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A Note on Homogenization of Advection-Diffusion Problems with Large Expected Drift

Patrick Henning and Mario Ohlberger

Abstract. This contribution is concerned homogenization of linear advection-diffusion problems with rapidly oscillating coefficient functions and large expected drift. Even though the homogenization of this type of problems is generally well known, there are several details that have not yet been treated explicitly or even not been treated at all. Here, we will have a special look at uniqueness, regularity, boundedness and equivalent formulations of the homogenized equation. In particular, we generalize results of Allaire and Raphael [C. R. Math. Acad. Sci. Paris 344 (2007)(8), 523 – 528] and Donato and Piatnitski [Multi Scale Problems and Asymptotic Analysis. Tokyo: Gakkōtosho 2006, pp. 153 – 165]. The results obtained in this contribution are of special interest for the numerical analysis of multi-scale schemes to approximate the analytic solutions.

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1. Introduction

This work concerning linear advection-diffusion problems with rapidly oscillating coefficient functions and a large expected drift, is devoted to the homogenization of problems of the kind

$$\begin{aligned} k\left(t,\frac{x}{\epsilon}\right)\partial_t u^\epsilon - \nabla\cdot\left(A\left(t,\frac{x}{\epsilon}\right)\nabla u^\epsilon\right) + \epsilon^{-1} b\left(t,\frac{x}{\epsilon}\right)\cdot\nabla u^\epsilon = 0 \quad \text{in } \mathbb{R}^d\times(0,T) \\ u^\epsilon(\cdot,0) = v_0 \quad \text{in } \mathbb{R}^d, \end{aligned} \tag{1}$$

P. Henning, M. Ohlberger: Institut für Numerische und Angewandte Mathematik, Fachbereich Mathematik und Informatik, Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany;
patrick.henning@uni-muenster.de; mario.ohlberger@uni-muenster.de

where the coefficients are assumed to be periodic in space. Here, ϵ denotes a very small parameter that should be regarded as a measure for the degree of fineness of the problem. Since all the results we achieve shall be used for a later numerical handling of this problem, we are especially concerned with questions of uniqueness, regularity, boundedness and equivalent formulations of the homogenized problem.

Equations of type (1) have a variety of applications such as reservoir displacement problems, the modeling of semi-conductor devices, polymer chemistry and especially the field dealing with models for transport of solutes in groundwater and surface water, where the process takes place in a porous medium.

The original interest behind equation (1) is the treatment of advection-diffusion-reaction problems with rapidly oscillating coefficient functions of the following type:

$$\begin{aligned} \partial_t \tilde{u^\epsilon} - \nabla\cdot\left(\tilde{A}\left(t,\frac{x}{\epsilon}\right)\nabla\tilde{u^\epsilon}\right) + \epsilon^{-1}\tilde{b}\left(t,\frac{x}{\epsilon}\right)\cdot\nabla\tilde{u^\epsilon} + \epsilon^{-2}\tilde{c}\left(t,\frac{x}{\epsilon}\right)\tilde{u^\epsilon} &= 0 \quad \text{in } \mathbb{R}^d\times(0,T) \\ \tilde{u^\epsilon}(0,\cdot) &= \tilde{v}_0 \quad \text{in } \mathbb{R}^d. \end{aligned} \tag{2}$$

Here the scaling corresponds to the standard ratio of Péclet and Damköhler numbers, where the period has a linear influence on the Péclet number and a quadratic influence on the Damköhler number (see for instance [6]). If the coefficient functions are independent of t, Allaire and Raphael [2, 4] show that, by means of so-called spectral cell problems, equation (2) can be transformed to a simple advection-diffusion problem with a divergence-free velocity field b. These types of equations are covered by problem (1). The transformation itself can be determined easily by solving the first spectral cell problem:

$$-\nabla_y\cdot(A(y)\nabla_y W_1) + b(y)\cdot\nabla_y W_1 + c(y)W_1 = \lambda_1 W_1 \quad \text{in } [0,1]^d, \ W_1 \text{ is } [0,1]^d\text{-periodic.}$$

Here λ_1 denotes the common first eigenvalue of the problem. After a normalization, the following relation between u^ϵ and $\tilde{u^\epsilon}$ holds true:

$$u^\epsilon(t,x) = e^{\lambda_1 t\epsilon^{-2}}\frac{\tilde{u^\epsilon}(t,x)}{W_1(\frac{x}{\epsilon})}.$$

In this case, the additional coefficient function k is a result of the described transformation, which can be stated easily in terms of the cell problem solution W_1 and the corresponding solution of the adjoint cell problem. Using this important result, we directly draw our focus on the observation of problem (1), since this also includes type (2) equations.

In general, the homogenization of such a problem is well known. The case with $c = 0$, $k = 1$ and $\int_Y b = 0$ has been for instance observed by Majda and Kramer in 1999 [10], whereas the more general case with nonlinear b has been treated by Marušić-Paloka and Piatnitski in 2005 [11] by means of a modified version of the two-scale convergence. Donato and Piatnitski [7] and independently Allaire and Raphael [2,4] (for porous media) were finally concerned with the case of advection-diffusion-reaction problems, where neither the restriction $\int_Y b = 0$ nor $\nabla \cdot b = 0$ was needed. To homogenize the equation, the cited authors use a factorization principle and the method of two-scale convergence. For a non-perforated medium, the very general case with all coefficient functions being allowed to vary also on the macro-scale was treated by Allaire and Orive in 2007 [3].

Besides all these mayor results, there are still several details about the case covered by problem (1) which have not yet been treated explicitly. Nevertheless, these questions concerning regularity, boundedness, the properties of the macro-problem and in particular uniqueness of the solutions in the homogenized two-scale problem are important for the numerical analysis of discretization schemes of such equations. Therefore, this contribution is engaged with the homogenization of problem (1) including time-dependent coefficients and the additional coefficient function k. For the homogenization we use the method of two-scale convergence with drift, introduced in [11] and later on used by Allaire and Raphael [4]. In this contribution we are in particular interested in the properties of the two-scale Cauchy problem, such as its structure and uniqueness of its solution. Moreover, we are concerned with some minor problems, produced by the occurrence of k^ϵ. With regard to a later numerical treatment, we prove and state regularity and boundedness results for the solutions u_0 and u_1 of the homogenized problem. On the basis of the two-scale equation we will be able to state an alternative proof for the homogenized macro-problem, which is of the type

$$\partial_t u_0 - \nabla \cdot (\bar{A} \nabla u_0) = 0 \quad \text{in} \ (0,T) \times \mathbb{R}^d$$
$$u_0(0,\cdot) = v_0 \ \text{in} \ \mathbb{R}^d.$$

Using this proof, we are able to obtain boundedness and especially coercivity of $\bar{A}(t)$ in a straight forward way.

The article is structured into three parts. In Section 2 we introduce several important assumptions and definitions. In Section 3 we state all the major results. This includes the derivation of the two-scale homogenized equation of problem (1), regularity and boundedness results, as well as the derivation of the homogenized macro problem. Proofs of the major theorems are finally given in Section 4.

2. General definitions and assumptions

Notation: Throughout the paper, we will sometimes use the notation $\int_\Omega f(\cdot)$ instead of $\int_\Omega f(x)dx$. This is done for the sake of readability and to avoid that expressions become too long. Note that this is only done, if the integration variable can be identified from the context.

For our analysis we introduce the following function spaces:

Definition 2.1 (Function spaces)**.** For $0 \leq m < \infty$, $1 \leq p < \infty$ and for any $Y' = \prod_{i=0}^{d}(a_i, b_i) \subset \mathbb{R}^d$ with $a_i < b_i$, we define:

$$\begin{aligned} C_\sharp^0(Y') &:= \{\phi \in C^0(Y') |\ \phi \text{ is } Y'\text{-periodic}\} \\ H_\sharp^{m,p}(Y') &:= \overline{C_\sharp^0(Y')}^{\|\cdot\|_{H^{m,p}(Y')}}, \\ \tilde{H}_\sharp^1(Y') &:= \left\{ v \in H_\sharp^{1,2}(Y') \;\middle|\; \fint_{Y'} v(y)\,dy = 0 \right\}. \end{aligned}$$

For $Y = (0,1)^d$ we furthermore define:

$$\begin{aligned} I &:= H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d, H_\sharp^1(Y)) \\ I_0 &:= H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d, \tilde{H}_\sharp^1(Y)) \\ X^0(0,T) &:= L^2(0,T; H^1(\mathbb{R}^d)) \times L^2((0,T) \times \mathbb{R}^d, \tilde{H}_\sharp^1(Y)) \\ X^1(0,T) &:= H^1(0,T; H^1(\mathbb{R}^d)) \times L^2((0,T) \times \mathbb{R}^d, \tilde{H}_\sharp^1(Y)). \end{aligned}$$

The semi-norm $|\cdot|_{L^2(\Omega, H^k(Y))}$ on the Bochner-space $L^2(\Omega, H^k(Y))$ is given by $|\Phi|_{L^2(\Omega,H^k(Y))} := \left(\int_\Omega |\Phi(x,\cdot)|^2_{H^k(Y)} \right)^{\frac{1}{2}}$.

For the coefficient functions of the advection-diffusion problem (1) we pose the following assumptions:

Assumption 2.2 (General analytical assumptions)**.** To assure existence and uniqueness of the solutions, we assume that $A \in \left(H^{1,\infty}(0,T; L_\sharp^\infty(Y))\right)^{d\times d}$ is an uniformly coercive matrix with corresponding ellipticity constant $\alpha > 0$, i.e.,

$$A(t,y)\xi \cdot \xi \geq \alpha|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^d \text{ and for almost every } (t,y) \in (0,T) \times Y.$$

Furthermore, we assume for the velocity field $b \in \left(H^{1,\infty}(0,T; H_\sharp^{1,\infty}(Y)\right)^d$ and that $b(t,\cdot)$ is divergence-free almost everywhere in $(0,T)$, i.e., $\nabla \cdot b(\cdot,t) = 0$.

In problem (1) the first coefficient function k takes a specific role, as it is the result of a transformation (see Introduction). Due to that transformation, k has certain properties. Since we make use of these properties, we state the following assumptions.

Assumption 2.3 (Assumptions on k)**.** We assume $k \in H^{1,\infty}(0,T; L_\sharp^\infty(Y))$,

$$\fint_Y k(t,\cdot) = 1 \quad \text{for almost every } t \in (0,T), \tag{3}$$

and that there exist constants $m, M \in \mathbb{R}$ such that

$$0 < m \le k(t,y) \le M < \infty, \quad \text{for almost every } (t,y) \in (0,T) \times Y. \tag{4}$$

Note that (3) is a normalization property and therefore not a strong assumption. Any of the following results can be stated (with slight modifications) without assumption (3).

In the following we use a generalized definition of the two-scale convergence. Since we expect a large drift beside the typical fine-scale oscillations, the test-functions in the original two-scale convergence are replaced by new test-functions which are expected to be in resonance with u^ϵ. The following formulation was initially introduced by Marušić-Paloka and Piatnitski [11]:

Definition 2.4 (Two-scale convergence with drift)**.** Let $B \in H^1(0,T)^d$ be a given drift, $(u^\epsilon)_{\epsilon>0}$ a sequence in $L^2((0,T)\times\mathbb{R}^d)$ and $u_0 \in L^2((0,T)\times\mathbb{R}^d \times Y)$. Then we say u^ϵ is two-scale convergent with drift B to u_0, if

$$\lim_{\epsilon\to 0} \int_0^T \int_{\mathbb{R}^d} u^\epsilon(t,x)\Phi\left(t, x - \frac{B(t)}{\epsilon}, \frac{x}{\epsilon}\right) dx\, dt = \int_0^T \int_{\mathbb{R}^d} \int_Y u_0(t,x,y)\Phi(t,x,y)\, dy\, dx\, dt$$

for all functions $\Phi \in L^2((0,T)\times\mathbb{R}^d; C^0_\sharp(Y))$.

This definition directly implies that the sequence $u^\epsilon\big(t, x + \frac{B(t)}{\epsilon}\big)$ converges to $\int_Y u_0(t,x,y)\, dy$ weakly in $L^2(\mathbb{R}^d \times (0,T))$. This suggests to use $\int_Y u_0\big(t, x - \frac{B(t)}{\epsilon}, y\big)\, dy$ as an approximation of u^ϵ. To see that even strong convergence can be expected, we refer to [3,4,11] for corresponding statements and theorems. A complete proof of an associated compactness result for the two-scale convergence with drift can be found in [5].

3. Homogenization of advection-diffusion equations with drift

In this section we state all major results of this contribution concerning the homogenization of advection-diffusion problems with drift. We start with the derivation of a two-scale homogenized limit equation for sequences of solutions u^ϵ of the Cauchy problem (1). We also show, that the limit equation admits a unique solution $(u_0, u_1) \in L^2(0,T; H^1(\mathbb{R}^d)) \times L^2((0,T)\times\mathbb{R}^d; \tilde{H}^1_\sharp(Y))$. A first regularity result for the two-scale homogenized solution (u_0, u_1) is then given in Proposition 3.3. By introducing suitable elliptic cell problems, we then see that the two-scale homogenized equation is equivalent to solving these cell problems in combination with a macro-scale equation for u_0. Further regularity results are derived from this equivalent formulation and properties of the macro-scale equation are given.

The first main theorem gives the convergence of sequences of solutions u^ϵ of the Cauchy problem (1) towards a two-scale homogenized equation with drift.

Theorem 3.1 (Two-scale homogenized equation with drift)**.** *Let* $(u^\epsilon)_{\epsilon>0}$ *be the sequence of solutions of Problem* (1) *and* $B(t) := \int_0^t \int_Y b(y,s)\,dy\,ds$ *the corresponding drift velocity we refer to. Defining* $\overline{b}(t) := \fint_Y b(y,t)\,dy$*, there exist functions* $u_0 \in L^2(0,T;H^1(\mathbb{R}^d)) \cap \left(H^{\frac{1}{2}}(0,T;L^2(\mathbb{R}^d)) + H^1(0,T;H^{-1}(\mathbb{R}^d))\right)$ *and* $u_1 \in L^2((0,T)\times\mathbb{R}^d;\tilde{H}^1_\sharp(Y))$

such that we have for a subsequence of u^ϵ*:*

$$\begin{aligned} u^\epsilon &\to u_0 && \textit{two-scale with drift } B(t) \textit{ and} \\ \nabla u^\epsilon &\to \nabla_x u_0 + \nabla_y u_1 && \textit{two-scale with drift } B(t). \end{aligned}$$

Moreover $(u_0,u_1) \in L^2(0,T;H^1(\mathbb{R}^d)) \times L^2((0,T)\times\mathbb{R}^d;\tilde{H}^1_\sharp(Y))$ *is the unique solution of the following homogenized problem:*

$$-\int_0^T (u_0, \partial_t \Phi_0)_{L^2(\mathbb{R}^d)} + \int_0^T E(\cdot)((u_0,u_1),(\Phi_0,\phi_1)) = (v_0, \Phi_0(0,\cdot))_{L^2(\mathbb{R}^d)}, \quad (5)$$

for all $\Phi_0 \in H^1(0,T;H^1(\mathbb{R}^d)$ *with* $\Phi_0(T,\cdot)=0$*;* $\phi_1 \in L^2\left((0,T)\times\mathbb{R}^d, H^1_\sharp(Y)\right)$*. Here, the parameter-dependend bilinearform* $E \in C^{0,1}([0,T],\mathcal{L}(I,I'))$ *by*

$$\begin{aligned} E(t)\left((u_0,u_1),(\Phi_0,\phi_1)\right) := &\int_{\mathbb{R}^d} \overline{b}(t)\cdot\nabla_x\Phi_0 \left(\int_Y k u_1\right) - \int_{\mathbb{R}^d}\int_Y \left(b(t,\cdot)\cdot\nabla_x\Phi_0\right) u_1 \\ &+ \int_{\mathbb{R}^d}\int_Y A(t,\cdot)\left(\nabla_x u_0 + \nabla_y u_1\right)\cdot\left(\nabla_x\Phi_0 + \nabla_y\phi_1\right) \\ &- \int_{\mathbb{R}^d} \overline{b}(t)\cdot\nabla_x u_0 \left(\int_Y k\phi_1\right) \\ &+ \int_{\mathbb{R}^d}\int_Y b(t,\cdot)\cdot\left(\nabla_x u_0 + \nabla_y u_1\right)\phi_1, \end{aligned}$$

where $\overline{b}(t)$ *denotes the average of* b *over* Y*.*

A detailed proof of Theorem 3.1 is given in Section 4. The proof is based on a generalized compactness theorem of Marušić-Paloka and Piatnitski [11] for sequences of bounded functions in $L^2(0,T;H^1(\mathbb{R}^d))$. The proof of the theorem also includes the uniqueness of solutions of the two-scale homogenized equations. Here it is important to note, that the bilinear form E is Lipschitz-continuous which follows from our assumptions $A \in \left(H^{1,\infty}(0,T;L^\infty(Y))\right)^{d\times d}$ and $b \in \left(H^{1,\infty}(0,T;H^{1,\infty}(Y))\right)^d$.

Remark 3.2 (Homogenization for general k)**.** If there is no such restriction as $k(t,\cdot)$ having average 1, the drift-velocity B needs to be generalized to

$$B(t) := \int_0^t \frac{\int_Y b(s,y)\,dy}{\int_Y k(s,y)\,dy}\,ds.$$

Then a similar result to Theorem 3.1 can be derived.

For completeness, we now state a general regularity results for solutions (u^0, u^1) of the two-scale homogenized equation.

Proposition 3.3 (Regularity of the homogenized equation)**.** *If the initial value v_0 belongs to the class $H^1(\mathbb{R}^d)$, we have*

$$u_0 \in H^1(0,T;H^1(\mathbb{R}^d)) \cap L^2(0,T;H^2(\mathbb{R}^d))$$
$$u_1 \in L^2\big(0,T;H^1(\mathbb{R}^d,\tilde{H}^1_\sharp(Y))\big)$$

and we rewrite problem (5) *to:*
Find $(u_0,u_1) \in H^1(0,T;H^1(\mathbb{R}^d)) \times L^2((0,T)\times\mathbb{R}^d, H^1_\sharp(Y))$*, such that*

$$\int_0^T (\partial_t u_0, \Phi_0)_{L^2(\mathbb{R}^d)} + \int_0^T E(\cdot)\left((u_0,u_1),(\Phi_0,\phi_1)\right) = 0 \tag{6}$$

for all $(\Phi_0,\phi_1) \in L^2(0,T;H^1(\mathbb{R}^d)) \times L^2((0,T)\times\mathbb{R}^d, H^1_\sharp(Y))$ *and* $u_0(0,\cdot) = v_0$ *in* $\mathbb{R}^d$*.*

If furthermore $A \in H^{1,\infty}(0,T;H^{1,\infty}_\sharp(Y))$ *then we even have*

$$u_1 \in L^2\big(0,T;H^1(\mathbb{R}^d,\tilde{H}^1_\sharp(Y))\big) \cap L^2((0,T)\times\mathbb{R}^d, H^2(Y)).$$

Proof. To prove the time regularity, i.e., $u_0 \in H^1(0,T;H^1(\mathbb{R}^d))$, one can proceed (on the basis of Theorem 3.1) in analogy to the proofs of the regularity theorems in [13], for the case of standard linear parabolic equations. The space regularity is obtained in analogy to the well known elliptic case. □

Next, we are concerned with the so called homogenized macro problem. In comparison to the two-scale equation where the microscopic behaviour is included by the fine-scale corrector u^1, in the macro problem this special behaviour will be accounted by the homogenized coefficient function $\bar{A}$. $\bar{A}$ will be defined in terms of the solutions w_i of a number of cell problems. These cell problems will take the role of fine-scale corrections, which is why u^1 can be expressed in dependence of these solutions. Moreover, we comment on the regularity of the w_i since it enables us to conclude on the regularity of u^1.

Definition 3.4 (Cell problems)**.** For $1 \le i \le d$, we call $w_i \in L^2(0,T;\tilde{H}^1_\sharp(Y))$ the solution of the i'th *cell problem*, if

$$\begin{aligned}&\int_Y A(t,y)\,(e_i+\nabla_y w_i(t,y))\cdot\nabla_y\phi(y)\,dy + \int_Y b(t,y)\cdot(e_i+\nabla_y w_i(t,y))\,\phi(y)\,dy\\ &= \int_Y k(t,y)(\bar{b}(t)\cdot e_i)\phi(y)\,dy, \quad \text{for all } \phi\in\tilde{H}^1_\sharp(Y).\end{aligned} \tag{7}$$

Remark 3.5. By means of the cell problems 3.4 we see that u_1 can be expressed as

$$u_1(t,x,y) = \sum_{i=1}^{d} \partial_{x_i} u_0(t,x) w_i(t,y). \tag{8}$$

Multiplying equation (7) with $\partial_{x_i} u_0(t,x)$ and summing up afterwards, immediately yields this relation.

Remark 3.6. Since the cell problem (7) implies that w_i solves a standard elliptic problem on the whole $\mathbb{R}^d$ with regular coefficient functions, we immediately have $w_i(t,\cdot) \in H^2(Y)$ and $|w_i|_{L^2(0,T;H^2(Y))} \leq C$, where C only depends on the coefficient functions. In particular this implies:

$$|u_1|_{L^2((0,T)\times\mathbb{R}^d,H^2(Y))} \leq C|u_0|_{L^2(0,T;H^1(\mathbb{R}^d)} \leq C\|v_0\|_{L^2(\mathbb{R}^d)}.$$

Since the solution of the cell problem (7) inherits the regularity of the coefficients, we even have $w_i \in C^{0,1}([0,T],H^1(Y))$. In particular we get

$$\|w_i\|_{H^{1,\infty}(0,T;H^1(Y))} \leq C,$$

where C only depends on A, b, k and its corresponding Lipschitz constants.

Corollary 3.7. *If $v_0 \in H^1(\mathbb{R}^d)$, we have that $u_1 \in H^1\big(0,T;L^2(\mathbb{R}^d,H^1(Y))\big)$.*

Proof. If $v_0 \in H^1(\mathbb{R}^d)$, Proposition 3.3 implies $u_0 \in H^1(0,T;H^1(\mathbb{R}^d))$. By Remark 3.6 we have $w_i \in H^{1,\infty}([0,T],H^1(Y))$. Using the identity (8) we finish the proof. □

Theorem 3.8 (Macro problem)**.** *Let the entries of the matrix $\bar{A}$ be defined by*

$$\bar{A}_{ij}(t) := \int_Y A(t,\cdot)\left(e_i + \nabla_y w_i(t,\cdot)\right)\cdot\left(e_j + \nabla_y w_j(t,\cdot)\right). \tag{9}$$

Then we have that u_0 is a weak solution of the following macro problem:

$$\begin{aligned} \partial_t u_0 - \nabla\cdot(\bar{A}\nabla u_0) &= 0 \quad \text{in } (0,T)\times\mathbb{R}^d \\ u_0(0,\cdot) &= v_0 \quad \text{in } \mathbb{R}^d. \end{aligned}$$

Moreover we have for $\bar{A}(t)$:

- *coercivity uniformly in t:*

$$\int_{\mathbb{R}^d} \bar{A}(t)\nabla\Phi(x)\cdot\nabla\Phi(x) \geq \alpha|\Phi|^2_{H^1(\mathbb{R}^d)} \quad \forall\ \Phi \in H^1(\mathbb{R}^d),$$

- *boundedness: $\bar{A} \in (H^{1,\infty}(0,T))^{d\times d}$ and in particular*

$$\|\bar{A}\|_{H^{1,\infty}(0,T)} \leq C, \tag{10}$$

  where C only depends on A, b, k and its corresponding Lipschitz constants.

The proof of this theorem is given at the end of Section 4.

4. Proofs of Theorem 3.1 and Theorem 3.8

The existence of the homogenized two-scale equation is derived via the following compactness result of Marušić-Paloka and Piatnitski [11] that guarantees two-scale convergence with drift up to a subsequence:

Theorem 4.1 (Generalized compactness theorem). *Let $(u^\epsilon)_{\epsilon>0}$ be a sequence in $L^2(0,T;H^1(\mathbb{R}^d))$. If there exists some $C \geq 0$ independent of ϵ with*

$$\|u^\epsilon\|_{L^2(0,T;H^1(\mathbb{R}^d))} \leq C,$$

then for any $B \in H^1(0,T)^d$, there exist functions $u_0 \in L^2(0,T;H^1(\mathbb{R}^d))$ and $u_1 \in L^2((0,T)\times\mathbb{R}^d;H^1_\sharp(Y))$ such that, up to a subsequence

$$\begin{aligned} u^\epsilon &\to u_0 && \text{two-scale with drift } B \text{ and} \\ \nabla u^\epsilon &\to \nabla_x u_0 + \nabla_y u_1 && \text{two-scale with drift } B. \end{aligned}$$

A detailed proof of the compactness result was given by Allaire (see [5]). In order to apply this theorem, we need boundedness of u^ϵ. An associated result is given in the next lemma.

Lemma 4.2 (Boundedness). *There exists a constant $C \geq 0$, independent of ϵ, such that $\|u^\epsilon\|_{L^2(0,T;H^1(\mathbb{R}^d))} \leq C$.*

Remark 4.3. In general it is not possible to show the corresponding boundedness of $\partial_t u^\epsilon$ independent of ϵ. This is a natural consequence, since in non-trivial cases a large drift is expected. Such drifts typically result in very large temporal gradients depending on $\frac{1}{\epsilon}$. Therefore, the sequence $\partial_t u^\epsilon$ will be unbounded in L^2.

Proof of Lemma 4.2. We have for all $\Phi \in H^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} k^\epsilon(t,\cdot)\partial_t u^\epsilon(t,\cdot)\Phi + \int_{\mathbb{R}^d} A^\epsilon(t,\cdot)\nabla u^\epsilon(t,\cdot)\cdot\nabla\Phi + \frac{1}{\epsilon}\int_{\mathbb{R}^d} b^\epsilon(t,\cdot)\cdot\nabla u^\epsilon(t,\cdot)\Phi = 0$$

almost everywhere in t. Without loss of generality, we assume that u^ϵ is sufficiently regular, i.e., $u^\epsilon \in C^1([0,T], H^1(\mathbb{R}^d))$. The general case is obtained by density arguments. Testing with $\Phi = \phi u^\epsilon$, where the function $\phi \in C^1[0,T]$ with $\phi \geq 0$ is given by $\phi(t) := e^{-ct}$, and $c := \frac{1}{m}\|\frac{\mathrm{d}}{\mathrm{dt}}k\|_{L^\infty(0,T;L^\infty(Y))}$, we get almost everywhere in t

$$\begin{aligned} &\int_{\mathbb{R}^d} k^\epsilon(t,\cdot)\partial_t u^\epsilon(t,\cdot)\phi(t)u^\epsilon(t,\cdot) + \int_{\mathbb{R}^d} A^\epsilon(t,\cdot)\nabla u^\epsilon(t,\cdot)\cdot\nabla\left(\phi(t)u^\epsilon(t,\cdot)\right) \\ &\quad + \frac{1}{\epsilon}\int_{\mathbb{R}^d} b^\epsilon(t,\cdot)\cdot\nabla u^\epsilon(t,\cdot)\left(\phi(t)u^\epsilon(t,\cdot)\right) \\ &= 0. \end{aligned}$$

Since $\min_t \phi(t) = \phi(T)$, we furthermore have

$$\int_{\mathbb{R}^d} \alpha |\nabla u^\epsilon(t,\cdot)|^2 e^{-cT} \le \int_{\mathbb{R}^d} A^\epsilon(t,\cdot)\nabla u^\epsilon(t,\cdot)\cdot\nabla\left(\phi(t)u^\epsilon(t,\cdot)\right).$$

Moreover, since b is divergence-free (and therefore also b^ϵ), we get

$$\begin{aligned}\int_{\mathbb{R}^d} b^\epsilon(t,\cdot)\cdot\nabla u^\epsilon(t,\cdot)\phi(t)u^\epsilon(t,\cdot) &= \int_{\mathbb{R}^d}\frac{1}{2}b^\epsilon(t,\cdot)\cdot\nabla\left(u^\epsilon(t,\cdot)^2\right)\phi(t)\\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\operatorname{div} b^\epsilon(t,\cdot)u^\epsilon(t,\cdot)^2\phi(t)\\ &= 0.\end{aligned}$$

This yields $\int_{\mathbb{R}^d}\frac{1}{2}k^\epsilon(t,\cdot)\phi(t)\frac{\mathrm{d}}{\mathrm{dt}}\left(u^\epsilon(t,\cdot)^2\right) + \int_{\mathbb{R}^d}\alpha|\nabla u^\epsilon(t,\cdot)|^2 e^{-cT} \le 0$ and

$$\int_{\mathbb{R}^d}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(k^\epsilon(t,\cdot)\phi(t)u^\epsilon(t,\cdot)^2\right) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(k^\epsilon(t,\cdot)\phi(t)\right)\, u^\epsilon(t,\cdot)^2 + \int_{\mathbb{R}^d}\alpha|\nabla u^\epsilon(t,\cdot)|^2 e^{-cT} \le 0.$$

With the definition of ϕ we see that

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(k^\epsilon\phi\right) = \left(\frac{\mathrm{d}}{\mathrm{dt}}k^\epsilon\right)\phi + k^\epsilon\phi' \le \left\|\frac{\mathrm{d}}{\mathrm{dt}}k\right\|_{L^\infty(Y_T)}\phi + m\phi' = 0.$$

Hence, we get $\frac{1}{2}\int_{\mathbb{R}^d}\frac{\mathrm{d}}{\mathrm{dt}}\left(k^\epsilon(t,\cdot)\phi(t)\right)u^\epsilon(t,\cdot)^2 \le 0$. All in all we obtain:

$$\int_{\mathbb{R}^d}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(k^\epsilon(t,\cdot)\phi(t)u^\epsilon(t,\cdot)^2\right) + \int_{\mathbb{R}^d}\alpha|\nabla u^\epsilon(t,\cdot)|^2 e^{-cT} \le 0.$$

Let c_1 be defined as $c_1 := \alpha\phi(T)$. Then we have by integration for arbitrary $t \in [0,T]$

$$\int_{\mathbb{R}^d}\frac{1}{2}k^\epsilon(t,\cdot)\phi(t)u^\epsilon(t,\cdot)^2 + c_1\int_0^t |u^\epsilon(s,\cdot)|^2_{H^1(\mathbb{R}^d)}\,ds \le \int_{\mathbb{R}^d}\frac{1}{2}k^\epsilon(0,\cdot)v_0(\cdot)^2. \quad (11)$$

Since we have the inequality for all t, we get $c_1\int_0^T|u^\epsilon(t,\cdot)|^2_{H^1(\mathbb{R}^d)}\,dt \le \int_{\mathbb{R}^d}\frac{1}{2}k^\epsilon(0,\cdot)v_0(\cdot)^2$ and since both summands in (11) are positive, we obtain

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d}\frac{1}{2}k^\epsilon(t,x)\phi(t)u^\epsilon(t,x)^2\,dt\,dx + c_1\int_0^T|u^\epsilon(t,\cdot)|^2_{H^1(\mathbb{R}^d)}\,dt\\ &\le \int_{\mathbb{R}^d}\frac{T+1}{2}k^\epsilon(0,x)v_0(x)^2\,dx.\end{aligned}$$

With $c_2 := \frac{1}{2}m\phi(T)$, we finally have

$$c_2\int_0^T\|u^\epsilon(t,\cdot)\|_{L^2(\mathbb{R}^d)}\,dt + c_1\int_0^T|u^\epsilon(t,\cdot)|^2_{H^1(\mathbb{R}^d)}\,dt \le \int_{\mathbb{R}^d}\frac{T+1}{2}k^\epsilon(0,\cdot)v_0(\cdot)^2.$$

Since k is bounded, this ends the proof. □

In order to pass to the two-scale limit in the weak formulation of the advection-diffusion equation (1), we need a result on the convergence of products of sequences of oscillating functions. Note that this unproblematic. If v^ϵ denotes a sequence of functions in $L^2((0,T)\times\mathbb{R}^d)$, which fulfills $v^\epsilon \to v$ two-scale with drift $B(t)$, then we have the following convergence for all $g \in L^\infty(0,T;L^\infty_\sharp(Y))$ and for all $\Phi \in C_0^\infty\left((0,T)\times\mathbb{R}^d, C_\sharp^\infty(Y)\right)$:

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^T\int_{\mathbb{R}^d} v^\epsilon(t,x)\,g\left(t,\frac{x}{\epsilon}\right)\Phi\left(t,x-\frac{B(t)}{\epsilon},\frac{x}{\epsilon}\right)dt\,dx\\ &=\int_0^T\int_{\mathbb{R}^d}\int_Y v(t,x,y)g(t,y)\Phi(t,x,y)\,dt\,dx\,dy.\end{aligned} \tag{12}$$

See for instance [1] for similar results without drift.

We are now prepared to prove Theorem 3.1. In the proof, we use the density of $C_0^\infty(\mathbb{R}^d)$ in $H^1(\mathbb{R}^d)$. Hence, we can work with test functions $\Phi \in C_0^\infty\left((0,T)\times\mathbb{R}^d, C_\sharp^\infty(Y)\right)$. Since the coefficient functions and their corresponding derivatives belong to $L^\infty(0,T;L^\infty_\sharp(Y))$, we apply (12) to pass to the limit in terms like

$$\int_0^T\int_{\mathbb{R}^d} A^\epsilon(t,x)\nabla u^\epsilon(t,x)\cdot\nabla\Phi\left(t,x-\frac{B(t)}{\epsilon},\frac{x}{\epsilon}\right)dt\,dx.$$

In the following proof of Theorem 3.1, this will be done without mentioning.

Proof of Theorem 3.1. To prove the first part of theorem 3.1, i.e., existence of the homogenized limit problem, we will proceed similar as in [11] (testing with functions of the kind $\Phi_0\big(t,x-\frac{B(t)}{\epsilon}\big)+\epsilon\phi_1\big(t,x-\frac{B(t)}{\epsilon},\frac{x}{\epsilon}\big)$ and forming the corresponding limits).

Let $(u^\epsilon)_{\epsilon>0}$ be the sequence of solutions of Problem (1). We assume that $u^\epsilon \in L^2(0,T;H^1(\mathbb{R}^d))\cap H^1(0,T,H^{-1}(\mathbb{R}^d))$ since this is the natural space of solutions. By means of Theorem 4.1, we are now able to extract a subsequence of $(u^\epsilon)_{\epsilon>0}$ such that

$$u^\epsilon \to u_0 \quad \text{two-scale with drift} \tag{13}$$

$$\nabla u^\epsilon \to \nabla_x u_0 + \nabla_y u_1 \quad \text{two-scale with drift.} \tag{14}$$

Assume that the functions Φ_0 and ϕ_1 are smooth:

$$\Phi_0 \in C_0^\infty\left((0,T)\times\mathbb{R}^d\right) \tag{15}$$

$$\phi_1 \in C_0^\infty\left((0,T)\times\mathbb{R}^d, C_\sharp^\infty(Y)\right). \tag{16}$$

Defining $\Omega_T := (0,T)\times\mathbb{R}^d$ and $\Phi^\epsilon(t,x) := \Phi_0\big(t, x-\frac{B(t)}{\epsilon}\big) + \epsilon\phi_1\big(t, x-\frac{B(t)}{\epsilon}, \frac{x}{\epsilon}\big)$ we have

$$\underbrace{\int_{\Omega_T} k^\epsilon \partial_t u^\epsilon \Phi^\epsilon}_{=:\mathrm{I}} + \underbrace{\int_{\Omega_T} A^\epsilon \nabla u^\epsilon \nabla \Phi^\epsilon}_{=:\mathrm{II}} + \underbrace{\frac{1}{\epsilon}\int_{\Omega_T} (b^\epsilon\cdot\nabla u^\epsilon)\,\Phi^\epsilon}_{=:\mathrm{III}} = 0.$$

We start with I and split the term again:

$$\begin{aligned}\mathrm{I} = &\underbrace{\int_{\Omega_T} k\Big(t,\frac{x}{\epsilon}\Big)\partial_t u^\epsilon(t,x)\Phi_0\left(t, x-\frac{B(t)}{\epsilon}\right)\,dt\,dx}_{=:\mathrm{I}_1}\\ &+\underbrace{\epsilon\int_{\Omega_T} k\Big(t,\frac{x}{\epsilon}\Big)\partial_t u^\epsilon(t,x)\phi_1\left(t, x-\frac{B(t)}{\epsilon},\frac{x}{\epsilon}\right)\,dt\,dx}_{=:\mathrm{I}_2}.\end{aligned}$$

Since Φ_0 and ϕ_1 have compact supports in $(0,T)$, we get

$$\begin{aligned}\mathrm{I}_1 = &-\int_{\Omega_T}\frac{\mathrm{d}}{\mathrm{dt}}\left(k\Big(t,\frac{x}{\epsilon}\Big)-1\right)u^\epsilon(t,x)\Phi_0\left(t,x-\frac{B(t)}{\epsilon}\right)\,dt\,dx\\ &-\int_{\Omega_T}\left(k\Big(t,\frac{x}{\epsilon}\Big)-1\right)u^\epsilon(t,x)\frac{\mathrm{d}}{\mathrm{dt}}\Phi_0\left(t,x-\frac{B(t)}{\epsilon}\right)\,dt\,dx\\ &-\int_{\Omega_T}u^\epsilon(t,x)\frac{\mathrm{d}}{\mathrm{dt}}\Phi_0\left(t,x-\frac{B(t)}{\epsilon}\right)\,dt\,dx\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{I}_1 = &-\int_{\Omega_T}\partial_t k\Big(t,\frac{x}{\epsilon}\Big)u^\epsilon(t,x)\Phi_0\left(t,x-\frac{B(t)}{\epsilon}\right)\,dt\,dx\\ &-\int_{\Omega_T}\left(k\Big(t,\frac{x}{\epsilon}\Big)-1\right)u^\epsilon(t,x)\\ &\times\left(\partial_t\Phi_0\left(t,x-\frac{B(t)}{\epsilon}\right)-\frac{B'(t)}{\epsilon}\cdot\nabla_x\Phi_0\left(t,x-\frac{B(t)}{\epsilon}\right)\right)dt\,dx\\ &-\int_{\Omega_T}u^\epsilon(t,x)\left(\partial_t\Phi_0\left(t,x-\frac{B(t)}{\epsilon}\right)-\frac{B'(t)}{\epsilon}\cdot\nabla_x\Phi_0\left(t,x-\frac{B(t)}{\epsilon}\right)\right)dt\,dx.\end{aligned}$$

Two of these terms need some further considerations since a possible convergence is not trivial. These are

$$\mathrm{i}_1 := \int_{\Omega_T}\left(k\Big(t,\frac{x}{\epsilon}\Big)-1\right)u^\epsilon(t,x)\left(\frac{B'(t)}{\epsilon}\cdot\nabla_x\Phi_0\left(t,x-\frac{B(t)}{\epsilon}\right)\right)dt\,dx \qquad (17)$$

and

$$\mathrm{i}_2 := \int_{\Omega_T} u^\epsilon(t,x) \left(\frac{B'(t)}{\epsilon} \cdot \nabla_x \Phi_0 \left(t, x - \frac{B(t)}{\epsilon} \right) \right) dt\, dx. \tag{18}$$

We will see that (17) converges, whereas (18) will neutralise a corresponding term which is part of the summand II.

Defining $k^*(t,y) := k(t,y) - 1$, we start with observing (17). Note that k^* is periodic and has zero average. Therefore, there exists some $K^* \in L^2(0,T; H^1_\sharp(Y))^d$ with

$$\begin{aligned} &\nabla_y \cdot K^*(t,\cdot) = k^*(t,\cdot) := k(t,\cdot) - 1 \quad \text{and} \\ &\int_Y K^*(t,\cdot) = 0. \end{aligned} \tag{19}$$

This implies that we have $\nabla \cdot \left(K^*\left(t, \frac{x}{\epsilon}\right)\right) = \frac{1}{\epsilon} k^*\left(t, \frac{x}{\epsilon}\right)$ and therefore

$$\begin{aligned} \mathrm{i}_1 =& \int_{\Omega_T} \nabla \cdot \left(K^*\left(t, \frac{x}{\epsilon}\right)\right) u^\epsilon(t,x) \left(\bar{b}(t) \cdot \nabla_x \Phi_0 \left(t, x - \frac{B(t)}{\epsilon} \right) \right) dt\, dx \\ =& - \int_{\Omega_T} \left(K^*\left(t, \frac{x}{\epsilon}\right) \cdot \nabla u^\epsilon(t,x) \right) \left(\bar{b}(t) \cdot \nabla_x \Phi_0 \left(t, x - \frac{B(t)}{\epsilon} \right) \right) dt\, dx \\ & - \int_{\Omega_T} u^\epsilon(t,x) K^*\left(t, \frac{x}{\epsilon}\right) \cdot \nabla \left(\bar{b}(t) \cdot \nabla_x \Phi_0 \left(t, x - \frac{B(t)}{\epsilon} \right) \right) dt\, dx. \end{aligned}$$

Note that $B'(t) = \bar{b}(t)$. With (13) and (14) we see now that

$$\begin{aligned} \mathrm{i}_1 \to& - \int_{\Omega_T} \int_Y K^*(t,y) \cdot (\nabla_x u_0(t,x) + \nabla_y u_1(t,x,y)) \left(\bar{b}(t) \cdot \nabla_x \Phi_0(t,x) \right) dt\, dx\, dy \\ & - \int_{\Omega_T} \int_Y u_0(t,x) K^*(t,y) \cdot \nabla_x \left(\bar{b}(t) \cdot \nabla_x \Phi_0(t,x) \right) dt\, dx\, dy \\ \overset{(19)}{=}& - \int_{\Omega_T} \int_Y K^*(t,y) \cdot \nabla_y u_1(t,x,y) \left(\bar{b}(t) \cdot \nabla_x \Phi_0(t,x) \right) dt\, dx\, dy \\ =& \int_{\Omega_T} \bar{b}(t) \cdot \nabla_x \Phi_0(t,x) \left(\int_Y \nabla_y \cdot K^*(t,y)\, u_1 \right) dt\, dx\, dy \\ =& \int_{\Omega_T} \bar{b}(t) \cdot \nabla_x \Phi_0(t,x) \left(\int_Y (k(t,y) - 1)\, u_1(t,x,y) \right) dt\, dx\, dy. \end{aligned}$$

I_2 is treated as follows:

$$\begin{aligned}
&\epsilon \int_{\Omega_T} k\Big(t, \frac{x}{\epsilon}\Big) \partial_t u^\epsilon(t,x) \phi_1\Big(t, x - \frac{B(t)}{\epsilon}, \frac{x}{\epsilon}\Big)\, dt\, dx \\
&= \int_{\Omega_T} k\Big(t, \frac{x}{\epsilon}\Big) u^\epsilon(t,x) \left(\overline{b}(t) \cdot \nabla_x \phi_1\Big(t, x - \frac{B(t)}{\epsilon}, \frac{x}{\epsilon}\Big)\right) dt\, dx + \mathrm{O}(\epsilon) \\
&\to \int_{\Omega_T} \int_Y k(t,y) u_0(t,x) \left(\overline{b}(t) \cdot \nabla_x \phi_1(t,x,y)\right) dy\, dt\, dx \\
&= -\int_{\Omega_T} \left(\overline{b}(t) \cdot \nabla_x u_0(t,x)\right) \left(\int_Y k(t,y) \phi_1(t,x,y)\, dy\right) dt\, dx.
\end{aligned}$$

For II we directly obtain with Definition 2.4:

$$\mathrm{II} \to \int_{\Omega_T} \int_Y A \left(\nabla_x u_0 + \nabla_y u_1\right) \cdot \left(\nabla_x \Phi_0 + \nabla_y \phi_1\right).$$

III can be separated by using the assumption that b is divergence-free:

$$\begin{aligned}
\mathrm{III} = &\underbrace{-\frac{1}{\epsilon} \int_{\Omega_T} b\Big(t, \frac{x}{\epsilon}\Big) \cdot \nabla \Phi_0\Big(t, x - \frac{B(t)}{\epsilon}\Big) u^\epsilon\, dt\, dx}_{=:\mathrm{III}_1} \\
&+ \underbrace{\int_{\Omega_T} b\Big(t, \frac{x}{\epsilon}\Big) \cdot \nabla u^\epsilon(t,x) \phi_1\Big(t, x - \frac{B(t)}{\epsilon}, \frac{x}{\epsilon}\Big)\, dt\, dx}_{=:\mathrm{III}_2}.
\end{aligned}$$

III_2 obviously converges:

$$\begin{aligned}
&\int_{\Omega_T} b\Big(t, \frac{x}{\epsilon}\Big) \cdot \nabla u^\epsilon(t,x) \phi_1\Big(t, x - \frac{B(t)}{\epsilon}, \frac{x}{\epsilon}\Big)\, dt\, dx \\
&\to \int_{\Omega_T} \int_Y b(t,y) \cdot \left(\nabla_x u_0(t,x) + \nabla_y u_1(t,x,y)\right) \phi_1(t,x,y)\, dt\, dx\, dy.
\end{aligned}$$

III_1 is treated together with (18): For our purpose we define $b^*(t,y) := \overline{b}(t) - b(t,y)$. With this definition, $b^*(t,\cdot)$ has zero-average and for any component b_i^* of b^*, there exists some $B_i^* \in L^2(0,T; H^1_\sharp(Y))^d$ with

$$\nabla_y \cdot B_i^*(t,\cdot) = b_i^*(t,\cdot) \quad \text{and} \tag{20}$$

$$\int_Y B_i^*(t,\cdot) = 0. \tag{21}$$

All in all we get:

$$\begin{aligned}
\mathrm{III}_1+\mathrm{i}_2 &= \frac{1}{\epsilon}\int_{\Omega_T} u^\epsilon(t,x) b^*\Big(t,\frac{x}{\epsilon}\Big)\cdot\nabla\Phi_0\Big(t,x-\frac{B(t)}{\epsilon}\Big)\,dt\,dx \\
&= \sum_{i=1}^d \int_{\Omega_T} u^\epsilon(t,x)\nabla\left(B_i^*\Big(t,\frac{x}{\epsilon}\Big)\right)\partial_{x_i}\Phi_0\Big(t,x-\frac{B(t)}{\epsilon}\Big)\,dt\,dx \\
&= -\sum_{i=1}^d \int_{\Omega_T}\Big(\nabla u^\epsilon(t,x)\cdot B_i^*\Big(t,\frac{x}{\epsilon}\Big)\Big)\,\partial_{x_i}\Phi_0\Big(t,x-\frac{B(t)}{\epsilon}\Big)\,dt\,dx \\
&\quad -\sum_{i=1}^d \int_{\Omega_T} u^\epsilon(t,x)\left(B_i^*\Big(t,\frac{x}{\epsilon}\Big)\cdot\nabla\left(\partial_{x_i}\Phi_0\Big(t,x-\frac{B(t)}{\epsilon}\Big)\right)\right)dt\,dx \\
&\to -\sum_{i=1}^d \int_{\Omega_T}\int_Y \big((\nabla_x u_0(t,x)+\nabla_y u_1(t,x,y))\cdot B_i^*(t,y)\big)\partial_{x_i}\Phi_0(t,x)\,dt\,dx\,dy \\
&\quad -\sum_{i=1}^d \int_{\Omega_T}\int_Y u_0(t,x)\big(B_i^*(t,y)\cdot\nabla_x\left(\partial_{x_i}\Phi_0(t,x)\right)\big)\,dt\,dx\,dy \\
&\overset{(21)}{=} -\sum_{i=1}^d \int_{\Omega_T}\int_Y (\nabla_y u_1(t,x,y)\cdot B_i^*(t,y))\,\partial_{x_i}\Phi_0(t,x)\,dt\,dx\,dy \\
&\overset{(20)}{=} \sum_{i=1}^d \int_{\Omega_T}\int_Y u_1(t,x,y) b_i^*(t,y)\partial_{x_i}\Phi_0(t,x)\,dt\,dx\,dy \\
&= \int_{\Omega_T}\int_Y u_1(t,x,y)\left((\bar b(t)-b(t,y))\cdot\nabla\Phi_0(t,x)\right)\,dt\,dx\,dy.
\end{aligned}$$

Combining the various terms yields

$$-\int_0^T (u_0,\partial_t\Phi_0)_{L^2(\mathbb{R}^d)} + \int_0^T E(\cdot)\big((u_0,u_1),(\Phi_0,\phi_1)\big) = 0$$

for all Φ_0 fulfilling (15) and any ϕ_1 fulfilling (16). If we assume that

$$\Phi_0 \in C^\infty\left([0,T];\left(C^\infty(\mathbb{R}^d)\cap H^1(\mathbb{R}^d)\right)\right) \tag{22}$$

with $\Phi_0(T,\cdot)=0$, we get some additional terms that can be treated analogously, since they have trivial limits. In this case we obtain by density:

$$-\int_0^T (u_0,\partial_t\Phi_0)_{L^2(\mathbb{R}^d)} + \int_0^T E(\cdot)\big((u_0,u_1),(\Phi_0,\phi_1)\big) = (v_0,\Phi_0(0))_{L^2(\mathbb{R}^d)}, \tag{23}$$

for all $\Phi_0 \in H^1(0,T;H^1(\mathbb{R}^d))$, with $\Phi_0(T,\cdot)=0$, $\phi_1 \in L^2\left((0,T)\times\mathbb{R}^d, H^1_\sharp(Y)\right)$. By this construction we conclude that the partial differential equation (23) has at least one solution $(u_0, u_1) \in L^2(0,T; H^1(\mathbb{R}^d)) \times L^2((0,T) \times \mathbb{R}^d; \tilde{H}^1_\sharp(Y))$. This ends the proof of existence.

To show uniqueness we need to verify that Problem (23) with $v_0 = 0$ has only the trivial solution. Therefore, we take a sequence $u_0^k \in C_0^\infty\left(0,T; H^1(\mathbb{R}^d)\right)$ that converges strongly in $L^2(0,T; H^1(\mathbb{R}^d))$ to u_0. We define $F_k \in (X^1(0,T))'$ by

$$F_k(\Phi_0, \phi_1) := -\int_0^T (u_0^k, \partial_t \Phi_0)_{L^2(\mathbb{R}^d)} + \int_0^T E(\cdot)\big((u_0^k, u_1), (\Phi_0, \phi_1)\big)$$

where $(\Phi_0, \phi_1) \in X^1(0,T)$. F_k is in the dual space $(X^1(0,T))'$ with respect to the norm $\|\cdot\|_{X^1(0,T)}$, but since u_0^k has a compact support in $(0,T)$, we get

$$F_k(\Phi_0, \phi_1) = \int_0^T (\partial_t u_0^k, \Phi_0)_{L^2(\mathbb{R}^d)} + \int_0^T E(\cdot)\big((u_0^k, u_1), (\Phi_0, \phi_1)\big),$$

which implies, that F_k is also continuous with respect to the norm $\|\cdot\|_{X^0(0,T)}$. So we conclude that $F_k \in (X^1(0,T)', \|\cdot\|_{X^0(0,T)})$ and therefore the Hahn-Banach theorem applies. Since (u_0^k, u_1) converges strongly in $X^0(0,T)$ to (u_0, u_1), which fulfils (23), we have $F_k(\Phi_0, \phi_1) \to 0$, for $k \to \infty$. So F_k is weak-star convergent to zero in $X^1(0,T)'$. Since the Hahn-Banach theorem yields some extension $\bar{F}_k \in X^0(0,T)'$ of F_k, we also have for arbitrary $(\Phi_0, \phi_1) \in X^1(0,T)$:

$$\bar{F}_k(\Phi_0, \phi_1) \to 0, \text{ for } k \to \infty.$$

Since $X^1(0,T)$ is a dense subset of $X^0(0,T)$ with regard to the norm $\|\cdot\|_{X^0(0,T)}$, $\bar{F}_k$ is determined by these values and we conclude that $\bar{F}_k$ is weak-star convergent to zero in $X^0(0,T)'$. Together with the strong convergence of (u_0^k, u_1) in $X^0(0,T)$, we obtain

$$\bar{F}_k(u_0^k, u_1) \to 0, \text{ for } k \to \infty. \tag{24}$$

With this construction of $\bar{F}_k$ and the regularity of u_0^k, we have

$$\bar{F}_k(u_0^k, u_1) = \int_0^T (\partial_t u_0^k, u_0^k)_{L^2(\mathbb{R}^d)} + \int_0^T E(\cdot)\big((u_0^k, u_1), (u_0^k, u_1)\big).$$

With the definition of $E(t)$ and using that

$$(\partial_t u_0^k(t,\cdot), u_0^k(t,\cdot))_{L^2(\mathbb{R}^d)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|u_0^k(t,\cdot)\|^2_{L^2(\mathbb{R}^d)},$$

we get

$$\int_{\Omega_T}\int_Y \frac{1}{2}\partial_t(u_0^k)^2 + A\left(\nabla_x u_0^k + \nabla_y u_1\right)\cdot\left(\nabla_x u_0^k + \nabla_y u_1\right) + \frac{1}{2} b \cdot \nabla_y\big((u_1)^2\big) \leq |\bar{F}_k(u_0^k, u_1)|.$$

Using the coercivity of A, the orthogonality of $\nabla_x u_0^k$ and $\nabla_y u_1$ and the assumption that b is divergence-free we finally have:

$$c_0 \int_{\Omega_T} \int_Y |\nabla_x u_0^k|^2 + |\nabla_y u_1|^2 \leq |\bar{F}_k(u_0^k, u_1)|.$$

Passing to the limit with (24), we get $\int_{\Omega_T} \int_Y |\nabla_x u_0|^2 + |\nabla_y u_1|^2 \leq 0$. Since the sole constant function in $H^1(\mathbb{R}^d)$ is equal to zero, we deduce $u_0 \equiv 0$. On the other hand, u^1 needs to be constant in y with zero average. This also yields $u_1 \equiv 0$ and the uniqueness is proved.

It remains to show that $u_0 \in \left(H^{\frac{1}{2}}(0,T;L^2(\mathbb{R}^d)) + H^1(0,T;H^{-1}(\mathbb{R}^d))\right)$. Choose $\phi^1 = 0$ and fixing u^1, we see that u^0 solves the following problem:

$$-\int_0^T \int_{\mathbb{R}^d} u_0 \partial_t \Phi_0 + \int_0^T \int_{\mathbb{R}^d} \bar{A} \nabla_x u_0 \cdot \nabla_x \Phi_0 = \int_{\mathbb{R}^d} v_0 \Phi_0(0,\cdot) + F(\Phi_0),$$

where $F \in L^2(0,T,H^{-1}(\mathbb{R}^d))$ is given by

$$\begin{aligned} F(\Phi_0) := &-\int_0^T \int_{\mathbb{R}^d} \bar{b} \cdot \nabla_x \Phi_0 \left(\int_Y k u_1 \right) \\ &+ \int_0^T \int_{\mathbb{R}^d} \int_Y (b \cdot \nabla_x \Phi_0)\, k u_1 - \int_0^T \int_{\mathbb{R}^d} \int_Y A \nabla_y u_1 \cdot \nabla_x \Phi_0 \end{aligned}$$

and $\bar{A}$ the average of A in y: $\bar{A}(t) := \int_Y A(t,y)\, dy$. Using that $\bar{A} \in H^{1,\infty}(0,T)$, standard existence results for linear parabolic Cauchy problems (see for instance [12]) yield that such a type of equation has a unique solution in the space $L^2(0,T;H^1(\mathbb{R}^d)) \cap \left(H^{\frac{1}{2}}(0,T;L^2(\mathbb{R}^d)) + H^1(0,T;H^{-1}(\mathbb{R}^d))\right)$. □

Remark 4.4 (Uniqueness for general k)**.** The general case with k not having average 1 yields the term $\int_{\Omega_T} \int_Y k \partial_t u_0 \Phi_0$ instead of $\int_{\Omega_T} \partial_t u_0 \Phi_0$. Therefore, the proof of uniqueness is not completely analogous. In this case we proceed as in Lemma 4.2 by defining $\phi(t) := e^{-ct}$, where $c := \frac{1}{m} \left\| \frac{\mathrm{d}}{\mathrm{dt}} k \right\|_{L^\infty(0,T;L^\infty(Y))}$ and m given by (4). Testing with $\Phi_0 = u_0^k \phi$ and $\phi_1 = u_1 \phi$ will proof uniqueness.

Finally, it remains to prove Theorem 3.8.

Proof of Theorem 3.8. We define the matrix $\tilde{A} = \tilde{A}(t)$ by

$$\tilde{A} := \int_Y \left(A + A(D_y w)^\top + W_b\right),$$

where $D_y w$ denotes the Jacobian matrix of $w = (w_1, ..., w_d)$ and W_b is given by

$$W_b := \begin{pmatrix} (k\bar{b}_1 - b_1)w_1 & ... & (k\bar{b}_1 - b_1)w_d \\ \vdots & & \vdots \\ (k\bar{b}_d - b_d)w_1 & ... & (k\bar{b}_d - b_d)w_d \end{pmatrix}.$$

Later we will define $\bar{A} := \frac{1}{2}(\tilde{A} + \tilde{A}^T)$ to prove the theorem. First, we show the claims of Theorem 3.8 for $\tilde{A}$ instead of $\bar{A}$, i.e., u_0 is a weak solution of the macro problem

$$\begin{aligned} \partial_t u_0 - \nabla \cdot (\tilde{A} \nabla u_0) &= 0 \quad \text{in } (0,T) \times \mathbb{R}^d \\ u_0(0,\cdot) &= v_0 \quad \text{in } \mathbb{R}^d \end{aligned} \tag{25}$$

and $\tilde{A}$ is uniformly coercive and an element of $(H^{1,\infty}(0,T))^{d\times d}$. To prove this, we are using equation (6) with $\phi_1 = 0$ to obtain:

$$\begin{aligned} &\int_{\mathbb{R}^d} \partial_t u_0 \Phi_0 + \int_{\mathbb{R}^d} \bar{b} \cdot \nabla_x \Phi_0 \left(\int_Y k u_1 \right) \\ &- \int_{\mathbb{R}^d} \int_Y (b \cdot \nabla_x \Phi_0)\, u_1 + \int_{\mathbb{R}^d} \int_Y A \left(\nabla_x u_0 + \nabla_y u_1 \right) \cdot \nabla_x \Phi_0 \\ = {}& 0. \end{aligned}$$

Remark 3.5 implies that $\nabla_y u_1 = (D_y w)^\top \nabla_x u_0$, where $(D_y w)^\top$ denotes the transposed of the Jacobian matrix of $w = (w_1, ..., w_d)$. Further calculations yield

$$\int_Y (k\bar{b} - b) u_1 = \int_Y \begin{pmatrix} (k\bar{b}_1 - b_1) w_1 & \dots & (k\bar{b}_1 - b_1) w_d \\ \vdots & & \vdots \\ (k\bar{b}_d - b_d) w_1 & \dots & (k\bar{b}_d - b_d) w_d \end{pmatrix} \nabla_x u_0.$$

All in all we obtain that

$$E(t)((u_0, u_1), (\Phi, 0)) = \int_{\mathbb{R}^d} \tilde{A} \nabla u_0 \nabla \Phi \tag{26}$$

for all $\Phi \in H^1(\mathbb{R}^d)$. This proves that u_0 solves the macro problem (25).

The assertion $\tilde{A} \in (H^{1,\infty}(0,T))^{d\times d}$ and the boundedness by the data is a direct result of Remark 3.6. To prove the coercivity we take an arbitrary $\Phi_0 \in H^1(\mathbb{R}^d)$ and define the operator K by

$$K(\Phi_0)(t,x,y) := \sum_{i=1}^d \partial_{x_i} \Phi_0(t,x) w_i(t,y). \tag{27}$$

In analogy to (26) we get

$$\int_{\mathbb{R}^d} \tilde{A} \nabla \Phi_0 \cdot \nabla \Phi_0 = E(t)((\Phi_0, K(\Phi_0)), (\Phi_0, 0)). \tag{28}$$

Since w_i is the solution of the i'th cell problem (7), we obtain by multiplication with $\partial_{x_i} \Phi_0$, summation afterwards and testing with $K(\Phi_0)$:

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_Y A \left(\nabla_x \Phi_0 + \nabla_y K(\Phi_0) \right) \cdot \nabla_y K(\Phi_0) \\ &+ \int_{\mathbb{R}^d} \int_Y b \cdot \nabla_x \Phi_0 K(\Phi_0) - \int_{\mathbb{R}^d} \bar{b} \cdot \nabla_x \Phi_0 \left(\int_Y k K(\Phi_0) \right) \\ = {}& 0. \end{aligned} \tag{29}$$

Here we used $\int_Y (b \cdot \nabla_y K(\Phi_0)) K(\Phi_0) = 0$. Adding (29) to (28) we get:

$$\begin{aligned}\int_{\mathbb{R}^d} \tilde{A}\nabla\Phi_0 \cdot \nabla\Phi_0 &= E(t)((\Phi_0, \phi_1), (\Phi_0, 0)) \\ &= E(t)\big((\Phi_0, \phi_1), (\Phi_0, K(\Phi_0))\big) \\ &= \int_{\mathbb{R}^d} A\left(\nabla_x \Phi_0 + \nabla_y K(\Phi_0)\right) \cdot \left(\nabla_x \Phi_0 + \nabla_y K(\Phi_0)\right) \\ &\geq \alpha \left(\int_{\mathbb{R}^d} |\nabla_x \Phi_0|^2 + \int_{\mathbb{R}^d} |\nabla_x K(\Phi_0)|^2 \right) \\ &\geq \alpha \int_{\mathbb{R}^d} |\nabla_x \Phi_0|^2 .\end{aligned}$$

This proves the claims for $\tilde{A}$. Since $\tilde{A}$ is not symmetric, we define the matrix $\bar{A} := \frac{1}{2}\big(\tilde{A} + \tilde{A}^\top\big)$. $\bar{A}$ is still coercive since transposing a matrix does not change this quality. The boundedness in $H^{1,\infty}$ and in particular assertion (10) are immediately inherited from $\tilde{A}$. Since $\tilde{A}$ is independent of x, we use

$$\int_{\mathbb{R}^d} \tilde{A}\nabla v \cdot \nabla\Psi = \int_{\mathbb{R}^d} \tilde{A}^\top \nabla v \cdot \nabla\Psi \quad \forall\ v, \Psi \in H^1(\mathbb{R}^d)$$

to conclude that $\tilde{A}$ can be replaced by $\bar{A}$ in (25). It remains to show that $\bar{A}$ is given by (9). To do so we use the definition (27), to get for arbitrary $\Phi, \Psi \in H^1(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} \tilde{A}\nabla\Phi\nabla\Psi = E(t)\big((\Phi, K(\Phi)), (\Psi, K(\Psi))\big)$$

and

$$\int_{\mathbb{R}^d} \tilde{A}^\top \nabla\Phi \cdot \nabla\Psi = \int_{\mathbb{R}^d} \nabla\Phi \cdot \tilde{A}\nabla\Psi = E(t)\big((\Psi, K(\Psi)), (\Phi, K(\Phi))\big).$$

This implies

$$\begin{aligned}\int_{\mathbb{R}^d} \bar{A}\nabla\Phi \cdot \nabla\Psi &= \frac{1}{2}\Big(E(t)\big((\Phi, K(\Phi)), (\Psi, K(\Psi))\big) + E(t)\big((\Psi, K(\Psi)), (\Phi, K(\Phi))\big)\Big) \\ &= \int_{\mathbb{R}^d} \int_Y A\left(\nabla_x \Phi + \nabla_y K(\Phi)\right) \cdot \left(\nabla_x \Psi + \nabla_y K(\Psi)\right),\end{aligned}$$

which ends the proof. □

5. Conclusion and outlook

In this contribution we gave a survey of the homogenization of advection-diffusion problems with time-dependent coefficient functions. Several known results were restated, partially generalized and proved. Particularly, we treated the properties of the homogenized two-scale problem and the homogenized macro problem. Moreover we observed the regularity of the homogenized solutions as well as its boundedness. All the results stated in this work, were especially given for a future numerical treatment of the problem. In particular, in [9] we formulate a heterogeneous multiscale finite elements method (HMM) for parabolic problems with large expected drift and we see that it is equivalent to a discretization of the two-scale equation (6) by means of a Discontinuous Galerkin Time Stepping Method with quadrature. Using the proved regularity of the solutions and corresponding upper bounds, we are able to determine a-priori and a-posteriori error estimates for this newly introduced version of the HMM (see [8, 9]).

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