Removable Sets for A-Harmonic Functions

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Abstract. We establish in this paper that sets of m-Hausdorff measure zero are removable for $C^{0,\alpha}$ A-harmonic functions with m depending only on n, α and A.

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1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, and μ a nonnegative Radon measure in Ω. We consider the equation

$$-\Delta_A u = -\operatorname{div}\left(\frac{a(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu \quad \text{in } \mathcal{D}'(\Omega), \tag{1.1}$$

where $A(t) = \int_0^t a(s)ds$, and a is an increasing C^1 function from $[0, +\infty)$ into $[0, +\infty)$ satisfying $a(0) = 0$ and

$$a_0 \leqslant \frac{ta'(t)}{a(t)} \leqslant a_1 \quad \forall t > 0, \quad a_0, a_1 \text{ positive constants.} \tag{1.2}$$

A first class of functions $a(t)$ is given by

$$a(t) = c_1 t^\alpha \chi_{[0,t_0)} + (c_2 t^\beta + c_3)\chi_{[t_0,\infty)},$$

where t_0, α and β are positive numbers, and c_1, c_2 and c_3 are real numbers such that $a(t)$ is a C^1 function. For this class (1.2) is satisfied with $a_0 = \min(\alpha,\beta)$ and $a_1 = \max(\alpha,\beta)$. A second class of functions $a(t)$ is given by $a(t) = t^\alpha \ln(\beta t + \gamma)$, with $\alpha, \beta, \gamma > 0$. In this case, we have $a_0 = \alpha$ and $a_1 = 1+\alpha$. In fact there exists a large class of functions a included in this work, since linear combinations with positive constants, products and compositions of C^1 functions satisfying (1.2)

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also satisfy this condition, with different positive constants a_0 and a_1. Note that we have $a_0 = a_1 = p - 1$ if and only if $a(t) = t^{a_0}$. In this case, we have $A(t) = \frac{t^p}{p}$ and Δ_A is the classical p-Laplace operator Δ_p.

Under assumption (1.2), A is an increasing C^2 convex function, which is an N-function satisfying the so called Δ_2-condition. We recall the definition of the Orlicz class $K^A(\Omega)$ associated with the function A

$$K^A(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega A(|u(x)|)\, dx < \infty \right\}.$$

We refer to [9, 13] for definitions.

The Orlicz space $L^A(\Omega)$ is defined as the linear hull of the class $K^A(\Omega)$. Thanks to the assumption (1.2), $L^A(\Omega)$ coincides with $K^A(\Omega)$ and is a reflexive and separable Banach space under the Luxembourg norm (see [1, 9, 13])

$$|u|_{L^A(\Omega)} = \inf \left\{ k > 0 : \int_\Omega A\left(\frac{|u(x)|}{k}\right) dx \leqslant 1 \right\}.$$

Since Ω is bounded, we have the imbeddings $L^\infty(\Omega) \subset L^A(\Omega) \subset L^1(\Omega)$. The latter one is due to the generalized Hölder inequality (1.4) in Orlicz spaces .

The dual of the space $L^A(\Omega)$ is $L^{\widetilde{A}}(\Omega)$, where $\widetilde{A}$ is the conjugate of the function A given by $\widetilde{A}(t) = \int_0^t a^{-1}(s)ds$, where a^{-1} is the inverse function of a which also satisfies the condition (1.2) with different constants. Indeed, since we have for $t > 0$

$$\frac{t(a^{-1})'(t)}{a^{-1}(t)} = \frac{\frac{t}{a'(a^{-1}(t))}}{a^{-1}(t)} = \frac{a(a^{-1}(t))}{a^{-1}(t)a'(a^{-1}(t))},$$

we deduce from (1.2) that we have

$$\frac{1}{a_1} \leqslant \frac{t(a^{-1})'(t)}{a^{-1}(t)} \leqslant \frac{1}{a_0} \quad \forall t > 0. \tag{1.3}$$

It follows from (1.3) that $\widetilde{A}$ is also an N-function satisfying the Δ_2-condition. Moreover, we have the Hölder inequality (see [9, 13])

$$\left| \int_\Omega uv\, dx \right| \leqslant 2|u|_{L^A(\Omega)}|v|_{L^{\widetilde{A}}(\Omega)}, \quad \forall u \in L^A(\Omega),\ \forall v \in L^{\widetilde{A}}(\Omega). \tag{1.4}$$

Another consequence of (1.2) is the following monotonicity inequality [4]

$$\left(\frac{a(|\xi|)}{|\xi|}\xi - \frac{a(|\zeta|)}{|\zeta|}\zeta \right) \cdot (\xi - \zeta) > 0 \quad \forall \xi, \zeta \in \mathbb{R}^n \setminus \{0\},\ \xi \neq \zeta, \tag{1.5}$$

which implies the weak maximum principle and, in particular, the uniqueness of the solution of (1.1) for Dirichlet boundary data.

We also recall the following inequalities that can be deduced easily from (1.2) (see [11, Lemma 1.1] and [12, Lemma 2.1, Remark 2.1])

$$\frac{ta(t)}{1+a_1} \leqslant A(t) \leqslant ta(t) \qquad\qquad \forall t \geqslant 0, \quad (1.6)$$

$$\min(s^{1+a_0}, s^{1+a_1})\frac{A(t)}{1+a_1} \leqslant A(st) \leqslant (1+a_1)\max(s^{1+a_0}, s^{1+a_1})A(t) \quad \forall s, t \geqslant 0. \quad (1.7)$$

The Orlicz-Sobolev space and its norm (see [1]) are defined as follows

$$W^{1,A}(\Omega) = \left\{\, u \in L^A(\Omega) \,:\, |\nabla u| \in L^A(\Omega)\, \right\},$$
$$|u|_{W^{1,A}(\Omega)} = |u|_{L^A(\Omega)} + |\nabla u|_{L^A(\Omega)}.$$

Under the assumption (1.2), $W^{1,A}(\Omega)$ is a reflexive and separable Banach space (see [1]).

We shall call a solution of (1.1) any function $u \in W^{1,A}_{loc}(\Omega)$ that satisfies

$$\int_\Omega \frac{a(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla\varphi\, dx = \int_\Omega \varphi\, d\mu \quad \forall \varphi \in \mathcal{D}(\Omega).$$

If $\mu \equiv 0$ in a domain $D \subset \Omega$, we say that u is A-harmonic in D.

In [7], Kilpeläinen studied equation (1.1) for the p-Laplacian, where p is a constant number greater than 1. He proved that if u is Hölder continuous with exponent α, then μ satisfies the growth condition $\mu(B_r(x)) \leq Cr^{n-p+\alpha(p-1)}$, for any open ball $B_r(x)$ with $B_{3r}(x) \subset \Omega$. Conversely, he proved that if μ satisfies this growth condition, then u is Hölder continuous with exponent β for all $\beta \in (0, \alpha)$. This result was improved later on in [8] for more general operators of p-Laplacian type, by showing that under the growth condition of μ, u is in fact Hölder continuous with exponent α. Moreover, the authors proved that a compact set is removable for $C^{0,\alpha}$ p-harmonic functions if and only if its $(n - p + \alpha(p - 1))$-Hausdorff measure is zero. A similar result was obtained by Trudinger and Wang [14]. For harmonic functions, this result is due to Carleson [3].

In this paper, we address the question of removability of closed sets for an A-harmonic function. We also investigate the relationship between Hölder continuity of the solutions of (1.1) and the growth of the measure μ. We first establish that bounded solutions of (1.1) belong to $C^{0,\gamma}_{loc}(\Omega)$ for all $\gamma \in (0, \gamma^*)$, where $\gamma^* = \frac{(m-n+1+a_0)}{a_1}$ provided that $\mu(B(x, r)) \leqslant Cr^m$ for any open ball $B(x, r) \subset\subset \Omega$ and $m > n - 1 - a_0$. This extends our previous work in [4] when $m = n - 1$. Our second main result shows that a function u is A-harmonic in Ω provided that it is continuous and A-harmonic in $\Omega \setminus E$, where the closed set E is of zero m-Hausdorff measure, and where

$$m = (\alpha - 1)\frac{a_0}{a_0 + 1}(1 + a_1) + \left(\frac{a_0}{a_0 + 1} + \frac{1}{a_1 + 1}\right)n - 1,$$

and α is such that

$$|u(x) - u(y)| \leqslant c|x - y|^\alpha \quad \forall x \in E \; \forall y \in \Omega.$$

Our work extends results in [7, 8, 14] established in the p-Laplacian framework.

We shall denote by $B_R(x_0)$ the open ball of center x_0 and radius $R > 0$. By $\mathrm{osc}(u, B_R(x_0))$, we denote the oscillation $M_R - m_R$ of u over $B_R(x_0)$, where $M_R = \sup_{B_R(x_0)} u$ and $m_R = \inf_{B_R(x_0)} u$.

2. Hölder continuity

In this section, we improve the local Hölder continuity established in [4] and generalize some of the results in [7, 8] for p-Laplacian type operators. First, we show that the measure μ satisfies a growth condition if there is a Hölder continuous solution to (1.1).

Theorem 2.1. *Suppose that $u \in W^{1,A}_{loc}(\Omega) \cap C^{0,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. Then there exist two positive constants $C = C(n, \alpha, a_0, a_1)$ and $R_0 = R_0(n, \alpha, a_0, a_1)$ such that for each $R \in (0, R_0)$, and each ball $B_R(x_0)$ such that $B_{3R}(x_0) \subset\subset \Omega$, we have*

$$\mu(B_R(x_0)) \leqslant CR^m,$$

where m is given by

$$m = \tau(\alpha) = (\alpha - 1)\frac{a_0}{a_0 + 1}(1 + a_1) + \left(\frac{a_0}{a_0 + 1} + \frac{1}{a_1 + 1}\right)n - 1.$$

Proof. Let $x_0 \in \Omega$, $R > 0$ such that $B_{3R}(x_0) \subset\subset \Omega$, and let $\eta \in \mathcal{D}(B_{2R}(x_0))$ be the usual cut-off function with $\eta = 1$ in $B_R(x_0)$ and $|\nabla \eta| \leqslant \frac{2}{R}$. We have

$$
\begin{aligned}
A(1)\mu(B_R(x_0)) &\leqslant \int_{B_{2R}(x_0)} A(\eta)\,d\mu \\
&= \int_{B_{2R}(x_0)} \frac{a(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla(A(\eta))\,dx \\
&= \int_{B_{2R}(x_0)} a(\eta)\frac{a(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla \eta\,dx \\
&\leqslant a(1) \int_{B_{2R}(x_0)} a(|\nabla u|)|\nabla \eta|\,dx.
\end{aligned}
$$

Using Hölder's inequality (1.4) and (1.6), we deduce that

$$\mu(B_R(x_0)) \leqslant 2(1 + a_1)|a(|\nabla u|)|_{L^{\tilde{A}}(B_{2R}(x_0))}\|\nabla \eta\|_{L^A(B_{2R}(x_0))}. \tag{2.1}$$

First we estimate $|a(|\nabla u|)|_{L^{\tilde{A}}(B_{2R}(x_0))}$: We recall the following inequalities (see [12, Lemmas 2.2, 2.3])

$$|v|_{L^A(\Omega)} \leqslant C(a_0, a_1) \max\left\{\left(\int_\Omega A(|v|)\, dx\right)^{\frac{1}{1+a_0}}, \left(\int_\Omega A(|v|)\, dx\right)^{\frac{1}{1+a_1}}\right\},$$

$$\tilde{A}(a(t)) \leqslant a_1 A(t).$$

Then, we obtain by taking into account (1.3),

$$|a(|\nabla u|)|_{L^{\tilde{A}}(B_{2R}(x_0))}$$
$$\leqslant C \max\left\{\left(\int_{B_{2R}(x_0)} \tilde{A}(a(|\nabla u|))\, dx\right)^{\frac{1}{1+\frac{1}{a_0}}}, \left(\int_{B_{2R}(x_0)} \tilde{A}(a(|\nabla u|))\, dx\right)^{\frac{1}{1+\frac{1}{a_1}}}\right\} \tag{2.2}$$
$$\leqslant C \max\left\{a_1^{\frac{a_0}{1+a_0}}\left(\int_{B_{2R}(x_0)} A(|\nabla u|)\, dx\right)^{\frac{a_0}{a_0+1}}, a_1^{\frac{a_1}{a_1+1}}\left(\int_{B_{2R}(x_0)} A(|\nabla u|)\, dx\right)^{\frac{a_1}{a_1+1}}\right\}.$$

To estimate the integral $\int_{B_{2R}(x_0)} A(|\nabla u|)\, dx$, we will apply the Cacciopoli inequality for A-subharmonic functions (see [12, Lemma 2.6]) to the function $M_{3R} - u$ in $B_{3R}(x_0)$. Indeed, since $\nabla(M_{3R} - u) = -\nabla u$, we have $\Delta_A(M_{3R} - u) = -\Delta_A u = \mu \geqslant 0$ in $\mathcal{D}'(B_{3R}(x_0))$. Consequently, we have

$$\int_{B_{2R}(x_0)} A(|\nabla u|)\, dx = \int_{B_{2R}(x_0)} A(|\nabla(M_{3R} - u)|)\, dx$$
$$\leqslant C(n, a_0, a_1) \int_{B_{3R}(x_0)} A\left(\frac{|M_{3R} - u|}{R}\right) dx$$
$$\leqslant C(n, a_0, a_1) \int_{B_{3R}(x_0)} A\left(\frac{\mathrm{osc}(u, B_{3R}(x_0))}{R}\right) dx.$$

Now, since $\mathrm{osc}_{B_{3R}(x_0)} u \leqslant CR^\alpha$, we deduce, by using (1.7) and assuming $R < 1$, that

$$\int_{B_{2R}(x_0)} A(|\nabla u|)\, dx \leqslant C(n, a_0, a_1) R^{(\alpha-1)(1+a_1)+n}. \tag{2.3}$$

Choosing $R < R_0$ with $R_0 = \min\left(1, \left(\frac{1}{C(n,a_0,a_1)}\right)^{\frac{1}{(\alpha-1)(1+a_1)+n}}\right)$, we ensure that $\int_{B_{2R}(x_0)} A(|\nabla u|)\, dx < 1$. It follows from (2.2), (2.3) that we have for $R < R_0$

$$|a(|\nabla u|)|_{L^{\tilde{A}}(B_{2R}(x_0))} \leqslant C \max\left(a_1^{\frac{a_1}{a_1+1}}, a_1^{\frac{a_0}{a_0+1}}\right)\left(\int_{B_{2R}(x_0)} A(|\nabla u|)\, dx\right)^{\frac{a_0}{a_0+1}} \tag{2.4}$$
$$\leqslant C(n, a_0, a_1) R^{\frac{a_0}{a_0+1}((\alpha-1)(1+a_1)+n)}.$$

Next, we estimate $\|\nabla\eta\|_{L^A(B_{2R}(x_0))}$:   By definition of the Luxembourg norm $|.|_{L^A}$, we have

$$\|\nabla\eta\|_{L^A(B_{2R}(x_0))} = \inf\left\{k > 0 : \int_{B_{2R}(x_0)} A\left(\frac{|\nabla\eta|}{k}\right) dx \leqslant 1\right\}$$
$$\leqslant \inf\left\{k > 0 : \int_{B_{2R}(x_0)} A\left(\frac{2}{Rk}\right) dx \leqslant 1\right\} \qquad (2.5)$$
$$= \frac{2}{RA^{-1}\left(\frac{1}{|B_2|R^n}\right)}.$$

Taking $t = 1$ and $s > 1$ in (1.7), we obtain $A(s) \leqslant A(1)(1 + a_1)s^{1+a_1}$ or equivalently $s \leqslant A^{-1}\left(A(1)(1+a_1)s^{1+a_1}\right)$. Setting $z = A(1)(1+a_1)s^{1+a_1}$, we get $s = \left(\frac{z}{A(1)(1+a_1)}\right)^{\frac{1}{1+a_1}}$. Hence we obtain

$$A^{-1}(z) \geqslant \left(\frac{z}{A(1)(1 + a_1)}\right)^{\frac{1}{1+a_1}}, \quad \text{for } z > A(1)(1 + a_1).$$

For $0 < R < \left(\frac{1}{|B_2|A(1)(1+a_1)}\right)^{\frac{1}{n}}$, we have $\frac{1}{|B_2|R^n} > A(1)(1 + a_1)$. We deduce that

$$A^{-1}\left(\frac{1}{|B_2|R^n}\right) \geqslant \left(\frac{1}{|B_2|R^n}\right)^{\frac{1}{1+a_1}}\left(\frac{1}{A(1)(1+a_1)}\right)^{\frac{1}{1+a_1}} = \frac{c}{R^{\frac{n}{1+a_1}}}. \qquad (2.6)$$

We deduce from (2.5), (2.6) that

$$\|\nabla\eta\|_{L^A(B_{2R}(x_0))} \leqslant \frac{2}{RA^{-1}\left(\frac{1}{|B_2|R^n}\right)} \leqslant \frac{2}{cR}R^{\frac{n}{a_1+1}} \leqslant C(n, a_0, a_1)R^{\frac{n}{a_1+1}-1}.$$

Finally, we conclude that we have for $R \in (0, R_0)$

$$\mu(B_R(x_0)) \leqslant C(n, a_0, a_1)R^{\frac{a_0}{a_0+1}(n+(\alpha-1)(a_1+1))}R^{\frac{n}{a_1+1}-1} = C(n, a_0, a_1)R^m. \qquad \square$$

In the next theorem, we establish a partial converse of Theorem 2.1.

Theorem 2.2. *Assume that μ satisfies for each ball $B_R(x_0) \subset\subset \Omega$,*

$$\mu(B_R(x_0)) \leqslant CR^m \quad \text{with } m = \tau(\alpha) \text{ as in Theorem 2.1.}$$

If $u \in W_{loc}^{1,A}(\Omega)$ is a solution of $-\Delta_A u = \mu$ in $\mathcal{D}'(\Omega)$, then we have $u \in C_{loc}^{0,\gamma}(\Omega)$ for all $\gamma \in (0, \gamma_*)$, with

$$\gamma_* = \frac{1}{a_1}(m-n+1+a_0) = \frac{1+a_1}{a_1}\left(\alpha\frac{a_0}{1+a_0} - \frac{a_1-a_0}{(1+a_0)(1+a_1)}\left(\frac{n}{1+a_1}+a_0\right)\right).$$

Proof. Arguing as in the proof of [4, Theorem 3.1], we obtain $u \in C_{loc}^{0,\gamma_0}(\Omega)$ with $\gamma_0 = \frac{m-n+1+a_0}{1+a_1}$. Now, using this result and arguing again as in [4], we obtain

$$u \in C_{loc}^{0,\gamma_1}(\Omega) \quad \text{with } \gamma_1 = \frac{m + \gamma_0 - n + 1 + a_0}{1 + a_1} = \gamma_0 + \frac{\gamma_0}{1 + a_1}.$$

Repeating this process, we get $u \in C_{loc}^{0,\gamma_k}(\Omega)$ with

$$\gamma_k = \frac{m + \gamma_{k-1} - n + 1 + a_0}{1 + a_1} = \gamma_0 + \frac{\gamma_{k-1}}{1 + a_1}.$$

It follows that we have $\gamma_k = \gamma_0 \sum_{j=0}^{k} \frac{1}{(1+a_1)^j}$, which leads to $\lim_{k\to\infty} \gamma_k = \frac{\gamma_0}{1 - \frac{1}{1+a_1}} = \gamma_0 \frac{1+a_1}{a_1} = \gamma_*$ and the result follows. □

Remark 2.3. We have $\gamma_* < \alpha$. Indeed

$$\gamma_* < \alpha \iff \alpha \frac{a_0}{1 + a_0} - \frac{a_1 - a_0}{(1 + a_0)(1 + a_1)} \left(\frac{n}{1 + a_1} + a_0 \right) < \alpha \frac{a_1}{1 + a_1}$$

$$\iff -\alpha \frac{a_1 - a_0}{(1 + a_0)(1 + a_1)} < \frac{a_1 - a_0}{(1 + a_0)(1 + a_1)} \left(\frac{n}{1 + a_1} + a_0 \right).$$

In [7], Kilpeläinen established in the p-Laplacian framework that u is Hölder continuous with exponent β for all $\beta \in (0, \alpha)$. In [8], this result was improved by showing that u is in fact Hölder continuous with exponent α. In the present situation, we remark that $\gamma_* \to \alpha$, as $a_1 - a_0 \to 0$.

3. Removable sets for A-harmonic functions

In this section, we give a sufficient condition for a closed set in order to be removable for A-harmonic functions. This extends [8, Theorem 1.6] in the p-Laplacian framework.

Theorem 3.1. *Let $E \subset \Omega$ be a closed set and $s > 0$. Assume that u is a continuous function in Ω, A-harmonic in $\Omega \setminus E$, and such that for some $\alpha \in (0,1)$*

$$|u(x) - u(y)| \leqslant L|x - y|^\alpha \quad \forall y \in \Omega, \ \forall x \in E.$$

If E is of m-Hausdorff measure zero, with $m = \tau(\alpha)$, then u is A-harmonic in Ω.

First, we introduce the following obstacle problem, where D is a smooth subdomain of Ω and $\phi \in W^{1,A}(D)$:

$$P(\phi, D) \begin{cases} \text{Find } v \in \mathcal{F} = \left\{ \zeta \in W^{1,A}(D) : \zeta \geqslant \phi \text{ in } \Omega \text{ and } \zeta - \phi \in W_0^{1,A}(D) \right\} \\ \int_D \frac{a(|\nabla v|)}{|\nabla v|} \nabla v \cdot \nabla(\zeta - v) \, dx \geqslant 0 \quad \text{for all } \zeta \in \mathcal{F}. \end{cases}$$

Then we have :

Theorem 3.2. *There exists a unique solution v to the obstacle problem $P(\phi, D)$. If ϕ is continuous in D, then v is continuous in D. Moreover $-\Delta_A v$ is a non-negative measure and v is A-harmonic in $[v > \phi]$.*

Proof. For the existence of a solution to $P(\phi, D)$, we refer to $[2, 6, 10]$ for a more general operator with A satisfying the Δ_2-condition. Moreover, using the monotonicity property (1.5), it is easy to see that the solution is unique. The proof of the continuity of v follows as for the one of Theorem 3.67 p. 78 in $[5]$ in the p-Laplacian case. Indeed it is essentially based on Harnack's inequalities which are also available in the A-Laplacian framework: $[11,$ Theorems 1.2, 1.3$]$. The last property in the theorem also follows as in the p-Laplacian situation (see the proof of $[5,$ Theorem 3.67$]$). $\qquad\square$

Next, we establish the following lemma :

Lemma 3.3. *Let Ω' be an open subset of Ω, and $K \subset \Omega'$ be a compact. Suppose that $\phi \in W^{1,A}_{loc}(\Omega)$ is a continuous function such that*

$$\forall y \in \Omega \ \forall x \in K : \ |\phi(x) - \phi(y)| \leqslant L|x - y|^\alpha, \quad \text{with } L > 0 \text{ and } \alpha = \tau^{-1}(m).$$

Let v be the solution of the obstacle problem $P(\phi, \Omega')$ and let $\mu = -\Delta_A v$. Then μ satisfies

$$\forall R < R_1 \ \forall x \in K : \ \mu(B_R(x)) \leqslant cR^m$$

for a constant $c = c(n, a_0, a_1, \alpha, L) > 0$ and $R_1 = \min\left(R_0, \frac{1}{9}\operatorname{dist}(K, \partial\Omega)\right)$.

Proof. Let $x_0 \in K$ and $0 < R < R_1$. We distinguish two cases:

Case 1: $v(x_0) = \phi(x_0)$
Let $\eta \in \mathcal{D}(B_{2R}(x_0))$ be a smooth cut-off function with $\eta = 1$ in $B_R(x_0)$ and $|\nabla\eta| \leqslant \frac{2}{R}$. We obtain as in the proof of Theorem 2.1

$$\mu(B_R(x_0)) \leqslant 2(1 + a_1) \, |a(|\nabla v|)|_{L^{\tilde{A}}(B_{2R}(x_0))} \, \|\nabla\eta\|_{L^A(B_{2R}(x_0))}$$

$$\leqslant C(n, a_0, a_1)\left(\int_{B_{3R}(x_0)} A\left(\frac{\operatorname{osc}(v, B_{3R}(x_0))}{R}\right) dx\right)^{\frac{a_0}{1+a_0}} R^{\frac{n}{1+a_1}-1}. \quad (3.1)$$

Without loss of generality, we can assume that $v(x_0) = \phi(x_0) = 0$. Let $\omega_0 = \operatorname{osc}(\phi, B_{6R}(x_0))$. Then $v + \omega_0$ is a nonnegative and A-superharmonic function in $B_{6R}(x_0)$. Indeed $\Delta_A(v + \omega_0) = \Delta_A v = -\mu \leqslant 0$, and $v + \omega_0 = v + M_{6R} - m_{6R} \geqslant M_{6R} \geqslant v(x_0) = 0$.

Moreover $(v - \omega_0)^+$ is a nonnegative and A-subharmonic function in $B_{6R}(x_0)$. Indeed let $\zeta \in \mathcal{D}(B_{6R}(x_0))$, $\zeta \geqslant 0$, and let $\epsilon > 0$. Using $v + \min\left(\zeta, \frac{(v-\omega_0)^+}{\epsilon}\right)$ as

a test function for the problem $P(\phi, \Omega')$, we get

$$\int_{B_{6R}(x_0)} \frac{a(|\nabla v|)}{|\nabla v|} \nabla v \cdot \nabla \left(\min \left(\zeta, \frac{(v - \omega_0)^+}{\epsilon} \right) \right) dx$$

$$= \int_{B_{6R}(x_0) \cap [v > \omega_0]} \frac{a(|\nabla v|)}{|\nabla v|} \nabla v \cdot \nabla \left(\min \left(\zeta, \frac{(v - \omega_0)^+}{\epsilon} \right) \right) dx \qquad (3.2)$$

$$= 0$$

since $B_{6R}(x_0) \cap [v > \omega_0] \subset B_{6R}(x_0) \cap [v > \phi]$ and $\Delta_A v = 0$ in $[v > \phi]$. We deduce from (3.2) that

$$\int_{B_{6R}(x_0) \cap [\epsilon \zeta \leqslant (v - \omega_0)^+]} \frac{a(|\nabla (v - \omega_0)^+|)}{|\nabla (v - \omega_0)^+|} \nabla (v - \omega_0)^+ \cdot \nabla \zeta \, dx$$

$$= -\frac{1}{\epsilon} \int_{B_{6R}(x_0) \cap [\epsilon \zeta > (v - \omega_0)^+]} |\nabla (v - \omega_0)^+| a(|\nabla (v - \omega_0)^+|) \, dx$$

$$\leqslant 0.$$

Letting $\epsilon \to 0$, we get $\int_{B_{6R}(x_0)} \frac{a(|\nabla (v - \omega_0)^+|)}{|\nabla (v - \omega_0)^+|} \nabla (v - \omega_0)^+ \cdot \nabla \zeta \, dx \leqslant 0$ which means that $\Delta_A (v - \omega_0)^+ \geqslant 0$ in $B_{6R}(x_0)$.

We are able now to use the inequalities of [11, Theorems 1.2, 1.3]. We obtain for some $C > 0$ and $p_0 > 0$

$$\sup_{B_{3R}(x_0)} (v - \omega_0)^+ \leqslant C \left(\frac{1}{|B_{6R}(x_0)|} \int_{B_{6R}(x_0)} ((v - \omega_0)^+)^{p_0} \, dx \right)^{\frac{1}{p_0}}$$

$$\leqslant C \left(\frac{1}{|B_{6R}(x_0)|} \int_{B_{6R}(x_0)} (v + \omega_0)^{p_0} \, dx \right)^{\frac{1}{p_0}}$$

$$\leqslant C \inf_{B_{\frac{9R}{2}}(x_0)} (v + \omega_0)$$

$$\leqslant C \omega_0$$

since $\inf_{B_{\frac{9R}{2}}(x_0)} v \leqslant v(x_0) = 0$. We deduce that

$$\sup_{B_{3R}(x_0)} v \leqslant \omega_0 + \sup_{B_{3R}(x_0)} (v - \omega_0) \leqslant \omega_0 + \sup_{B_{3R}(x_0)} (v - \omega_0)^+ \leqslant (1 + C)\omega_0.$$

Since $M_{6R} \geqslant v(x_0) = 0$, we have

$$-\inf_{B_{3R}(x_0)} v \leqslant -\inf_{B_{6R}(x_0)} \phi = -m_{6R} \leqslant M_{6R} - m_{6R} = \omega_0.$$

It follows by the assumption of the lemma, since $x_0 \in K$, that

$$\operatorname{osc}(v, B_{3R}(x_0)) = \sup_{B_{3R}(x_0)} v - \inf_{B_{3R}(x_0)} v \leqslant (2 + C)\omega_0 \leqslant 2L(2 + C)(6R)^\alpha.$$

Using (3.1), we conclude as in the proof of Theorem 2.1, that

$$\mu(B_R(x_0)) \leqslant C(n, a_0, a_1, \alpha, L)R^m.$$

Case 2: $v(x_0) > \phi(x_0)$
If $B_R(x_0) \cap [v = \phi] = \emptyset$, then $\mu(B_R(x_0)) = 0$. If there exists $y_0 \in B_R(x_0) \cap [v = \phi]$, then we have as in Case 1,

$$\mu(B_R(x_0)) \leqslant \mu(B_{2R}(y_0)) \leqslant C(n, a_0, a_1, \alpha, L)R^m. \qquad \square$$

Proof of Theorem 3.1. Let E be a closed subset of Ω. Assume that E has m-Hausdorff measure zero, i.e., $\mathcal{H}^m(E) = 0$, and let u be a continuous function in Ω, that is A-harmonic in $\Omega \setminus E$, and such that for all $y \in \Omega$ and $x \in E$

$$|u(x) - u(y)| \leqslant L|x - y|^\alpha \quad \text{with } \alpha = \tau^{-1}(m).$$

We would like to prove that u is A-harmonic in Ω. Consider a smooth domain $D \subset\subset \Omega$ and let v be the solution of the Obstacle problem $P(u, D)$. Then $\mu = -\Delta_A v$ is a nonnegative Radon measure. Let K be a compact subset of $E \cap D$. From Lemma 3.1, we have for a positive constant $c = c(n, a_0, a_1, \alpha, L)$

$$\mu(B_R(x)) \leqslant cR^m \quad \forall x \in K, \ \forall R < R_1 = \min\left(R_0, \frac{1}{9}\operatorname{dist}(K, \partial\Omega)\right).$$

Let us recall the definition of m-Hausdorff measure of a set F

$$H^m(F) = \lim_{\delta \to 0} H_\delta^m(F) = \sup_{\delta > 0} H_\delta^m(F), \quad \text{where for } \delta > 0,$$

$$H_\delta^m(F) = \inf\left\{\sum_{j=1}^\infty \alpha(m)\left(\frac{\operatorname{diam}(C_j)}{2}\right)^m : F \subset \bigcup_{j=1}^\infty C_j, \ \operatorname{diam}(C_j) \leqslant \delta\right\},$$

$\alpha(m) = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)}$ and $\Gamma(m) = \int_0^\infty e^{-t}t^{m-1}dt$ is the usual Gamma function.

Let $\epsilon > 0$. Since $\mathcal{H}^m(E) = 0$ and $K \subset E$, we have $\mathcal{H}^m(K) = 0$. So there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, $0 \leqslant H_\delta^m(K) \leqslant \epsilon$. We deduce that for each $\delta \in (0, \delta_0)$, there exists a family of sets (C_j^δ) such that $K \subset \bigcup_{j=1}^\infty C_j$, $\operatorname{diam}(C_j^\delta) \leqslant \delta$ and

$$H_\delta^m(K) \leqslant \sum_{j=1}^\infty \alpha(m)\left(\frac{\operatorname{diam}(C_j)}{2}\right)^m < \epsilon.$$

We assume naturally that for each j, $C_j^\delta \cap K \neq \emptyset$. So for each j, there exists $x_j \in C_j^\delta \cap K$. This leads to $C_j^\delta \subset B_{R_j}(x_j)$, with $R_j = \operatorname{diam}(C_j^\delta)$. It follows that $\mu(C_j^\delta) \leqslant \mu(B_{R_j}(x_j)) \leqslant C(\operatorname{diam}(C_j^\delta))^m$, and hence,

$$\mu(K) \leqslant \sum_{j=1}^\infty \mu(C_j^\delta) \leqslant C\frac{2^m}{\alpha(m)}\sum_{j=1}^\infty \alpha(m)\left(\frac{\operatorname{diam}(C_j)}{2}\right)^m \leqslant C\frac{2^m}{\alpha(m)}\epsilon.$$

Letting $\epsilon \to 0$, we get $\mu(K) = 0$. This leads to $\mu(E \cap D) = 0$.

Next, we prove that $\mu(D \setminus E) = 0$. Let $\zeta \in \mathcal{D}(D \setminus E)$, $\zeta \geqslant 0$, $\epsilon > 0$, and set $\zeta_\epsilon = \min\left(\zeta, \frac{v-u}{\epsilon}\right)$. Note that we have $\mu = -\Delta_A v = 0$ in $[v > u]$ and $\zeta_\epsilon = 0$ in $[v = u]$. We deduce that

$$\int_{D \setminus E} \frac{a(|\nabla v|)}{|\nabla v|} \nabla v \cdot \nabla \zeta_\epsilon \, dx = \int_{(D \setminus E) \cap [v > u]} \frac{a(|\nabla v|)}{|\nabla v|} \nabla v \cdot \nabla \zeta_\epsilon \, dx = 0. \qquad (3.3)$$

Since $\Delta_A u = 0$ in $D \setminus E$, we have

$$\int_{D \setminus E} \frac{a(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \zeta_\epsilon \, dx = 0. \qquad (3.4)$$

Subtracting (3.4) from (3.3), we get $\int_{D \setminus E} \left(\frac{a(|\nabla v|)}{|\nabla v|} \nabla v - \frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) \cdot \nabla \zeta_\epsilon \, dx = 0$. Then, by taking into account the monotonicity inequality (1.5), we deduce

$$\int_{(D \setminus E) \cap [\epsilon \zeta \leqslant v - u]} \left(\frac{a(|\nabla v|)}{|\nabla v|} \nabla v - \frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) \cdot \nabla \zeta \, dx$$
$$= -\frac{1}{\epsilon} \int_{(D \setminus E) \cap [\epsilon \zeta > v - u]} \left(\frac{a(|\nabla v|)}{|\nabla v|} \nabla v - \frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) \cdot (\nabla v - \nabla u) \, dx \qquad (3.5)$$
$$\leqslant 0.$$

Writing

$$\int_{(D \setminus E) \cap [\epsilon \zeta \leqslant v - u]} \left(\frac{a(|\nabla v|)}{|\nabla v|} \nabla v - \frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) \cdot \nabla \zeta \, dx = \int_{D \setminus E} f_\epsilon \, dx \qquad (3.6)$$

with $f_\epsilon = \chi_{[\epsilon \zeta \leqslant v - u]} \left(\frac{a(|\nabla v|)}{|\nabla v|} \nabla v - \frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) \cdot \nabla \zeta$, and remarking that $\nabla v = \nabla u$ a.e. in $[v = u]$, we obtain

$$\begin{cases} f_\epsilon = 0 & \text{a.e. in } [v = u] \\ f_\epsilon \to \left(\frac{a(|\nabla v|)}{|\nabla v|} \nabla v - \frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) \cdot \nabla \zeta & \text{as } \epsilon \to 0, \quad \text{a.e. in } [v > u] \end{cases} \qquad (3.7)$$

Letting $\epsilon \to 0$ in (3.5) and using (3.6), (3.7), we get by the Lebesgue theorem

$$\int_{D \setminus E} \left(\frac{a(|\nabla v|)}{|\nabla v|} \nabla v - \frac{a(|\nabla u|)}{|\nabla u|} \nabla u \right) \cdot \nabla \zeta \, dx \leqslant 0.$$

Since $\Delta_A u = 0$ in $D \setminus E$, we obtain $\int_{D \setminus E} \frac{a(|\nabla v|)}{|\nabla v|} \nabla v \cdot \nabla \zeta \, dx \leqslant 0$, which means that $\mu = -\Delta_A v \leqslant 0$ in $D \setminus E$. Since μ is nonnegative, we obtain $\mu(D \setminus E) = 0$. We conclude that $\mu(D) = 0$ and therefore $\Delta_A v = 0$ in D.

Similarly, we consider the solution w of the obstacle problem $P(-u, D)$. In the same way, we prove that $\Delta_A w = 0$ in D. Now we have

$$\begin{cases} \Delta_A v = \Delta_A(-w) = 0, & \text{in } D \\ \quad v = -w, & \text{on } \partial D. \end{cases}$$

By the maximum principle, thanks to the inequality (1.5), we deduce that $v = -w$ in D. Since we have $-w \leqslant u \leqslant v$ in D, we obtain $u = v = -w$ in D and $\Delta_A u = 0$ in D. $\qquad\square$

As an immediate consequence of Theorem 3.1, we obtain the following corollary.

Corollary 3.4. *Suppose that $u \in W^{1,A}_{loc}(\Omega) \cap C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ is A-harmonic in $\Omega \setminus E$, where E is a closed subset of Ω. If $m = \tau(\alpha)$ and $\mathcal{H}^m(E) = 0$, then u is A-harmonic in Ω.*

Remark 3.5. Corollary 3.1 was proved by Carleson for the Laplacian [3]. For p-Laplacian like operators, it was proved by Kilpeläinen and Zhong [8], and by Trudinger and Wang [14].

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