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Symmetries of the Generalized Variational Functional of Herglotz for Several Independent Variables

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Abstract. This paper provides a method for calculating the symmetry groups of the functional defined by the generalized variational principle of Herglotz in the case of several independent variables. Examples of calculating variational symmetry groups are given, including those for the non-conservative nonlinear Klein-Gordon equation, and for the equations describing the propagation of electromagnetic fields in a conductive medium.
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1. Introduction

The generalized variational principle, proposed by Herglotz [6, 7], defines the functional whose extrema are sought by a differential equation rather than by an integral. This variational principle is uniquely useful for the description of nonconservative processes. It is more general than the classical variational principle with one independent variable and contains it as a special case. The paper of Furta et al. [2] shows a close link between the Herglotz variational principle and control and optimal control theories. It is also related to contact transformations, see Guenther et al. [5]. Georgieva et al. [4] formulated and proved a Noether-type theorem which yields conservation laws corresponding to the symmetries of the functional defined by the Herglotz variational principle. Georgieva et al. [3] extended the Herglotz principle to a variational principle with several independent variables which contains as special cases both the classical variational principle and the Herglotz variational principle. This variational principle can describe not only all physical processes which the classical

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variational principle can, but also many others for which the classical variational principle is not applicable. For example, it can give a variational description of nonconservative processes even when the Lagrangian is not dependent on time, something which can not be done with the classical variational principle. In the same paper the authors formulate and prove a theorem of Noether-type which gives an identity corresponding to each symmetry of the functional defined by this new variational principle. From this identity a first integral is readily obtained.

In the present paper we formulate and prove a theorem which provides a method for calculating the symmetry groups of the functional defined by the generalized variational principle of Herglotz in the case of several independent variables.

Historically, the question of calculating the symmetries of a given Lagrangian functional was answered by W. Killing [8] in 1892 in the context of describing the motions of a n-dimensional manifold with fundamental form given by

$$L = \frac{1}{2} g_{kl} \dot{x}^k \dot{x}^l$$

(see Eisenhart [1] and Logan [11]). In the case of a classical variational functional, some authors refer to the system of partial differential equations for the unknown symmetry group generators as the *generalized Killing equations.* For the derivation of these equations in the case of the classical variational principle see Logan [11].

Note. The summation convention on repeated indices is used throughout this paper.

2. The variational principle of Herglotz

The variational principle of Herglotz defines the functional z, whose extrema are sought, by the differential equation

$$\frac{dz}{dt} = L\Big(t, x(t), \frac{dx(t)}{dt}, z\Big) \tag{2.1}$$

where L is a known function, differentiable in its four arguments, t is the independent variable, and $x(t) \equiv (x_1(t), \ldots, x_n(t))$ stands for the argument functions. In order for the equation (2.1) to define a functional $z = z[x]$ of $x(t)$ equation (2.1) must be solved with the same fixed initial condition $z(0)$ for all argument functions $x(t)$, and the solution $z(t)$ must be evaluated at the same fixed final time $t = T$ for all argument functions $x(t)$. L is called the *Lagrangian* in analogy with the classical case.

The equations whose solutions produce the extrema of this functional are

$$\frac{\partial L}{\partial x_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}_k} = 0, \quad k = 1, \ldots, n, \tag{2.2}$$

where $\dot{x}_k$ denotes $\frac{dx_k}{dt}$. Herglotz called them the *generalized Euler-Lagrange equations.* See Guenther et al. [5] for a derivation of this system. The solutions of these equations, when written in terms of the dependent variables x_k and the associated momenta $p_k = \frac{dL}{d\dot{x}_k}$, determine a family of *contact transformations.* See Guenther et al. [5].

Below are a few examples of ordinary differential equations which can be given a variational description via Herglotz principle. To the right of each equation is the Herglotz Lagrangian which produces it: $(a, k = \text{const.})$

- the damped harmonic oscillator $\ddot{x} + a\dot{x} + kx = 0$, $L = \frac{1}{2}\big(\dot{x}^2 - kx^2\big) - az$,
- the Lienard equation $\ddot{x} + g(t)\dot{x} + kx = 0$, $L = \frac{1}{2}\big(\dot{x}^2 - kx^2\big) - g(t)z$,
- the Lane-Emden equation $\ddot{x} + \frac{2}{t}\dot{x} + x^n = 0$, $L = \frac{1}{2}\dot{x}^2 - \frac{x^{n+1}}{n+1} - \frac{2}{t}z$, and
- the Liouville equation $\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} = 0$, $L = \frac{1}{2}\dot{x}^2 - \big(2f(x)\dot{x} + g(t)\big)z$.

These are all special cases of the equation

$$\ddot{x} + f(x)\dot{x}^2 + g(t)\dot{x} + h(x) = 0$$

which can be obtained via the Herglotz variational principle, by letting L in the defining equation (2.1) be

$$L = \frac{1}{2}\dot{x}^2 - \big(2f(x)\dot{x} + g(t)\big)z - U(x)$$

where $U(x)$ is any solution of the ODE

$$\frac{dU(x)}{dx} + 2f(x)U(x) = h(x).$$

For equations which can be obtained from Herglotz variational principle as (2.1) one can systematically derive conserved quantities, as shown in Georgieva et al. [4], by applying the first Noether-type theorem formulated and proven in the same paper.

The Generalized Variational Principle with Several Independent Variables is as follows:
Let the functional $z = z[u; s]$ of $u = u(t, x)$ be given by an integro-differential equation of the form

$$\frac{dz}{dt} = \int_{\Omega} \mathcal{L}(t, x, u, u_t, u_x, z)\, d^n x, \quad 0 \le t \le s \tag{2.3}$$

where $x \equiv (x^1, \dots, x^n)$, $d^n x \equiv dx^1 \cdots dx^n$, $u \equiv (u^1, \dots, u^m)$, $u_x \equiv (u_x^1, \dots, u_x^m)$, $u_t \equiv (u_t^1, \dots, u_t^m)$ *and* $u_x^i \equiv (u_{x^1}^i, \dots, u_{x^n}^i)$, $i = 1, \dots, m$, *and where the function* $\mathcal{L}$ *is at least twice differentiable with respect to* u_{x^i}, u_t *and once differentiable with respect to* t, x, z. *Let* $\eta = (\eta^1(t,x), \dots, \eta^m(t,x))$ *have continuous first derivatives and otherwise be arbitrary except for the boundary conditions:*

$$\eta(0,x) = \eta(s,x) = 0$$
$$\eta(t,x) = 0 \quad \text{for } x \in d\Omega,\ 0 \le t \le s$$

where $d\Omega$ *is the boundary of* Ω. *Then, the value of the functional* $z[u;s]$ *is an extremum for functions* u *which satisfy the condition*

$$\frac{d}{d\varepsilon} z[u + \varepsilon\eta; s]\Big|_{\varepsilon=0} = 0. \tag{2.4}$$

The function $\mathcal{L}$ will be called, just as in the classical case, the *Lagrangian density*. The notation u_t, u_x etc. is used to denote the partial derivatives with respect to t, x, etc. It should be observed that when a variation $\varepsilon\eta$ is applied to u the equation (2.3), defining the functional z, must be solved with the same fixed initial condition $z(0)$ at $t = 0$ and the solution evaluated at the same fixed final time $t = s$ for all varied argument functions $u + \varepsilon\eta$.

Theorem 2.1. *Every function* $u \equiv (u^1, \dots, u^m)$, *for which the functional* z *defined by the integro-differential equation* (2.3) *has an extremum, is a solution of*

$$\frac{\partial \mathcal{L}}{\partial u^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t^i} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}^i} + \frac{\partial \mathcal{L}}{\partial u_t^i}\int_\Omega \frac{\partial \mathcal{L}}{\partial z}\, dx = 0, \quad i = 1, \dots, m. \tag{2.5}$$

Because of the obvious correspondence with the classical case, we call these equations the *generalized Euler-Lagrange equations with several independent variables.*

The following theorem provides an identity which corresponds to each symmetry of the functional z defined by the integro-differential equation (2.3). We call it a first Noether-type theorem for the generalized variational principle with several independent variables because this theorem contains as a special case the classical first Noether theorem.

Theorem 2.2. *Let*

$$v = \tau(t)\frac{\partial}{\partial t} + \xi^k(t,x,u)\frac{\partial}{\partial x^k} + \eta^i(t,x,u)\frac{\partial}{\partial u^i}, \quad k = 1, \dots, n,\ i = 1, \dots, m$$

be the generator of a given symmetry group of the functional $z[u;s]$ *defined by* (2.3). *Then the identity*

$$\int_D \left(\frac{d}{dt}\left(E\left((\tau u_t^i + \xi^j u_{x^j}^i - \eta^i)\frac{\partial \mathcal{L}}{\partial u_t^i} - \tau\mathcal{L}\right)\right) + \frac{d}{dx^k}\left(E\left((\tau u_t^i + \xi^j u_{x^j}^i - \eta^i)\frac{\partial \mathcal{L}}{\partial u_{x^k}^i} - \xi^k\mathcal{L}\right)\right)\right) d^n x = 0$$

holds on solutions of the generalized Euler-Lagrange equations (2.5)*. Here D is any subdomain of Ω, including Ω itself, whose closure $D^{cl} \subset \Omega^{cl}$ and $E = E(t)$ is*

$$E = \exp\left(-\int_0^t\int_D \frac{\partial \mathcal{L}}{\partial z}\, d^n x\, d\theta\right).$$

Corollary 2.3. *Theorem* 2.2 *reduces to the classical first Noether theorem when the generalized variational principle with several independent variables reduces to the classical variational principle.*

The proofs of Theorems 2.1, 2.2 and Corollary 2.3 can be found in Georgieva et al. [3].

Two examples of partial differential equations which can be given a variational description via the generalized variational principle of Herglotz with several independent variables are presented next. They were published in Georgieva et al. [3]. We restate them here for the convenience of the reader.

The first is the set of equations which describe the propagation of electromagnetic fields in a conductive medium.

$$c^2\nabla^2\mathbf{E} - \frac{\partial^2\mathbf{E}}{\partial t^2} - \frac{\sigma}{\varepsilon}\frac{\partial \mathbf{E}}{\partial t} = 0 \tag{2.6}$$

where $\mathbf{E} = (E^1, E^2, E^3)$ is the electric field vector, c is the velocity of the electromagnetic waves, σ is the electrical conductivity and ε is the dielectric constant of the medium. Exactly the same equation holds for the magnetic field vector $\mathbf{B} = (B^1, B^2, B^3)$. These equations are direct consequence of the Maxwell's equations in conjunction with the medium's property equations $\mathbf{J} = \sigma\mathbf{E}$ and $\rho = 0$, where $\mathbf{J} = (J^1, J^2, J^3)$ is the current density and ρ is the charge density.

One can verify that equation (2.6) is the generalized Euler-Lagrange equation of the Lagrangian

$$\mathcal{L} = c^2\,\frac{\partial E^i}{\partial x^j}\,\frac{\partial E^i}{\partial x^j} - \frac{\partial E^i}{\partial t}\frac{\partial E^i}{\partial t} + \alpha(x)z, \quad i, j = 1, 2, 3 \tag{2.7}$$

where $\frac{\sigma}{\varepsilon} = \int_\Omega \alpha(x)\, d^3x = \text{const}$. As a second example consider the equation

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + G(uu^*)u = 0 \tag{2.8}$$

describing the real or complex field $u = u(x, t)$, where u^* denotes the complex conjugate of u, G is a differentiable function and v is a constant. This equation is known as the nonlinear Klein-Gordon equation. Its linear version, with $G = \text{const.}$ plays an important role in relativistic field theories. The one-dimensional version of (2.8) with real u and $G(u^2)u = \sin u$ is the sine-Gordon equation. The field equations of the form (2.8) can be derived from the Lagrangian density

$$\mathcal{L}(u, u_t, \nabla u) = \nabla u \cdot \nabla u^* - \frac{1}{v^2}\frac{\partial u}{\partial t}\frac{\partial u^*}{\partial t} - F(uu^*)$$

where $\frac{dF(\rho)}{d\rho} = G(\rho)$ and $F(0) = 0$. We consider as physically meaningful only those solutions of (2.8) which are free of singularities and for which

$$\left| \int_\Omega \mathcal{L}(t, x, u, u_t, u_x)\, d^n x \right| < \infty$$

holds over the entire time domain. The processes described with an equation of the form (2.8) are conservative.

One is also interested in nonconservative processes involving fields. The simplest modification of (2.8) which makes it suitable to describe nonconservative processes is to include in it a term proportional to the time-derivative of the field. Thus, a physically meaningful nonconservative version of (2.8) is

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + k\,\frac{\partial u}{\partial t} + G(uu^*)\,u = 0 \tag{2.9}$$

where k is a constant. With $k > 0$ the process described by (2.9) is generative, and with $k < 0$ it is dissipative. When u is a real field equation (2.9) can be derived via the present generalized variational principle from the Lagrangian density

$$\mathcal{L} = \nabla u \cdot \nabla u - \frac{1}{v^2}\left(\frac{\partial u}{\partial t}\right)^2 - F(u^2) + \alpha(x) z \tag{2.10}$$

where $\frac{dF(\rho)}{d\rho} = G(\rho)$, and $\alpha = \alpha(x)$ is a given function of the coordinates $x \equiv (x^1, \ldots, x^n)$ which satisfies the condition $\left|\int_\Omega \alpha(x)\, d^n x\right| < \infty$. Indeed, inserting the Lagrangian (2.10) into the generalized Euler-Lagrange equations (2.5)

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u_{x^k}} + \frac{\partial \mathcal{L}}{\partial u_t}\int_\Omega \frac{\partial \mathcal{L}}{\partial z}\, d^n x \\
&= -2u\,\frac{\partial F}{\partial (u^2)} + \frac{2}{v^2}\frac{\partial^2 u}{\partial t^2} - 2\,\nabla^2 u - \frac{2}{v^2}\frac{\partial u}{\partial t}\int_\Omega \alpha(x)\, d^n x \\
&= 0
\end{aligned}$$

we see that the last expression is the same as (2.9) with $k = \frac{1}{v^2}\int_\Omega \alpha(x)\, d^n x =$ const.

3. The symmetries of the variational functional of Herglotz

The above discussion should have made apparent the need for a method of finding symmetries of the functional defined by the Herglotz principle with several indendent variables. In this section we formulate and prove a theorem which provides such a method.

Consider a one-parameter group of transformations of the independent variables t, $x \equiv (x^1, \dots, x^n)$ and the dependent variables $u \equiv (u^1, \dots, u^m)$, i.e.,

$$\begin{aligned} \bar{t} &= \phi(t, x, u; \varepsilon) \\ \bar{x}^k &= \varphi^k(t, x, u; \varepsilon), \quad k = 1, \dots, n \\ \bar{u}^i &= \psi^i(t, x, u; \varepsilon), \quad i = 1, \dots, m. \end{aligned} \tag{3.1}$$

To find the transformed functions $\bar{u}^i = \bar{u}^i(\bar{t}, \bar{x}; \varepsilon)$ of the functions $u^i = u^i(t, x)$ we insert the latter into ϕ and φ^k of (3.1) to get a system of $n+1$ equations with $n+1$ unknowns $t, x^1, \dots, x^n$ and a parameter ε. We invert this system to obtain t and $x^1, \dots, x^n$ as functions of $\bar{t}$ and $\bar{x}^1, \dots, \bar{x}^n$. These we substitute into the last m equations of (3.1) to get $\bar{u}^i$ as a function of $\bar{t}$ and $\bar{x}^1, \dots, \bar{x}^n$ and ε, which we denote by $\bar{u}^i = \bar{u}^i(\bar{t}, \bar{x}; \varepsilon)$.

Definition 3.1. The transformed functional $\bar{z}$, of a functional z defined by (2.3), is the solution of the transformed integro-differential equation

$$\frac{d\bar{z}}{d\bar{t}} = \int_{\bar{\Omega}} \mathcal{L}(\bar{t}, \bar{x}, \bar{u}(\bar{t}, \bar{x}), \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{z})\, d^n\bar{x} \tag{3.2}$$

where $\bar{\Omega}$ is the transformed domain of the domain Ω.

Observation. *The most general one-parameter group of transformations of the independent and the dependent variables admitted by equation* (2.3) *is*

$$\begin{aligned} \bar{t} &= \phi(t; \varepsilon) \\ \bar{x}^k &= \varphi^k(t, x, u; \varepsilon), \quad k = 1, \dots, n \\ \bar{u}^i &= \psi^i(t, x, u; \varepsilon), \quad i = 1, \dots, m. \end{aligned} \tag{3.3}$$

The proof of this observation can be found in Georgieva et al. [3].

Definition 3.2. Let Φ, Ω and Ψ^i be the sets on which t, x and $u^i(t, x)$ vary. A local group of transformations G acting on the independent and the dependent variables is a *symmetry group* of the functional z defined by the integro-differential equation (2.3) if whenever D is a sub-domain with closure $D^{cl} \subset \Omega$ and $u^i = u^i(t, x)$ are functions defined over $\Phi \times D$ whose graphs lie in $\Phi \times \Omega \times \Psi^i$ with continuous second partial derivatives, and $g \in G$ is such that

$$\bar{u}^i = \bar{u}^i(\bar{t}, \bar{x}) = g \circ u^i(t, x), \quad i = 1, \dots, m$$

are single valued functions defined over $\bar{\Phi} \times \bar{D} \subset \Phi \times \Omega$, then the functional defined by the transformed integro-differential equation

$$\frac{d\bar{z}}{d\bar{t}} = \int_{\bar{D}} \mathcal{L}(\bar{t}, \bar{x}, \bar{u}(\bar{t}, \bar{x}), \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{z})\, d^n\bar{x}$$

is equal to the functional defined by the original integro-differential equation

$$\frac{dz}{dt} = \int_{D} \mathcal{L}(t, x, u(t, x), u_t, u_x, z)\, d^n x$$

for all t. Here $\bar{D}$ denotes the transformed D under G.

Theorem 3.3. *Let*

$$v = \tau(t)\frac{\partial}{\partial t} + \xi^k(t, x, u)\frac{\partial}{\partial x^k} + \eta^i(t, x, u)\frac{\partial}{\partial u^i}, \quad k = 1, \dots, n,\ i = 1, \dots, m \tag{3.4}$$

be the infinitesimal generator of a one-parameter group (3.3) *of symmetries of the functional* $z[u; s]$ *defined by* (2.3)*. Then* τ*,* ξ^k*,* η^i *are solutions to the system of partial differential equations obtained from the identity*

$$\begin{aligned}\int_0^s\!\!\int_D E\bigg(&\frac{\partial\mathcal{L}}{\partial t}\tau + \frac{\partial\mathcal{L}}{\partial x^k}\xi^k + \frac{\partial\mathcal{L}}{\partial u^i}\eta^i + \mathcal{L}\left(\frac{\partial\xi^j}{\partial x^j} + \frac{\partial\xi^j}{\partial u^i}u^i_{x^j} + \frac{d\tau}{dt}\right)\\ &+ \frac{\partial\mathcal{L}}{\partial {u^i}_t}\left(\frac{\partial\eta^i}{\partial t} + \frac{\partial\eta^i}{\partial u^j}u^j_t - u^i_t\frac{d\tau}{dt} - u^i_{x^k}\left(\frac{\partial\xi^k}{\partial t} + \frac{\partial\xi^k}{\partial u^j}u^j_t\right)\right)\\ &+ \frac{\partial\mathcal{L}}{\partial u^i_{x^k}}\left(\frac{\partial\eta^i}{\partial x^k} + \frac{\partial\eta^i}{\partial u^j}u^j_{x^k} - u^i_{x^j}\left(\frac{\partial\xi^j}{\partial x^k} + \frac{\partial\xi^j}{\partial u^l}u^l_{x^k}\right)\right)\bigg)d^n x dt = 0\end{aligned} \tag{3.5}$$

by equating to zero the coefficients in front of z*, derivatives of* u*, powers of derivatives of* u*, and products of those in the integrand. Here* D *is any subdomain of* Ω*, including* Ω *itself, whose closure* $D^{cl} \subset \Omega^{cl}$ *and* $E = E(t)$ *is*

$$E = \exp\left(-\int_0^t\!\!\int_D \frac{\partial\mathcal{L}}{\partial z}\, d^n x\, d\theta\right). \tag{3.6}$$

Proof. We write the integro-differential equation (2.3) for any subdomain D of Ω and apply the transformation (3.3) to it, i.e.,

$$\frac{d\bar{z}}{d\bar{t}} = \int_{\bar{D}} \mathcal{L}\big(\bar{t}, \bar{x}, \bar{u}(\bar{t}, \bar{x}), \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{z}\big)\, d^n\bar{x}, \quad \phi(0;\varepsilon) \le \bar{t} \le \phi(s;\varepsilon). \tag{3.7}$$

Here $d^n\bar{x} \equiv d\bar{x}^1 \cdots d\bar{x}^n$ and $\bar{D} = \bar{D}(\bar{t}, \bar{u}, \varepsilon)$ denotes the result of transforming D with (3.3) which, in general, depends on $\bar{t}$, $\bar{u}$ and ε. Now we change the independent variables $\bar{t}$ and $\bar{x}^k$ in (3.7) (but not the dependent variables) back to the original independent variables t and x^k. The resulting equation is

$$\frac{d\bar{z}}{dt} = \frac{d\bar{t}}{dt}\int_{D} \mathcal{L}\big(\bar{t}, \bar{x}, \bar{u}(\bar{t}, \bar{x}), \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{z}\big) \det\Big(\frac{\partial\bar{x}}{\partial x}\Big) d^n x, \quad 0 \le t \le s. \tag{3.8}$$

where $\frac{\partial\bar{x}}{\partial x}$ stands for the Jacobian matrix of the transformation of the x-variables. Differentiating equation (3.8) with respect to ε

$$\frac{d}{dt}\frac{d\bar{z}}{d\varepsilon}=\frac{d\bar{t}}{dt}\int_D\left(\frac{d\mathcal{L}}{d\varepsilon}\det\left(\frac{\partial\bar{x}}{\partial x}\right)+\mathcal{L}\,\frac{d}{d\varepsilon}\det\left(\frac{\partial\bar{x}}{\partial x}\right)\right)d^nx+\frac{d}{d\varepsilon}\frac{d\bar{t}}{dt}\int_D\mathcal{L}\det\left(\frac{\partial\bar{x}}{\partial x}\right)d^nx \quad (3.9)$$

and observing that $\frac{d\bar{t}}{dt}\Big|_{\varepsilon=0}=1,\ \det\left(\frac{\partial\bar{x}}{\partial x}\right)\Big|_{\varepsilon=0}=1,\ \frac{d}{d\varepsilon}\frac{d\bar{t}}{dt}\Big|_{\varepsilon=0}=\frac{d\tau}{dt}$ produces

$$\frac{d\zeta}{dt}=\int_D\frac{d\mathcal{L}}{d\varepsilon}\bigg|_{\varepsilon=0}d^nx+\int_D\mathcal{L}\,\frac{d}{d\varepsilon}\det\left(\frac{\partial\bar{x}}{\partial x}\right)\bigg|_{\varepsilon=0}d^nx+\frac{d\tau}{dt}\int_D\mathcal{L}\,d^nx \quad (3.10)$$

where, by definition, the variation $\zeta=\zeta(t)$ of $\bar{z}$ is $\zeta(t)\equiv\frac{d\bar{z}}{d\varepsilon}\Big|_{\varepsilon=0}$. Now, we need to express the first and the second integrands in (3.10) in terms of known functions. The calculations are lengthy and are given in the Appendix. When the results

$$\begin{aligned}\frac{d\mathcal{L}}{d\varepsilon}\bigg|_{\varepsilon=0}&=\frac{\partial\mathcal{L}}{\partial t}\tau+\frac{\partial\mathcal{L}}{\partial x^k}\xi^k+\frac{\partial\mathcal{L}}{\partial u^i}\eta^i+\frac{\partial\mathcal{L}}{\partial u^i_t}\left(\frac{d\eta^i}{dt}-u^i_t\frac{d\tau}{dt}-u^i_{x^k}\frac{d\xi^k}{dt}\right)\\&\quad+\frac{\partial\mathcal{L}}{\partial u^i_{x^k}}\left(\frac{d\eta^i}{dx^k}-u^i_{x^j}\frac{d\xi^j}{dx^k}\right)+\frac{\partial\mathcal{L}}{\partial z}\zeta\end{aligned} \quad (3.11)$$

and

$$\frac{d}{d\varepsilon}\det\left(\frac{\partial\bar{x}}{\partial x}\right)\bigg|_{\varepsilon=0}=\frac{d\xi^k}{dx^k} \quad (3.12)$$

are inserted into (3.10) we obtain the equation for the variation $\zeta(t)$, namely,

$$\begin{aligned}\frac{d\zeta(t)}{dt}&=\int_D\left(\frac{\partial\mathcal{L}}{\partial t}\tau+\frac{\partial\mathcal{L}}{\partial x^k}\,\xi^k+\frac{\partial\mathcal{L}}{\partial u^i}\eta^i+\frac{\partial\mathcal{L}}{\partial u^i_t}\left(\frac{d\eta^i}{dt}-u^i_t\frac{d\tau}{dt}-u^i_{x^k}\frac{d\xi^k}{dt}\right)\right.\\&\quad\left.+\frac{\partial\mathcal{L}}{\partial u^i_{x^k}}\left(\frac{d\eta^i}{dx^k}-u^i_{x^j}\frac{d\xi^j}{dx^k}\right)+\mathcal{L}\frac{d\xi^j}{dx^j}\right)d^nx+\frac{d\tau}{dt}\int_D\mathcal{L}\,d^nx+\zeta(t)\int_D\frac{\partial\mathcal{L}}{\partial z}\,d^nx.\end{aligned} \quad (3.13)$$

Its solution $\zeta(s)$, evaluated at $t=s$, is given by

$$\begin{aligned}&E(s)\,\zeta(s)-\zeta(0)\\&=\int_0^s\!\!\int_D E(t)\left(\frac{\partial\mathcal{L}}{\partial t}\tau+\frac{\partial\mathcal{L}}{\partial x^k}\xi^k+\frac{\partial\mathcal{L}}{\partial u^i}\eta^i+\frac{\partial\mathcal{L}}{\partial u^i_t}\left(\frac{d\eta^i}{dt}-u^i_t\frac{d\tau}{dt}-u^i_{x^k}\frac{d\xi^k}{dt}\right)\right.\\&\qquad\left.+\frac{\partial\mathcal{L}}{\partial u^i_{x^k}}\left(\frac{d\eta^i}{dx^k}-u^i_{x^j}\frac{d\xi^j}{dx^k}\right)+\mathcal{L}\left(\frac{d\xi^j}{dx^j}+\frac{d\tau}{dt}\right)\right)d^nx\,dt\end{aligned} \quad (3.14)$$

where $E(t)$ is the expression (3.10) and s is the value of t at which the solution $z(t)$ of equation (2.3) was evaluated in order to obtain the functional $z[u;s]$.

By definition, $\zeta(0) = 0$. By hypothesis, the transformation group (3.3) leaves the functional $\bar{z}$ invariant, so $\zeta(s) = 0$. Thus, (3.14) becomes

$$\int_0^s\int_D E\left(\frac{\partial \mathcal{L}}{\partial t}\tau + \frac{\partial \mathcal{L}}{\partial x^k}\xi^k + \frac{\partial \mathcal{L}}{\partial u^i}\eta^i + \frac{\partial \mathcal{L}}{\partial u^i_t}\left(\frac{d\eta^i}{dt} - u^i_t\frac{d\tau}{dt} - u^i_{x^k}\frac{d\xi^k}{dt}\right)\right.$$
$$\left. + \frac{\partial \mathcal{L}}{\partial u^i_{x^k}}\left(\frac{d\eta^i}{dx^k} - u^i_{x^j}\frac{d\xi^j}{dx^k}\right) + \mathcal{L}\left(\frac{d\xi^j}{dx^j} + \frac{d\tau}{dt}\right)\right)d^nx dt = 0. \tag{3.15}$$

Next we expand the total derivatives of η and ξ in (3.15), obtaining identity (3.5). It is satisfied if the coefficients in front of z, derivatives of u, powers of derivatives of u, and products of those in the integrand are zero. To calculate τ, ξ^k, and η^i we form the system of partial differential equations obtained by equating those coefficients to zero. Each solution of this system corresponds to one symmetry group generator of the form (3.4). □

4. Applications

In this section we show how Theorem 3.3 can be used to calculate symmetries of the variational functional of Herglotz in the case of several independent variables. Having obtained such symmetries one can readily apply the first Noether-type Theorem 2.2 to obtain the corresponding first integrals.

As a first example consider the one-dimensional nonconservative sine-Gordon equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} + \sin u = 0. \tag{4.1}$$

This equation can be obtained as the generalized Euler-Lagrange equation (2.5) for the variational functional z defined by (2.3) with

$$\mathcal{L} = \left(\frac{\partial u}{\partial x}\right)^2 - \frac{1}{v^2}\left(\frac{\partial u}{\partial t}\right)^2 - 2\cos u + \alpha(x)z.$$

Here k and v are constants, v represents the velocity of the wave and α is a differentiable function such that $\int_\Omega \alpha(x)dx = kv^2$. In this case identity (3.5) becomes

$$\int_0^s\int_D E\left(\frac{d\alpha}{dx}\xi z + 2\eta\sin u - \frac{2}{v^2}u_t\Big(\eta_t + \eta_u u_t - u_t\frac{d\tau}{dt} - u_x\big(\xi_t + \xi_u u_t\big)\Big)\right.$$
$$+ 2u_x\big(\eta_x + \eta_u u_x - u_x\big(\xi_x + \xi_u u_x\big)\big) \tag{4.2}$$
$$\left. + \left(u_x^2 - \frac{1}{v^2}u_t^2 - 2\cos u + \alpha(x)z\right)\left(\xi_x + \xi_u u_x + \frac{d\tau}{dt}\right)\right)dxdt = 0.$$

We form the coefficients in front of z, u_t, u_t^2, u_x, u_x^2, u_x^3, u_tu_x, $u_t^2u_x$, and zu_x in the integrand of (4.2) and equate them to zero. The solution of this system is

$\tau = a = \text{const.}$, $\xi = b = \text{const.}$ and $\eta = 0$. Thus, the infinitesimal generator of the symmetry group of the functional z is $v = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}$.

As a second example let us find the variational symmetries, which this method produces, for the non-conservative two dimensional Klein-Gordon equation for a real field $u(x^1, x^2, t)$

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} + G(u^2)u + k\frac{\partial u}{\partial t} = 0. \tag{4.3}$$

It is the generalized Euler-Lagrange equation for the functional z defined by the integro-differential equation (2.3) with

$$\mathcal{L} = \left(\frac{\partial u}{\partial x^1}\right)^2 + \left(\frac{\partial u}{\partial x^2}\right)^2 - \frac{1}{v^2}\left(\frac{\partial u}{\partial t}\right)^2 - F(u^2) + \alpha(x)z.$$

In this case, the generalized Killing identity (3.5) is

$$\begin{aligned}\int_0^s\!\!\int_D E\Big(&\frac{\partial \alpha}{\partial x^1}\xi^1 z + \frac{\partial \alpha}{\partial x^2}\xi^2 z - 2uF'\eta \\ &- \frac{2}{v^2}u_t\Big(\eta_t + \eta_u u_t - u_t\frac{d\tau}{dt} - u_{x^1}\left(\xi^1_t + \xi^1_u u_t\right) - u_{x^2}\left(\xi^2_t + \xi^2_u u_t\right)\Big) \\ &+ 2u_{x^1}\left(\eta_{x^1} + \eta_u u_{x^1} - u_{x^1}\left(\xi^1_{x^1} + \xi^1_u u_{x^1}\right) - u_{x^2}\left(\xi^2_{x^1} + \xi^2_u u_{x^1}\right)\right) \\ &+ 2u_{x^2}\left(\eta_{x^2} + \eta_u u_{x^2} - u_{x^1}\left(\xi^1_{x^2} + \xi^1_u u_{x^2}\right) - u_{x^2}\left(\xi^2_{x^2} + \xi^2_u u_{x^2}\right)\right) \\ &+ \left(u^2_{x^1} + u^2_{x^2} - \frac{1}{v^2}u^2_t - F(u^2) + \alpha(x)z\right) \\ &\times \left(\xi^1_{x^1} + \xi^1_u u_{x^1} + \xi^2_{x^2} + \xi^2_u u_{x^2} + \frac{d\tau}{dt}\right)\Big)dx^1dx^2dt = 0.\end{aligned}$$

We form the coefficients in front of z, zu_{x^1}, zu_{x^2}, u_t, u_t^2, u_{x^1}, u_{x^2}, $u_tu_{x^1}$, $u_tu_{x^2}$, $u_t^2u_{x^1}$, $u_t^2u_{x^2}$, $u^2_{x^1}$, $u^2_{x^2}$, $u^3_{x^1}$, $u^3_{x^2}$, $u_{x^1}u_{x^2}$, $u^2_{x^1}u_{x^2}$, and $u^2_{x^2}u_{x^1}$ in the integrand and equate them to zero. Then consider the system which these equations together with the equation of the terms not multiplied by z or derivatives of u form. There are several different cases for the solution of this system depending on the functions F and α. If $F(\rho) = s\rho$ with $s = \text{const.}$, we obtain $\tau = \tau^1 = \text{const.}$, $\xi^1 = ax^1 - cx^2 + c^2$, $\xi^2 = ax^2 + cx^1 + c^1$, $\eta = -au$, where a, c, c^1 and c^2 are constants which satisfy the identity

$$\frac{\partial \alpha}{\partial x^1}(ax^1 - cx^2 + c^2) + \frac{\partial \alpha}{\partial x^2}(ax^2 + cx^1 + c^1) + 2a\alpha(x) = 0. \tag{4.4}$$

(i) If $\alpha = \text{const.}$ then $a = 0$ and we obtain the variational symmetries $\tau = \tau^1 = \text{const.}$, $\xi^1 = -cx^2 + c^2$, $\xi^2 = cx^1 + c^1$, $\eta = 0$ where c, c^1 and c^2 are arbitrary constants.

(ii) If $\alpha = b^1x^1 + b^2x^2 + b^3$ with constant b^1, b^2, b^3, $b^1 \neq 0$, then from the system obtained by equating to zero the coefficients in identity (4.4), we get $a = 0$, $c = 0$ and $c^2 = -\frac{c^1b^2}{b^1}$, thus the symmetries in this case are $\tau = \tau^1$, $\xi^1 = -\frac{c^1b^2}{b^1}$, $\xi^2 = c^1$, $\eta = 0$ with τ^1 and c^1 being arbitrary constants.

(iii) If $\alpha = (x^1)^2 + (x^2)^2$ the same procedure as in the previous cases produces $\tau = \tau^1$, $\xi^1 = -cx^2$, $\xi^2 = cx^1$, $\eta = 0$ with τ^1 and c being arbitrary constants.

(iv) If $\alpha = (x^1)^{-2} + (x^2)^{-2}$ then we get the variational symmetry group of the nonlinear nonconservative Klein-Gordon equation with infinitesimal generator (3.4) where $\tau = \tau^1$, $\xi^1 = ax^1$, $\xi^2 = ax^2$, $\eta = -au$ with τ^1 and a being arbitrary constants.

As a last application let us find variational symmetries of the equation

$$c^2\nabla^2\mathbf{E} - \frac{\partial^2\mathbf{E}}{\partial t^2} - \frac{\sigma}{\varepsilon}\frac{\partial\mathbf{E}}{\partial t} = 0 \tag{4.5}$$

which describes the propagation of electromagnetic fields in a conductive medium. Here $\mathbf{E} = (E^1, E^2, E^3)$ is the electric field vector, c is the velocity of the electromagnetic waves, σ is the electrical conductivity and ε is the dielectric constant of the medium. As mentioned in the introduction, exactly the same equation holds for the magnetic field vector $\mathbf{B} = (B^1, B^2, B^3)$. These equations are direct consequence of the Maxwell's equations in conjunction with the medium's property equations $\mathbf{J} = \sigma\mathbf{E}$ and $\rho = 0$, where $\mathbf{J} = (J^1, J^2, J^3)$ is the current density and ρ is the charge density. We write the generalized Killing identity (3.5) for the Lagrangian

$$\mathcal{L} = c^2\frac{\partial E^i}{\partial x^j}\frac{\partial E^i}{\partial x^j} - \frac{\partial E^i}{\partial t}\frac{\partial E^i}{\partial t} + \alpha(x)z, \quad i, j = 1, 2, 3$$

where $\frac{\sigma}{\varepsilon} = \int_\Omega \alpha(x)\, d^3x = \text{const}$.

$$\begin{aligned}
\int_0^s\!\!\int_\Omega e^{-\frac{\sigma t}{\varepsilon}}\Bigg(&\frac{\partial\alpha}{\partial x^k}\xi^k z \\
&- 2\frac{\partial E^i}{\partial t}\left(\frac{\partial\eta^i}{\partial t} + \frac{\partial\eta^i}{\partial E^j}\frac{\partial E^j}{\partial t} - \frac{\partial E^i}{\partial t}\frac{d\tau}{dt} - \frac{\partial E^i}{\partial x^k}\left(\frac{\partial\xi^k}{\partial t} + \frac{\partial\xi^k}{\partial E^j}\frac{\partial E^j}{\partial t}\right)\right) \\
&+ 2c^2\frac{\partial E^i}{\partial x^k}\left(\frac{\partial\eta^i}{\partial x^k} + \frac{\partial\eta^i}{\partial E^j}\frac{\partial E^j}{\partial x^k} - \frac{\partial E^i}{\partial x^1}\left(\frac{\partial\xi^1}{\partial x^k} + \frac{\partial\xi^1}{\partial E^j}\frac{\partial E^j}{\partial x^k}\right)\right. \\
&\left.- \frac{\partial E^i}{\partial x^2}\left(\frac{\partial\xi^2}{\partial x^k} + \frac{\partial\xi^2}{\partial E^j}\frac{\partial E^j}{\partial x^k}\right) - \frac{\partial E^i}{\partial x^3}\left(\frac{\partial\xi^3}{\partial x^k} + \frac{\partial\xi^3}{\partial E^j}\frac{\partial E^j}{\partial x^k}\right)\right) \\
&+ \left(c^2\frac{\partial E^i}{\partial x^j}\frac{\partial E^i}{\partial x^j} - \frac{\partial E^i}{\partial t}\frac{\partial E^i}{\partial t} + \alpha(x)z\right) \\
&\times \left(\frac{\partial\xi^k}{\partial x^k} + \frac{\partial\xi^k}{\partial E^j}\frac{\partial E^j}{\partial x^k} + \frac{d\tau}{dt}\right)\Bigg)\, dx^1dx^2dx^3dt = 0.
\end{aligned}$$

As usual we form the system of partial differential equations for the unknowns τ, ξ^i and η^i, $i = 1, 2, 3$, by equating to zero the coefficients in the Killing's identity. The solution is: τ is an arbitrary constant, $\eta^1 = aE^2 - bE^3 + p^1$, $\eta^2 = kE^3 - aE^1 + p^2$, $\eta^3 = bE^1 - kE^2 + p^3$ where a, b, k, p^1, p^2, p^3 are arbitrary constants, $\xi^1 = qx^2 - sx^3 + r^1$, $\xi^2 = px^3 - qx^1 + r^2$, $\xi^3 = sx^1 - px^2 + r^3$ where q, s, p, r^1, r^2, r^3 are arbitrary constants restricted by the condition $\xi^i \frac{\partial \alpha}{\partial x^i} = 0$. If, for example, α is a constant, $\alpha = \frac{\sigma}{\varepsilon V}$, where V is the volume of Ω then this condition is satisfied.

5. Appendix

5.1. Derivation of the relation (3.11). We differentiate the transformed Lagrangian density $\mathcal{L}\big(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{z}\big)$ in equation (3.8) with respect to ε and set $\varepsilon = 0$

$$\frac{d\mathcal{L}}{d\varepsilon}\bigg|_{\varepsilon=0} = \left(\frac{\partial \mathcal{L}}{\partial \bar{t}}\frac{d\phi}{d\varepsilon} + \frac{\partial \mathcal{L}}{\partial \bar{x}^k}\frac{d\varphi^k}{d\varepsilon} + \frac{\partial \mathcal{L}}{\partial \bar{u}^i}\frac{d\psi^i}{d\varepsilon} + \frac{\partial \mathcal{L}}{\partial \bar{u}^i_{\bar{t}}}\frac{d}{d\varepsilon}\frac{\partial \bar{u}^i}{\partial \bar{t}} + \frac{\partial \mathcal{L}}{\partial \bar{u}^i_{\bar{x}^k}}\frac{d}{d\varepsilon}\frac{\partial \bar{u}^i}{\partial \bar{x}^k} + \frac{\partial \mathcal{L}}{\partial \bar{z}}\frac{d\bar{z}}{d\varepsilon}\right)\bigg|_{\varepsilon=0}$$

which, when written with ζ and the infinitesimal generators of the group, becomes

$$\begin{aligned} \frac{d\mathcal{L}}{d\varepsilon}\bigg|_{\varepsilon=0} &= \frac{\partial \mathcal{L}}{\partial t}\tau + \frac{\partial \mathcal{L}}{\partial x^k}\xi^k + \frac{\partial \mathcal{L}}{\partial u^i}\eta^i + \frac{\partial \mathcal{L}}{\partial u^i_t}\frac{d}{d\varepsilon}\left(\frac{\partial \bar{u}^i}{\partial \bar{t}}\right)\bigg|_{\varepsilon=0} \\ &+ \frac{\partial \mathcal{L}}{\partial u^i_{x^k}}\frac{d}{d\varepsilon}\left(\frac{\partial \bar{u}^i}{\partial \bar{x}^k}\right)\bigg|_{\varepsilon=0} + \frac{\partial \mathcal{L}}{\partial z}\zeta. \end{aligned} \tag{5.1}$$

To calculate $\frac{d}{d\varepsilon}\big(\frac{\partial \bar{u}^i}{\partial \bar{t}}\big)\big|_{\varepsilon=0}$ differentiate the equation $\bar{u}^i(\bar{t}, \bar{x}; \varepsilon) = \psi^i(t, x, u; \varepsilon) \equiv \bar{u}^i(t, x, u; \varepsilon)$ with respect to t

$$\frac{\partial \bar{u}^i}{\partial \bar{t}}\frac{d\bar{t}}{dt} + \frac{\partial \bar{u}^i}{\partial \bar{x}^k}\left(\frac{\partial \bar{x}^k}{\partial t} + \frac{\partial \bar{x}^k}{\partial u^j}u^j_t\right) = \frac{\partial \bar{u}^i}{\partial t} + \frac{\partial \bar{u}^i}{\partial u^j}u^j_t. \tag{5.2}$$

Set $\varepsilon = 0$ and take into account the identities:

$$\frac{\partial \bar{u}^i}{\partial t}\bigg|_{\varepsilon=0} = 0, \quad \frac{\partial \bar{u}^i}{\partial u^j}\bigg|_{\varepsilon=0} = \delta^i_j, \quad \frac{\partial \bar{t}}{\partial t}\bigg|_{\varepsilon=0} = 1, \quad \frac{\partial \bar{x}^k}{\partial t}\bigg|_{\varepsilon=0} = 0, \quad \frac{\partial \bar{x}^k}{\partial u^j}\bigg|_{\varepsilon=0} = 0.$$

Substitute these in (5.2) and solve the resulting equation for $\bar{u}^i_{\bar{t}}\big|_{\varepsilon=0}$ to find

$$\bar{u}^i_{\bar{t}}\big|_{\varepsilon=0} = u^i_t. \tag{5.3}$$

Differentiate the equation $\bar{u}^i(\bar{t}, \bar{x}; \varepsilon) = \psi^i(t, x, u; \varepsilon) \equiv \bar{u}^i(t, x, u; \varepsilon)$ with respect to x^k

$$\frac{\partial \bar{u}^i}{\partial \bar{x}^l}\left(\frac{\partial \bar{x}^l}{\partial x^k} + \frac{\partial \bar{x}^l}{\partial u^j}u^j_{x^k}\right) = \frac{\partial \bar{u}^i}{\partial x^k} + \frac{\partial \bar{u}^i}{\partial u^j}u^j_{x^k}. \tag{5.4}$$

Set $\varepsilon = 0$ and substitute the identities:

$$\left.\frac{\partial \bar{u}^i}{\partial x^k}\right|_{\varepsilon=0} = 0, \quad \left.\frac{\partial \bar{u}^i}{\partial u^j}\right|_{\varepsilon=0} = \delta^i_j, \quad \left.\frac{\partial \bar{x}^l}{\partial x^k}\right|_{\varepsilon=0} = \delta^l_k, \quad \left.\frac{\partial \bar{x}^l}{\partial u^j}\right|_{\varepsilon=0} = 0$$

in (5.4). Then solve the resulting equation for $\bar{u}^i_{\bar{x}^k}\big|_{\varepsilon=0}$ to obtain

$$\bar{u}^i_{\bar{x}^k}\big|_{\varepsilon=0} = u^i_{x^k}. \tag{5.5}$$

Differentiate (5.2) with respect to ε to get

$$\begin{aligned} &\bar{u}^i_{\bar{t}}\frac{d}{d\varepsilon}\frac{d\bar{t}}{dt} + \frac{d\bar{t}}{dt}\frac{d\bar{u}^i_{\bar{t}}}{d\varepsilon} + \bar{u}^i_{\bar{x}^k}\left(\frac{d}{d\varepsilon}\frac{\partial \bar{x}^k}{\partial t} + \frac{d}{d\varepsilon}\left(\frac{\partial \bar{x}^k}{\partial u^j}\right)u^j_t\right) + \left(\frac{\partial \bar{x}^k}{\partial t} + \frac{\partial \bar{x}^k}{\partial u^j}u^j_t\right)\frac{d\bar{u}^i_{\bar{x}^k}}{d\varepsilon} \\ &= \frac{d}{d\varepsilon}\left(\frac{\partial \bar{u}^i}{\partial t} + \frac{\partial \bar{u}^i}{\partial u^j}u^j_t\right). \end{aligned} \tag{5.6}$$

Set $\varepsilon = 0$ in (5.6) and substitute (5.3) and (5.5) in it. Then take into account the identities

$$\begin{aligned} &\left.\frac{d}{d\varepsilon}\frac{\partial \bar{u}^i}{\partial t}\right|_{\varepsilon=0} = \frac{\partial \eta^i}{\partial t}, \quad \left.\frac{d}{d\varepsilon}\frac{\partial \bar{u}^i}{\partial u^j}\right|_{\varepsilon=0} = \frac{\partial \eta^i}{\partial u^j}, \quad \left.\frac{d}{d\varepsilon}\frac{\partial \bar{t}}{\partial t}\right|_{\varepsilon=0} = \frac{\partial \tau}{\partial t}, \quad \left.\frac{\partial \bar{t}}{\partial t}\right|_{\varepsilon=0} = 1, \\ &\left.\frac{d}{d\varepsilon}\frac{\partial \bar{x}^k}{\partial t}\right|_{\varepsilon=0} = \frac{\partial \xi^k}{\partial t}, \quad \left.\frac{d}{d\varepsilon}\frac{\partial \bar{x}^k}{\partial u^j}\right|_{\varepsilon=0} = \frac{\partial \xi^k}{\partial u^j}, \quad \left.\frac{\partial \bar{x}^k}{\partial t}\right|_{\varepsilon=0} = 0, \quad \left.\frac{\partial \bar{x}^k}{\partial u^j}\right|_{\varepsilon=0} = 0. \end{aligned}$$

Consequently, equation (5.6) becomes

$$\frac{\partial \eta^i}{\partial t} + \frac{\partial \eta^i}{\partial u^j}u^j_t = u^i_t\frac{\partial \tau}{\partial t} + \left.\frac{d}{d\varepsilon}\bar{u}^i_{\bar{t}}\right|_{\varepsilon=0} + u^i_{x^k}\left(\frac{\partial \xi^k}{\partial t} + \frac{\partial \xi^k}{\partial u^j}u^j_t\right) \tag{5.7}$$

from which we obtain

$$\left.\frac{d}{d\varepsilon}\bar{u}^i_{\bar{t}}\right|_{\varepsilon=0} = \frac{d\eta^i}{dt} - u^i_t\frac{d\tau}{dt} - u^i_{x^k}\frac{d\xi^k}{dt}. \tag{5.8}$$

We must now calculate $\frac{d}{d\varepsilon}\left(\frac{\partial \bar{u}^i}{\partial \bar{x}^k}\right)\Big|_{\varepsilon=0}$ which appears in (5.1). For this purpose differentiate (5.4) with respect to ε

$$\frac{d}{d\varepsilon}\frac{\partial \bar{u}^i}{\partial x^k} + \frac{d}{d\varepsilon}\frac{\partial \bar{u}^i}{\partial u^j}u^j_{x^k} = \frac{d}{d\varepsilon}\bar{u}^i_{\bar{x}^l}\left(\frac{\partial \bar{x}^l}{\partial x^k} + \frac{\partial \bar{x}^l}{\partial u^j}u^j_{x^k}\right) + \bar{u}^i_{\bar{x}^l}\left(\frac{d}{d\varepsilon}\frac{\partial \bar{x}^l}{\partial x^k} + u^j_{x^k}\frac{d}{d\varepsilon}\frac{\partial \bar{x}^l}{\partial u^j}\right). \tag{5.9}$$

Set $\varepsilon = 0$ in (5.9), substitute (5.3) and (5.5) into (5.9) and observe that

$$\begin{aligned} &\left.\frac{d}{d\varepsilon}\frac{\partial \bar{u}^i}{\partial x^k}\right|_{\varepsilon=0} = \frac{\partial \eta^i}{\partial x^k}, \quad \left.\frac{d}{d\varepsilon}\frac{\partial \bar{u}^i}{\partial u^j}\right|_{\varepsilon=0} = \frac{\partial \eta^i}{\partial u^j}, \quad \left.\frac{\partial \bar{x}^l}{\partial x^k}\right|_{\varepsilon=0} = \delta^l_k, \\ &\left.\frac{\partial \bar{x}^l}{\partial u^j}\right|_{\varepsilon=0} = 0, \quad \left.\frac{d}{d\varepsilon}\frac{\partial \bar{x}^l}{\partial x^k}\right|_{\varepsilon=0} = \frac{\partial \xi^l}{\partial x^k}, \quad \left.\frac{d}{d\varepsilon}\frac{\partial \bar{x}^l}{\partial u^j}\right|_{\varepsilon=0} = \frac{\partial \xi^l}{\partial u^j}. \end{aligned}$$

Then (5.9) becomes

$$\frac{\partial \eta^i}{\partial x^k} + \frac{\partial \eta^i}{\partial u^j} u^j_{x^k} = \frac{d}{d\varepsilon} \bar{u}^i_{\bar{x}^l}\Big|_{\varepsilon=0} \delta^l_k + u^i_{x^l}\left(\frac{\partial \xi^l}{\partial x^k} + \frac{\partial \xi^l}{\partial u^j} u^j_{x^k}\right), \tag{5.10}$$

from which we get

$$\frac{d}{d\varepsilon} \bar{u}^i_{\bar{x}^k}\Big|_{\varepsilon=0} = \frac{d\eta^i}{dx^k} - u^i_{x^l} \frac{d\xi^l}{dx^k}. \tag{5.11}$$

Substituting (5.8) and (5.11) into (5.1) produces the relation (3.11)

$$\frac{d\mathcal{L}}{d\varepsilon}\Big|_{\varepsilon=0} = \frac{\partial \mathcal{L}}{\partial t}\tau + \frac{\partial \mathcal{L}}{\partial x^k}\xi^k + \frac{\partial \mathcal{L}}{\partial u^i}\eta^i + \frac{\partial \mathcal{L}}{\partial u^i_t}\left(\frac{d\eta^i}{dt} - u^i_t \frac{d\tau}{dt} - u^i_{x^k}\frac{d\xi^k}{dt}\right)$$
$$+ \frac{\partial \mathcal{L}}{\partial u^i_{x^k}}\left(\frac{d\eta^i}{dx^k} - u^i_{x^l}\frac{d\xi^l}{dx^k}\right) + \frac{\partial \mathcal{L}}{\partial z}\zeta.$$

5.2. Derivation of the relation (3.12). We use the formula for the derivative of a determinant, according to which

$$\frac{d}{d\varepsilon} \det\left(\frac{\partial \bar{x}}{\partial x}\right) = A_k{}^j \frac{d}{d\varepsilon}\left(\frac{\partial \bar{x}^k}{\partial x^j}\right) \tag{5.12}$$

where $A_k{}^j$ is the cofactor of the determinant's entry $\frac{\partial \bar{x}^k}{\partial x^j}$. Next,

$$\frac{d}{d\varepsilon}\left(\frac{\partial \bar{x}^k}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\frac{d\bar{x}^k}{d\varepsilon} = \frac{\partial^2 \bar{x}^k}{\partial x^j\,\partial \varepsilon} + \frac{\partial^2 \bar{x}^k}{\partial u^i\,\partial \varepsilon}\frac{\partial u^i}{\partial x^j}$$

because $\bar{x}^k = \bar{x}^k(t, x, u(t,x); \varepsilon)$. Hence (5.12) becomes

$$\frac{d}{d\varepsilon} \det\left(\frac{\partial \bar{x}}{\partial x}\right) = A_k{}^j \left(\frac{\partial^2 \bar{x}^k}{\partial x^j\,\partial \varepsilon} + \frac{\partial^2 \bar{x}^k}{\partial u^i\,\partial \varepsilon}\frac{\partial u^i}{\partial x^j}\right).$$

Setting $\varepsilon = 0$ in the above expression and observing that $A_k{}^j\big|_{\varepsilon=0} = \delta_k{}^j$ is a cofactor of the identity matrix, we get the relation (3.12)

$$\frac{d}{d\varepsilon} \det\left(\frac{\partial \bar{x}}{\partial x}\right)\Big|_{\varepsilon=0} = \left(\frac{\partial \xi^k}{\partial x^j} + \frac{\partial \xi^k}{\partial u^i}\frac{\partial u^i}{\partial x^j}\right)\delta_k{}^j = \frac{d\xi^k}{dx^j}\,\delta_k{}^j = \frac{d\xi^k}{dx^k}.$$

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