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Global Existence and Nonexistence of Solution to a Nonlinear Wave Equation

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Abstract. In this paper we focus on a nonlinear wave equation, we show that the solution blows up in finite time under certain conditions, and we obtain two results on the global existence of solution and large time behavior.

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1. Introduction

In this essay, we consider the following initial-boundary-value problem,

$$\begin{cases} u_{tt} - \Delta u + g(u_t) + |u_t|^{m-1}u_t = u|u|^{p-1}, & (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0, & (x,t) \in \Gamma \times (0,T) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{1}$$

where $p > 1$, $m \geq 1$, and $\Omega \subseteq R^n$ with smooth boundary. The function g satisfies the following properties:

$$\begin{cases} g : R \to R \text{ is a } C^1 \text{ function, nondecreasing, and } g(0) = 0 \\ sg(s) > 0, \text{ for all } s \neq 0 \\ \exists\ k_0, k_1, \text{ such that } k_0 s \leq |g(s)| \leq k_1|s|, \text{ for all } s \in R. \end{cases} \tag{2}$$

For (1), a special case with $g(u_t) = au_t$, $(a > 0)$ was considered in [4], where it is shown that the energy of the solution decays exponentially for $m > 2$.

Some similar equations were studied recently in [3, 5–7, 9]. In particular, in [7], Zhou established blow-up result and time decay rate for the following equation:

$$u_{tt} + a|u|^{m-1} - \phi\Delta u = b|u|^{p-1}u - \mu u, \quad x \in R^n,\ t > 0.$$

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In [2], it is proved that the solution to (1) exists globally as long as $\|\nabla u_0\|_{L^2}$ is small. The main purpose of this paper is to establish global nonexistence result and decay rate for (1) by using the argument and method in [7].

In Section 2, we recall some preliminary results about equation (1). In Section 3, we establish global nonexistence criteria, and we discuss global existence and large time behavior in Section 4.

2. Preliminary

First, let us recall the following local existence theorem:

Theorem 2.1 (Theorem 2.1 from [2]). *Assume that $m \geq 1$ and when $p > 1$, as $n = 1, 2$; when $1 < p \leq \frac{n}{n-2}$, as $n \geq 3$. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given, then the first equation of (1) has a unique solution $u \in C([0,T); H_0^1(\Omega))$, $u_t \in C([0,T); L^2(\Omega)) \cap L^m(\Omega \times [0,T))$, for some T small enough.*

Remark 2.2. We can use Galerkin's method to prove the result (see [1]).

The supremum of all T's for which the solution exists on $\Omega \times [0,T)$ is called the lifespan of the solution to (1). The lifespan is denoted by T^*, if $T^* = \infty$, we say the solution is global, while it is nonglobal if $T^* < \infty$. We say that the solution blows up in finite time.

Lemma 2.3. *Let $p > 1$, as $n = 1, 2$; $1 < p \leq \frac{n}{n-2}$, as $n \geq 3$. Then there exists a positive constant $C > 1$ depending only on Ω (C denotes a generic positive constant, which may be different from line to line), such that*

$$\|u\|_{L^{p+1}}^s \leq C\left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1}\right), \tag{3}$$

with $2 \leq s \leq p+1$, for any $u \in H_0^1(\Omega)$, if u is a solution constructed as in Theorem 2.1, then

$$\|u\|_{L^{p+1}}^s \leq C\left(|H(t)| + \|u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1}\right), \tag{4}$$

with $2 \leq s \leq p+1$ on $[0,T)$ where $H(t) = -E(t)$.

Proof. Suppose $\|u\|_{L^{p+1}} \leq 1$, by Sobolev embedding $\|u\|_{L^{p+1}} \leq C\|\nabla u\|_{L^2}$, then $\|u\|_{L^{p+1}}^s \leq \|u\|_{L^{p+1}}^2 \leq C^2\|\nabla u\|_{L^2}^2$. When $\|u\|_{L^{p+1}} > 1$, then $\|u\|_{L^{p+1}}^s \leq \|u\|_{L^{p+1}}^{p+1}$. So (3) and (4) follows from the definition of the energy corresponding to the solution. □

If we let $l(t) = \frac{1}{2}\|u(.,t)\|_{L^2}^2$, where u is a solution of problem (1). We can get the derivative of $l(t)$ with respect to time

$$l'(t) = \int_{R^n} u u_t \, dx, \tag{5}$$

which is well defined and one can get

$$l''(t) = \|u_t\|_{L^2}^2 - \|\nabla u\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1} - (u, u_t|u_t|^{m-1})_{L^2} - \int_{R^n} ug(u_t)\,dx, \qquad (6)$$

almost everywhere in $[0, T)$.

Now we define the energy $E(t)$ for (1)

$$E(t) = \frac{1}{2}(\|u_t(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2) - \frac{1}{p+1}\|u(t)\|_{L^{p+1}}^{p+1}. \qquad (7)$$

Using condition (2), one can compute directly that

$$\frac{d}{dt}E(t) = -\int_{\Omega} \big(\|u_t\|^{m+1} + u_t g(u_t)\big)\,dx \le 0. \qquad (8)$$

Inequality (8) tells us that the energy for the system is nonincreasing. We use $E(0)$ to denote the initial energy.

3. Global nonexistence

The first global nonexistence result for linear damping case, we can establish finite time blow up with nonpositive initial energy.

Theorem 3.1. *Suppose $p > 1$, as $n = 1, 2$; $1 < p \le \frac{n}{n-2}$, when $n \ge 3$. If $E(0) \le 0$, $\int_{R^n} u_0 u_1\,dx \ge 0$, then the corresponding solution blows up in finite time.*

Before going to the proof, we write down the following technique lemma.

Lemma 3.2. *Suppose that $\Psi(t)$ is a twice continuously differential satisfying*

$$\begin{cases} \Psi''(t) + \gamma\Psi'(t) \ge C_0(t+L)^{\beta}\Psi^{1+\alpha}(t), \quad t > 0,\ C_0 > 0,\ \alpha > 0. \\ \Psi(0) > 0,\ \Psi'(0) \ge 0. \end{cases}$$

where C_0, $L > 0$, $-1 < \beta \le 0$ are constants. Then $\Psi(t)$ blows up in finite time. Moreover the blow up time can be estimated explicitly.

Remark 3.3. The proof of this lemma is easy, for simplicity, we omit it here. One can see [8] for a similar proof.

Proof of Theorem 3.1. Now we consider $\Psi(t) = \frac{1}{2}\int_{R^n} u^2(x,t)\,dx$, one has $\Psi'(t) = \int_{R^n} uu_t\,dx$. From (5) and (6) we have

$$\begin{aligned}
\Psi''(t) &= \int_{R^n} |u_t|^2\,dx + \int_{R^n} uu_{tt}\,dx \\
&= \|u_t\|_{L^2}^2 - \|\nabla u\|_{L^2}^2 - \int_{R^n} ug(u_t)\,dx + \|u\|_{L^{p+1}}^{p+1} - \int_{R^n} uu_t\,dx \\
&= -2E(t) + 2\|u_t\|_{L^2}^2 + \frac{p-1}{p+1}\|u\|_{L^{p+1}}^{p+1} - \Psi'(t) - \int_{R^n} ug(u_t)\,dx \\
&\geq -2E(0) + \frac{p-1}{p+1}\|u\|_{L^{p+1}}^{p+1} - \Psi'(t) - C\Psi'(t) \quad \text{(by condition (2))} \\
&\geq -C_1\Psi'(t) + \frac{p-1}{p+1}\|u\|_{L^{p+1}}^{p+1},
\end{aligned}$$

where $C_1 = C+1$. By Hölder's inequality, we obtain

$$\int_{R^N} |u|^2\,dx \leq \left(\int_{R^N} |u|^{p+1}\right)^{\frac{2}{p+1}} \left(\int_{B(t+r)} 1\,dx\right)^{\frac{p-1}{p+1}},$$

where r satisfies $\operatorname{supp}(u_0,u_1) \subset B(r)$, $B(t+r)$ represents the ball with radius $t+r$. Therefore we have

$$\Psi''(t) \geq -C_1\Psi'(t) + \frac{p-1}{p+1} \cdot 2^{\frac{p+1}{2}} R_N^{\frac{1-p}{2}} (t+r)^{\frac{(1-p)N}{2}} \Psi(t)^{\frac{1+p}{2}},$$

where R_N denotes the volume of the unit sphere in R^N. Set $C = \frac{p-1}{p+1} \cdot 2^{\frac{p+1}{2}} R_N^{\frac{1-p}{2}}$, it is obviously that

$$\Psi(t) > 0, \quad \text{for all } t \geq 0; \quad \Psi'(0) = \int_{R^n} u_0u_1 \geq 0.$$

Then by Lemma 3.2, $\Psi(t)$ blows up in finite time. The proof is complete. □

The second blow up result is

Theorem 3.4. *Suppose $m > 1$ and $1 < p \leq \frac{n}{n-2}$, as $n \geq 2$ or $p > 1$, as $n = 2$. For $p \leq m$, the solution for* (1) *blows up in finite time if the initial energy is negative.*

Proof. By the definition $H(t) = -E(t)$, we have

$$0 < H(0) \leq H(t) \leq \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}. \tag{9}$$

We use the method of that in [7] and define $M(t) = H^{1-\alpha}(t) + \theta \int_{R^n} uu_t\,dx$, for $0 < \alpha < \frac{p-1}{2(p+1)}$ and θ can be determined later. We can compute

$$\begin{aligned} M'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \theta\|u_t\|_{L^2}^2 + \theta\|u\|_{L^{p+1}}^{p+1} - \theta\|\nabla u\|_{L^2}^2 \\ &\quad - \theta\int_{R^n} ug(u_t)\,dx - \theta\int_{R^n} u|u_t|^{m-1}u_t\,dx. \end{aligned}$$

By Young's inequality, we have

$$\int_{R^n} u|u_t|^{m-1}u_t\,dx \le \frac{\delta^{m+1}}{m+1}\|u\|_{L^{m+1}}^{m+1} + \frac{m}{m+1}\delta^{-\frac{m+1}{m}}\|u_t\|_{L^{m+1}}^{m+1}. \tag{10}$$

According to (10), it follows that

$$\begin{aligned} M'(t) &\ge (1-\alpha)H^{-\alpha}H'(t) - \frac{\theta\delta^{m+1}}{m+1}\|u\|_{L^{m+1}}^{m+1} - \frac{\theta m}{m+1}\delta^{-\frac{m+1}{m}}\|u_t\|_{L^{m+1}}^{m+1} \\ &\quad + \theta\|u_t\|_{L^2}^2 + \theta\|u\|_{L^{p+1}}^{p+1} - \theta\|\nabla u\|_{L^2}^2 - \theta\int_{R^n} ug(u_t)\,dx - \theta\int_{R^n} u|u_t|^{m-1}u_t\,dx \\ &\ge \left((1-\alpha)H^{-\alpha} - \theta\frac{m}{m+1}\delta^{-\frac{m+1}{m}}\right)H'(t) + \theta\|u_t\|_{L^2}^2 \\ &\quad + \theta\|u\|_{L^{p+1}}^{p+1} - \theta\|\nabla u\|_{L^2}^2 - \theta\int_{R^n} ug(u_t)\,dx - \frac{\theta\delta^{m+1}}{m+1}\|u\|_{L^{m+1}}^{m+1}. \end{aligned}$$

If we let $\delta^{-\frac{m+1}{m}} = KH^{-\alpha}$, i.e., $\delta^{m+1} = K^{-m}H^{\alpha m}$, $K > 0$ to be determined later. By (9) we obtain

$$H^{\alpha m}\|u\|_{L^{m+1}}^{m+1} \le C\left(\frac{1}{p+1}\right)^{\alpha m}\|u\|_{L^{p+1}}^{m+1+\alpha m(p+1)}. \tag{11}$$

Therefore, from (11), the following inequalities hold true:

$$\begin{aligned} M'(t) &\ge \left((1-\alpha) - \frac{\theta m}{m+1}K\right)H^{-\alpha}H'(t) + \theta(p+1)H(t) + \frac{\theta(p-1)}{2}\|\nabla u\|_{L^2}^2 \\ &\quad + \frac{\theta(p+3)}{2}\|u_t\|_{L^2}^2 - C\theta\int_{R^n} uu_t\,dx - \frac{\theta(\delta^{m+1})}{m+1}\|u\|_{L^{m+1}}^{m+1} \\ &\ge \left((1-\alpha) - \frac{\theta m}{m+1}K\right)H^{-\alpha}H'(t) + \theta(p+1)H(t) + \frac{\theta(p-1)}{2}\|\nabla u\|_{L^2}^2 \\ &\quad + \frac{\theta(p+3)}{2}\|u_t\|_{L^2}^2 - C\frac{\theta\delta^2}{2}\|u\|_{L^2}^2 - C\frac{\theta}{2\delta^2}\|u_t\|_{L^2}^2 \\ &\quad - \theta C_1K^{-m}\left(H(t) + \|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1}\right) \end{aligned}$$

and hence

$$M'(t) \geq \left((1-\alpha) - \frac{\theta m}{m+1}K\right)H^{-\alpha}H'(t) + \theta\left(\frac{p+7}{4} - C\frac{\delta^{-2}}{2} - C_1K^{-m}\right)\|u_t\|_{L^2}^2$$
$$+ \theta\left(\frac{p-1}{4} - C_1K^{-m}\right)\|\nabla u\|_{L^2}^2 + \theta\left(\frac{p+3}{2} - C_1K^{-m}\right)H(t)$$
$$+ \theta\left(\frac{p-1}{2(p+1)} - C\frac{\delta^2}{2} - C_1K^{-m}\right)\|u\|_{L^{p+1}}^{p+1}.$$

Letting K large enough, there exists a constant $C_2 > 0$ and $\frac{p-1}{2(p+1)} - \frac{C\delta^2}{2} - C_1K^{-m} \geq C_2$, where $C_1 = \frac{C}{(p+1)^{\alpha m}(m+1)}$. Then we choose θ so small that

$$1 - \alpha - \frac{\theta m}{m+1}K \geq 0, \quad \text{and} \quad M(0) = H^{1-\alpha}(0) + \theta\int_{R^n} u_0u_1\,dx > 0, \tag{12}$$

therefore, $M'(t) \geq \theta C_2\left(H(t) + \|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1}\right)$.

On the other hand, by Young's inequality with $s = \frac{2}{1-2\alpha} \leq p+1$, we get

$$\begin{aligned}\left|\int_{R^n} uu_t\,dx\right|^{\frac{1}{1-\alpha}} &\leq \|u\|_{L^2}^{\frac{1}{1-\alpha}}\|u_t\|_{L^2}^{\frac{1}{1-\alpha}}\\ &\leq C\|u\|_{L^{p+1}}^{\frac{1}{1-\alpha}}\|u_t\|_{L^2}^{\frac{1}{1-\alpha}}\\ &\leq C\left(\|u\|_{L^{p+1}}^s + \|u_t\|_{L^2}^2\right)\\ &\leq C\left(H(t) + \|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1}\right),\end{aligned}$$

then

$$\begin{aligned}M^{\frac{1}{1-\alpha}}(t) &= \left(H^{1-\alpha}(t) + \theta\int_{R^n} uu_t\,dx\right)^{\frac{1}{1-\alpha}}\\ &\leq 2^{\frac{1}{1-\alpha}}\left(H(t) + \left|\int_{R^n} uu_t\,dx\right|^{\frac{1}{1-\alpha}}\right)\\ &\leq C\left(H(t) + \|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1}\right).\end{aligned}$$

So it follows that $M'(t) \geq C_0M^{\frac{1}{1-\alpha}}(t)$, where C_0 is a constant depending on C, C_2 and θ. For (12), $M(t)$ goes to infinity as t tends to $\frac{1-\alpha}{C_0\alpha}M^{\frac{\alpha}{\alpha-1}}(0)$. □

4. Global existence and large time behavior

In order to establish the decay rate for a solution with positive initial energy, let us recall the lemma first:

Lemma 4.1 (Lemma 5.2 from [7]). *Let $\phi(t)$ be a nonincreasing and nonnegative function defined on $[0,T]$, $T>1$, satisfying $\phi^{1+r}(t)\le k_0(\phi(t)-\phi(t+1))$, for all $t\in[0,T]$, $k_0>1$ and $r\ge 0$. Then we have for each $t\in[0,T]$,*

$$\begin{cases} \phi(t)\le \phi(0)e^{-k(t-1)^+}, & r=0,\\ \phi(t)\le (\phi(0)^{-r}+k_0 r(t-1))^{-\frac{1}{r}}, & r>0,\end{cases}$$

where $(t-1)^+=max(t-1,0)$ and $k=\ln\left(\frac{k_0}{k_0-1}\right)$.

The main theorem in this section reads:

Theorem 4.2. *Assume that $m\ge 1$ and $p>1$, as $n=1,2$; $1<p\le\frac{n}{n-2}$, as $n\ge 3$. Suppose that $\|\nabla u_0\|_{L^2}^2<\lambda_0$ and $E(0)<E_0$, where $\lambda_0=k_0^{\frac{-2(p+1)}{p-1}}$, $E_0=\left(\frac{1}{2}-\frac{1}{p+1}\right)\lambda_0$, here k_0 is the constant of the Sobolev embedding $\|u\|_{L^{p+1}}\le k_0\|\nabla u\|_{L^2}$, for $u\in H_0^1(\Omega)$. Then the solution is global and the energy of problem (1) decays as*

$$\begin{cases} E(t)\le E(0)e^{-k(t-1)^+}, & t\ge 0,\ for\ m=1\\ E(t)\le\left(E(0)^{\frac{m-1}{2}}+\dfrac{(m-1)C}{2}(t-1)^+\right)^{-\frac{2}{m-1}}, & t\ge 0,\ for\ m>1.\end{cases}\tag{13}$$

Remark 4.3. In [7], an argument to show the solution for problem (1) exists globally and decays under some condition. Theorem 4.2 also shows that the solution exists globally under some similar conditions, and the method used here is simpler than that in [2].

Proof. First, by the decreasing of energy E(t). We have $E(t)\le E(0)<E_0=\left(\frac{1}{2}-\frac{1}{p+1}\right)\lambda_0$. We claim that

$$\|\nabla u(t)\|_{L^2}^2<\lambda_0,\quad\text{and}\quad \|\nabla u(t)\|_{L^2}^2+\|u_t\|_{L^2}^2\le\frac{2(p+1)}{p-1}E(t)\le\frac{2(p+1)}{p-1}E(0),$$

for all $t\ge 0$.

By the definition of $E(t)$ and Sobolev embedding, we can conclude that

$$E(t)\ge\frac{1}{2}\|\nabla u(t)\|_{L^2}^2-\frac{1}{p+1}\|u(t)\|_{L^{p+1}}^{p+1}\ge\frac{1}{2}\|\nabla u(t)\|_{L^2}^2-\frac{k_0^{p+1}}{p+1}\|\nabla u(t)\|_{L^2}^{p+1}.$$

Now if we let $f(\xi)=\frac{1}{2}\xi-\frac{k_0^{p+1}}{p+1}\xi^{\frac{p+1}{2}}$, then $E(t)\ge f(\xi)=\frac{1}{2}\xi-\frac{k_0^{p+1}}{p+1}\xi^{\frac{p+1}{2}}$, with $\xi=\|\nabla u(t)\|_{L^2}^2$. It is easily to verify that the function $f(\xi)$ have the following properties:

$$\begin{cases} f(\xi)\ \text{is strictly increasing on}\ [0,\lambda_0)\\ f(\xi)\ \text{takes its maximum value}\ E_0\equiv\left(\frac{1}{2}-\frac{1}{p+1}\right)\lambda_0\ \text{at}\ \lambda_0\\ f(\xi)\ \text{is strictly decreasing on}\ (\lambda_0,+\infty).\end{cases}\tag{14}$$

Since $E_0 > E(0) \geq E(t) \geq f(\|\nabla u(t)\|_{L^2}^2)$, for all $t \geq 0$. By virtue of (14), there is no time t^*, such that $\|\nabla u(t^*)\|_{L^2}^2 = \lambda_0$. By the continuity of $\|\nabla u(t)\|_{L^2}^2$-norm, we have

$$\|\nabla u(t)\|_{L^2}^2 < \lambda_0, \quad \text{for all } t \geq 0. \tag{15}$$

From (15) and Sobolev embedding, we have

$$\frac{1}{p+1}\|u(t)\|_{L^{p+1}}^{p+1} \leq \frac{k_0^{p+1}}{p+1}[\|\nabla u(t)\|_{L^2}^2]^{\frac{p-1}{2}} \leq \frac{1}{p+1}\|\nabla u(t)\|_{L^2}^2.$$

Moreover

$$E(t) \geq \frac{1}{2}\|u_t\|_{L^2}^2 + \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\nabla u(t)\|_{L^2}^2 \geq \frac{p-1}{2(p+1)}\left(\|\nabla u(t)\|_{L^2}^2 + \|u_t\|_{L^2}^2\right).$$

By continuation argument, we get that the local solution constructed by Theorem 2.1 exists globally.

Then we pay attention to large time behavior. By Sobolev embedding and the initial condition, we have

$$\|u(\cdot,t)\|_{L^{p+1}}^{p+1} \leq k_0^{p+1}\|\nabla u(t)\|_{L^2}^{p+1} < k_0^{p+1}(\lambda_0)^{\frac{p-1}{2}}\|\nabla u(t)\|_{L^2}^2 < \theta\|\nabla u(t)\|_{L^2}^2,$$

for all $t \geq 0$, where we define $0 \leq \theta < 1$ as $\theta = k_0^{p+1}(\lambda_0)^{\frac{p-1}{2}}$. Therefore, if we let $I(t) = \|\nabla u(t)\|_{L^2}^2 - \|u(t)\|_{L^{p+1}}^{p+1}$, then due to Sobolev embedding inequality, it follows that $I(t) > (1-\theta)\|\nabla u(t)\|_{L^2}^2$, for all $t \geq 0$. Now we set

$$F^{m+1}(t) = \int_t^{t+1} \|u_t(,s)\|_{L^{m+1}}^{m+1} ds + \int_t^{t+1}\int_\Omega u_t g(u_t)\,dxds = E(t) - E(t+1),$$

and

$$G^{m+1}(t) = F^{m+1}(t) - \int_t^{t+1}\int_\Omega u_t g(u_t)\,dxds = \int_t^{t+1} \|u_t(,s)\|_{L^{m+1}}^{m+1} ds < F^{m+1}(t).$$

Integrating $I(t)$ on $[t_1, t_2]$, we have

$$\begin{aligned}
\int_{t_1}^{t_2} I(s)ds &= \int_{t_1}^{t_2} \left(\|\nabla u(t)\|_{L^2}^2 - \|u(t)\|_{L^{p+1}}^{p+1}\right) ds \\
&= \int_{t_1}^{t_2} \left(2E(s) - \|u_t\|_{L^2}^2 + \frac{1}{p+1}\|u(t)\|_{L^{p+1}}^{p+1} - \|u(t)\|_{L^{p+1}}^{p+1}\right) ds \\
&= \int_{t_1}^{t_2} 2E(s)ds - \int_{t_1}^{t_2} \|u_t\|_{L^2}^2 - \frac{p-1}{p+1}\int_{t_1}^{t_2} \|u(t)\|_{L^{p+1}}^{p+1} \\
&\leq \int_{t_1}^{t_2} 2E(s)ds + CG^2(t) \quad (C \text{ is a generic constant}) \\
&\leq C\left[\int_{t_1}^{t_2} E(s)ds + F^2(t)\right].
\end{aligned}$$

Due to (7), the following inequalities hold

$$\begin{aligned}
\int_{t_1}^{t_2} E(s)ds &= \frac{1}{2}\int_{t_1}^{t_2} \|u_t\|_{L^2}^2 ds + \frac{1}{2}\int_{t_1}^{t_2} \|\nabla u\|_{L^2}^2 ds - \frac{1}{p+1}\int_{t_1}^{t_2} \|u(t)\|_{L^{p+1}}^{p+1} ds \\
&= \frac{1}{2}\int_{t_1}^{t_2} \|u_t\|_{L^2}^2 ds + \frac{1}{2}\int_{t_1}^{t_2} \|\nabla u\|_{L^2}^2 ds + \frac{1}{p+1}\int_{t_1}^{t_2} \|\nabla u\|_{L^2}^2 ds \\
&\quad - \frac{1}{p+1}\int_{t_1}^{t_2} \|\nabla u\|_{L^2}^2 ds - \frac{1}{p+1}\int_{t_1}^{t_2} \|u(t)\|_{L^{p+1}}^{p+1} ds \\
&= \frac{1}{2}\int_{t_1}^{t_2} \|u_t\|_{L^2}^2 ds + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{t_1}^{t_2} \|\nabla u\|_{L^2}^2 ds + \frac{1}{p+1}\int_{t_1}^{t_2} I(s)ds \\
&\le CG^2(t) + \left(\frac{1}{p+1} + \frac{p-1}{2(p+1)(1-\theta)}\right)\int_{t_1}^{t_2} I(s)ds \\
&\le CG^2(t) + C\int_{t_1}^{t_2} I(s)ds
\end{aligned}$$

So $\int_{t_1}^{t_2} E(s)ds \le CG^2(t) < CF^2(t)$. On the other hand, from the nonincreasing property of $E(t)$, one has $\int_{t_1}^{t_2} E(s)ds \ge \frac{1}{2}E(t_2)$. Therefore,

$$\begin{aligned}
E(t) &= E(t_2) + \int_t^{t_2} \|u_t\|_{L^{m+1}}^{m+1} ds + \int_t^{t_2}\int_\Omega u_t g(u_t) ds \\
&\le E(t_2) + \int_t^{t_2} \|u_t\|_{L^{m+1}}^{m+1} ds + C\int_t^{t_2} \|u_t\|_{L^2}^2 ds \\
&\le 2\int_{t_1}^{t_2} E(s)ds + \int_t^{t_2} \|u_t\|_{L^{m+1}}^{m+1} ds + C\int_t^{t_2} \|u_t\|_{L^2}^2 ds \\
&\le C\left(G^2(t) + G^{m+1}(t) + G^2(t)\right) \\
&\le C\left(F^2(t) + F^{m+1}(t)\right).
\end{aligned} \tag{16}$$

If $m = 1$, (16) gives

$$E(t) \le CF^2(t) = C\left(E(t) - E(t+1)\right), \tag{17}$$

then the first inequality in (13) follows from (17) and Lemma 4.1.

If $m > 1$, since $F^{m+1}(t) = E(t) - E(t+1) \le E(0)$, inequality (16) gives

$$E(t) \le C\left(F^2(t) + F^{m+1}(t)\right) \le C\left(1 + E(0)^{\frac{m-1}{m+1}}\right)F^2(t) \le CF^2(t),$$

which implies

$$E^{\frac{m+1}{2}}(t) \le CF^{m+1}(t) \le C\left(E(t) - E(t+1)\right). \tag{18}$$

Then the second inequality in (13) following from (18) and Lemma 4.1. This finishes the proof. □

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