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The Riesz Potential Operator in Optimal Couples of Rearrangement Invariant Spaces

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Abstract. We prove continuity of the Riesz potential operator in optimal couples of rearrangement invariant function spaces defined in $\mathbf{R}^n$ with the Lebesgue measure. An application is given to the Hardy-Littlewood maximal operator.

Keywords. Riesz potential operator, rearrangement invariant function spaces, real interpolation

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1. Introduction

Let L_{loc} be the space of all locally integrable functions f on $\mathbf{R}^n$ with the Lebesgue measure. Analogously, let L be the space of all locally integrable functions $g \geq 0$ on $(0, \infty)$ with the Lebesgue measure that are in $L^1 + L^\infty$. The Riesz potential operator R^s, $0 < s < n$, $n \geq 1$ is defined formally by

$$R^s f(x) = \int_{\mathbf{R}^n} f(y)|x-y|^{s-n} dy, \quad f \in L_{loc}.$$

We shall consider rearrangement invariant quasi-Banach spaces E, continuously embedded in $L^1(\mathbf{R}^n) + L^\infty(\mathbf{R}^n)$, such that the quasi-norm $\|f\|_E$ in E

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is generated by a quasi-norm ρ_E, defined on L with values in $[0,\infty]$, in the sense that $\|f\|_E = \rho_E(f^*)$. In this way equivalent quasi-norms ρ_E give the same space E. We suppose that E is nontrivial. Here f^* is the decreasing rearrangement of f, given by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \le t\}, \quad t > 0,$$

where μ_f is the distribution function of f, defined by

$$\mu_f(\lambda) = \left|\{x \in \mathbf{R}^n : |f(x)| > \lambda\}\right|_n,$$

$|\cdot|_n$ denoting the Lebesgue $n-$measure.

There is an equivalent quasi-norm ρ_p that satisfies the triangle inequality $\rho_p^p(g_1 + g_2) \le \rho_p^p(g_1) + \rho_p^p(g_2)$ for some $p \in (0,1)$ that depends only on the space E (see [20]). We say that the quasi-norm ρ_E is K-monotone (cf. [6, p. 84] and also [5, p. 305]) if

$$\int_0^t g_1^*(s)\,ds \le \int_0^t g_2^*(s)\,ds \quad \text{implies} \quad \rho_E(g_1^*) \le \rho_E(g_2^*),\ g_1 \in L,\ g_2 \in L. \qquad (1)$$

Then ρ_E is monotone, i.e., $g_1 \le g_2$ implies $\rho_E(g_1) \le \rho_E(g_2)$.

We use the notations $a_1 \lesssim a_2$ or $a_2 \gtrsim a_1$ for nonnegative functions or functionals to mean that the quotient a_1/a_2 is bounded; also, $a_1 \approx a_2$ means that $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$. We say that a_1 is equivalent to a_2 if $a_1 \approx a_2$.

Recall that the relation $g_1^{**} \le g_2^{**}$, $g_1, g_2 \in L$ is equivalent to $g_1 = Cg_2$, where C is a positive contraction in the couple (L^1, L^∞) (see [21, Theorem 3.4, p. 89]).

We say that the quasi-norm ρ_E satisfies Minkovski inequality if for the equivalent quasi-norm ρ_p,

$$\rho_p^p\left(\sum g_j\right) \lesssim \sum \rho_p^p(g_j), \quad g_j \in L. \qquad (2)$$

For example, if E is a rearrangement invariant Banach function space as in [5], then by the Luxemburg representation theorem $\|f\|_E = \rho_E(f^*)$ for some norm ρ_E satisfying (1) and (2). More general example is given by the Riesz-Fischer monotone spaces as in [5, p. 305].

Recall the definition of the lower and upper Boyd indices α_E and β_E. Let

$$h_E(u) = \sup\left\{\frac{\rho_E(g_u^*)}{\rho_E(g^*)} : g \in L\right\}, \quad g_u(t) := g\left(\frac{t}{u}\right)$$

be the dilation function generated by ρ_E. Then

$$\alpha_E := \sup_{0<t<1} \frac{\log h_E(t)}{\log t} \quad \text{and} \quad \beta_E := \inf_{1<t<\infty} \frac{\log h_E(t)}{\log t}.$$

If ρ_E is monotone, then the function h_E is submultiplicative, increasing, $h_E(1) = 1$, $h_E(u)h_E(\frac{1}{u}) \geq 1$, hence $0 \leq \alpha_E \leq \beta_E$. If ρ_E is K-monotone, then by interpolation, (analogously to [5, p. 148]) we see that $h_E(s) \leq \max(1, s)$. Hence in this case we have also $\beta_E \leq 1$.

Using the Minkovski inequality for the equivalent quasi-norm ρ_p and monotonicity of f^*, we see that

$$\rho_E(f^*) \approx \rho_E(f^{**}) \quad \text{if} \quad \beta_E < 1, \tag{3}$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$.

Consider the Gamma spaces $\Gamma^q(w)$, $0 < q \leq \infty$, w - positive weight, i.e., positive function from L (see also [15] for a class of generalized Gamma spaces), with a quasi-norm $\|f\|_{\Gamma^q(w)} := \rho_{w,q,\Gamma}(f^*)$, where

$$\rho_{w,q,\Gamma}(g) := \left(\int_0^\infty [g^{**}(t)w(t)]^q \,\frac{dt}{t}\right)^{\frac{1}{q}}.$$

The condition $\left(\int_0^\infty \min(1, t^{-q})w^q(t)\,\frac{dt}{t}\right)^{\frac{1}{q}} < \infty$ should be satisfied (otherwise the space will be trivial). Then this space is continuously embedded in the sum $L^1 + L^\infty$. Using this embedding, the completeness of the space can be established in a standard way. The space $E = \Gamma^q(w)$ with $\rho_E = \rho_{w,q,\Gamma}$ satisfies the conditions (1), (2).

Consider the classical Lorentz spaces $\Lambda^q(w)$, $0 < q \leq \infty$; $f \in \Lambda^q(w)$ if

$$\|f\|_{\Lambda^q_w} := \rho_{w,q}(f^*), \quad \rho_{w,q}(g) := \left(\int_0^\infty [g^*(t)w(t)]^q \,\frac{dt}{t}\right)^{\frac{1}{q}} < \infty.$$

This is a quasi-normed space if $w(2t) \approx w(t)$. If $w(t) = t^{\frac{1}{r}}$, $1 \leq r < \infty$, the usual notation $L^{r,q}$ is used. Then $\alpha_E = \beta_E = \frac{1}{r}$. In some cases the Lorentz space $E = \Lambda^q(w)$, $1 \leq q < \infty$ also satisfies the conditions (1), (2). For example, if $\frac{w^q(t)}{t}$ is not increasing, then (see [5, p. 218]), the functional $\rho_{w,q}$ is a K-monotone norm. It is easy to check that this space is continuously embedded in $L^q + L^\infty$.

We have the equivalence

$$\|f\|_{\Gamma^q(w)} \approx \|f\|_{\Lambda^q(w)} \tag{4}$$

in the following cases.

If $1 \leq q < \infty$ then (4) is satisfied if and only if w is such that (see [2])

$$t^q \int_t^\infty s^{-q}[w(s)]^q \,\frac{ds}{s} \lesssim \int_0^t [w(s)]^q \,\frac{ds}{s}. \tag{5}$$

If $q = \infty$ then (4) is valid if and only if (see [9])

$$\frac{1}{t}\int_0^t \frac{1}{w(s)}\,ds \lesssim \frac{1}{w(t)}, \quad \text{where} \quad w(t) := \int_0^t v(s)\,ds \text{ for some } v.$$

For weights satisfying $\int_0^t w(s)\,\frac{ds}{s} \lesssim w(t)$, the condition (5) with $q = 1$ is equivalent to

$$\int_t^\infty \frac{w(s)}{s}\,\frac{ds}{s} \lesssim \frac{w(t)}{t}. \tag{6}$$

Indeed, the condition (6) implies (5) with $q = 1$ by integration and Fubini's theorem . Conversely, (6) follows from (5) with $q = 1$ if $\int_0^t w(s)\,\frac{ds}{s} \lesssim w(t)$. Note that (4) is equivalent to $\beta_E < 1$ (see [5, p. 150]).

The main goal of this paper is to prove continuity of the Riesz potential operator $R^s : E \mapsto G$ in optimal couples of rearrangement invariant function spaces E and G, where $\|f\|_G := \rho_G(f^*)$. It is convenient to introduce the following classes of quasi-norms:

- N_d consists of all quasi-norms ρ_E that are monotone, rearrangement invariant, and such that $\alpha_E \geq \frac{s}{n}$, $\beta_E \leq 1$;
- $N_{d,1}$ consists of all quasi-norms ρ_E that are monotone, rearrangement invariant, satisfy Minkovski inequality and $\alpha_E \geq \frac{s}{n}$, $\beta_E < 1$;
- $N_{d,2}$ consists of all quasi-norms ρ_E that are monotone, rearrangement invariant, satisfy Minkovski inequality and $\alpha_E > \frac{s}{n}$, $\beta_E \leq 1$;
- $N_{d,3}$ consists of all quasi-norms ρ_E that are K-monotone, rearrangement invariant, satisfy Minkovski inequality, and $\alpha_E \geq \frac{s}{n}$;
- N_t consists of all quasi-norms ρ_G that are monotone and $\beta_G \leq 1 - \frac{s}{n}$;
- $N_{t,1}$ consists of all quasi-norms ρ_G that are monotone, satisfy Minkovski inequality and $\beta_G < 1 - \frac{s}{n}$;
- $N_{t,2}$ consists of all quasi-norms ρ_G that are monotone, satisfy Minkovski inequality and $\alpha_G > 0$, $\beta_G \leq 1 - \frac{s}{n}$.

Definition 1.1 (admissible couple)**.** We say that the couple $\rho_E \in N_d$, $\rho_G \in N_t$ is admissible for the Riesz potential if the following estimate is valid:

$$\rho_G((R^s f)^{**}) \lesssim \rho_E(f^*). \tag{7}$$

Moreover, ρ_E (respectively E) is called domain quasi-norm (domain space), and ρ_G (respectively G) is called target quasi-norm (target space).

For example, by Theorem 2.2 below (the sufficient part), the couple $E = \Lambda^q\left(t^{\frac{s}{n}} w\right)$, $G = \Lambda^q(v)$, $1 \leq q \leq \infty$, is admissible if $\beta_E < 1$ and v is related to w by the Muckenhoupt condition [27]:

$$\left(\int_0^t [v(s)]^q\,\frac{ds}{s}\right)^{\frac{1}{q}} \left(\int_t^\infty [w(s)]^{-r}\,\frac{ds}{s}\right)^{\frac{1}{r}} \lesssim 1, \quad \frac{1}{q} + \frac{1}{r} = 1.$$

Definition 1.2 (optimal target quasi-norm)**.** Given the domain quasi-norm $\rho_E \in N_d$, the optimal target quasi-norm, denoted by $\rho_{G(E)}$, is the strongest target quasi-norm, i.e.,

$$\rho_G(g^*) \lesssim \rho_{G(E)}(g^*), \quad g \in L,$$

for any target quasi-norm $\rho_G \in N_t$ such that the couple ρ_E, ρ_G is admissible.

Definition 1.3 (optimal domain quasi-norm)**.** Given the target quasi-norm $\rho_G \in N_t$, the optimal domain quasi-norm, denoted by $\rho_{E(G)}$, is the weakest domain quasi-norm, i.e.,

$$\rho_{E(G)}(g^*) \lesssim \rho_E(g^*), \quad g \in L,$$

for any domain quasi-norm $\rho_E \in N_d$ such that the couple ρ_E, ρ_G is admissible.

Definition 1.4 (optimal couple)**.** The admissible couple ρ_E, ρ_G is said to be optimal if $\rho_E = \rho_{E(G)}$ and $\rho_G = \rho_{G(E)}$.

We prove that optimal quasi-norms are uniquely determined up to equivalence, while the corresponding optimal quasi-Banach spaces are unique. We give a characterization of all admissible couples, optimal target quasi-norms, optimal domain quasi-norms, and optimal couples. It is convenient to consider two cases: subcritical and critical.

Definition 1.5 (subcritical case)**.** The subcritical case is defined by the condition

$$\int_0^1 u^{-p\frac{s}{n}} h_E^p(u)\,\frac{du}{u} < \infty, \quad \text{or equivalently,} \quad \frac{s}{n} < \alpha_E. \tag{8}$$

The equivalence in (8) can be established as in [5, p. 147]. For example, if $E = L^r$, $1 \le r < \infty$, then the condition (8) means that $s < \frac{n}{r}$.

In the subcritical case and if $\beta_E < 1$ we prove that the optimal target quasi-norm satisfies $\rho_{G(E)}(g^*) \approx \rho_E\left(t^{-\frac{s}{n}} g^*(t)\right)$, $g \in L$. Moreover, the couple $\rho_E, \rho_{G(E)}$ is optimal.

Definition 1.6 (critical case)**.** The critical case is defined by the condition $\frac{s}{n} = \alpha_E$.

In the critical case we use real interpolation similarly to [11], but in a simpler way [1] and consider domain quasi-norms $\rho_E(g) := \rho_H\Big(\big(t^{\frac{s}{n}} b(t) g^*(t)\big)^{**}_\mu\Big)$, where ρ_H is K-monotone quasi-norm on $(0,\infty)$, satisfying $\beta_H < 1$, and h^*_μ means the rearrangement of h with respect to the Haar measure on $(0,\infty)$, $d\mu := \frac{dt}{t}$, $h^{**}_\mu(t) := \frac{1}{t}\int_0^t h^*_\mu(u)\,du$. In this case the optimal target quasi-norm $\rho_{G(E)}$ is

$$\rho_{G(E)}(g) := \rho_H\left((cg)^{**}_\mu\right). \tag{9}$$

Here b and c belong to a large class of Muckenhoupt slowly varying weights (see Theorem 4.1 below).

Recall that w is slowly varying on $(1,\infty)$ (in the sense of Karamata), if for all $\varepsilon > 0$ the function $t^\varepsilon w(t)$ is equivalent to a non-decreasing function, and the function $t^{-\varepsilon}w(t)$ is equivalent to a non-increasing function. By symmetry, we say that w is slowly varying on $(0,1)$ if the function $t \mapsto w\left(\frac{1}{t}\right)$ is slowly varying on $(1,\infty)$. Finally, w is slowly varying if it is slowly varying on $(0,1)$ and $(1,\infty)$.

For example, if $\rho_H(g) := \left(\int_0^\infty [g(t)]^q\,dt\right)^{\frac{1}{q}}$, $1 < q \le \infty$, then $\beta_H = \frac{1}{q} < 1$, and

$$\rho_E(g^*) \approx \left(\int_0^\infty \left[\left(t^{\frac{s}{n}}b(t)g^*(t)\right)^*_\mu(u)\right]^q du\right)^{\frac{1}{q}} = \left(\int_0^\infty \left[t^{\frac{s}{n}}b(t)g^*(t)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

Hence $E = \Lambda^q\left(t^{\frac{s}{n}}b(t)\right)$ and $G(E) = \Lambda^q(c)$.

The problem of the optimal target space for potential type operators defined on L^p is considered in [17] by different methods. The case $s = 2$ is treated in [10]. Since $f = cR^k(\nabla^k f)$, $f \in C_0^\infty$, where ∇^k is the kth order gradient (see for example [13]), the results about the optimal couples for the Riesz potential imply optimal embeddings for the homogeneous Sobolev space w^kE with a quasi-norm $\|f\|_{w^kE} = \|\nabla^k f\|_E$, $f \in C_0^\infty$. A direct approach to the same problem for the homogeneous Sobolev space with a norm $\sum_{|\alpha|=k}\|D^\alpha f\|_E$ is used in [1] and similar results are proved. The problem of optimal embeddings of inhomogeneous Sobolev spaces, defined on a bounded domain in $\mathbf{R}^n$, is treated by somewhat different methods in [13, 14, 16, 18, 20, 22–26].

In this paper we will use the standard notation for the Hardy operators

$$Pg(t) = \frac{1}{t}\int_0^t g(u)\,du, \quad Qg(t) = \int_t^\infty g(u)\,\frac{du}{u}.$$

2. Admissible couples

Here we give a characterization of all admissible couples ρ_E, ρ_G. It is convenient to define the case $\beta_E = 1$ as limiting and the case $\beta_E < 1$ as sublimiting.

Theorem 2.1 (general case $\beta_E \le 1$)**.** *The couple $\rho_E \in N_d$, $\rho_G \in N_t$ is admissible if and only if*

$$\rho_G(Sg) \lesssim \rho_E(g), \quad g \in L, \tag{10}$$

where

$$S = T + T', \quad Tg(t) := \int_t^\infty u^{\frac{s}{n}} g(u)\,\frac{du}{u}, \quad t > 0,\ 0 < s < n,\ n \ge 1, \tag{11}$$

and $T'g(t) = t^{\frac{s}{n}-1}\int_0^t g(u)\,du$ is the operator adjoint to T.

Proof. First we prove

$$(R^s f)^{**} \lesssim Sf^*. \tag{12}$$

We are going to use real interpolation for quasi-Banach spaces. First we recall some basic definitions. Let (A_0, A_1) be a couple of two quasi-Banach spaces (see [6, 7]) and let

$$K(t, f) = K(t, f; A_0, A_1) = \inf_{f=f_0+f_1} \{\|f_0\|_{A_0} + t\,\|f_1\|_{A_1}\}, \quad f \in A_0 + A_1,$$

be the K-functional of Peetre (see [6]). By definition, the K-interpolation space $A_\Phi = (A_0, A_1)_\Phi$ has a quasi-norm $\|f\|_{A_\Phi} = \|K(t, f)\|_\Phi$, where Φ is a quasi-normed function space with a monotone quasi-norm on $(0, \infty)$ with the Lebesgue measure and such that $\min\{1, t\} \in \Phi$. Then (see [7])

$$A_0 \cap A_1 \hookrightarrow A_\Phi \hookrightarrow A_0 + A_1.$$

where by $X \hookrightarrow Y$ we mean that X is continuously embedded in Y. If $\|g\|_\Phi = \left(\int_0^\infty t^{-\theta q} g^q(t)\, \frac{dt}{t}\right)^{\frac{1}{q}}$, $0 < \theta < 1$, $0 < q \le \infty$, we write $(A_0, A_1)_{\theta,q}$ instead of $(A_0, A_1)_\Phi$ (see [6]).

Using the Hardy-Littlewood inequality $\int_{\mathbf{R}^n} |f(x)g(x)|dx \le \int_0^\infty f^*(t)g^*(t)dt$, we get the well known mapping property $R^s\colon \Lambda^1\left(t^{\frac{s}{n}}\right) \mapsto L^\infty$. and by the Minkovski inequality for the norm f^{**} we get $R^s\colon L^1 \mapsto \Lambda^\infty\left(t^{1-\frac{s}{n}}\right)$. Hence $t^{1-\frac{s}{n}}(R^s f)^{**}(t) \lesssim K\left(t^{1-\frac{s}{n}}, f; L^1, \Lambda^1\left(t^{\frac{s}{n}}\right)\right)$, therefore (see [6, Section 5.7])

$$(R^s f)^{**}(t) \lesssim S f^*(t).$$

It is clear that (7) follows from (12) and (10).

Now we prove that (7) implies (10). To this end we choose the test function in the form $f(x) = g(c|x|^n)$, $g \in L$, so that $f^*(t) = g^*(t)$ for some positive constant c (cf. [10]). Then

$$R^s f(x) = \int_{|y|<|x|} g(c|y|^n)|x - y|^{s-n}\, dy + \int_{|y|>|x|} g(c|y|^n)|x - y|^{s-n}\, dy,$$

whence

$$|R^s f(x)| \gtrsim |x|^{s-n} \int_0^{c|x|^n} g(u)\, du + \int_{c|x|^n}^\infty u^{\frac{s}{n}-1} g(u)\, du \gtrsim (Sg)(c|x|^n).$$

Note that $S = \frac{n}{n-s} Q T'$, hence Sg is decreasing, therefore

$$|R^s f|^*(t) \gtrsim Sg(t), \quad S = T + T'. \tag{13}$$

Thus, if (7) is given, then (13) implies (10). □

In the sublimiting case $\beta_E < 1$ we can simplify the condition (10), replacing S by T.

Theorem 2.2 (Sublimiting case $\beta_E < 1$)**.** *The couple $\rho_E \in N_{d,1}$, $\rho_G \in N_t$ is admissible if and only if*

$$\rho_G(Tg) \lesssim \rho_E(g), \quad g \in M, \tag{14}$$

where $M := \{g \in L : t^m g(t)$ is increasing for some $m > 0\}$.

Proof. We need to prove sufficiency only. But from (12) it follows that $(R^s f)^{**} \lesssim Tf^{**}$, therefore (7) follows from (14) and (3) since $\beta_E < 1$. □

In the case $\alpha_E > \frac{s}{n}$ we have another simplification of (10).

Theorem 2.3 (case $\alpha_E > \frac{s}{n}$)**.** *The couple $\rho_E \in N_d$, $\rho_G \in N_{t,2}$ is admissible if and only if*

$$\rho_G(T'g) \lesssim \rho_E(g), \quad g \in M_1 := \{g \in L : g \text{ is decreasing}\}. \tag{15}$$

Proof. It is enough to check that (15) implies (10). First we prove the estimate for $\rho_E \in N_d$:

$$\rho_E(t^{-a}Qg(t)) \lesssim \rho_E(t^{-a}g(t)), \quad 0 \le a < 1,\ g \in M,\ \alpha_E > a, \tag{16}$$

where $Qg(t) = \int_t^\infty g(u)\,\frac{du}{u}$. The proof is standard, we just have to use the Minkovski inequality for the equivalent quasi-norm ρ_p and that $\alpha_E > a$ is equivalent to $\int_0^1 u^{-ap} h_E^p(u)\,\frac{du}{u} < \infty$. (cf. [5])

On the other hand, it is $Tg \lesssim T'\left(t^{-\frac{s}{n}}Tg(t)\right)$, hence by (15), $\rho_G(Tg) \lesssim \rho_E\left(t^{-\frac{s}{n}}Tg\right)$, $g \in M$, therefore if $\alpha_E > \frac{s}{n}$, then (16) implies $\rho_G(Tg) \lesssim \rho_E(g)$. □

For example, the couple $E = \Gamma^q(tw)$, $G = \Lambda^q\left(t^{1-\frac{s}{n}}v\right)$, $1 \le q \le \infty$, is admissible if $\alpha_G > 0$ and v is related to w by [27]

$$\left(\int_t^\infty [v(u)]^q\,\frac{du}{u}\right)^{\frac{1}{q}} \left(\int_0^t [w(u)]^{-r}\,\frac{du}{u}\right)^{\frac{1}{r}} \lesssim 1, \quad \frac{1}{q} + \frac{1}{r} = 1.$$

2.1. Optimal quasi-norms. Here we give a characterization of the optimal domain and optimal target quasi-norms. Let

$$E \hookrightarrow L^1 + \Lambda^1\left(t^{\frac{s}{n}}\right) \quad \text{and} \quad G \hookrightarrow \Lambda^\infty\left(t^{1-\frac{s}{n}}\right) + L^\infty.$$

We can define an optimal target quasi-norm by using Theorem 2.1.

Definition 2.4 (construction of the optimal target quasi-norm)**.** For a given domain quasi-norm $\rho_E \in N_d$ we set

$$\rho_{G(E)}(g) := \inf\{\rho_E(h) : g \le Sh,\ h \in L\}. \tag{17}$$

Note that $\alpha_{G(E)} \geq \alpha_E - \frac{s}{n}$, $\beta_{G(E)} \leq \beta_E - \frac{s}{n}$. In particular, if $\alpha_E = \beta_E$ then $\alpha_{G(E)} = \beta_{G(E)} = \alpha_E - \frac{s}{n}$.

Proposition 2.5. *The couple $\rho_E \in N_d$, $\rho_{G(E)} \in N_t$ is admissible and the target quasi-norm is optimal. Also,*

$$G(E) \hookrightarrow \Lambda^\infty(t^{1-\frac{s}{n}}) + L^\infty. \tag{18}$$

Proof. Since ρ_E is a monotone quasi-norm it follows that $\rho_{G(E)}$ is also monotone quasi-norm. The couple is admissible due to $\rho_{G(E)}(Sh) \leq \rho_E(h),\ h \in L$ and Theorem 2.1. Suppose that the couple $\rho_E \in N_d$, $\rho_G \in N_t$ is admissible. Then by Theorem 2.1, $\rho_G(Sg) \lesssim \rho_E(g),\ g \in L$. Therefore if $g^* \leq Sh$, $h \in L$, then $\rho_G(g^*) \leq \rho_G(Sh) \lesssim \rho_E(h)$, whence $\rho_G(g^*) \lesssim \rho_{G(E)}(g^*)$.

It remains to check (18). Note that (see [6])

$$\|f\|_{\Lambda^\infty(t^{1-\frac{s}{n}})+L^\infty} \approx \sup_{0<t<1} t^{1-\frac{s}{n}} f^*(t) + \sup_{t>1} f^*.$$

Let $f^* \leq Sh$, $h \in L$. Then $f^* \lesssim Sh^*$ and

$$\sup_{0<t<1} t^{1-\frac{s}{n}} Sh^*(t) \leq \int_0^1 h^*(u)\,du + \int_1^\infty u^{\frac{s}{n}-1} h^*(u)\,du.$$

Since (see [6]) $\|f\|_{L^1+\Lambda^1\left(t^{\frac{s}{n}}\right)} \approx \int_0^1 f^*(u)\,du + \int_1^\infty u^{\frac{s}{n}-1} f^*(u)\,du$, we obtain $\sup_{0<t<1} t^{1-\frac{s}{n}} Sh(t) \lesssim \rho_E(h)$. Analogously, $\sup_{t>1} Sh(t) \lesssim \rho_E(h)$. Therefore, $\|f\|_{\Lambda^\infty\left(t^{1-\frac{s}{n}}\right)+L^\infty} \lesssim \rho_E(h)$. Taking the infimum, we get (18). □

In the sublimiting case $\beta_E < 1$ we can simplify the optimal target quasi-norm.

Proposition 2.6. *Let $\rho_E \in N_{d,1}$. Then*

$$\rho_{G(E)}(g^*) \approx \rho(g^*), \quad \rho(g) := \inf\{\rho_E(h) : g \leq Th,\ h \in M\}. \tag{19}$$

Proof. If $g^* \leq Th$, $h \in M$, then $g^* \leq Sh$, therefore $\rho(g^*) \geq \rho_{G(E)}(g^*)$. For the reverse, if $g^* \leq Sh$, $h \in L$, then $g^* \lesssim Sh^*$ and using $T'h \leq t^{\frac{s}{n}} h^{**}$, we get $g^* \lesssim Th^{**}$. Hence $\rho(g^*) \lesssim \rho_E(h^{**})$ and because $\beta_E < 1$, we derive $\rho(g^*) \lesssim \rho_E(h)$. Taking the infimum, we get $\rho(g^*) \lesssim \rho_{G(E)}(g^*)$. □

A simplification of the optimal target quasi-norm is possible also in the subcritical case $\alpha_E > \frac{s}{n}$.

Proposition 2.7. *Let $\rho_E \in N_{d,2}$. Then*

$$\rho_{G(E)}(g^*) \approx \rho_1(g^*), \quad \rho_1(g) := \inf\{\rho_E(h) : g \leq T'h,\ h \in M_1\}. \tag{20}$$

Proof. If $g^* \leq T'h$, $h \in M_1$, then $g^* \leq Sh$, therefore $\rho_1(g^*) \geq \rho_{G(E)}(g^*)$. For the reverse, if $g^* \leq Sh$, $h \in L$, then using $Th(t) \lesssim T'\left(t^{-\frac{s}{n}}Th\right)$, we get $g^* \lesssim T'(h^* + t^{-\frac{s}{n}}Th)$. Hence $\rho_1(g^*) \lesssim \rho_E(h) + \rho_E\left(t^{-\frac{s}{n}}Th\right) \lesssim \rho_E(h)$ using $\alpha_E > \frac{s}{n}$ and (16). Taking the infimum, we get $\rho_1(g^*) \lesssim \rho_{G(E)}(g^*)$. □

We can construct an optimal domain quasi-norm $\rho_{E(G)}$ by Theorem 2.1 as follows.

Definition 2.8 (construction of an optimal domain quasi-norm)**.** For a given target quasi-norm $\rho_G \in N_t$, we construct an optimal domain quasi-norm $\rho_{E(G)} \in N_d$ by

$$\rho_{E(G)}(g) := \sup\{\rho_G(Sh) : h^{**} \leq g^{**},\ h \in L\}, \quad g \in L.$$

Note that $\alpha_{E(G)} \geq \frac{s}{n} + \alpha_G$ and $\beta_{E(G)} \leq \frac{s}{n} + \beta_G$.

Theorem 2.9. *Let $\rho_G \in N_t$. Then $\rho_{E(G)} \in N_{d,3}$, the couple $\rho_{E(G)}, \rho_G$ is admissible and the domain quasi-norm $\rho_{E(G)}$ is optimal. Also*

$$E(G) \hookrightarrow L^1 + \Lambda^1\left(t^{\frac{s}{n}}\right). \tag{21}$$

Proof. First we prove a few properties of the functional $\rho_{E(G)}$. We suppose that $\rho_G \in N_t$. To prove that $\rho_{E(G)}$ satisfies the triangle inequality for quasi-norms, let $h^{**} \leq (g_1 + g_2)^{**}$. Then there is ([5, 21]) a positive contraction C in the couple (L^1, L^∞) such that $h = C(g_1 + g_2)$. If $h_j := Cg_j$ then $h_j \in L$ and $h = h_1 + h_2$. Since $\rho_G(Sh) \lesssim \rho_G(Sh_1) + \rho_G(Sh_2) \lesssim \rho_{E(G)}(g_1) + \rho_{E(G)}(g_2)$, we have $\rho_{E(G)}(g_1 + g_2) \lesssim \rho_{E(G)}(g_1) + \rho_{E(G)}(g_2)$. Evidently, the quasi-norm $\rho_{E(G)}$ is rearrangement invariant. It is K-monotone, because if $g_1^{**} \leq g_2^{**}$ and $h^{**} \leq g_1^{**}$, then $\rho_{E(G)}(g_2) \geq \rho_G(Sh)$, whence $\rho_{E(G)}(g_1) \leq \rho_{E(G)}(g_2)$.

Further, the couple $\rho_{E(G)}, \rho_G$ is admissible since $\rho_{E(G)}(g) \geq \rho_G(Sg)$. Moreover, $\rho_{E(G)}$ is optimal, since for any admissible couple ρ_E, ρ_G we have $\rho_G(Sh) \lesssim \rho_E(h) = \rho_E(h^*)$, where $h \in L$. Therefore,

$$\rho_{E(G)}(g) \leq \sup\{\rho_E(h) : h^{**} \leq g^{**},\ h \in L\} \lesssim \rho_E(g)$$

by K-monotonicity of ρ_E. To prove the property (21), we notice that

$$\rho_{E(G)}(f^*) \geq \rho_G(Sf^*) \gtrsim \int_0^1 f^*(t)\,dt + \int_1^\infty u^{\frac{s}{n}-1} f^*(u)\,du \approx \|f\|_{L^1+\Lambda^1\left(t^{\frac{s}{n}}\right)}. \qquad \square$$

Remark 2.10. If ρ_G satisfies Minkovski inequality then $\rho_{E(G)}$ also satisfies Minkovski inequality. Indeed, let $g = \sum g_j$, $g, g_j \in L$. If $h^{**} \leq g^{**}$, then $h = Cg$. Define $h_j := Cg_j$. Then $h = \sum h_j$ and $Sh = \sum Sh_j$, therefore (for the equivalent quasi-norms)

$$\rho_G^p(Sh) \lesssim \sum \rho_G^p(Sh_j) \lesssim \sum \rho_{E(G)}^p(g_j),$$

whence $\rho_{E(G)}^p(g) \leq \sum \rho_{E(G)}^p(g_j)$.

In the case $\alpha_G > 0$ we can simplify the formula for the optimal domain quasi-norm.

Proposition 2.11. *Let* $\rho_G \in N_{t,2}$. *Then* $\rho_{E(G)} \in N_{d,2} \cap N_{d,3}$ *and*

$$\rho_{E(G)}(g) \approx \rho_G\left(t^{\frac{s}{n}} g^{**}(t)\right) \approx \rho_G(Tg^{**}).$$

Proof. We have $\rho_{E(G)}(g^*) \geq \rho_G(Sg^*) \geq \rho_G(T'g^*) = \rho_G\left(t^{\frac{s}{n}} g^{**}\right)$. On the other hand, the quasi-norm $\rho_1(g) := \rho_G\left(t^{\frac{s}{n}} g^{**}(t)\right)$ is in the class $N_{d,2} \cap N_{d,3}$ and the couple ρ_1, ρ_G is admissible since $\rho_G(T'h) \leq \rho_G(T'h^*) = \rho_1(h)$. Therefore, $\rho_{E(G)} \lesssim \rho_1$. □

A simplification is possible also in the case $\beta_G < 1 - \frac{s}{n}$.

Theorem 2.12. *Let* $\rho_G \in N_{t,1}$. *Then* $\rho_{E(G)} \in N_{d,1} \cap N_{d,3}$ *and* $\rho_{E(G)}(g) \approx \rho_2(g) := \rho_G(Tg^{**})$. *Moreover, the couple* $\rho_{E(G)}, \rho_G$ *is optimal.*

Also, if $\alpha_G > 0$, *then* $\rho_{E(G)}(g) \approx \rho_G\left(t^{\frac{s}{n}} g^{**}(t)\right)$.

Proof. We have $\rho_{E(G)}(g^*) \geq \rho_G(Sg^*) \geq \rho_G(Tg^*) \approx \rho_2(g^*)$, since the upper Boyd index for ρ_2 is $\leq \frac{s}{n} + \beta_G < 1$. On the other hand the couple ρ_2, ρ_G is admissible since for $g \in M$, $\rho_G(Tg) \lesssim \rho_G(Tg^{**}) = \rho_2(g)$. Therefore, $\rho_{E(G)} \lesssim \rho_2$.

Now we check that the couple $\rho_{E(G)}$, ρ_G is optimal. We need only to prove that ρ_G is an optimal target quasi-norm, i.e., $\rho(g^*) \lesssim \rho_G(g^*)$, where $\rho = \rho_{G(E(G))}$ is defined by (19) since $\beta_{E(G)} < 1$. We have $g^{**} = Th$, $h(t) = t^{-\frac{s}{n}}[g^{**}(t) - g^*(t)] \in M$, therefore

$$\rho(g^*) \leq \rho_{E(G)}(h) \approx \rho_G(Th^{**}).$$

Since $h^* \lesssim Qh$, we have $h^{**} = Ph^* \lesssim QPh$, therefore $Th^{**} \lesssim TQ(Ph) \lesssim T(Ph)$. Also $T(Ph) \approx Th + t^{\frac{s}{n}} Ph$ and $Ph \leq h^{**}$. Therefore,

$$\rho(g^*) \lesssim \rho_G(Th) + \rho_G\left(t^{\frac{s}{n}} h^{**}\right).$$

Since $h(t) \leq t^{-\frac{s}{n}} g^{**}(t)$ we have $h^*(t) \leq t^{-\frac{s}{n}} g^{**}$, thus we have $\rho_G\left(t^{\frac{s}{n}} h^{**}(t)\right) \leq \int_0^1 u^{-\frac{s}{n}} h_G\left(\frac{1}{u}\right) du \leq \rho_G(g^{**})$. Therefore $\rho(g^*) \lesssim \rho_G(g^{**}) \approx \rho_G(g^*)$. □

Now we give some examples.

Example 2.13. Consider the space $G = \Lambda^1(v)$ and let $\beta_G < 1 - \frac{s}{n}$. This is true in the particular case when v is slowly varying. Using Theorem 2.12, we can construct the optimal couple E, G, where

$$\rho_E(g) = \rho_G(Tg^{**}) = \int_0^\infty t^{\frac{s}{n}} w(t) g^{**}(t)\, \frac{dt}{t},$$

and $w(t) = \int_0^t v(u)\, \frac{du}{u}$, $w(\infty) = \infty$. Hence $E = \Gamma^1\left(t^{\frac{s}{n}} w\right)$. Also $\alpha_E = \beta_E = \frac{s}{n}$ if v is slowly varying.

Example 2.14. If $G = C_0$ consists of all bounded functions such that $f^*(\infty) = 0$ and $\rho_G(g) = g^*(0) = g^{**}(0)$, then $\alpha_G = \beta_G = 0$ and $\rho_{E(G)}(g) \approx \int_0^\infty t^{\frac{s}{n}} g^{**} \frac{dt}{t}$, i.e., $E = \Gamma^1\big(t^{\frac{s}{n}}\big)$ and the couple E, G is optimal.

Example 2.15. Let $G = \Lambda^\infty(v)$ with $v(\infty) = \infty$, $\beta_G < 1 - \frac{s}{n}$ and let

$$\rho_E(g) = \sup v(t) \int_t^\infty u^{\frac{s}{n}} g^{**}(u) \,\frac{du}{u}.$$

Then by Theorem 2.12, the couple E, G is optimal and $\beta_E < 1$. In particular, this is true if v is slowly varying since then $\alpha_G = \beta_G = 0$ and $\alpha_E = \beta_E = \frac{s}{n} < 1$.

Example 2.16. Let $E = \Lambda^\infty\left(t^{\frac{s}{n}} w(t)\right)$, where w is slowly varying. If

$$\frac{1}{v(t)} = \int_t^\infty \frac{1}{w(u)} \frac{du}{u},$$

then $G = \Lambda^\infty(v)$ is optimal target space. Indeed, $\beta_E = \frac{s}{n} < 1$, and $\rho_G(Tg) \lesssim \rho_E(g)$, which means that the couple is admissible. In order to prove that ρ_G is optimal, take any $g \in L$, and define h from $t^{\frac{s}{n}} w(t)h(t) = \sup_{0<u\le t} v(u)g^*(u)$. Then $h \in M$, $t^{\frac{s}{n}} w(t)h(t) \le \rho_G(g^*)$, therefore $t^{\frac{s}{n}} w(t)h^*(t) \lesssim \rho_G(g^*)$, whence $\rho_E(h) \lesssim \rho_G(g^*)$. On the other hand

$$Th(t) = \int_t^\infty \sup_{0<x\le u} v(x)g^*(x)\frac{1}{w(u)}\frac{du}{u} \ge \sup_{0<u\le t} v(u)g^*(u)\frac{1}{v(t)} \ge g^*(t).$$

Hence $\rho_{G(E)}(g^*) \le \rho_E(h) \lesssim \rho_G(g^*)$, therefore ρ_G is optimal.

Example 2.17. Let $\rho_E(g) := \rho_F\left(tw(t)g^{**}(t)\right)$, $\rho_G(g) := \rho_F\left(t^{1-\frac{s}{n}} w(t)g(t)\right)$, where w is slowly varying and ρ_F is a monotone quasi-norm with $\alpha_F = \beta_F = 0$. Then $\alpha_E = \beta_E = 1$, $\alpha_G = \beta_G = 1 - \frac{s}{n}$, the couple ρ_E, ρ_G is admissible since $\rho_G(T'g) \le \rho_E(g)$ and according to Proposition 2.11 the domain quasi-norm is optimal. For example we can take $\rho_F(g) = \left(\int_0^\infty g^q(t) \frac{dt}{t}\right)^{\frac{1}{q}}$, $0 < q \le \infty$. But this couple is not optimal. If $w\chi_{(0,1)} \in F$ and $f^*(t) = t^{\frac{s}{n}-1}$, then $f \in G$ and $\rho_{G(E)}(f^*) = \inf \rho_E(h)$, where infimum is taken with respect to all $h \in M_1$ such that $1 \le \int_0^t h(u)\,du$. Hence $\int_0^\infty h(u)\,du = \infty$. Since we have $\rho_E(h) \ge \rho_F(w\chi_{(0,1)}) \int_0^t h(u)\,du$ for $t > 1$, it follows $\rho_E(h) = \infty$, therefore $f \notin G(E)$.

3. Subcritical case

Here we suppose that $\frac{s}{n} < \alpha_E$.

Theorem 3.1 (sublimiting case). *Let $\beta_E < 1$ and $\rho_E \in N_{d,3}$. Then the optimal target quasi-norm $\rho_{G(E)}$ is equivalent to $\rho_1(g) := \rho_E\left(t^{-\frac{s}{n}} g\right)$, $\rho_{G(E)}(g^*) \approx \rho_1(g^*)$. Moreover, the couple $\rho_E, \rho_{G(E)}$ is optimal.*

Proof. Let $g \le T'h$, $h \in M_1$. Then $t^{-\frac{s}{n}} g(t) \le Ph(t) := t^{-1}\int_0^t h(u)\,du$, whence $\rho_1(g) \lesssim \rho_E(h)$, since $\beta_E < 1$. Taking the infimum, we get $\rho_1(g) \lesssim \rho_{G(E)}(g)$. On the other hand, $g^* \lesssim T'\left(t^{-\frac{s}{n}} g^*(t)\right)$, therefore $\rho_{G(E)}(g^*) \lesssim \rho_E\left(t^{-\frac{s}{n}} g^*(t)\right) = \rho_1(g^*)$. Finally, the couple $\rho_E, \rho_{G(E)}$ is optimal, since

$$\rho_{E(G(E))}(g^*) \ge \rho_{G(E)}(Tg^*) \approx \rho_E\left(t^{-\frac{s}{n}} Tg^*\right) \gtrsim \rho_E(g^*). \qquad \square$$

Example 3.2. Let $E = \Lambda^q(t^\alpha w_1) \bigcap \Lambda^r\left(t^\beta w_2\right)$, $\frac{s}{n} < \alpha \le \beta < 1$, $0 < q, r \le \infty$, where w_1, w_2 are slowly varying. Then $G(E) = \Lambda^q\left(t^{\alpha-\frac{s}{n}} w_1\right) \bigcap \Lambda^r\left(t^{\beta-\frac{s}{n}} w_2\right)$ and this couple is optimal. In particular, we can take $E = L^p$, $1 < p < \frac{n}{s}$. Then $G(E) = L^{q,p}$, $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$. This is a classical result, see [17]. Now we know that the domain space L^p is also optimal.

In the limiting case $\beta_E = 1$ we do not know how to simplify the formula (20) for the optimal target quasi-norm. In the next example we provide a construction of an optimal target quasi-norm.

Example 3.3. Let $E = \Lambda^\infty(tw)$, w is slowly varying and $\frac{1}{v(t)} = \int_0^t \frac{1}{w(u)}\frac{du}{u}$. Then $\beta_E = 1$. If $G = \Lambda^\infty\left(t^{1-\frac{s}{n}} v\right)$ then $\alpha_G > 0$ and the couple E, G is admissible since $\rho_G(T'g) \lesssim \rho_E(g)$, $g \in M_1$. Moreover, the target space is optimal. Indeed, choose h so that $tw(t)h(t) = \sup_{u>t} u^{1-\frac{s}{n}} v(u) g^*(u)$. Then $tw(t)h(t) \le \rho_G(g^*)$, hence $tw(t)h^*(t) \lesssim \rho_G(g^*)$ and $\rho_E(h) \lesssim \rho_G(g^*)$. On the other hand,

$$T'h(t) \ge t^{\frac{s}{n}-1}\int_0^t \frac{1}{w(u)}\frac{du}{u} \quad \sup_{u>t} u^{1-\frac{s}{n}} v(u) g^*(u) \ge g^*(t).$$

Therefore $\rho_{G(E)}(g^*) \lesssim \rho_G(g^*)$.

4. Critical case

Here we are going to use real interpolation for quasi-normed spaces, similarly to [11, 12].

First we construct the needed couples of Muckenhoupt weights. Let the function b satisfy the following properties:

$$\text{It is non-decreasing, slowly varying on } (0,\infty),\ b(t^2) \approx b(t), \tag{22}$$

$$\text{for some } \varepsilon > 0 \text{ the function } (1+\ln t)^{-1-\varepsilon} b(t) \text{ is increasing for } t > 1. \tag{23}$$

Let

$$c(t) = \frac{b(t)}{1 + |\ln t|}. \tag{24}$$

Then

$$\int_t^\infty \frac{1}{b(u)}\frac{du}{u} \lesssim \frac{1}{c(t)}, \quad t > 0. \tag{25}$$

Indeed, if $0 < t < 1$ we can write:

$$\int_t^\infty \frac{1}{b(u)}\frac{du}{u} = \int_t^1 \frac{1}{b(u)}\frac{du}{u} + \int_1^\infty \frac{(1+\ln u)^{-1-\varepsilon}}{b(u)(1+\ln u)^{-1-\varepsilon}}\frac{du}{u}.$$

Using monotonicity properties (22), (23) and $c(t) \lesssim 1$ for $0 < t < 1$, we get (25). The case $t > 1$ is analogous, but simpler.

We denote by $L^r_*(v)$, $1 \le r \le \infty$, v - positive weight, the weighted Lebesgue space on $(0,\infty)$ with the Haar measure $d\mu = \frac{dt}{t}$ and norm

$$\|g\|_{L^r_*(v)} := \left(\int_0^\infty |g(t)v(t)|^r \frac{dt}{t}\right)^{\frac{1}{r}}.$$

We write L^r_* when $v = 1$. Let L^∞_v be the weighted Lebesgue space on $(0,\infty)$ with the Lebesgue measure and a norm $\|g\|_{L^\infty_v} := \sup |g(t)v(t)|$.

Theorem 4.1. *Let ρ_H be a K-monotone quasi-norm on L and let H be the corresponding quasi-Banach space with $\beta_H < 1$. Let b, c be given by (22), (24). Let ρ_E be defined by*

$$\rho_E(g) := \rho_F\left(t^{\frac{s}{n}} b(t) g^*(t)\right), \tag{26}$$

$$F := (L^1_*, L^\infty_*)_{H\left(\frac{1}{t}\right)}, \tag{27}$$

and $H\left(\frac{1}{t}\right)$ has a quasi-norm $\|g\|_{H\left(\frac{1}{t}\right)} := \rho_H\left(\frac{g(t)}{t}\right)$. Then the optimal target quasi-norm is given by $\rho_{G(E)}(g) := \rho_F(gc)$.

Proof. The operator T, defined by (11) is bounded in the following couple of spaces:

$$T : L^1_*\left(t^{\frac{s}{n}} b(t)\right) \mapsto L^\infty_b \quad \text{and} \quad T : L^\infty_*\left(t^{\frac{s}{n}} b(t)\right) \mapsto L^\infty_c,$$

where b, c are given by (22), (24). Define F by (27). It is well known [6] that

$$\rho_F(g) = \rho_H(g^{**}_\mu) \approx \rho_H(g^*_\mu), \tag{28}$$

where $g^{**}_\mu(t) = \frac{1}{t}\int_0^t g^*_\mu(s)\,ds$. The equivalence in (28) is true because $\beta_H < 1$. By interpolation, $T : E_1 \mapsto G_1$, where

$$E_1 := \left(L^1_*\left(t^{\frac{s}{n}} b(t)\right), L^\infty_*\left(t^{\frac{s}{n}} b(t)\right)\right)_{H\left(\frac{1}{t}\right)}, \quad G_1 := (L^\infty_b, L^\infty_c)_{H\left(\frac{1}{t}\right)}.$$

Denote the quasi-norm in E_1 by ρ_1 and let $\rho_E(g) = \rho_1(g^*)$. We have

$$\rho_E(g) = \rho_F\left(t^{\frac{s}{n}} b(t) g^*(t)\right) = \rho_H\left(\left(t^{\frac{s}{n}} b(t) g^*(t)\right)^{**}_\mu\right) \approx \rho_H\left(\left(t^{\frac{s}{n}} b(t) g^*(t)\right)^{*}_\mu\right).$$

Hence ρ_E is a K-monotone quasi-norm with both Boyd indices equal to $\frac{s}{n} < 1$ (here we are using the fact that b is slowly varying).

Now we characterize the space G_1. Since (see [6])

$$K\left(t, g; L_b^\infty, L_c^\infty\right) = tK\left(\tfrac{1}{t}, g; L_c^\infty, L_b^\infty\right) = t \sup_s |g(s)| \min\left(c(s), \tfrac{b(s)}{t}\right),$$

we get the formula

$$\rho_{G_1}(g) = \rho_H(h_g), \quad h_g(u) := \sup_s |g(s)| \min\left(c(s), \tfrac{b(s)}{t}\right). \tag{29}$$

Also, since $L_b^\infty \hookrightarrow L_c^\infty$ it follows $h_g(u) \approx \sup |g(s)|c(s)$ if $0 < u < 1$. Let

$$H_g(t) := h_g(1 + |\ln t|), \quad 0 < t < \infty.$$

Then $(H_g)_\mu^*(t) \le h_g\left(\frac{t}{2}\right)$, hence, by (28) and (29), $\rho_F(H_g) \lesssim \rho_{G_1}(g)$. Note that $H_g \gtrsim gc$, hence, if we define the quasi-norm $\rho_G(g) := \rho_F(gc)$, we get the relation

$$\rho_G(Tg) \lesssim \rho_{G_1}(Tg) \lesssim \rho_E(g), \quad g \in M.$$

Since $\beta_E < 1$ Theorem 2.2 shows that the couple ρ_E, ρ_G is admissible.

Now we want to prove that ρ_G is an optimal target quasi-norm. It is sufficient to see that $\rho_G(g) \approx \rho_{G(E)}(g)$, $g \in L$, g decreasing, where $\rho_{G(E)}$ is defined by (17). And since the quasi-norm $\rho_{G(E)}$ is optimal, we need only to prove that $\rho_{G(E)}(g) \lesssim \rho_G(g)$, $g \in L$, g decreasing . To this end first for any such g we construct a decreasing h such that $g \lesssim Th$ and $\rho_E(h) \lesssim \rho_G(g)$. Let $t^{\frac{s}{n}} b(t) h_1(t) = g_1(t)$, where $g_1(t) = g\big(\frac{t^2}{e^2}\big)c(t^2)$ for $0 < t < 1$ and $g_1(t) = g\big(\frac{\sqrt{t}}{\sqrt{e}}\big)c(\sqrt{t})$ if $t > 1$. Note that $h_1 \approx h_1^*$. Then $\rho_E(h_1) \approx \rho_F\left(t^{\frac{s}{n}} b(t) h_1^*(t)\right) \approx \rho_F(gc) = \rho_G(g)$. On the other hand, for $0 < t < 1$,

$$Th_1(t) \ge \int_t^{\sqrt{te}} g\left(\frac{u^2}{e^2}\right) \frac{c(u^2)}{b(u)} \frac{du}{u} \ge g(t)A(t) \gtrsim g(t),$$

since

$$A(t) = \int_t^{\sqrt{te}} \frac{c(u^2)}{b(u)} \frac{du}{u} \approx \int_t^{\sqrt{te}} \frac{1}{1 + |\ln u|} \frac{du}{u} \gtrsim 1.$$

Similarly, for $t > 1$ we obtain

$$Th_1(t) \gtrsim \int_t^{et^2} g\left(\frac{\sqrt{u}}{\sqrt{e}}\right) \frac{1}{1 + \ln u} \frac{du}{u} \gtrsim g(t).$$

Therefore we can find $h \approx h_1$ such that $\rho_E(h) \approx \rho_G(g)$ and $Th \ge g$. This means that $\rho_{G(E)} \lesssim \rho_G$. □

The above result suggests the following construction. Let

$$\rho_E(g) = \rho_F\left(t^{\frac{s}{n}} b(t) g^*(t)\right), \quad \rho_G(g) = \rho_F(cg),$$

where ρ_F is a monotone quasi-norm satisfying $\rho_F(g(t^2)) \lesssim \rho_F(g(t))$ with $\alpha_F = \beta_F = 0$ and b, c are slowly varying weights such that $c(t^2) \approx c(t)$ and

$$\int_0^\infty m(t) \frac{c(u)}{b(u)} \frac{du}{u} \gtrsim 1, \quad m = \begin{cases} \chi_{(t,\sqrt{te})} & \text{if } t < 1 \\ \chi_{(t,et^2)} & \text{if } t > 1. \end{cases} \tag{30}$$

Then if $\rho_F(cQg) \lesssim \rho_F(bg)$ the couple ρ_E, ρ_G is admissible and the same argument as above shows that the target quasi-norm is optimal.

Example 4.2. Let $G = \Lambda^q(c)$, $1 \leq q \leq \infty$, $E = \Lambda^q\left(t^{\frac{s}{n}} b(t)\right)$, where b, c are slowly varying on $(0, \infty)$, satisfying (30), $c(t^2) \approx c(t)$ and

$$\left(\int_0^t c^q(s) \frac{ds}{s} \right)^{\frac{1}{q}} \left(\int_t^\infty [b(s)]^{-r} \frac{ds}{s} \right)^{\frac{1}{r}} \lesssim 1, \quad \frac{1}{q} + \frac{1}{r} = 1.$$

Then the couple E, G is admissible and G is an optimal target space. In particular, if $b(t) = 1$, $0 < t < 1$ and $b(t) = (1 + \ln t)^2$, $t > 1$, then $c(t) = (1 - \ln t)^{-1}$, $0 < t < 1$ and $c(t) = 1 + \ln t$, $t > 1$. Therefore $E = \Lambda^{\frac{n}{s}}\left(t^{\frac{s}{n}} b(t)\right)$, $G(E) = \Lambda^{\frac{n}{s}}(c)$, the result that corresponds to the optimal embedding proved in [17] in the critical case.

5. An application to the maximal operator

The Hardy-Littlewood maximal operator M (see, e.g., [5, p. 117]) is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy, \quad f \in L_{loc}$$

where the supremum extends over all balls B containing $x \in \mathbf{R}^n$. The methods from the previous sections can be applied to the problem of optimal couples of rearrangement invariant spaces for M. More precisely, we consider optimality in the following classes:

- $\mathcal{N}_d$ consists of all quasi-norms ρ_E that are monotone, rearrangement invariant, and such that $0 \leq \alpha_E \leq \beta_E \leq 1$;
- $\mathcal{N}_{d,1}$ consists of all quasi-norms $\rho_E \in \mathcal{N}_d$ such that $\beta_E < 1$;
- $\mathcal{N}_t$ consists of all quasi-norms ρ_G that are monotone and $0 \leq \alpha_G \leq \beta_G \leq 1$.

We say that the couple $\rho_E \in \mathcal{N}_d$, $\rho_G \in \mathcal{N}_t$ is admissible for M if $\rho_G((Mf)^*) \lesssim \rho_E(f^*)$ Since $(Mf)^* \approx f^{**}$ (see, e.g., [5, Theorem 3.8 p. 122]), admissibility is equivalent to $\rho_G(g^{**}) \lesssim \rho_E(g)$, $g \in L$. Using the same arguments as in Section 3, we can prove the following results. Let $\int_0^a g(t)dt \lesssim \rho_E(g)$, $g \in L$, $0 < a < \infty$. Given the domain quasi-norm $\rho_E \in \mathcal{N}_d$, the optimal target quasi-norm $\rho_{G(E)}$ satisfies (cf. Proposition 2.5)

$$\rho_{G(E)}(g) := \inf\{\rho_E(h) : g \leq h^{**},\ h \in L\}.$$

Evidently, $E \hookrightarrow G(E)$. If $\rho_G \in \mathcal{N}_t$ is given, then the optimal domain quasi-norm $\rho_{E(G)}$ satisfies $\rho_{E(G)}(g) = \rho_G(g^{**})$ and $E(G) \hookrightarrow G$. Moreover, if $\rho_E \in \mathcal{N}_{d,1}$ then $G(E) = E$ and the couple (E, E) is optimal in the class $\mathcal{N}_{d,1}$. In the limiting case $\beta_E = 1$ we have the following example, that corresponds to Example 3.3. If $E = \Lambda^\infty(tw)$, w is slowly varying and $\frac{1}{v(t)} = \int_0^t \frac{1}{w(u)} \frac{du}{u}$, then $G(E) = \Lambda^\infty(tv)$ and $G(E) \supsetneq E$.

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