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# Multiplicity Results for Classes of Infinite Positone Problems

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**Abstract.** We study positive solutions to the singular boundary value problem

$$\begin{cases} -\Delta_p u = \lambda \dfrac{f(u)}{u^\beta} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u = \text{div}\ (|\nabla u|^{p-2}\nabla u)$, $p > 1, \lambda > 0, \beta \in (0,1)$ and $\Omega$ is a bounded domain in $\mathbb{R}^N, N \geq 1$. Here $f:\ [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function such that $\lim_{u\to\infty} \frac{f(u)}{u^{\beta+p-1}} = 0$. We establish the existence of multiple positive solutions for certain range of $\lambda$ when $f$ satisfies certain additional assumptions. A simple model that will satisfy our hypotheses is $f(u) = e^{\frac{\alpha u}{\alpha+u}}$ for $\alpha \gg 1$. We also extend our results to classes of systems when the nonlinearities satisfy a combined sublinear condition at infinity. We prove our results by the method of sub-supersolutions.

**Keywords.** Singular boundary value problems, infinite positone problems, multiplicity of positive solutions, sub-supersolutions

**Mathematics Subject Classification (2000).** 35J25, 35J55

## 1. Introduction

We first consider the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda \dfrac{f(u)}{u^\beta} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian of $u$, $p > 1, \beta \in (0,1), \lambda$ is a positive parameter and $\Omega$ is a bounded domain with a smooth boundary in $\mathbb{R}^N, N \geq 1$. We assume that $f$ is a $C([0,\infty))$-function satisfying the following assumptions:

(H1) $f(u) > 0$ for all $u \geq 0$,

(H2) $\lim_{u\to\infty} \frac{f(u)}{u^{\beta+p-1}} = 0$.

We note that $\lim_{u\to 0} \frac{f(u)}{u^\beta} = \infty$, and hence (1) is a singular boundary value problem which we call here as an infinite positone problem. Our results in this paper are motivated by the problem:

$$\begin{cases} -\Delta_p u = \lambda \dfrac{\exp[\frac{\alpha u}{\alpha+u}]}{u^\beta} & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

When $\beta = 0$ for every $\alpha > 0$ and $\lambda > 0$ it is known that there exist a positive solution and when $\alpha \gg 1$ there exists a range of $\lambda$ for which there exist at least three positive solutions [8]. In this paper, we extend this study to the singular case when $0 < \beta < 1$. In particular, we establish the existence of a positive solution for all $\alpha > 0$ and for all $\lambda > 0$ and a multiplicity result for certain range of $\lambda$ when $\alpha \gg 1$. However, our multiplicity result is restricted two positive solutions. In [1], the author studied this singular problem (2) when $p = 2$ by treating it as a limit problem of the class of non-singular problems defined by $-\Delta u_\epsilon = \lambda \frac{e^{\frac{\alpha u}{\alpha+u}}}{(u+\epsilon)^\beta}$ in $\Omega$ and $u_\epsilon = 0$ on $\partial\Omega$. Here we establish the our results for all $p > 1$ directly by method of sub- and supersolutions associated with such singular problems. Also our proofs easily extend to classes of system where the nonlinearities satisfy a combined sublinear condition at infinity.

By a subsolution of (1) we mean a function $\psi : \overline{\Omega} \to \mathbb{R}$ such that $\psi \in W^{1,p}(\Omega) \bigcap C(\overline{\Omega})$ and satisfies:

$$\begin{aligned} -\Delta_p \psi \leq \lambda \frac{f(\psi)}{\psi^\beta} & \quad \text{in } \Omega \\ \psi > 0 & \quad \text{in } \Omega \\ \psi = 0 & \quad \text{on } \partial\Omega \end{aligned}$$

and by a supersolution of (1) we mean a function $\phi : \overline{\Omega} \to \mathbb{R}$ such that $\phi \in W^{1,p}(\Omega) \bigcap C(\overline{\Omega})$ and satisfies:

$$\begin{aligned} -\Delta_p \phi \geq \lambda \frac{f(\phi)}{\phi^\beta} & \quad \text{in } \Omega \\ \phi > 0 & \quad \text{in } \Omega \\ \phi = 0 & \quad \text{on } \partial\Omega. \end{aligned}$$

Then we have the following lemma:

**Lemma 1.1** (See [2, 5, 9])**.** *If there exist a subsolution* $\psi$ *and a supersolution* $\phi$ *of* (1) *such that* $\psi \leq \phi$ *on* $\overline{\Omega}$, *then* (1) *has at least one solution* $u \in W^{1,p}(\Omega) \bigcap C(\overline{\Omega})$ *satisfying* $\psi \leq u \leq \phi$ *on* $\overline{\Omega}$.

We first establish:

**Theorem 1.2.** *Assume* (H1) – (H2). *Then* (1) *has a positive solution for all* $\lambda > 0$.

We refer to [7] for a more general existence result for (1). However, for certain classes of $f$ we can get at least two positive solutions for certain range of $\lambda$. To state this multiplicity result, for any $0 < a < d$ we define

$$Q(a, d) := \frac{a^{\beta+p-1}}{f(a)} \frac{f(d)}{d^{\beta+p-1}}.$$

Further, let

$$A := \left( \frac{(N + p - 1)^{N+p-1}}{N^N} \right)^{\frac{1}{p-1}}.$$

Throughout this paper, $w \in W^{1,p}(\Omega) \bigcap C(\bar{\Omega})$ (see [4, Lemma 3.1]) is the unique solution of

$$\begin{cases} -\Delta_p w = \dfrac{1}{w^\beta} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

We now assume that $f$ further satisfies:

(H3) $f(u)$ is nondecreasing for all $u \geq 0$

(H4) There exist $a$ and $b$ such that $0 < a < \frac{p}{A} b$ and $\frac{f(u)}{u^\beta}$ is nondecreasing on $(a, b)$.

We establish:

**Theorem 1.3.** *Assume* (H1) – (H4). *Further assume that there exists* $d$ *such that*

$$a < d < \frac{p}{A} b \quad \textit{and} \quad Q(a, d) > \frac{A^{p-1} N \|w\|_\infty^{\beta+p-1}}{(p-1)^{p-1} R^p} := C(\beta, N, \Omega),$$

*where* $R$ *is the radius of the largest inscribed ball* $B_R$ *in* $\Omega$. *Then* (1) *has at least two positive solutions for* $\lambda_* < \lambda < \lambda^*$, *where*

$$\lambda_* = \frac{d^{\beta+p-1}}{f(d)} \frac{A^{p-1} N}{(p-1)^{p-1} R^p} \quad \textit{and}$$

$$\lambda^* = \min \left\{ \frac{d^\beta}{f(d)} \frac{N}{R^p} \left( \frac{p}{p-1} \right)^{p-1} b^{p-1}, \ \frac{a^{\beta+p-1}}{f(a)} \frac{1}{\|w\|_\infty^{\beta+p-1}} \right\}.$$

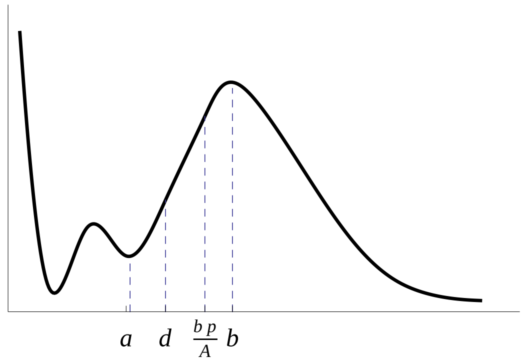


Figure 1: Graph of the function $\frac{f(u)}{u^\beta}$

**Remark 1.4.** Since $d < \frac{p}{A}b$, we have $\frac{d^{\beta+p-1}}{f(d)}\frac{A^{p-1}N}{(p-1)^{p-1}R^p} < \frac{d^\beta}{f(d)}\frac{N}{R^p}(\frac{p}{p-1})^{p-1}b^{p-1}$ and since $Q(a,d) > \frac{A^{p-1}N\|w\|_\infty^{\beta+p-1}}{(p-1)^{p-1}R^p}$, we obtain $\frac{d^{\beta+p-1}}{f(d)}\frac{A^{p-1}N}{(p-1)^{p-1}R^p} < \frac{a^{\beta+p-1}}{f(a)}\frac{1}{\|w\|_\infty^{\beta+p-1}}$. Therefore, $(\lambda_*, \lambda^*)$ is not empty.

**Remark 1.5.** A simple example satisfying the hypotheses of Theorem 1.2 and Theorem 1.3 is

$$\begin{cases} -\Delta_p u = \lambda \dfrac{e^{\frac{\alpha u}{\alpha+u}}}{u^\beta} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly, $f(u) := e^{\frac{\alpha u}{\alpha+u}}$ satisfies hypotheses (H1) – (H3). Choosing $a = 1, d = \alpha$ and $b = \frac{\alpha^2}{2}$, we can easily show that $\frac{f(u)}{u^\beta}$ is nondecreasing on $(a,b)$ for $\alpha \gg 1$. Further $Q(a,d) = \frac{a^{\beta+p-1}}{f(a)}\frac{f(d)}{d^{\beta+p-1}} = [\frac{1}{\alpha}]^{\beta+p-1}\exp[\frac{\alpha}{2} - \frac{\alpha}{\alpha+1}]$ and hence, for any given $\Omega$, we have $a < d < \frac{p}{A}b$ and $Q(1,\alpha) > C(\beta, N, \Omega)$ for $\alpha$ large.

Next we note that the method of sub- and supersolutions discussed in Lemma 1.1 extends to the system:

$$\begin{cases} -\Delta_p u = \lambda \dfrac{f(v)}{u^\beta} & \text{in } \Omega \\ -\Delta_p v = \lambda \dfrac{g(u)}{v^\beta} & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

This follows by using the result in [5]. For the system (4) by a subsolution we mean a pair of functions $(\psi, \bar{\psi}) : \overline{\Omega} \to \mathbb{R}\times\mathbb{R}$ such that $(\psi, \bar{\psi}) \in$

$\big(W^{1,p}(\Omega)\cap C(\overline{\Omega})\big)\times\big(W^{1,p}(\Omega)\cap C(\overline{\Omega})\big)$ and satisfy

$$\begin{aligned} -\Delta_p\psi &\le \lambda\frac{f(\bar\psi)}{\psi^\beta} && \text{in } \Omega \\ -\Delta_p\bar\psi &\le \lambda\frac{g(\psi)}{\bar\psi^\beta} && \text{in } \Omega \\ \psi > 0,\ \ \bar\psi &> 0 && \text{in } \Omega \\ \psi = \bar\psi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

By a supersolution we mean a pair of functions $(\phi,\bar\phi):\overline{\Omega}\to\mathbb{R}\times\mathbb{R}$ such that $(\phi,\bar\phi)\in(W^{1,p}(\Omega)\cap C(\overline{\Omega}))\times(W^{1,p}(\Omega)\cap C(\overline{\Omega}))$ and satisfy

$$\begin{aligned} -\Delta_p\phi &\ge \lambda\frac{f(\bar\phi)}{\phi^\beta} && \text{in } \Omega \\ -\Delta_p\bar\phi &\ge \lambda\frac{g(\phi)}{\bar\phi^\beta} && \text{in } \Omega \\ \phi > 0,\ \ \bar\phi &> 0 && \text{in } \Omega \\ \phi = \bar\phi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

We now assume that $f$ and $g$ are $C([0,\infty))$ functions satisfying the following assumptions:

(G1) $f$ and $g$ are nondecreasing and $f(0)>0$ and $g(0)>0$

(G2) $\lim_{x\to\infty}\frac{f(Mg(x))}{x^{\beta+p-1}}=0$ for all $M>0$ (a combined sublinear condition at infinity).

We establish:

**Theorem 1.6.** *Assume* (G1) – (G2). *Then* (4) *has a positive solution for all* $\lambda>0$.

Next, under certain combined nonlinear effects of $\frac{x^{\beta+p-1}}{f(x)}$ and $\frac{x^{\beta+p-1}}{g(x)}$ we study the existence of multiple positive solutions to (4). To state the multiplicity result, for any $0<a<d$ we define

$$Q_1(a,d):=\frac{a^{\beta+p-1}}{g(a)}\frac{f(d)}{d^{\beta+p-1}}.$$

We also assume:

(G3) $f(u)\le g(u)$ for all $u\ge 0$

(G4) There exist $a$ and $b$ with $0<a<b$ such that $a<\frac{p}{A}b$ and $\frac{f(u)}{u^\beta}$ is nondecreasing on $(a,b)$.

We establish:

**Theorem 1.7.** *Assume* (G1) – (G4). *Further assume there exists* $d$ *such that* $a < d < \frac{p}{A}b$ *and* $Q_1(a,d) > C(\beta,N,\Omega)$, *where* $C(\beta,N,\Omega)$ *is as defined in Theorem* 1.3. *Then* (4) *has at least two positive solutions for* $\lambda_* < \lambda < \lambda^*$, *where*

$$\lambda_* = \frac{d^{\beta+p-1}}{f(d)}\frac{A^{p-1}N}{(p-1)^{p-1}R^p} \quad \textit{and}$$
$$\lambda^* = \min\left\{\frac{d^\beta}{f(d)}\frac{N}{R^p}\left(\frac{p}{p-1}\right)^{p-1}b^{p-1},\ \frac{a^{\beta+p-1}}{g(a)}\frac{1}{\|w\|_\infty^{\beta+p-1}}\right\}.$$

**Remark 1.8.** Since $d < \frac{p}{A}b$, we have $\frac{d^{\beta+p-1}}{f(d)}\frac{A^{p-1}N}{(p-1)^{p-1}R^p} < \frac{d^\beta}{f(d)}\frac{N}{R^p}(\frac{p}{p-1})^{p-1}b^{p-1}$ and since $Q_1(a,d) > \frac{A^{p-1}N\|w\|_\infty^{\beta+p-1}}{(p-1)^{p-1}R^p}$, we obtain $\frac{d^{\beta+p-1}}{f(d)}\frac{A^{p-1}N}{(p-1)^{p-1}R^p} < \frac{a^{\beta+p-1}}{g(a)}\frac{1}{\|w\|_\infty^{\beta+p-1}}$. Therefore, $(\lambda_*, \lambda^*)$ is not empty.

**Remark 1.9.** A simple example satisfying the hypotheses of Theorem 1.6 and Theorem 1.7 is

$$\begin{aligned} -\Delta_p u &= \lambda\frac{e^{\frac{\alpha v}{\alpha+v}}}{u^\beta} && \text{in } \Omega\\ -\Delta_p v &= \lambda\frac{u^q+M}{v^\beta} && \text{in } \Omega\\ u &= 0 = v && \text{on } \partial\Omega, \end{aligned}$$

where $q > 0$ and $M \gg 1$ so that (G3) is satisfied. Clearly, $f(u) := e^{\frac{\alpha u}{\alpha+u}}$ and $g(u) := u^q + M$ satisfy hypotheses (G1) – (G3). Choosing $a = 1, d = \alpha$ and $b = \frac{\alpha^2}{2}$, we can easily show that $\frac{f(u)}{u^\beta}$ is nondecreasing on $(a,b)$ for $\alpha \gg 1$. Further $Q_1(a,d) = \frac{a^{\beta+p-1}}{g(a)}\frac{f(d)}{d^{\beta+p-1}} = \left(\frac{1}{1+M}\right)\left(\frac{1}{\alpha}\right)^{\beta+p-1}\exp[\frac{\alpha}{2}]$ and hence, for any given $\Omega$ we have $a < d < \frac{p}{A}b$ and $Q_1(1,\alpha) > C(\beta,N,\Omega)$ for $\alpha$ large.

We will prove Theorems 1.2 and 1.3 in the Section 2 and the Theorem 1.6 and 1.7 in Section 3.

## 2. Proof of Theorem 1.2 and Theorem 1.3

*Proof of Theorem* 1.2. We construct a positive supersolution $\phi_1$ of (1).

Let $f^*(u) = \max_{0\le x\le u} f(x)$. Then $f^*(u)$ is nondecreasing and $\frac{f^*(u)}{u^{\beta+p-1}} \to 0$ as $u \to \infty$, since $\frac{f(u)}{u^{\beta+p-1}} \to 0$ as $u \to \infty$. So there exists $M_\lambda \gg 1$ such that

$$\frac{f^*(M_\lambda\|w\|_\infty)}{(M_\lambda\|w\|_\infty)^{\beta+p-1}} \le \frac{1}{\lambda\|w\|_\infty^{\beta+p-1}}.$$

Let $\phi_1 = M_\lambda w$, where $w$ is defined in (3). We have

$$-\Delta_p\phi_1 = \frac{M_\lambda^{p-1}}{w^\beta} \ge \lambda\frac{f^*(M_\lambda\|w\|_\infty)}{(M_\lambda w)^\beta} \ge \lambda\frac{f^*(M_\lambda w)}{(M_\lambda w)^\beta} \ge \lambda\frac{f(M_\lambda w)}{(M_\lambda w)^\beta} = \lambda\frac{f(\phi_1)}{\phi_1^\beta},$$

showing that $\phi_1$ is a positive supersolution of (1).

Now we construct a positive subsolution $\psi_1$. Let $\lambda_1$ be the first eigenvalue of $-\Delta_p$ with Dirichlet boundary condition and $e > 0$ be a corresponding eigenfunction. Hence $e$ and $\lambda_1$ satisfy:

$$\begin{cases} -\Delta_p e = \lambda_1 e^{p-1} & \text{in } \Omega \\ e = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\frac{f(u)}{u^\beta} \to \infty$ as $u \to 0$, there exists a sufficiently small $m_\lambda$ such that

$$\lambda_1 (m_\lambda e)^{p-1} \leq \lambda \frac{f(m_\lambda e)}{(m_\lambda e)^\beta} \quad \text{for all} \quad \lambda > 0.$$

Let $\psi_1 = m_\lambda e$. We have $-\Delta_p \psi_1 = \lambda_1 (m_\lambda e)^{p-1} \leq \lambda \frac{f(m_\lambda e)}{(m_\lambda e)^\beta} = \lambda \frac{f(\psi_1)}{\psi_1^\beta}$. Thus $\psi_1$ is subsolution of (1), and if $m_\lambda$ is chosen sufficiently small, then $\psi_1 \leq \phi_1$. Hence, Theorem 1.2 is proven. □

*Proof of Theorem* 1.3. Here we construct a second positive supersolution $\phi_2$ of (1) with $\|\phi_2\|_\infty = a$ when $\lambda \leq \frac{a^{\beta+p-1}}{f(a)} \frac{1}{\|w\|_\infty^{\beta+p-1}}$. Let $\phi_2 = a \frac{w}{\|w\|_\infty}$, where $w$ is defined in (3). Since $\lambda \leq \frac{a^{\beta+p-1}}{f(a)} \frac{1}{\|w\|_\infty^{\beta+p-1}}$,

$$-\Delta_p \phi_2 = \frac{a^{p-1}}{\|w\|_\infty} \frac{1}{w^\beta} = \frac{\|w\|_\infty^\beta}{a^\beta w^\beta} \frac{a^{\beta+p-1}}{\|w\|_\infty^{\beta+p-1}} \geq \lambda \frac{f(a)}{\phi_2^\beta} \geq \lambda \frac{f\left(a \frac{w}{\|w\|_\infty}\right)}{\phi_2^\beta} = \lambda \frac{f(\phi_2)}{\phi_2^\beta}.$$

Next we construct a second positive subsolution $\psi_2$ of (1) when

$$\frac{d^{\beta+p-1}}{f(d)} \frac{A^{p-1} N}{(p-1)^{p-1} R^p} < \lambda < \frac{d^\beta}{f(d)} \frac{N}{R^p} \left( \frac{p}{p-1} \right)^{p-1} b^{p-1}.$$

Let $a^* \in (0, a]$ be such that $f(a^*) = \min_{0<x\leq a} f(x)$ and define $h \in C([0, \infty))$ such that

$$h(u) = \begin{cases} \dfrac{f(a^*)}{(a^*)^\beta}, & u \leq a^* \\ \dfrac{f(u)}{u^\beta}, & u \geq a, \end{cases}$$

so that $h$ is nondecreasing on $(0, a]$ and $h(u) \leq \frac{f(u)}{u^\beta}$ for all $u \geq 0$ (See Figure 2).

Consider the following nonsingular problem:

$$\begin{cases} -\Delta_p u = \lambda h(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

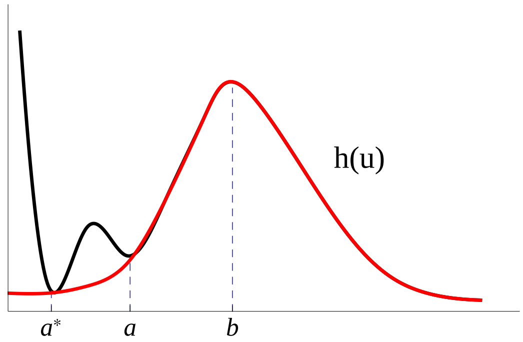


Figure 2: Graph of the function $h(u)$ below $\frac{f(u)}{u^\beta}$

Let $R$ be the radius of the largest inscribed ball $B_R$ of $\Omega$. For $0 < \epsilon < R$, and $\delta, \mu > 1$, define $\rho(r) : [0, R] \to [0, 1]$ by

$$\rho(r) = \begin{cases} 1, & 0 \le r \le \epsilon \\ 1 - \left(1 - \left(\dfrac{R-r}{R-\epsilon}\right)^{\mu}\right)^{\delta}, & \epsilon < r \le R. \end{cases}$$

Then

$$\rho'(r) = \begin{cases} 0, & 0 \le r \le \epsilon \\ -\dfrac{\delta\mu}{R-\epsilon}\left(1 - \left(\dfrac{R-r}{R-\epsilon}\right)^{\mu}\right)^{\delta-1}\left(\dfrac{R-r}{R-\epsilon}\right)^{\mu-1}, & \epsilon < r \le R. \end{cases}$$

Let $v(r) = d\rho(r)$. Here note that $|v'(r)| \le \frac{d\delta\mu}{R-\epsilon}$ since $| \rho'(r) | \le \frac{\delta\mu}{R-\epsilon}$.

Define $\psi$ as the radially symmetric solution of

$$\begin{cases} -\Delta_p \psi(x) = \lambda h(v(|x|)) & \text{in } B(0, R) \\ \psi = 0 & \text{on } \partial B(0, R). \end{cases}$$

Then $\psi$ satisfies

$$\begin{cases} -(r^{N-1} G(\psi'(r)))' = \lambda r^{N-1} h(v(r)) \\ \psi'(0) = 0, \;\; \psi(R) = 0, \end{cases}$$

where $G(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$. Integrating once, for $0 < r < R$, we get

$$-G(\psi'(r)) = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} h(v(s))\, ds. \tag{6}$$

Since $G$ is monotone, $G^{-1}$ is also continuous and monotone. Hence, we have

$$-\psi'(r) = G^{-1}\left(\frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} h(v(s))\, ds\right). \tag{7}$$

We claim that

$$\psi(r) \geq v(r), \quad \forall\, 0 \leq r \leq R \tag{8}$$

and

$$\|\psi\|_\infty \leq b, \tag{9}$$

when $\frac{d^{\beta+p-1}}{f(d)} \frac{A^{p-1}N}{(p-1)^{p-1}R^p} < \lambda < \frac{d^\beta}{f(d)} \frac{N}{R^p} (\frac{p}{p-1})^{p-1} b^{p-1}$. If our claim is true, $\psi$ is a positive subsolution of the nonsingular problem (5) since $-\Delta_p \psi = \lambda h(v) \leq \lambda h(\psi)$. In order to show (8), since $\psi(R) = v(R) = 0$, it is enough to show that

$$\psi'(r) \leq v'(r), \quad \forall\, 0 \leq r \leq R. \tag{10}$$

Note that for $0 \leq r \leq \epsilon$, clearly $\psi'(r) \leq 0 = v'(r)$. Now for $r > \epsilon$, from (6)

$$\begin{aligned} -G(\psi'(r)) &= \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} h(v(s))\, ds \\ &> \frac{\lambda}{R^{N-1}} \int_0^\epsilon s^{N-1} h(v(s))\, ds \\ &= \frac{\lambda}{R^{N-1}} h(d) \frac{\epsilon^N}{N} \\ &= \frac{\lambda}{R^{N-1}} \frac{f(d)}{d^\beta} \frac{\epsilon^N}{N}. \end{aligned}$$

So, we have $-\psi'(r) > G^{-1}\left(\frac{\lambda}{R^{N-1}} \frac{f(d)}{d^\beta} \frac{\epsilon^N}{N}\right)$. Thus, (10) will hold for all $\epsilon \geq r \geq R$, if $G^{-1}\left(\frac{\lambda}{R^{N-1}} \frac{f(d)}{d^\beta} \frac{\epsilon^N}{N}\right) \geq \frac{\delta\mu}{R-\epsilon} d$, which is same as

$$\frac{\lambda}{R^{N-1}} \frac{f(d)}{d^\beta} \frac{\epsilon^N}{N} \geq G\left(\frac{\delta\mu}{R-\epsilon} d\right) = \left(\frac{\delta\mu}{R-\epsilon} d\right)^{p-1}.$$

Thus, if $\lambda \geq \frac{d^{\beta+p-1}}{f(d)} \frac{NR^{N-1}(\delta\mu)^{p-1}}{\epsilon^N (R-\epsilon)^{p-1}}$, inequality (10) will hold for all $\epsilon \leq r \leq R$. Note that

$$\inf \frac{d^{\beta+p-1}}{f(d)} \frac{NR^{N-1}(\delta\mu)^{p-1}}{\epsilon^N (R-\epsilon)^{p-1}} = \frac{d^{\beta+p-1}}{f(d)} \frac{A^{p-1}N}{(p-1)^{p-1}R^p} (\delta\mu)^{p-1}$$

and is achieved at $\epsilon = \frac{NR}{N+p-1}$. Hence, if $\lambda > \frac{d^{\beta+p-1}}{f(d)} \frac{A^{p-1}N}{(p-1)^{p-1}R^p}$, then in the definition of the function $\rho$ we can choose $\epsilon = \frac{NR}{N+p-1}$ and values for $\delta (> 1)$ and $\mu (> 1)$ so that $\lambda \geq \frac{d^{\beta+p-1}}{f(d)} \frac{NR^{N-1}(\delta\mu)^{p-1}}{\epsilon^N (R-\epsilon)^{p-1}}$ and hence (10) will hold for all $\epsilon \leq r \leq R$.

In order to obtain (9), integrating (7) from $t$ to $R$, we have

$$\int_t^R -\psi'(r) dr = \int_t^R G^{-1}\left(\frac{\lambda}{r^{N-1}} \left(\int_0^r s^{N-1} h(v(s))\, ds\right)\right) dr$$

for $0 \le t \le R$. Hence

$$\begin{aligned}
\psi(t) &= \int_t^R G^{-1}\left(\frac{\lambda}{r^{N-1}}\left(\int_0^r s^{N-1}h(v(s))\,ds\right)\right)dr \\
&\le \int_t^R G^{-1}\left(\frac{\lambda}{r^{N-1}}h(d)\frac{r^N}{N}\right)dr \\
&= \int_t^R \left(\frac{\lambda}{N}h(d)\right)^{\frac{1}{p-1}} r^{\frac{1}{p-1}}\,dr \\
&\le \left(\frac{\lambda}{N}h(d)\right)^{\frac{1}{p-1}} \int_0^R r^{\frac{1}{p-1}}\,dr \\
&= \frac{p-1}{p}\left(\frac{\lambda R^p}{N}\frac{f(d)}{d^\beta}\right)^{\frac{1}{p-1}}.
\end{aligned}$$

from which we have $\|\psi\|_\infty \le \frac{p-1}{p}\left(\frac{\lambda R^p}{N}\frac{f(d)}{d^\beta}\right)^{\frac{1}{p-1}}$. Since $\lambda < \frac{d^\beta}{f(d)}\frac{N}{R^p}\left(\frac{p}{p-1}\right)^{p-1} b^{p-1}$, we obtain $\|\psi\|_\infty \le b$. Thus $\psi$ satisfies

$$\begin{cases} -\Delta_p \psi \le \lambda h(\psi) & \text{in } B(0,R) \\ \psi = 0 & \text{on } \partial B(0,R) \end{cases}$$

and $d \le \|\psi\|_\infty \le b$.

Now, let $z(x) = \psi(x)$, if $x \in B_R$ and $z(x) = 0$, if $x \in \Omega - B_R$. Then $z \in W^{1,p}(\Omega)\bigcap C(\bar{\Omega})$ and $z = 0$ on $\partial\Omega$, which is subsolution of the nonsingular problem (5) in $\Omega$. However, $z$ is not strictly positive in $\Omega$. To obtain a strictly positive subsolution of (5) in $\Omega$ we iterate this subsolution $z$ once in a suitable manner. By the properties of $h$, there exists $\sigma_\lambda > 0$ such that $\lambda h(z) + \sigma_\lambda G(z)$ is increasing for all $z \ge 0$. Define $\psi_2$ to be the solution of

$$\begin{cases} -\Delta_p \psi_2 + \sigma_\lambda G(\psi_2) = \tilde{h}(z) & \text{in } \Omega \\ \psi_2 = 0 & \text{on } \partial\Omega \end{cases}$$

with $\tilde{h}(z) = \lambda h(z) + \sigma_\lambda G(z)$. Then since the operator $-\Delta_p + \sigma_\lambda G$ satisfies the weak comparison principle (see [3]), we can have $z \le \psi_2$ (see [6]). Further we get $\psi_2(x) > 0$ for all $x \in \Omega$ since $\tilde{h}(0) > 0$. Hence by the monotonicity of $\tilde{h}$ we have

$$-\Delta_p \psi_2 + \sigma_\lambda G(\psi_2) = \tilde{h}(z) \le \tilde{h}(\psi_2) = \lambda h(\psi_2) + \sigma_\lambda G(\psi_2),$$

which implies that $\psi_2$ is a subsolution of the nonsingular problem (5) such that $\psi_2 > 0$ in $\Omega$. Since $h(u) \le \frac{f(u)}{u^\beta}$ for all $u \ge 0$, we have $-\Delta_p \psi_2 \le \lambda h(\psi_2) \le \lambda \frac{f(\psi_2)}{\psi_2^\beta}$, showing that $\psi_2$ is a positive subsolution of our singular problem (1). Therefore, for

$$\frac{d^{1+\beta}}{f(d)}\frac{A^{p-1}N}{(p-1)^{p-1}R^p} < \lambda < \min\left\{\frac{d^\beta}{f(d)}\frac{N}{R^p}\left(\frac{p}{p-1}\right)^{p-1} b^{p-1},\ \frac{a^{\beta+p-1}}{f(a)}\frac{1}{\|w\|_\infty^{\beta+p-1}}\right\}$$

we obtain a positive subsolution $\psi_2$ and a positive supersolution $\phi_2$ be such that $\psi_2 \not\leq \phi_2$.

From the proof of Theorem 1.2 we note that we have a sufficiently small positive subsolution $\psi_1$ such that $\psi_1 \leq \phi_2$ and a sufficiently large positive supersolution $\phi_1$ such that $\psi_2 \leq \phi_1$. Hence, there exist a positive solution $u_1$ of (1) such that $\psi_1 \leq u_1 \leq \phi_2$ and a positive solution $u_2$ of (1) such that $\psi_2 \leq u_2 \leq \phi_1$. Since $\psi_2 \not\leq \phi_2$, we have $u_1 \neq u_2$. Therefore, there exist at least two positive solutions of (1) for $\lambda \in (\lambda_*, \lambda^*)$ and Theorem 1.3 is proven. □

## 3. Proof of Theorem 1.6 and Theorem 1.7

*Proof of Theorem* 1.6. We construct a positive supersolution $(\phi_1, \bar{\phi_1})$ of (4).

If both $f$ and $g$ are bounded, let $(\phi_1, \bar{\phi_1}) = (\lambda M_\lambda w, \lambda M_\lambda w)$ and choose $M_\lambda$ so large that $M_\lambda^{p-1} \geq \frac{1}{\lambda^{p-2}} \max\{\|f\|_\infty, \|g\|_\infty\}$. Then for $M_\lambda \gg 1$ we have

$$-\Delta_p \phi_1 = \lambda^{p-1} M_\lambda^{p-1} \frac{1}{w^\beta} \geq \lambda \frac{\|f\|_\infty}{w^\beta} \geq \lambda \frac{f(\lambda M_\lambda w)}{(\lambda M_\lambda w)^\beta} = \lambda \frac{f(\bar{\phi_1})}{{\phi_1}^\beta}$$

and

$$-\Delta_p \bar{\phi_1} = \lambda^{p-1} M_\lambda^{p-1} \frac{1}{w^\beta} \geq \lambda \frac{\|g\|_\infty}{w^\beta} \geq \lambda \frac{g(\lambda M_\lambda w)}{(\lambda M_\lambda w)^\beta} = \lambda \frac{g(\phi_1)}{\bar{\phi_1}^\beta},$$

showing that $(\phi_1, \bar{\phi_1})$ is a positive supersolution of (4). Suppose that $g(x) \to \infty$ as $x \to \infty$, let $(\phi_1, \bar{\phi_1}) = \left(M_\lambda w, \lambda^{\frac{1}{\beta+p-1}} g(M_\lambda \|w\|_\infty)^{\frac{1}{\beta+p-1}} w\right)$. Then by (G2), we can choose $M_\lambda$ large so that

$$\frac{f\left(\lambda^{\frac{1}{\beta+p-1}} \|w\|_\infty g(M_\lambda \|w\|_\infty)^{\frac{1}{\beta+p-1}}\right)}{(M_\lambda \|w\|_\infty)^{\beta+p-1}} \leq \frac{1}{\lambda \|w\|_\infty^{\beta+p-1}}.$$

Then we have

$$\begin{aligned}
-\Delta_p \phi_1 &= \frac{M_\lambda^{p-1}}{w^\beta} \\
&\geq \lambda \frac{f\left(\lambda^{\frac{1}{\beta+p-1}} \|w\|_\infty g(M_\lambda \|w\|_\infty)^{\frac{1}{\beta+p-1}}\right)}{(M_\lambda w)^\beta} \\
&\geq \lambda \frac{f\left(\lambda^{\frac{1}{\beta+p-1}} g(M_\lambda \|w\|_\infty)^{\frac{1}{\beta+p-1}} w\right)}{(M_\lambda w)^\beta} \\
&= \lambda \frac{f(\bar{\phi_1})}{{\phi_1}^\beta}.
\end{aligned}$$

We also have

$$\begin{aligned}
-\Delta_p \bar{\phi}_1 &= \lambda^{\frac{p-1}{\beta+p-1}} g(M_\lambda \|w\|_\infty)^{\frac{p-1}{\beta+p-1}} \frac{1}{w^\beta} \\
&= \lambda \frac{g(M_\lambda \|w\|_\infty)}{\lambda^{\frac{\beta}{\beta+p-1}} g(M_\lambda \|w\|_\infty)^{\frac{\beta}{\beta+p-1}} w^\beta} \\
&\geq \lambda \frac{g(M_\lambda w)}{\left(\lambda^{\frac{1}{\beta+p-1}} g(M_\lambda \|w\|_\infty)^{\frac{1}{\beta+p-1}} w\right)^\beta} \\
&= \lambda \frac{g(\phi_1)}{\bar{\phi_1}^\beta},
\end{aligned}$$

showing that $(\phi_1, \bar{\phi}_1)$ is a supersolution of (4). (If $g$ is bounded and $f(x) \to \infty$ as $x \to \infty$, then $\lim_{x\to\infty} \frac{g(Mf(x))}{x^{\beta+p-1}} = 0$ for all $M > 0$ and we can prove that $(\phi_1, \bar{\phi}_1) = \left(\lambda^{\frac{1}{\beta+p-1}} f(M_\lambda \|w\|_\infty)^{\frac{1}{\beta+p-1}} w, M_\lambda w\right)$ is a supersolution of (4)).

Now, we construct a positive subsolution $(\psi_1, \bar{\psi}_1)$ of (4). Let $e$ and $\lambda_1$ be as in the proof of Theorem 1.2. Since $\lim_{x\to 0} \frac{f(0)}{x^\beta} = \infty = \lim_{x\to 0} \frac{g(0)}{x^\beta}$, there exist sufficiently small $m_\lambda$ and $m'_\lambda$ such that

$$\lambda_1 (m_\lambda e)^{p-1} \leq \lambda \frac{f(0)}{(m_\lambda e)^\beta} \quad \text{and} \quad \lambda_1 (m'_\lambda e)^{p-1} \leq \lambda \frac{g(0)}{(m'_\lambda e)^\beta}.$$

Let $(\psi_1, \bar{\psi}_1) = (m_\lambda e, m'_\lambda e)$. Since $f$ and $g$ are nondecreasing, we have

$$-\Delta_p \psi_1 = \lambda_1 (m_\lambda e)^{p-1} \leq \lambda \frac{f(0)}{(m_\lambda e)^\beta} \leq \lambda \frac{f(m'_\lambda e)}{(m_\lambda e)^\beta} = \lambda \frac{f(\bar{\psi}_1)}{\psi_1^\beta}$$

and

$$-\Delta_p \bar{\psi}_1 = \lambda_1 (m'_\lambda e)^{p-1} \leq \lambda \frac{g(0)}{(m'_\lambda e)^\beta} \leq \lambda \frac{g(m_\lambda e)}{(m'_\lambda e)^\beta} = \lambda \frac{g(\psi_1)}{\bar{\psi_1}^\beta}.$$

Thus $(\psi_1, \bar{\psi}_1)$ is a positive subsolution of (4), and if $m_\lambda$ and $m'_\lambda$ are sufficiently small then $(\psi_1, \bar{\psi}_1) \leq (\phi_1, \bar{\phi}_1)$. Hence Theorem 1.6 is proven. □

*Proof of Theorem* 1.7. We construct a second positive supersolution $(\phi_2, \bar{\phi}_2)$ of (4) when $\lambda \leq \frac{a^{\beta+p-1}}{g(a)} \frac{1}{\|w\|_\infty^{\beta+p-1}}$. Let $(\phi_2, \bar{\phi}_2) = \left(a\frac{w}{\|w\|_\infty}, a\frac{w}{\|w\|_\infty}\right)$. Since $\lambda \leq \frac{1}{\|w\|_\infty^{\beta+p-1}} \frac{a^{\beta+p-1}}{g(a)}$ and $g(x) \geq f(x)$ for all $x \geq 0$, we have

$$-\Delta_p \phi_2 = \frac{a^{p-1}}{\|w\|_\infty^{p-1}} \frac{1}{w^\beta} \geq \lambda \frac{g(a)}{\left(a\frac{w}{\|w\|_\infty}\right)^\beta} \geq \lambda \frac{f\left(a\frac{w}{\|w\|_\infty}\right)}{\left(a\frac{w}{\|w\|_\infty}\right)^\beta} = \lambda \frac{f(\bar{\phi_2})}{\phi_2^\beta}$$

and

$$-\Delta_p \bar{\phi_2} = \frac{a^{p-1}}{\|w\|_\infty^{p-1}} \frac{1}{w^\beta} \geq \lambda \frac{g(a)}{\left(a \frac{w}{\|w\|_\infty}\right)^\beta} \geq \lambda \frac{g\left(a \frac{w}{\|w\|_\infty}\right)}{\left(a \frac{w}{\|w\|_\infty}\right)^\beta} = \lambda \frac{g(\phi_2)}{\bar{\phi_2}^\beta}.$$

Hence, $(\phi_2, \bar{\phi_2})$ is a positive supersolution of (4) with $\|\phi_2\|_\infty = a$ and $\|\bar{\phi_2}\|_\infty = a$ when $\lambda \leq \frac{a^{\beta+p-1}}{g(a)} \frac{1}{\|w\|_\infty^{\beta+p-1}}$.

Now, when $\frac{d^{\beta+p-1}}{f(d)} \frac{A^{p-1}N}{(p-1)^{p-1}R^p} < \lambda < \frac{d^\beta}{f(d)} \frac{N}{R^p} \left(\frac{p}{p-1}\right)^{p-1} b^{p-1}$, we construct a second positive subsolution $(\psi_2, \bar{\psi_2})$ of (4). Let $h, \rho, v, \psi, z$ and consequently $\psi_2$ be as defined in the proof of Theorem 1.3. We note that $\psi_2 > 0$ in $\Omega$ and for this range of $\lambda$ it satisfies

$$\begin{cases} -\Delta_p \psi_2 \leq \lambda \dfrac{f(\psi_2)}{\psi_2^\beta} & \text{in } \Omega \\ \psi_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Now choosing $\bar{\psi_2} = \psi_2$, we have

$$-\Delta_p \psi_2 \leq \lambda \frac{f(\psi_2)}{\psi_2^\beta} = \lambda \frac{f(\bar{\psi_2})}{\psi_2^\beta}$$

and

$$-\Delta_p \bar{\psi_2} \leq \lambda \frac{f(\bar{\psi_2})}{\bar{\psi_2}^\beta} \leq \lambda \frac{g(\psi_2)}{\bar{\psi_2}^\beta}$$

since $f(u) \leq g(u)$ for all $u \geq 0$. Hence, $(\psi_2, \bar{\psi_2})$ is a positive subsolution of (4), when $\frac{d^{\beta+p-1}}{f(d)} \frac{A^{p-1}N}{(p-1)^{p-1}R^p} < \lambda < \frac{d^\beta}{f(d)} \frac{N}{R^p} \left(\frac{p}{p-1}\right)^{p-1} b^{p-1}$. Therefore, we obtain a positive supersolution $(\phi_2, \bar{\phi_2})$ and a positive subsolution $(\psi_2, \bar{\psi_2})$ such that for

$$\frac{d^{\beta+p-1}}{f(d)} \frac{A^{p-1}N}{(p-1)^{p-1}R^p} < \lambda < \min\left\{ \frac{d^\beta}{f(d)} \frac{N}{R^p} (\frac{p}{p-1})^{p-1} b^{p-1},\ \frac{1}{\|w\|_\infty^{\beta+p-1}} \frac{a^{\beta+p-1}}{g(a)} \right\},$$

$(\psi_2, \bar{\psi_2}) \not\leq (\phi_2, \bar{\phi_2})$.

From the proof of Theorem 1.6 we note that we have a sufficiently small positive subsolution $(\psi_1, \bar{\psi_1})$ such that $(\psi_1, \bar{\psi_1}) \leq (\phi_2, \bar{\phi_2})$ and a sufficiently large positive supersolution $(\phi_1, \bar{\phi_1})$ such that $(\psi_2, \bar{\psi_2}) \leq (\phi_1, \bar{\phi_1})$. Hence, there exist a positive solution $(u_1, \bar{u}_1)$ of (4) such that $(\psi_1, \bar{\psi_1}) \leq (u_1, \bar{u}_1) \leq (\phi_2, \bar{\phi_2})$ and a positive solution $(u_2, \bar{u}_2)$ of (4) such that $(\psi_2, \bar{\psi_2}) \leq (u_2, \bar{u}_2) \leq (\phi_1, \bar{\phi_1})$. Since $(\psi_2, \bar{\psi_2}) \not\leq (\phi_2, \bar{\phi_2})$, we have $(u_1, \bar{u}_1) \neq (u_2, \bar{u}_2)$. Therefore, there exist at least two positive solutions of (4) for $\lambda \in (\lambda_*, \lambda^*)$ and Theorem 1.7 is proven. □

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