dx \;\middle|\; \eta \in C_c^\infty(B;\mathbb{R}^L) \text{ with } \|\eta\|_{W^{1,p'}} \leq 1\right\}\\
&\leq \sup\left\{\int_{\mathbb{R}^N} u_n\cdot\mathcal{A}^*\eta\,dx \;\middle|\; \eta \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^L) \text{ with } \|\nabla\eta\|_{L^{p'}} \leq 1\right\}\\
&\leq C\sup\left\{\int_{\mathbb{R}^N} u_n\cdot\mathcal{A}^*\eta\,dx \;\middle|\; \eta \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^L) \text{ with } \|(-\Delta)^{\frac{1}{2}}\eta\|_{L^{p'}} \leq 1\right\}\\
&\leq C\sup\left\{\int_{\mathbb{R}^N} u_n\cdot\mathcal{A}^*(-\Delta)^{-\frac{1}{2}}\psi\,dx \;\middle|\; \psi \in L^{p'}(\mathbb{R}^N;\mathbb{R}^L) \text{ with } \|\psi\|_{L^{p'}} \leq 1\right\},
\end{aligned}$$

since $\frac{\xi}{|\xi|}$ gives rise to a continuous Fourier multiplier on $L^{p'}$and thus $\|(-\Delta)^{\frac{1}{2}}\eta\|_{L^{p'}} \leq C\,\|\nabla\eta\|_{L^{p'}}$. Thus,

$$\|\mathcal{A}u_n\|_{W^{-1,p}(B;\mathbb{R}^M)} = C\left\|(-\Delta)^{-\frac{1}{2}}\mathcal{A}u_n\right\|_{L^p(\mathbb{R}^N;\mathbb{R}^M)} \underset{n\to\infty}{\longrightarrow} 0.$$

Hence, [7, Proposition 3.8] can be applied to u_n restricted to B (which generates ν restricted to B), yielding the assertion. □

As an immediate consequence, we have the following.

Proposition 5.4. *Let $\Omega \subset \mathbb{R}^N$ be open, let $1 \leq p < \infty$ and let $\nu = (\nu_x)$ be a Young measure generated a bounded sequence $(v_n) \subset L^p(\mathbb{R}^N;\mathbb{R}^M) \cap \ker\mathcal{A}$ such that (v_n) is equiintegrable in L^p and $v_n \rightharpoonup 0$ weakly in L^p, and suppose that f satisfies* (f:0) *and* (f:1)*. Then, for a.e. $a \in \Omega$ such that $f(a,\cdot)$ is $\mathcal{A}$-quasiconvex at $\xi \in \mathbb{R}^M$, we have*

$$\int_{\mathbb{R}^M} f(a,\xi+\mu)\,d\nu_a(\mu) \geq f(a,\xi) \tag{5.1}$$

[4] Beware that the notion of equiintegrability used in [7] is equivalent to what we term "does not concentrate" and hence coincides with our definition only on domains with finite measure.

Moreover, for a.e. $a \in \Omega$ *such that* $f(a, \cdot)$ *is strongly p-$\mathcal{A}$-quasiconvex at* ξ*,*

$$\text{equality in (5.1) implies that } \nu_a = \delta_0, \tag{5.2}$$

where δ_0 *denotes the Dirac mass concentrated at the point* $0 \in \mathbb{R}^M$*. In particular, given* $u \in L^p(\Omega; \mathbb{R}^M)$ *such that* $f(x, \cdot)$ *is $\mathcal{A}$-quasiconvex at* $u(x)$ *for a.e.* $x \in \Omega$*, we have that*

$$\liminf_{n\to\infty} \int_\Omega f(x, u(x) + v_n(x))\, dx \geq \int_\Omega f(x, u(x))\, dx \tag{5.3}$$

and if $f(x, \cdot)$ *is strongly p-$\mathcal{A}$-quasiconvex at* $u(x)$ *for a.e.* $x \in \Omega$*, then*

$$\text{equality in (5.3) implies that } v_n \to 0 \text{ in } L^p. \tag{5.4}$$

Proof. The first assertion (5.1) is a simple consequence of Proposition 5.3, Theorem 5.1 and the definition of $\mathcal{A}$-quasiconvexity. Here, note that we may assume that the sequence w_n of Proposition 5.3 actually belongs to $C^\infty_\sharp(\mathbb{R}^N; \mathbb{R}^M)$, because if not, we can replace it with a mollified sequence $\tilde{w}_n$ (mollified as usual by convolution with a smooth kernel with small support) such that $\tilde{w}_n - w_n \to 0$ strongly in $L^p_\sharp$, whence $\tilde{w}_n$ inherits all properties of w_n. To show (5.2), we again employ Proposition 5.3 to choose a sequence w_n of smooth functions which is equiintegrable in L^p_{per} and generates ν_a. If (5.1) holds with equality, Theorem 5.1 and the strong p-$\mathcal{A}$-quasiconvexity of f imply that $g(t_n, T) \to 0$ with $t_n := \int_Q |w_n|\, dx$ and $T := \sup_n \int_Q |w_n|^p\, dx$ (recall that g is decreasing in its second variable). This is possible only if $t_n \to 0$, whence $w_n \to 0$ in $L^1_\sharp$ and $\nu_a = \delta_0$ due to Corollary 5.2. Finally, (5.1) and (5.2) imply (5.3) and (5.4), respectively, by Theorem 5.1. As to (5.4), we first get that $v_n \to 0$ locally in measure, which in turn implies that $v_n \to 0$ in L^p by Vitali's theorem, since v_n is equiintegrable in L^p. □

Remark 5.5. In fact, the converse of Proposition 5.4 is also true. More precisely, if f satisfies (f:0) and (f:1) then the following holds for a.e. $a \in \Omega$ and every $\xi \in \mathbb{R}^M$: If (5.1) is valid for every homogeneous Young measure ν_a generated by an $\mathcal{A}$-free, bounded sequence $(w_n) \in L^p_\sharp(\mathbb{R}^N; \mathbb{R}^M)$ such that $\langle \nu_a, id \rangle = 0$, then f is $\mathcal{A}$-quasiconvex at ξ, and if (5.1) and (5.2) hold for every such ν_a, then f is strongly p-$\mathcal{A}$-quasiconvex at ξ. Corresponding converse statements of (5.3) and (5.4) also hold, at least if f is bounded from below by a constant: If (5.3) is satisfied for every $(v_n) \subset L^p(\Omega; \mathbb{R}^M)$ which is $\mathcal{A}$-free, equiintegrable in L^p and satisfies $v_n \rightharpoonup 0$ weakly in L^p, then $f(x, \cdot)$ is $\mathcal{A}$-quasiconvex at $u(x)$ for a.e. x, and if (5.3) and (5.4) hold for every such (v_n), then $f(x, \cdot)$ is strongly p-$\mathcal{A}$-quasiconvex at $u(x)$ for a.e. x. The proof is omitted. It is not entirely trivial as it involves a problem of measurable selection on the level of the associated Young measures (cf. the concluding remark in [20]).

Remark 5.6. Given $\Omega \subset \mathbb{R}^N$ open, any Young measure generated by a bounded, $\mathcal{A}$-free sequence $(u_n) \subset L^p(\Omega;\mathbb{R}^M)$ is also generated by a bounded, $\mathcal{A}$-free sequence $(\tilde{u}_n) \subset L^p(\Omega;\mathbb{R}^M)$ which is equiintegrable in L^p. For instance, one may take $\tilde{u}_n := u + v_n$ with u and v_n defined in Lemma 4.4.

Below, (f:2) is used exclusively in form of the following simple observation.

Proposition 5.7. *Let $1 \leq p < \infty$, let $\Omega \subset \mathbb{R}^N$ be open and suppose that f satisfies* (f:0) *and* (f:2). *Then the map $u \mapsto f(\cdot,u)$, $L^p(\Omega;\mathbb{R}^M) \to L^1(\Omega)$, is uniformly continuous on bounded subsets of $L^p(\Omega;\mathbb{R}^M)$.*

Proof. By (f:2) and Hölder's inequality, we have

$$\int_\Omega |f(x,u) - f(x,v)|\, dx \leq C \left(\|u\|_{L^p}^{p-1} + \|v\|_{L^p}^{p-1} + \|h\|_{L^p}^{p-1} \right) \|u-v\|_{L^p}$$

for any $u, v \in L^p(\Omega;\mathbb{R}^M)$. □

5.2. Domains with compact boundary. As we shall see, the proof of Theorem 2.14 heavily relies on the corresponding decomposition lemma of Section 4, Lemma 4.5. In a sense made precise below, it exploits that the component sequences do not interact with each other in f, essentially due to Proposition 5.7.

Proposition 5.8. *Let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be open with compact boundary and suppose that f satisfies* (f:0)–(f:2). *Moreover, let u_n be an $\mathcal{A}$-free, bounded sequence which weakly converges to a function u in $L^p(\Omega;\mathbb{R}^M)$, and let $u_n = u + w_n^0 + \dots + w_n^5$ be a decomposition as in Lemma* 4.5. *Then for any $j \in \{1,\dots,5\}$, we have*

$$f(\cdot,u_n) - f(\cdot,u_n - w_n^j) - \left[f(\cdot,w_n^j) - f(\cdot,0) \right] \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^1(\Omega). \tag{5.5}$$

In particular,

$$f(\cdot,u_n) - f(\cdot,u + w_n^0) - \sum_{j=1}^{5} \left[f(\cdot,w_n^j) - f(\cdot,0) \right] \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^1(\Omega). \tag{5.6}$$

This kind of result is fairly standard in the context of bounded domains, where only two component sequences appear in the decomposition lemma besides the weak limit (i.e., oscillations and concentrations); in particular, it is implicitely used in [8]. For a sequence of gradients on an unbounded domain, a corresponding result was obtained in [12]. In our present context, it would still be possible to give a proof relying on the abstract framework developed in [12], which provides a way to handle the numerous different properties of the component sequences w_n^j in a more systematic way. However, the case of functionals is somewhat simpler than that of operators mapping into a Banach space which allows a reasonably-sized self-contained proof "by hand", although our proof of (5.5) below only discusses the case $j = 5$ in full detail, the other cases being more or less analogous.

Proof of Proposition 5.8. For $\delta > 0$ let $\Omega_\delta := \{x \in \Omega \mid \operatorname{dist}(x;\partial\Omega) > \delta\}$. We first show (5.5) for $j = 5$. Fix $\varepsilon > 0$ and define $E = E(\delta) := \Omega \setminus \Omega_\delta$, and choose $\delta = \delta(\varepsilon) \in (0,1)$ small enough such that

$$\sup_{n\in\mathbb{N}} \left\|\chi_{E(\delta)}(u_n - w_n^5)\right\|_{L^p} \leq \left\|\chi_{E(\delta)} u\right\|_{L^p} + \sum_{i=0}^{4} \sup_{n\in\mathbb{N}} \left\|\chi_{E(\delta)} w_n^i\right\|_{L^p} < \varepsilon. \tag{5.7}$$

Note that such a choice of δ is possible because the constant sequence u, as well as $\chi_{\Omega\setminus\Omega_1} w_n^1, \ldots, \chi_{\Omega\setminus\Omega_1} w_n^4$, are Ω-tight in L^p, the latter as a consequence of their properties (a)–(e) listed in Lemma 4.5. In addition, we have

$$\chi_{\Omega\setminus E(\delta)} w_n^5 = \chi_{\Omega_\delta} w_n^5 \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^p \quad \text{for any fixed } \delta \in (0,1), \tag{5.8}$$

by definition of w_n^5. Together with the uniform continuity of $v \mapsto f(\cdot, v)$, $L^p \to L^1$, on bounded subsets of L^p as derived in Proposition 5.7, (5.7) and (5.8) imply that

$$\begin{aligned}
&\limsup_{n\to\infty} \int_\Omega \left|f(x,u_n) - f(x,u_n - w_n^5) - \left[f(x,w_n^5) - f(x,0)\right]\right| dx \\
&\leq \limsup_{\varepsilon\to 0} \sup_{n\in\mathbb{N}} \int_E \left|f(x,(u_n - w_n^5) + w_n^5) - f(x,w_n^5)\right| dx \\
&\quad + \limsup_{\varepsilon\to 0} \sup_{n\in\mathbb{N}} \int_E \left|f(x,0) - f(x,u_n - w_n^5)\right| dx \\
&\quad + \limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \int_{\Omega\setminus E} \left|f(x,u_n) - f(x,u_n - w_n^5)\right| dx \\
&\quad + \limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \int_{\Omega\setminus E} \left|f(x,w_n^5) - f(x,0)\right| dx \\
&= 0,
\end{aligned} \tag{5.9}$$

with $E = E(\delta(\varepsilon))$. This concludes the proof of (5.5) for $j = 5$. Essentially, we exploited that $w_n^5 \to 0$ in $L^p(\Omega \setminus E; \mathbb{R}^M)$ while at the same time the remaining components u and $w_n^0, \ldots, w_n^4$ are uniformly close to zero in $L^p(E;\mathbb{R}^M)$ by their properties obtained in Lemma 4.5.

The same kind of argument also yields (5.5) for $j = 1, \ldots, 4$, employing different choices for E which now also depend on n, adapted to the properties of the component sequence w_n^j which is separated from the rest. More precisely, we use

$$\begin{aligned}
E_n &= E_n(\delta) := \Omega_\delta \cap \left\{ \left|w_n^3\right| < \delta \right\} \cap \left\{ \left|w_n^2\right| < 1 \right\} \setminus B_{\frac{1}{\delta}}(0) && \text{if } j = 4, \\
E_n &= E_n(\delta) := \Omega_\delta \cap \left\{ \left|w_n^3\right| > \delta \right\} \cap \left\{ \left|w_n^2\right| < 1 \right\} \setminus B_{\frac{1}{\delta}}(0) && \text{if } j = 3, \\
E_n &= E_n(\delta) := \Omega_\delta \cap \left\{ \left|w_n^2\right| > \tfrac{1}{\delta} \right\} \setminus B_{\frac{1}{\delta}}(0) && \text{if } j = 2, \\
E_n &= E_n(\delta) := \Omega_\delta \cap \left\{ \left|w_n^1\right| > \tfrac{1}{\delta} \right\} \cap B_{\frac{1}{\delta}}(0) && \text{if } j = 1,
\end{aligned}$$

where in each case $\delta = \delta(\varepsilon, j)$ is chosen small enough such that

$$\sup_{n\in\mathbb{N}} \left\|\chi_{E_n}(u_n - w_n^j)\right\|_{L^p} < \varepsilon \quad \text{and} \quad \left\|\chi_{\Omega\setminus E_n} w_n^j\right\|_{L^p} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for fixed } \delta.$$

As before, it is not difficult to see that the choice is possible due to the properties (a)–(e) of w_n^j obtained in Lemma 4.5, and these also yield that $\chi_{\Omega\setminus E_n} w_n^j \to 0$ in L^p in each case; we omit the (lengthy) details. Repeating (5.9) in each case then gives (5.5) for $j = 1, \ldots, 4$. Finally, note that (5.6) can be obtained by applying (5.5) successively to the sequences $\tilde{u}_n^{(j)} := u + w_n^0 + \sum_{i=j}^5 w_n^i$, for $j = 1, \ldots, 5$. □

Remark 5.9. In the preceding proof, we exploited that $u \mapsto f(\cdot, u)$, $L^p \to L^1$, is uniformly continuous on bounded subsets of L^p, and not just on bounded subsets on $\mathcal{U}_\mathcal{A}$ (as a closed subspace of L^p). In view of the fact that we are only interested in F defined on $\mathcal{U}_\mathcal{A}$, one may wonder if it is possible to get away with an assumption significantly weaker that (f:2), whose only purpose is the proof of uniform continuity on bounded subsets of L^p. If $\Omega = \mathbb{R}^N$, we can apply the projector $\mathcal{P}$ of Section 4 to any sequences in the proof without having to face the problem of $\mathcal{A}$-free extension. In this case, it is possible to work under the assumption that $F_E(u) := \int_E f(x, u)\, dx$ is uniformly continuous on bounded subsets of $\mathcal{U}_\mathcal{A}$ for any $E \subset \mathbb{R}^N$ measurable, with a modulus of continuity which is also uniform in E. There is still no obvious way to do the proof just using uniform continuity of F on bounded subsets of $\mathcal{U}_\mathcal{A}$, though.

If f is $\mathcal{A}$-quasiconvex and F is bounded from below, the assertion of Proposition 5.8 can be enhanced. As a byproduct, we get weak lower semicontinuity of F on $\mathcal{U}_\mathcal{A}$.

Proposition 5.10. *Let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be open with compact boundary, let u_n be an $\mathcal{A}$-free, bounded sequence which weakly converges to a function u in $L^p(\Omega; \mathbb{R}^M)$, and let $u_n = u + w_n^0 + \cdots + w_n^5$ be a decomposition as in Lemma 4.5. Moreover, suppose that f satisfies (f:0)–(f:2), that $f(x, \cdot)$ is $\mathcal{A}$-quasiconvex at $u(x)$ for a.e. $x \in \Omega$ and that $\inf\{F(v) \mid v \in \mathcal{U}_\mathcal{A}\} > -\infty$. Then*

$$\begin{aligned} \liminf_{n\to\infty} \left[F(u + w_n^0) - F(u)\right] &\geq 0, \\ \liminf_{n\to\infty} \left[F(w_n^j) - F(0)\right] &\geq 0 \quad \text{for } j = 1, \ldots, 5, \end{aligned} \tag{5.10}$$

and

$$\liminf_{n\to\infty} F(u_n) \geq F(u). \tag{5.11}$$

If, in addition, $\limsup_{n\to\infty} F(u_n) \leq F(u)$, then we even have that

$$F(u + w_n^0) \underset{n\to\infty}{\longrightarrow} F(u) \quad \text{and} \quad F(w_n^j) \underset{n\to\infty}{\longrightarrow} F(0) \quad \text{for } j = 1, \ldots, 5. \tag{5.12}$$

Proof. The first inequality in (5.10) is an immediate consequence of Proposition 5.4, since w_n^0 is equiintegrable and $f(\cdot, u(x))$ is $\mathcal{A}$-quasiconvex at $u(x)$, for a.e. $x \in \Omega$. To check the remaining inequalities, fix an $\varepsilon > 0$ and choose $u_\varepsilon^* \in \mathcal{U}_\mathcal{A}$ such that $\inf\{F(v) \mid v \in \mathcal{U}_\mathcal{A}\} + \varepsilon \geq F(u_\varepsilon^*)$. In particular,

$$\liminf_{n\to\infty} \big[F(u_\varepsilon^* + w_n^j) - F(u_\varepsilon^*)\big] \geq -\varepsilon. \tag{5.13}$$

Moreover, by applying Proposition 5.8 to the sequences $\tilde{u}_n = \tilde{u}_n(\varepsilon, j) := u_\varepsilon^* + w_n^j$ (which is also an admissible decomposition of $\tilde{u}_n$) for fixed ε and j, we get that

$$\big[F(u_\varepsilon^* + w_n^j) - F(u_\varepsilon^*)\big] - \big[F(w_n^j) - F(0)\big] \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for } j = 1, \dots, 5.$$

Using this to replace $F(u_\varepsilon^* + w_n^j) \geq F(u_\varepsilon^*)$ in (5.13), we infer that $\liminf_{n\to\infty} \big[F(w_n^j) - F(0)\big] \geq -\varepsilon$ for $j = 1, \dots, 5$. Since this is true for any $\varepsilon > 0$, this concludes the proof of (5.10). As to the remaining assertions, first note that by (5.6) in Proposition 5.8,

$$\begin{aligned} &\liminf_{n\to\infty} \big[F(u_n) - F(u)\big] \\ &\geq \liminf_{n\to\infty} \big[F(u + w_n^0) - F(u)\big] + \sum_{j=1}^{5} \liminf_{n\to\infty} \big[F(w_n^j) - F(0)\big], \end{aligned} \tag{5.14}$$

whence $\liminf_{n\to\infty} F(u_n) \geq F(u)$ due to (5.10). Finally, assume that $\limsup_{n\to\infty} F(u_n) \leq F(u)$. Proposition 5.8 then allows us to replace (5.14) by

$$0 \geq \limsup_{n\to\infty} \big[F(u + w_n^0) - F(u)\big] + \sum_{j=1}^{5} \liminf_{n\to\infty} \big[F(w_n^j) - F(0)\big], \tag{5.15}$$

where each of the six summands is nonnegative due to (5.10). Hence

$$0 \geq \limsup_{n\to\infty} \big[F(u + w_n^0) - F(u)\big] \geq \liminf_{n\to\infty} \big[F(u + w_n^0) - F(u)\big] \geq 0,$$

which implies the first part of (5.12). The other parts can be obtained analogously, with suitable variants of (5.15). □

Proof of Theorem 2.14. From any given subsequence of u_n (not relabeled, specified later), we can extract another subsequence $u_{k(n)}$ such that $u_{k(n)} = u + w_n^0 + \dots + w_n^5$ according to Lemma 4.5. Since $\limsup_{n\to\infty} F(u_n) \leq F(u)$ by assumption, Proposition 5.10 yields that

$$F(u + w_n^0) \to F(u) \quad \text{and} \quad F(w_n^j) \to F(0) \quad \text{for } j = 1, \dots, 5. \tag{5.16}$$

With (5.16) as a starting point, we are now ready to prove (i)–(iv). Throughout, we argue by contradiction.

(i) Suppose that u_n does not converge to u locally in measure. Hence it has a subsequence (not relabeled) such that

$$\liminf_{n\to\infty} |\Omega' \cap \{|u_n - u| > \delta\}| \geq \varepsilon \tag{5.17}$$

for an $\varepsilon > 0$, a $\delta > 0$ and a bounded, open set $\Omega' \subset \Omega$. The properties of w_n^j obtained in the decomposition lemma entail that for $j = 1, \ldots, 5$, $w_n^j \to 0$ locally in measure. In particular, we can replace u_n by $u + w_n^0$ in (5.17). But by (5.16) for w_n^0 and Proposition 5.4, $w_n^0 \to 0$ in L^p and thus also locally in measure, contradicting (5.17).

(ii) Suppose that u_n does concentrate in L^p. Then

$$\liminf_{n\to\infty} \left\|\chi_{E_n} u_n\right\|_{L^p} > 0 \tag{5.18}$$

for suitable measurable sets $E_n \subset \Omega$ with $|E_n| \to 0$, at least up to a subsequence of (u_n) (not relabeled). Recall that by the properties of the component sequences in Lemma 4.5, w_n^3 and w_n^4 do not concentrate in L^p, while (w_n^1), (w_n^2) and (w_n^5) are elements of Φ_c. Hence, by assumption, (5.16) for $j = 1, 2, 5$ implies that w_n^1, w_n^2 and w_n^5 converge to zero strongly in L^p. In particular, $u_{k(n)} = u + w_n^0 + \cdots + w_n^5$ does not concentrate in L^p, which contradicts (5.18).

(iii) Suppose that $\chi_{\{s^{-1}<|u_n|<s\}} u_n$ is not $\mathbb{R}^N$-tight in L^p for an $s > 1$. Then,

$$\liminf_{n\to\infty} \left\|\chi_{\{s^{-1}<|u_n|<s\}\setminus B_{R_n}} u_n\right\|_{L^p} > 0 \tag{5.19}$$

for a suitable sequence of balls B_{R_n} centered at zero with radius $R_n \to \infty$, at least up to a subsequence of (u_n) (not relabeled). By the properties of the component sequences in Lemma 4.5, w_n^j for $j \neq 3$ cannot contribute to (5.19), and neither can u. Moreover, $(w_n^3) \in \Phi_{mov}$, whence by assumption, (5.16) for $j = 3$ implies that $w_n^3 \to 0$ in L^p. Consequently, (5.19) cannot hold along the subsequence $u_{k(n)}$.

(iv) Suppose that u_n does spread out in L^p. Then

$$\liminf_{n\to\infty} \left\|\chi_{\{|u_n|<\delta_n\}} u_n\right\|_{L^p} > 0. \tag{5.20}$$

for a suitable sequence $\delta_n \to 0^+$, at least up to a subsequence of (u_n) (not relabeled). By the properties of w_n^j in Lemma 4.5, w_n^j for $j \neq 4$ does not spread out in L^p, and of course the constant sequence u does not spread out in L^p. In addition, by assumption, (5.16) for $j = 4$ implies that $w_n^4 \to 0$, whence w_n^4 does not spread out in L^p. As a consequence, $u_{k(n)}$ does not spread out in L^p, contradicting (5.20). Here, note that if a_n and b_n are bounded sequences that do not spread out in L^p, then $a_n + b_n$ does not spread out in L^p:

For every $0<\delta<1$,

$$\begin{aligned}&\int_{\{|a_n+b_n|<\delta^2\}}|a_n+b_n|^p\,dx\\&\le\int_{\{|a_n|<\delta\}\cap\{|b_n|<\delta\}}|a_n+b_n|^p\,dx+\big|\{|a_n|\ge\delta\}\cup\{|b_n|\ge\delta\}\big|\delta^{2p}\\&\le 2^p\int_{\{|a_n|<\delta\}}|a_n|^p\,dx+2^p\int_{\{|b_n|<\delta\}}|b_n|^p\,dx+\big(\,\|a_n\|_{L^p}^p+\|b_n\|_{L^p}^p\,\big)\delta^p,\end{aligned}$$

and the terms in the last line all converge to zero as $\delta\to 0^+$, uniformly in n.

Last but not least, observe that if the conclusions of (i)–(iv) all hold, then u_n is equiintegrable in L^p and $u_n\to u$ locally in measure. By Vitali's theorem, this entails that $u_n\to u$ strongly in L^p. □

Proof of Corollary 2.15. Essentially, (ii.1)–(ii.3) can be obtained by arguing as in (ii) in the proof of Theorem 2.14. We omit the details. □

5.3. General domains. In complete analogy to Proposition 5.8 and Proposition 5.10, using Lemma 4.4 instead of Lemma 4.5, we have the following.

Proposition 5.11. *Let* $1<p<\infty$*, let* $\Omega\subset\mathbb{R}^N$ *be open and suppose that* f *satisfies* (f:0)–(f:2)*. Moreover, let* u_n *be an* $\mathcal{A}$*-free, bounded sequence which weakly converges to a function* u *in* $L^p(\Omega;\mathbb{R}^M)$*, and let* $u_n=u+v_n+w_n+z_n$ *be a decomposition as in Lemma* 4.4*. Then for* $q_n=w_n$ *as well as for* $q_n=z_n$*, we have that*

$$f(\cdot,u_n)-f(\cdot,u_n-q_n)-\big[f(\cdot,q_n)-f(\cdot,0)\big]\underset{n\to\infty}{\longrightarrow}0\ \text{ in } L^1(\Omega).$$

In particular,

$$f(\cdot,u_n)-f(\cdot,u+v_n)-\big[f(\cdot,w_n)-f(\cdot,0)\big]-\big[f(\cdot,z_n)-f(\cdot,0)\big]\underset{n\to\infty}{\longrightarrow}0\ \text{ in } L^1(\Omega).$$

Proposition 5.12. *Let* $1<p<\infty$*, let* $\Omega\subset\mathbb{R}^N$ *be open, let* u_n *be an* $\mathcal{A}$*-free, bounded sequence which weakly converges to a function* u *in* $L^p(\Omega;\mathbb{R}^M)$*, and let* $u_n=u+v_n+w_n+z_n$ *be a decomposition as in Lemma* 4.4*. Moreover, suppose that* f *satisfies* (f:0)–(f:2)*, that* $f(x,\cdot)$ *is* $\mathcal{A}$*-quasiconvex at* $u(x)$ *for a.e.* $x\in\Omega$ *and that* $\inf\{F(v)\mid v\in\mathcal{U}_\mathcal{A}\}>-\infty$*. Then we have that*

$$\begin{aligned}\liminf_{n\to\infty}\big[F(u+v_n)-F(u)\big]&\ge 0\\\liminf_{n\to\infty}\big[F(w_n)-F(0)\big]&\ge 0\\\liminf_{n\to\infty}\big[F(z_n)-F(0)\big]&\ge 0,\end{aligned}\tag{5.21}$$

and

$$\liminf_{n\to\infty} F(u_n) \geq F(u). \tag{5.22}$$

If, in addition, $\limsup_{n\to\infty} F(u_n) \leq F(u)$, *then we even have that*

$$F(u+v_n) \underset{n\to\infty}{\longrightarrow} F(u), \quad F(w_n) \underset{n\to\infty}{\longrightarrow} F(0) \quad \text{and} \quad F(z_n) \underset{n\to\infty}{\longrightarrow} F(0). \tag{5.23}$$

Proof of Theorem 2.6. As already observed in Remark 2.7, it suffices to show that F is lower semicontinuous along sequences in $\mathcal{U}_{\mathcal{A}}$ which weakly converge in L^p, and this is due to Proposition 5.12. □

Proof of Theorem 2.17. The proof is analogous to the one of Theorem 2.14, substituting Lemma 4.4 for Lemma 4.5 and and Proposition 5.12 for Proposition 5.10. □

5.4. Proof of Proposition 2.3 and Proposition 2.16. To prove the characterization of strong p-$\mathcal{A}$-quasiconvexity of Proposition 2.3, we need a decomposition lemma for $\mathcal{A}$-free sequences of periodic functions on $\mathbb{R}^N$.

Lemma 5.13 (cf. [7, Lemma 2.15])**.** *Let* $Q := (0,1)^N$, *let* $1 < p < \infty$, *let* $(u_n) \subset L^p(Q;\mathbb{R}^M)$ *be a bounded sequence with* $\int_Q u_n\,dx = 0$, *and suppose that* $\mathcal{A}u_n = 0$ *on* $\mathbb{R}^N$. *Here, functions in* $L^p(Q;\mathbb{R}^M)$ *are identified with their* Q*-periodic extension to* $\mathbb{R}^N$. *Then there exists a subsequence* $u_{k(n)}$ *of* u_n *and a bounded sequence* $(v_n) \subset L^p(Q;\mathbb{R}^M)$ *such that*

$$\mathcal{A}v_n = 0 \text{ on } \mathbb{R}^N,\ \int_Q v_n\,dx = 0,\ (v_n) \text{ is equiintegrable in } L^p$$
$$\text{and } u_{k(n)} - v_n \to 0 \text{ locally in measure.}$$

Proof. To be precise, [7, Lemma 2.15] is stated for functions u_n defined on a bounded domain $\Omega \subset \mathbb{R}^N$ instead of periodic functions on $\mathbb{R}^N$, but the construction in the proof actually yields a sequence $v_n \in L^p(\tilde{Q};\mathbb{R}^M)$, bounded and equiintegrable in L^p with $\chi_\Omega(u_n - v_n) \to 0$ locally in measure, which is defined on a given open cube $\tilde{Q} \in \mathbb{R}^N$ containing Ω. In addition, v_n is $\mathcal{A}$-free on $\mathbb{R}^N$ if extended $\tilde{Q}$-periodically. Since any open cube $\tilde{Q}$ containing Ω is admissible in [7], we may use $\tilde{Q} := \Omega := Q$ in our context. (In fact, some of the steps in the proof could be simplified as well, as in our case there is no need to extend from Ω to $\tilde{Q}$-periodic functions.) □

Proposition 5.14. *Let* $Q := (0,1)^N$, *let* $1 < p < \infty$, *and suppose that* $f : \mathbb{R}^M \to \mathbb{R}$ *is a continuous function which satisfies* (f:1) *and* (f:2) *where* $h(x)$ *is replaced by a constant. If* $u_{k(n)}$ *and* v_n *denote the sequences of Lemma* 5.13, *then we have*

$$f(u_{k(n)}) - f(v_n) - \left[f(u_{k(n)} - v_n) - f(0)\right] \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^1(Q).$$

Proof. This is analogous to the proof of Proposition 5.8. □

Proof of Proposition 2.3. We want to show that $f(x,\cdot)$ is strongly p-$\mathcal{A}$-quasiconvex at $\xi \in \mathbb{R}^M$ if and only if (2.1) holds. To shorten notation, we set $\tilde{f}(\mu) := f(x, \xi + \mu)$ for $\mu \in \mathbb{R}^M$, with fixed $x \in \Omega$ and $\xi \in \mathbb{R}^M$. It now suffices to study strong p-$\mathcal{A}$-quasiconvexity of $\tilde{f}$ at 0.

``only if'': Obviously, strong p-$\mathcal{A}$-quasiconvexity at 0 implies $\mathcal{A}$-quasiconvexity at 0. Now suppose that $\int_Q \big[\tilde{f}(\varphi_n(y)) - \tilde{f}(0)\big]\, dy \to 0$ for a sequence $(\varphi_n) \in \Phi_{osc}$, i.e., $(\varphi_n) \in C^\infty_\sharp(Q;\mathbb{R}^M)$ is $\mathcal{A}$-free as well as bounded and equiintegrable in $L^p(Q;\mathbb{R}^M)$ with weak limit zero. By the definition of strong p-$\mathcal{A}$-quasiconvexity at 0, we infer that

$$g\Big(\int_Q |\varphi_n|\, dx, \bar{T}\Big) \to 0 \quad \text{ with } \bar{T} := \sup_{n\in\mathbb{N}} \int_Q |\varphi_n|^p\, dx.$$

Since $g(t,\bar{T})$ is increasing in t and nonzero whenever $t > 0$, this is possible only if $\int_Q |\varphi_n|\, dx \to 0$, which in turn implies that $\varphi_n \to 0$ locally in measure.

``if'': For $t, T \geq 0$ define

$$g(t,T) := \inf\left\{ \int_Q \big[\tilde{f}(\varphi(y)) - \tilde{f}(0)\big]\, dy \;\middle|\; \varphi \in \phi_{\mathcal{A}}, \int_Q |\varphi|\, dx \geq t, \int_Q |\varphi|^p\, dx \leq T \right\},$$

with the convention that $g(t,T) = +\infty$ if no admissible φ exists. Here, recall that $\phi_{\mathcal{A}} := \{\varphi \in C^\infty_\sharp(\mathbb{R}^N;\mathbb{R}^M) \mid \mathcal{A}\varphi = 0 \text{ on } \mathbb{R}^N \text{ and } \int_Q \varphi\, dx = 0\}$. Note that g depends on x and ξ, just like $\tilde{f}$. By construction, the inequality required in the definition of strong p-$\mathcal{A}$-quasiconvexity at 0 is satisfied. Moreover, g is increasing in t and decreasing in T, and since $\tilde{f}$ is $\mathcal{A}$-quasiconvex at 0, we have $g \geq 0$. It remains to show that $g(t,T) > 0$ for all $t > 0$, $T \geq 0$. Assume by contradiction that there is a $t_0 > 0$ and a $T_0 \geq 0$ such that $g(t_0, T_0) = 0$. In particular, $T_0 > 0$ as $g(t_0, 0) = +\infty$, and there is a sequence $(\tilde{\varphi}_n) \subset \phi_{\mathcal{A}}$ such that $\int_Q |\tilde{\varphi}_n|\, dx \geq t_0$, $\int_Q |\tilde{\varphi}_n|^p\, dx \leq T_0$ and

$$\int_Q \big[\tilde{f}(\tilde{\varphi}_n(y)) - \tilde{f}(0)\big]\, dy \underset{n\to\infty}{\longrightarrow} 0. \tag{5.24}$$

For $y \in \mathbb{R}^N$ define $\hat{\varphi}_n(y) := \tilde{\varphi}_n(ny)$, which inherits all the properties of $\tilde{\varphi}_n$ stated above. In particular, (5.24) turns into

$$\int_Q \big[\tilde{f}(\hat{\varphi}_n(y)) - \tilde{f}(0)\big]\, dy \underset{n\to\infty}{\longrightarrow} 0. \tag{5.25}$$

In addition, $\hat{\varphi}_n \rightharpoonup 0$ weakly in $L^p(Q;\mathbb{R}^M)$, as $\int_Q \tilde{\varphi}_n\, dx = 0$. By Lemma 5.13 applied to $u_n := \hat{\varphi}_n$, we get an $\mathcal{A}$-free sequence φ_n which is bounded and equiintegrable in $L^p(Q;\mathbb{R}^M)$ and which still satisfies $\int_Q \varphi_n\, dx = 0$ and $\varphi_n \rightharpoonup 0$

weakly in L^p. Moreover, $\lim_{n\to\infty} \int_Q |\varphi_n|\, dx = \lim_{n\to\infty} \int_Q \left|\hat{\varphi}_{k(n)}\right| dx = t_0 > 0$ since $\hat{\varphi}_{k(n)} - \varphi_n \to 0$ locally in measure and thus in L^1 as $\hat{\varphi}_{k(n)} - \varphi_n$ is bounded in L^p and $p > 1$. Hence $(\varphi_n) \in \Phi_{osc}$ and φ_n does not converge to zero locally in measure. Due to Proposition 5.14, (5.25) gives

$$\int_Q \left[\tilde{f}(\varphi_n) - \tilde{f}(0)\right] dx + \int_Q \left[\tilde{f}(\hat{\varphi}_n - \varphi_n) - \tilde{f}(0)\right] dx \underset{n\to\infty}{\longrightarrow} 0, \tag{5.26}$$

Since f is $\mathcal{A}$-quasiconvex at 0, both terms on the left hand side of (5.26) are nonnegative for every n, whence (5.26) implies that $\int_Q \left[\tilde{f}(\varphi_n) - \tilde{f}(0)\right] dx \to 0$, as $n \to \infty$, contradicting (2.1). □

Proof of Proposition 2.16. We want to show that (2.7) is equivalent to (2.3) with $\Psi = \Phi_{c\infty} \cup \Phi_{mov} \cup \Phi_{spr}$. First assume that $\Omega = \mathbb{R}^N$. Due to (2.6),

$$\chi_{\mathbb{R}^N \setminus B_R} |f_\infty(u) - f(\cdot, u)| \underset{R\to\infty}{\longrightarrow} 0 \quad \text{in } L^1(\mathbb{R}^N), \text{ uniformly in } u \in U, \tag{5.27}$$

where U may be any subset of $\mathcal{U}_\mathcal{A}$ which is bounded in L^p. In the following, let

$$\Phi_\infty := \left\{ (\varphi_n) \in \mathcal{U}_\mathcal{A} \;\middle|\; \begin{array}{l} \varphi_n \text{ is bounded in } L^p \text{ and satisfies } \chi_B \varphi_n \to 0 \text{ in } L^p \\ \text{for every bounded, open set } B \subset \mathbb{R}^N \end{array} \right\}$$

Note that $\Phi_{c\infty} \cup \Phi_{mov} \cup \Phi_{spr} \subset \Phi_\infty$. Since f_∞ satisfies the same growth conditions as f (i.e., (f:1), with $h(x)$ replaced by $0 = \liminf_{|x|\to\infty} h(x)$), we have that $f_\infty(0) = 0$, and (5.27) implies that

$$f(\cdot, \varphi_n) - f(\cdot, 0) - f_\infty(\varphi_n) \underset{n\to\infty}{\longrightarrow} 0 \text{ in } L^1(\mathbb{R}^N), \quad \text{for every } (\varphi_n) \in \Phi_\infty. \tag{5.28}$$

As a consequence of (5.28), f can be replaced by f_∞ in (2.3) for any $\Psi \subset \Phi_\infty$, whence (2.7) implies (2.3) for $\Psi = \Phi_{c\infty} \cup \Phi_{mov} \cup \Phi_{spr}$. It remains to show that the converse is also true. First suppose that there exists a $t_0 > 0$ such that $\{u \in \mathcal{U}_\mathcal{A} \mid \|u\|_{L^p} = t_0\} = \emptyset$. In this case, $\{u \in \mathcal{U}_\mathcal{A} \mid \|u\|_{L^p} = t\} = \emptyset$ for all $t > 0$ since $\mathcal{U}_\mathcal{A}$ is invariant under multiplication with scalars. Hence $\mathcal{U}_\mathcal{A} = \{0\}$ and there is nothing to show. Otherwise, for $t \in [0, \infty)$ define

$$g(t) := \inf \left\{ \int_{\mathbb{R}^N} f_\infty(\varphi)\, dx \;\middle|\; \varphi \in L^p(\mathbb{R}^N; \mathbb{R}^M),\ \mathcal{A}\varphi = 0,\ \|u\|_{L^p} = t \right\}.$$

Since f_∞ also inherits the p-Lipschitz property (f:2) (with $\liminf_{|x|\to\infty} h(x) = 0$ instead of h), $F_\infty(u) := \int_{\mathbb{R}^N} f_\infty(u)\, dx$ is uniformly continuous on bounded subsets of L^p by Proposition 5.7, which implies that g is continuous. It remains to show that $g > 0$ on $(0, \infty)$. Suppose by contradiction that $g(t_0) = 0$ for a $t_0 > 0$. Then there exists a sequence $(\eta_n) \subset \mathcal{U}_\mathcal{A}$ with $\|\eta_n\|_{L^p} = t_0$ such that $\int_{\mathbb{R}^N} f_\infty(\eta_n)\, dx \to 0$. Since η_n is bounded in L^p, there exists a subsequence $k(n)$

of n and a sequence of points $(x_n) \subset \mathbb{R}^N$ such that $\chi_{B_n(x_n)}\eta_{k(n)} \to 0$ in L^p. For $x \in \mathbb{R}^N$ let $\varphi_n(x) := \eta_{k(n)}(x - x_n)$. By construction, $(\varphi_n) \in \Phi_\infty$, $\|\varphi_n\|_{L^p} = t_0 > 0$ and $\int_{\mathbb{R}^N} f_\infty(\varphi_n)\, dx \to 0$. By (5.28), the latter entails that

$$\int_{\mathbb{R}^N} f(x, \varphi_n)\, dx \to \int_{\mathbb{R}^N} f(x, 0)\, dx. \tag{5.29}$$

This already contradicts (2.3) for $\Psi = \Phi_\infty$. To get the contradiction also with the smaller set $\Psi = \Phi_{c\infty} \cup \Phi_{mov} \cup \Phi_{spr}$, decompose $\varphi_n = \sum_{j=0}^4 \varphi_n^j$ according to Lemma 4.1 (or a suitable subsequence, not relabeled; note that φ_n weakly converges to zero). We have that $\varphi_n^0 + \varphi_n^1 \to 0$ in L^p, since $\varphi_n^0 + \varphi_n^1 = \varphi_n - \sum_{j=2}^4 \varphi_n^j$ is $\mathbb{R}^N$-tight and converges to zero in L^p_{loc}. Since $\|\varphi_n\|_{L^p} = t_0 > 0$, this means that at least one of the three sequences φ_n^j, $j = 2, 3, 4$, does not converge to zero strongly in L^p. Moreover, Proposition 5.8 and (5.29) imply that

$$\int_{\mathbb{R}^N} f(x, \varphi_n^j)\, dx \to \int_{\mathbb{R}^N} f(x, 0)\, dx \quad \text{for } j = 2, 3, 4. \tag{5.30}$$

As $(\varphi_n^2) \in \Phi_{c\infty}$, $(\varphi_n^3) \in \Phi_{mov}$ and $(\varphi_n^4) \in \Phi_{spr}$, this contradicts (2.3) for $\Psi = \Phi_{c\infty} \cup \Phi_{mov} \cup \Phi_{spr}$.

The general case where $\Omega \subset \mathbb{R}^N$ is the complement of some compact set is essentially analogous. The only additional difficulty occurs while showing that (2.7) implies (2.3) for $\Psi = \Phi_\infty$, because (2.7) just applies to functions defined on the whole space while the sequences in $\Phi_{c\infty} \cup \Phi_{mov} \cup \Phi_{spr}$ and Φ_∞ now are defined only on Ω. However, any sequence $(\varphi_n) \in \Phi_\infty$ converges to zero strongly in L^p on any bounded set, in particular on any bounded vicinity of $\partial\Omega$. Using smooth cut-off functions as in the proof of Lemma 4.5 to extend before projecting back onto $\mathcal{A}$-free fields allows us to replace φ_n with an $\mathcal{A}$-free sequence $\tilde{\varphi}_n$ such that $\varphi_n - \tilde{\varphi}_n \to 0$ in $L^p(\Omega; \mathbb{R}^M)$ and $\tilde{\varphi}_n \to 0$ in $L^p(\mathbb{R}^N \setminus \Omega; \mathbb{R}^M)$. □

6. Concluding remarks

Remark 6.1. While the main results of this paper and the decomposition lemmas of Section 4 are stated for the space L^p, the method presented here can actually handle more general spaces without significant additional difficulties. In fact, the results of Section 3 are already stated in a form more general than needed if we only study L^p. In particular, it is easy to adapt the decompositions lemmas and the main results to $L^p + L^q$ and $L^p \cap L^q$, respectively, with $1 < q < p < \infty$. This generalization is particularly useful for functionals on domains with infinite measure whose integrand does not have the same behavior near zero and near infinity, which is actually quite natural (e.g., $f(x, \mu) \approx |\mu|^2$ as $|\mu| \to 0$ and $f(x, \mu) \approx |\mu|^p$ as $\mu \to \infty$). In addition, the results can be extended to weighted Lebesgue spaces, as long as the suitable results for the continuity of

Fourier multipliers in these spaces are still available. Beware though that even if one is interested in one specific space only, Fourier multiplier results are still needed for a suitable family of related spaces to use the arguments employed in the proof of Lemma 3.4 (iii) and Lemma 4.1.

Remark 6.2. If $\Omega = \mathbb{R}^N$, all of the results of this paper involving a given bounded, $\mathcal{A}$-free sequence $(u_n) \subset L^p(\mathbb{R}^N;\mathbb{R}^M)$ stay true if instead of $\mathcal{A}u_n = 0$, we only require the weaker condition $\|(-\Delta)^{-\frac{1}{2}}\mathcal{A}u_n\|_{L^p} \to 0$. To see this, simply replace u_n with the $\mathcal{A}$-free sequence $\tilde{u}_n := \mathcal{P}u_n$, where $\mathcal{P}$ is the projection on $\mathcal{A}$-free fields defined in Section 3. Since $u_n - \tilde{u}_n = (I - \mathcal{P})u_n \to 0$ strongly in L^p by Lemma 3.4 (iv), the uniform continuity of F on bounded sets shown in Proposition 5.7 implies that $F(u_n) - F(\tilde{u}_n) \to 0$, which means that any assumption on $F(u_n)$ used in our results will not be affected. Unfortunately, it is not clear if this also works on domains with unbounded boundary if

$$\|\mathcal{A}u_n\|_{L^{-1,p}} := \sup\left\{ \int_\Omega u_n\mathcal{A}^*\varphi\,dx \,\middle|\, \varphi \in C_c^\infty(\Omega;\mathbb{R}^L) \text{ with } \int_\Omega |\nabla\varphi|^{\frac{p}{p-1}}\,dx \leq 1 \right\}$$

is used to replace $\|(-\Delta)^{-\frac{1}{2}}\mathcal{A}u_n\|_{L^p}$.

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