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Global Non-Small Data Existence of Spherically Symmetric Solutions to Nonlinear Viscoelasticity in a Ball

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Abstract. We consider some initial-boundary value problems for non-linear equations of the three dimensional viscoelasticity. We examine the Dirichlet and the Neumann boundary conditions. We assume that the stress tensor is a nonlinear tensor valued function depending on the strain tensor fulfilling the rules of the continuum mechanics. We consider the initial-boundary value problems in a ball B_R with radius R. Since, we are interested in proving global existence the spherically symmetric solutions are considered. Therefore we have to examine the spherically symmetric viscoelasticity system in spherical coordinates. Applying the energy method implies estimates in weighted anisotropic Sobolev spaces, where the weight is a power function of radius. Hence the origin of coordinates becomes a singular point. First the existence of weak solutions is proved. Next having appropriate estimates the weak solutions appear bounded and continuous. We have to emphasize that non-small data problem is considered.
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1. Introduction

Before starting to present our results, we recall some most important facts from the nonlinear theory of viscoelasticity. Among the papers devoted to

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nonlinear viscoelasticity we mention below some of them. The global solution (in time) for sufficiently small and smooth data are proved by Ponce (cf. [16]), Kawashima and Shibata (cf. [9]) for quasilinear hyperbolic system of 2-nd order with viscosity. The one-dimensional viscoelasticity was considered by Andrews (see [1]).

In paper [10], Kobayashi, Pecher and Shibata proved global in time solution to a nonlinear wave equation with viscoelasticity under the special assumption about nonlinearity. In paper [15], Pawłow and Zajączkowski showed the existence, uniqueness of global in time, regular solutions to the Cahn-Hilliard system coupled with viscoelasticity.

In our paper we consider more general nonlinear system of viscoelasticity with the boundary and initial conditions because the stress tensor is a general nonlinear function depending on a strain. We assume that the stress tensor is a function of a strain at a given instant of time t, but it does not depend on strains at time $t' < t$. It is worth to emphasize that our constitutive relation for the stress tensor and another constitutive relation satisfy the rules of continuum mechanics.

In order to prove the global (in time) solution for non-small data for nonlinear system of viscoelasticity (cf. formulae (1.1)–(1.3)) we consider the spherically symmetric case and use anisotropic Sobolev spaces with weights.

Speaking precisely more, we consider the motion of viscoelastic medium described by the following system of equations (cf. [3–5, 7, 8, 14])

$$\varrho u_{,tt} = \operatorname{div}\sigma + \varrho f, \tag{1.1}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ is the displacement vector, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is a given system of Cartesian coordinates, $t \in \mathbb{R}_+ \cup \{0\}$, ϱ is the mass density, $\sigma = \sigma(x,t) \in \mathbb{R}^9$ the stress tensor, $f = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ the external force field.

We examine system (1.1) in a bounded domain $\Omega \subset \mathbb{R}^3$ with the boundary conditions

$$\text{either} \quad \bar{n}\cdot\sigma|_S = 0 \quad \text{or} \quad u|_S = 0, \tag{1.2}$$

where $S = \partial\Omega$, $\bar{n}$ is the unit outward normal to S vector.

Moreover we add the initial conditions

$$u|_{t=0} = u_0, \quad u_{,t}|_{t=0} = u_1. \tag{1.3}$$

We shall assume that

$$\sigma = \frac{\partial F}{\partial \varepsilon}(\varepsilon) + \mu_0 \varepsilon_{,t}, \tag{1.4}$$

where $\varepsilon = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the linearized strain tensor, $F = F(\varepsilon)$ is some function which will be specified later and μ_0 is a positive constant.

Our aim is to prove the global existence of solutions to problem (1.1)–(1.4) for non-small data.

Since we do not know how to show the existence in a general case we restrict our considerations to the spherically symmetric case. We assume that Ω is a ball B_R with radius R centered at the origin of the introduced Cartesian coordinates. We introduce the spherical coordinates r, φ, ϑ by the relations

$$x_1 = r\cos\varphi\sin\vartheta, \quad x_2 = r\sin\varphi\sin\vartheta, \quad x_3 = r\cos\vartheta.$$

With these coordinates we connect the orthonormal vectors

$$\begin{aligned}\bar{e}_r &= (\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta),\\ \bar{e}_\vartheta &= (\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, -\sin\vartheta),\\ \bar{e}_\varphi &= (-\sin\varphi, \cos\varphi, 0).\end{aligned}$$

Then we define $u_r = u\cdot\bar{e}_r$, $u_\vartheta = u\cdot\bar{e}_\vartheta$, $u_\varphi = u\cdot\bar{e}_\varphi$, $\varepsilon_{rr} = \bar{e}_r\cdot\varepsilon\cdot\bar{e}_r = u_{r,r}$, $\varepsilon_{\varphi\varphi} = \frac{u_r}{r}$, $\varepsilon_{\vartheta\vartheta} = \frac{u_r}{r}$. Since the spherically symmetric case is considered we have $u_\vartheta = u_\varphi = 0$.

To simplify the notation we introduce

$$w = u_r. \tag{1.5}$$

Assuming $\varrho = 1$ and transforming equations (1.1) to the spherical coordinates we obtain

$$w_{,tt} = \frac{1}{r^2}(\sigma_{rr}r^2)_{,r} - \frac{1}{r}(\sigma_{\vartheta\vartheta} + \sigma_{\varphi\varphi}) + f_r, \tag{1.6}$$

where

$$\sigma_{rr} = \frac{\partial F}{\partial\varepsilon_{rr}} + \mu_0\varepsilon_{rr,t}, \quad \sigma_{\vartheta\vartheta} = \frac{\partial F}{\partial\varepsilon_{\vartheta\vartheta}} + \mu_0\varepsilon_{\vartheta\vartheta,t}, \quad \sigma_{\varphi\varphi} = \frac{\partial F}{\partial\varepsilon_{\varphi\varphi}} + \mu_0\varepsilon_{\varphi\varphi,t}. \tag{1.7}$$

Let us introduce the quantity

$$F(\varepsilon) = \psi(w_{,r}, \eta) \tag{1.8}$$

where $\eta = \frac{w}{r}$. Then (1.6) takes the form

$$w_{,tt} = \frac{1}{r^2}\left[\left(\frac{\partial\psi}{\partial w_{,r}} + \mu_0 w_{,rt}\right)r^2\right]_{,r} - \frac{1}{r}\left(\frac{\partial\psi}{\partial\eta} + \mu_0\frac{w_{,t}}{r}\right) \tag{1.9}$$

and in view of (1.3) we have the initial conditions

$$w|_{t=0} = w_0, \quad w_{,t}|_{t=0} = w_1, \tag{1.10}$$

and in view of (1.2), (1.7) the boundary condition

$$\left(\frac{\partial\psi}{\partial w_{,r}} + \mu_0 w_{,rt}\right)\Big|_{S_R} = 0, \tag{1.11}$$

where $S_R = \partial B_R$.

In this paper we also consider the Dirichlet boundary condition

$$w|_{S_R} = 0. \tag{1.12}$$

To formulate the main results of this paper we need

Assumptions. *Let us introduce the notation* $\vartheta = w_{,r}$, $\eta = \frac{w}{r}$.

1. $\psi(\vartheta, \eta) = \psi_1(\vartheta) + \psi_2(\eta)$
2. *There exist positive constants* α_1, α_2, β_1, β_2 *such that*

$$\frac{\partial^2 \psi_1}{\partial \vartheta^2} \geq \alpha_1, \quad \frac{\partial^2 \psi_2}{\partial \eta^2} \geq \alpha_2, \quad \left|\frac{\partial \psi_1}{\partial \vartheta}\right| \leq \beta_1 |\vartheta|, \quad \left|\frac{\partial \psi_2}{\partial \eta}\right| \leq \beta_2 |\eta|.$$

3. $w_0 \in H^2_{\mu_1}(0, R)$, $w_1 \in H^2_{\mu_1}(0, R)$, $\mu_1 \in \big(\frac{7}{4}, 1 + \frac{\sqrt{5}}{2}\big)$
4. $w_1 \in L_{2,\mu_2}(0, R)$, $w_0 \in H^1_{\mu_2}(0, R)$, $\mu_2 \in \big(0, \frac{3}{2}\big)$
5. $w_1 \in L_{2,\mu_3}(0, R)$, $w_0 \in H^1_{\mu_3}(0, R)$, $\mu_3 \in \big(0, \frac{1}{2}\big)$.

Main Theorem. *Let Assumptions* 1–3 *hold. Let Assumption* 4 *for* $\mu_2 = \mu_1 - 1$ *be satisfied. Then there exists a solution to problem* (1.9), (1.10), (1.12) *such that*

$$w_{,tt} \in B\big(0, T; L_{2,\mu_1}(0, R)\big), \ w_{,t} \in B\big(0, T; H^1_{\mu_1}(0, R)\big), \ w_{,r} \in B\big(0, T; H^2_{\mu_1}(0, R)\big)$$
$$w_{,tt} \in L_2\big(0, T; H^1_{\mu_1}(0, R)\big), \ w_{,t} \in L_2\big(0, T; H^1_{\mu_1 - 1}(0, R)\big).$$

Let Assumption 5 *be additionally satisfied. Then*

$$w \in L_\infty\big((0, R) \times (0, T)\big), \quad w \in B\Big(0, T; C^{\frac{1}{2} - \frac{\mu_3}{2}}(0, R)\Big),$$
$$w \in L_\beta\Big(0, T; C^{\frac{\beta - 1}{\beta}}(0, T)\Big), \quad \beta \in \left(1, \frac{2}{2\mu_3 + 1}\right), \quad w_{,t} \in L_2\Big(0, T; C^{\frac{1}{2} - \frac{\mu_3}{2}}(0, R)\Big).$$

Our paper is organized as follows. In the introduction the formulation of the considered problem and the main results were presented. In Section 2 the notation is introduced. Mainly, we define anisotropic Sobolev spaces with weights. Section 3 is devoted to the proof of energy type estimates to solutions of problem (1.9), (1.10), (1.12).

In Section 4 the existence of the global solution for non-small data of the problem (1.9), (1.10), (1.12) is proved. Finally Section 5 contains some concluding remarks.

2. Notation and auxiliary results

By c we denote the generic constant which changes from formula to formula. By $c(\sigma)$, $\sigma > 0$, we denote a generic function which is always positive and increasing.

We replace forms right-hand side (left-hand side) by the abbreviation r.h.s. (l.h.s.). We mark $w_{,t} = \partial_t w$, $w_{,r} = \partial_r w$ and so on. By $B(I)$ we denote the space of bounded functions on the interval I. By $H^k_\mu(0,R)$, $\mu \in \mathbb{R}$, $k \in \mathbb{N} \cup \{0\}$ we denote a weighted Sobolev space with the finite norm

$$\|u\|_{H^k_\mu(0,R)} = \left(\sum_{\alpha=0}^{k} \int_0^R |\partial_r^\alpha u|^2 r^{2(\mu-k+|\alpha|)} dr \right)^{\frac{1}{2}}$$

By $C^\alpha(I)$, $\alpha \in (0,1)$ we denote the Hölder space with the finite norm

$$\|u\|_{C^\alpha(I)} = \sup_{\tau\in I} |u(\tau)| + \sup_{\tau',\tau''\in I} \frac{|u(\tau') - u(\tau'')|}{|\tau' - \tau''|^\alpha}.$$

Next we recall the Hardy inequality (see [2, Chapter 1, Section 2.15])

$$\left| \frac{1}{p'} - \mu \right|^p \int_0^\infty r^{p(\mu-1)} |f|^p dr \le \int_0^\infty r^{p\mu} |f_{,r}|^p dr, \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $\mu \in \mathbb{R}$ and $\mu \ne \frac{1}{p'}$. The inequality holds also for functions with compact support. Assuming that $\operatorname{supp} f \subset [0,R]$ we introduce $F(x) = \int_x^\infty f(y)dy$ and repeat the proof from [2, Chapter 1, Section 2.15]. From [12, Chapter 2, Section 3] we have the imbedding

$$\|u\|_{L_q(0,T;L_p(0,R))} \le c\Big(\|u\|_{L_\infty(0,T;L_2(0,R))} + \|u_{,r}\|_{L_2(0,T;L_2(0,R))} \Big) \tag{2.2}$$

where $\frac{1}{p} + \frac{2}{q} \ge \frac{1}{2}$.

Finally we consider the problem

$$\begin{aligned} u_{,t} - u_{,rr} &= f, \\ u|_{t=0} &= u_0, \\ u|_{r=R} &= 0. \end{aligned} \tag{2.3}$$

To examine nonstationary problems (2.3) we need anisotropic weighted Sobolev spaces $V^{2,1}_{p,\nu}((0,R)\times(0,T))$, $p \in (1,\infty)$, $\nu \in \mathbb{R}$, of functions with the finite norm

$$\|u\|_{V^{2,1}_{p,\nu}((0,R)\times(0,T))} = \left(\sum_{\alpha+2a\le 2} \int_0^T \int_0^R |\partial_r^\alpha \partial_t^a u|^p r^{p(\nu-2+\alpha+2a)} dr \right)^{\frac{1}{p}}.$$

Spaces $V^2_{p,\nu}(0,R)$ appropriate for elliptic problems were introduced in [13]. The following result is valid.

Lemma 2.1. *Let us assume that* $f \in L_{p,\nu}((0,R)\times(0,T))$, $u_0 \in V_{p,\nu}^{2-\frac{2}{p}}(0,R)$. *Then there exists a solution to problem* (2.3) *such that* $u \in V_{p,\nu}^{2,1}((0,R)\times(0,T))$ *and*

$$\|u\|_{V_{p,\nu}^{2,1}((0,R)\times(0,T))} \le c\Big(\|f\|_{L_{p,\nu}((0,R)\times(0,T))} + \|u_0\|_{V_{p,\nu}^{2-\frac{2}{p}}(0,R)}\Big). \tag{2.4}$$

In the case of elliptic equations such result was proved in [11] for $p=2$ and in [13] for any $p\in(1,\infty)$. The weighted Sobolev spaces with fractional derivatives are introduced in [13]. In the nonstationary case, Lemma 2.1 follows from [19] in the case $p=2$. For the general p, Lemma 2.1 results from considerations in [17–19].

Finally, we introduce spaces used in this paper. We shall define them by introducing finite norms:

1. Besov space $B_{p,\nu}^l(B_R)$, $l\in\mathbb{R}_+$, $p\in(1,\infty)$, $\nu\in\mathbb{R}$,

$$\|u\|_{B_{p,\nu}^l(B_R)} = \|u\|_{\dot{B}_{p,\nu}^{l-[l]}(B_R)} + \|u\|_{V_{p,\nu}^{[l]}(B_R)}$$

where $[l]$ is the integer part of l,

$$\|u\|_{\dot{B}_{p,\nu}^{\alpha}(B_R)} = \left(\int_0^R\int_0^R \frac{|u(r_1)r_1^\nu - u(r_2)r_2^\nu|^p}{|r_1-r_2|^{1+p\alpha}}dr_1dr_2\right)^{\frac{1}{p}},$$

where $\alpha\in(0,1)$ and

$$\|u\|_{V_{p,\nu}^k(B_R)} = \left(\sum_{\alpha\le k}\int_0^R |\partial_r^\alpha u|^p r^{p(\nu-k+\alpha)}dr\right)^{\frac{1}{p}}.$$

2. $B(B_R\times(0,T))$ is the space of bounded functions.
3. $C^\alpha(\bar{B}_R)$, $C^{\alpha,\beta}(\bar{B}_R\times[0,T])$, $\alpha,\beta\in(0,1)$, are the Hölder spaces with the finite norms

$$\|u\|_{C^\alpha(\bar{B}_R)} = \|u\|_{B(\bar{B}_R)} + \sup_{r_1,r_2\in\bar{B}_R}\frac{|u(r_1)-u(r_2)|}{|r_1-r_2|^\alpha}$$

$$\|u\|_{C^{\alpha,\beta}(\bar{B}_R\times[0,T])} = \|u\|_{B(\bar{B}_R\times[0,T])} + \sup_{t\in[0,T]}\sup_{r_1,r_2\in\bar{B}_R}\frac{|u(r_1,t)-u(r_2,t)|}{|r_1-r_2|^\alpha} + \sup_{r\in\bar{B}_R}\sup_{t',t''\in[0,T]}\frac{|u(r,t')-u(r,t'')|}{|t'-t''|^\beta}.$$

4. By $L_q^l(0,T;W_p^k(B_R))$, $l,k\in\mathbb{N}\cup\{0\}$, $p,q\in[1,\infty]$ we denote a space with finite norm $\|\partial_t^l u\|_{L_q(0,T;W_p^k(B_R))}$.

Let $B_R=\{r\in\mathbb{R}:\ r<R\}$ and $B_R^T=B_R\times(0,T)$. We use the Sobolev-Slobodetski spaces $W_p^{l,\frac{l}{2}}(B_R^T)$, $l\in\mathbb{R}_+$, $p\in(1,\infty)$ with the finite norm

$$\|u\|_{W_p^{l,\frac{l}{2}}(B_R^T)}=\left(\sum_{\alpha+2a\le[l]}\int\limits_{B_R^T}|\partial_r^\alpha\partial_t^a u|^p drdt\right.$$
$$+\sum_{\alpha+2a=[l]}\int\limits_0^T\int\limits_{B_R}\int\limits_{B_R}\frac{|\partial_{r'}^\alpha\partial_t^a u-\partial_{r''}^\alpha\partial_t^a u|^p}{|r'-r''|^{1+p(l-[l])}}dr'dr'dt$$
$$\left.+\sum_{\alpha+2a=[l]}\int\limits_{B_R}\int\limits_0^T\int\limits_0^T\frac{|\partial_r^\alpha\partial_{t'}^a u-\partial_r^\alpha\partial_{t''}^a u|^p}{|t'-t'|^{1+p\left(\frac{l}{2}-\left[\frac{l}{2}\right]\right)}}dt'dt''dr\right)^{\frac{1}{p}}.$$

In the case of l, $\frac{l}{2}$ integer the last two terms in the above norm disappear.

Moreover, $B_{p,0}^l(B_R)=B_p^l(B_R)$. Similarly as $B_p^l(B_R)$ we define the space $B_p^l(0,T)$ introducing the finite norm

$$\|u\|_{B_p^l(0,T)}=\|u\|_{W_p^{[l]}(0,T)}+\left(\int\limits_0^T\int\limits_0^T\frac{|\partial_{t'}^{[l]}u(t')-\partial_{t''}^{[l]}u(t'')|^p}{|t'-t''|^{1+p(l-[l])}}dt'dt''\right)^{\frac{1}{p}}.$$

For noninteger l we have

$$\|u\|_{W_p^l(B_R)}=\|u\|_{B_p^l(B_R)},\quad \|u\|_{W_p^l(0,T)}=\|u\|_{B_p^l(0,T)}.$$

Similar equivalence we have for the weighted spaces.

3. Estimates

First, we have

Lemma 3.1. *Let us assume that*

$$\int\limits_{B_R}\left[\frac{1}{2}w_1^2+\psi\left(w_{0,r},\frac{w_0}{r}\right)\right]r^2dr\equiv c_0^2<\infty \tag{3.1}$$

Then, solutions to problems (1.9)–(1.11) *and* (1.9), (1.10), (1.12) *satisfy*

$$\begin{aligned}&\int\limits_{B_R}\left[\frac{1}{2}w_{,t}^2+\psi\left(w_{,r},\frac{w}{r}\right)\right]r^2dr+\int\limits_{B_R^t}\left(w_{,rt}^2+\frac{w_{,t}^2}{r^2}\right)r^2drdt'\\&=\int\limits_{B_R}\left[\frac{1}{2}w_1^2+\psi\left(w_{0,r},\frac{w_0}{r}\right)\right]r^2dr=c_0^2.\end{aligned}\tag{3.2}$$

Proof. Multiplying (1.9) by $w_{,t}r^2$ and integrating over B_R yields

$$\frac{1}{2}\frac{d}{dt}\int\limits_{B_R} w_{,t}^2 r^2 dr + \int\limits_{B_R}\left(\frac{\partial\psi}{\partial w_{,r}} + \mu_0 w_{,rt}\right) w_{,rt} r^2 dr + \int\limits_{B_R}\left(\frac{\partial\psi}{\partial\eta} + \mu_0\frac{w_{,t}}{r}\right) w_{,t} r dr = 0,$$

where the boundary condition either (1.11) or (1.12) were used. Hence

$$\frac{d}{dt}\int\limits_{B_R}\left[\frac{1}{2}w_{,t}^2 + \psi\left(w_{,r}, \frac{w}{r}\right)\right] r^2 dr + \mu_0\int\limits_{B_R}\left(w_{,rt}^2 + \frac{w_{,t}^2}{r^2}\right) r^2 dr = 0. \tag{3.3}$$

Integrating (3.3) with respect to time and using the initial condition (1.10) implies (3.2). This concludes the proof. □

In the proof of Lemma 3.1 the crucial step is integration by parts which can be performed under both boundary conditions (1.11) and (1.12).

Problems (1.9)–(1.11) and (1.9), (1.10), (1.12) are considered in ball B_R, so the energy estimate (3.2) suggests that weighted Sobolev spaces are natural to treat them. This is connected with the fact that the transformation of the original problems (1.1)–(1.3) to the spherically symmetric cases generates the weight r^2 which is the Jacobian of the mapping from the Cartesian to the spherical coordinates. This also suggests an existence of some singularity of solutions at the origin of coordinates. Therefore we shall use weighted Sobolev spaces to control the behaviour of solutions to problems (1.9)–(1.11) and (1.9), (1.10), (1.12) at the origin of coordinates. We shall restrict our considerations to the L_2-approach because energy type estimates are very natural for problems (1.9)–(1.11) and (1.9), (1.10), (1.12). First we shall derive an analogue of Lemma 3.1 in the case of weighted Sobolev spaces.

Lemma 3.2. *Let us assume that* $\vartheta = w_{,r}$, $\eta = \frac{w}{r}$; α_1, α_2, β_1, β_2 *are positive constants. Assume*

$$\psi(\vartheta,\eta) = \psi_1(\vartheta) + \psi_2(\eta), \quad \psi_1(\vartheta) \geq \alpha_1\vartheta^2, \quad \psi_2(\eta) \geq \alpha_2\eta^2, \tag{3.4}$$

$$\left|\frac{\partial\psi_1}{\partial\vartheta}\right| \leq \beta_1|\vartheta|, \quad \left|\frac{\partial\psi_2}{\partial\eta}\right| \leq \beta_2|\eta|, \tag{3.5}$$

$$w_1 \in L_{2,\mu}(0,R), \quad w_0 \in H_\mu^1(0,R), \quad \mu \in \left(0, \frac{3}{2}\right). \tag{3.6}$$

Then, for solutions to problems (1.9)–(1.11) *and* (1.9), (1.10), (1.12), *the following estimate holds*

$$\begin{aligned}\int\limits_0^R\left(\frac{1}{2}w_{,t}^2 + \alpha_1 w_{,r}^2 + \alpha_2 w^2 r^{-2}\right) r^{2\mu} dr + \frac{\mu_0}{2}\int\limits_0^t\int\limits_0^R w_{,rt}^2 r^{2\mu} dr dt' + \int\limits_0^t\int\limits_0^R w_{,t}^2 r^{2\mu-2} dr dt' \\ \leq c(R,\mu,t)\int\limits_0^R\left(\frac{1}{2}w_1^2 + \alpha_1 w_{0,r}^2 + \alpha_2 w_0^2 r^{-2}\right) r^{2\mu} dr \equiv c_1^2\end{aligned} \tag{3.7}$$

where $\mu \in \big(0, \frac{3}{2}\big)$.

Proof. Multiplying (1.9) by $w_{,t}r^{2\mu}$ and integrating over B_R we obtain

$$\int_0^R w_{,tt}w_{,t}r^{2\mu}dr+\int_0^R\left(\frac{\partial\psi}{\partial\vartheta}+\mu_0 w_{,rt}\right)r^2(r^{2\mu-2}w_{,t})_{,r}dr+\int_0^R\left(\frac{1}{r}\frac{\partial\psi}{\partial\eta}+\mu_0\frac{w_{,t}}{r^2}\right)w_{,t}r^{2\mu}dr=0.$$

Performing calculations imply

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^R w_{,t}^2r^{2\mu}dr+\int_0^R\frac{\partial\psi}{\partial\vartheta}\vartheta_{,t}r^{2\mu}dr+\int_0^R\frac{\partial\psi}{\partial\eta}\eta_{,t}r^{2\mu}dr\\ &+\mu_0\int_0^R w_{,rt}^2r^{2\mu}dr+\mu_0\int_0^R w_{,t}^2r^{2\mu-2}dr\\ =&-(2\mu-2)\mu_0\int_0^R\frac{\partial\psi}{\partial\vartheta}w_{,t}r^{2\mu-1}dr-(2\mu-2)\mu_0\int_0^R w_{,rt}w_{,t}r^{2\mu-1}dr.\end{aligned}\tag{3.8}$$

In view of assumption (3.4) we have

$$\int_0^R\frac{\partial\psi}{\partial\vartheta}\vartheta_{,t}r^{2\mu}dr=\frac{d}{dt}\int_0^R\psi_1(\vartheta)r^{2\mu}dr,\quad\int_0^R\frac{\partial\psi}{\partial\eta}\eta_{,t}r^{2\mu}dr=\frac{d}{dt}\int_0^R\psi_2(\eta)r^{2\mu}dr.$$

Moreover, the first integral on the r.h.s. of (3.8) can be estimated by

$$|2\mu-2|\beta_1\int_0^R|w_{,r}|\,|w_{,t}|r^{2\mu-1}dr\le\frac{\varepsilon_1}{2}\int_0^R w_{,t}^2r^{2\mu-2}dr+\frac{2(1-\mu)^2\beta_1^2}{\varepsilon_1}\int_0^R w_{,r}^2r^{2\mu}dr$$

and the second equals

$$\begin{aligned}&-(\mu-1)\mu_0\int_0^R\partial_r w_{,t}^2r^{2\mu-1}dr\\ =&-(\mu-1)\mu_0\int_0^R\partial_r(w_{,t}^2r^{2\mu-1})dr+(\mu-1)(2\mu-1)\mu_0\int_0^R w_{,t}^2r^{2\mu-2}dr\equiv I_1.\end{aligned}$$

The first integral in I_1 equals

$$(1-\mu)\mu_0\big[w_{,t}^2(R)R^{2\mu-1}-\lim_{r\to0}w_{,t}^2(r)r^{2\mu-1}\big]\tag{3.9}$$

and the second must be absorbed by the last term on the l.h.s. of (3.8). For this purpose we need

$$(\mu-1)(2\mu-1)<1-\frac{1}{2}\varepsilon_1, \quad \text{so} \quad \frac{3-\sqrt{9-4\varepsilon_1}}{4}<\mu<\frac{3+\sqrt{9-4\varepsilon_1}}{4} \tag{3.10}$$

In view of the above considerations we obtain from (3.8) the inequality

$$\begin{aligned}&\frac{d}{dt}\int\limits_0^R\left[\frac{1}{2}w_{,t}^2+\psi_1(\vartheta)+\psi_2(\eta)\right]r^{2\mu}dr+\mu_0\int\limits_0^R w_{,rt}^2r^{2\mu}dr\\&+\mu_0\left[1-\frac{1}{2}\varepsilon_1-(\mu-1)(2\mu-1)\right]\int\limits_0^R w_{,t}^2r^{2\mu-2}dr\\&\le\frac{2(1-\mu)^2\beta_1^2}{\varepsilon_1\alpha_1^2}\int\limits_0^R\psi_1(\vartheta)r^{2\mu}dr+(1-\mu)\mu_0\big[w_{,t}^2(R)R^{2\mu-1}-\lim_{r\to0}w_{,t}^2(r)r^{2\mu-1}\big].\end{aligned} \tag{3.11}$$

For $\mu\le\frac{1}{2}$, the coefficient near the last integral on the l.h.s. of (3.11) equals $\mu_0\big[1-\frac{1}{2}\varepsilon_1+(1-\mu)(1-2\mu)\big]$, so it is positive for ε_1 sufficiently small without other restrictions on μ.

For $\mu<1$ and because $w_{,t}^2(r)r^{2\mu-1}$ is positive for any $r>0$ we can omit the last term in the second expression on the r.h.s. of (3.11).

For $\mu>1$, the first term in the second expression on the r.h.s. of (3.11) can be omitted. In this case equality (3.2) implies that $w_{,t}$ behaves as $r^{-\varepsilon}$ for $\varepsilon<\frac{1}{2}$ for small r. Then $\lim_{r\to0}w_{,t}^2(r)r^{2\mu-1}\le\lim_{r\to0}r^{-2\varepsilon+2\mu-1}=0$ because $\mu>1$.

Finally for ε_1 close to 0 restriction (3.10) implies

$$0<\mu<\frac{3}{2}. \tag{3.12}$$

Integrating (3.11) with respect to time and using assumptions (3.5) we obtain

$$\begin{aligned}&\int\limits_0^R\left(\frac{1}{2}w_{,t}^2+\alpha_1w_{,r}^2+\alpha_2w^2r^{-2}\right)r^{2\mu}dr\\&+\mu_0\int\limits_0^t\int\limits_0^R w_{,rt}^2r^{2\mu}drdt'+\mu_1\int\limits_0^t\int\limits_0^R w_{,t}^2r^{2\mu-2}drdt'\\&\le c(t)\left[\int\limits_0^t w_{,t}^2(R)R^{2\mu-1}dt'+\int\limits_0^R\left(\frac{1}{2}w_{,t}^2(0)+\alpha_1w_{,r}^2(0)+\alpha_2w^2(0)r^{-2}\right)r^{2\mu}dr\right],\end{aligned} \tag{3.13}$$

where $\mu_1 = \mu_0\left[1-\frac{1}{2}\varepsilon_1+(1-\mu)(2\mu-1)\right]$. We estimate the first integral on the r.h.s. of (3.13) by

$$\int_0^t w_{,t}^2(R)dt' \le \int_0^t\int_{\frac{R}{2}}^R\left(\varepsilon_2 w_{,rt}^2 + c\left(\frac{1}{\varepsilon_2}\right)w_{,t}^2\right)drdt' \le c(R)\int_0^t\int_0^R\left(\varepsilon_2 w_{,rt}^2 + c\left(\frac{1}{\varepsilon_2}\right)w_{,t}^2\right)r^{2\mu}drdt'.$$

Hence for ε_2 sufficiently small we obtain from (3.13) the inequality

$$\begin{aligned}
&\int_0^R\left(\frac{1}{2}w_{,t}^2+\alpha_1 w_{,r}^2+\alpha_2 w^2 r^{-2}\right)r^{2\mu}dr\\
&\quad+\frac{\mu_0}{2}\int_0^t\int_0^R w_{,rt}^2 r^{2\mu}drdt' + \mu_1\int_0^t\int_0^R w_{,t}^2 r^{2\mu-2}drdt'\\
&\le c(t,R)\int_0^t\int_0^R w_{,t}^2 r^{2\mu}drdt'\\
&\quad+c(t)\int_0^R\left(\frac{1}{2}w_{,t}^2(0)+\alpha_1 w_{,r}^2(0)+\alpha_2 w^2(0) r^{-2}\right)r^{2\mu}dr.
\end{aligned}\tag{3.14}$$

Finally, applying the Gronwall inequality we obtain from (3.14) estimate (3.7). This concludes the proof. □

Next we have:

Lemma 3.3. *Let us assume that*

$$\frac{\partial^2\psi_1}{\partial\vartheta^2}\ge\alpha_1,\quad \frac{\partial^2\psi_2}{\partial\eta^2}\ge\alpha_2,\quad \left|\frac{\partial^2\psi_1}{\partial\vartheta^2}\right|\le\beta_1,\quad \left|\frac{\partial^3\psi_1}{\partial\vartheta^3}\right|\le c_1,\quad \left|\frac{\partial^3\psi_2}{\partial\eta^3}\right|\le c_2 \tag{3.15}$$

$$\begin{aligned}
&\eta_{,t},\vartheta_{,t}\in L_{3,\frac{2}{3}\mu}((0,R)\times(0,T)),\quad w_{,t}(0)=w_1\in L_{2,\mu-1}(0,R),\\
&w_{,rt}(0)=w_{1,r}\in L_{2,\mu}(0,R),\quad w_{,tt}(0)\in L_{2,\mu}(0,R)\quad \textit{and}\\
&w_{,tt}(0)=\frac{1}{r^2}\left[\left(\frac{\partial\psi}{\partial\vartheta}\left(w_{0,r},\frac{w_0}{r}\right)+\mu_0 w_{1,r}\right)r^2\right]_{,r}-\frac{1}{r}\left(\frac{\partial\psi}{\partial\eta}\left(w_{0,r},\frac{w_0}{r}\right)+\frac{w_1}{r}\right).
\end{aligned}\tag{3.16}$$

Then solutions to (1.9), (1.10) *and either* (1.11) *or* (1.12) *satisfy the inequalities:*

For $\mu \in \left(0, \frac{3}{2}\right)$,

$$\int_0^R \left(w_{,tt}^2 + w_{,tr}^2 + w_{,t}^2 r^{-2}\right) r^{2\mu} dr + \mu_0 \int_0^t \int_0^R w_{,rtt}^2 r^{2\mu} dr dt'$$

$$+ \mu_0 \int_0^t \int_0^R w_{,tt}^2 r^{2\mu-2} dr dt' \leq c(t) \Bigg[\int_0^t \int_0^R \left(|\vartheta_{,t}|^3 r^{2\mu} + |\eta_{,t}|^3 r^{2\mu}\right) dr dt' \tag{3.17}$$

$$+ \int_0^R \left(w_{,tt}^2(0) + w_{,tr}^2(0) + w_{,t}^2(0) r^{-2}\right) r^{2\mu} dr \Bigg];$$

for $\mu \in \left(1, \frac{2+\sqrt{5}}{2}\right)$,

$$\int_0^R \left(w_{,tt}^2 + w_{,tr}^2 + w_{,t}^2 r^{-2}\right) r^{2\mu} dr + \mu_0 \int_0^t \int_0^R \left(w_{,rtt}^2 + w_{,tt}^2 r^{-2}\right) r^{2\mu} dr dt'$$

$$\leq c(t) \Bigg[\int_0^t \int_0^R \left(|\vartheta_{,t}|^3 + |\eta_{,t}|^3\right) r^{2\mu} dr dt' \tag{3.18}$$

$$+ \int_0^R \left(w_{,tt}^2(0) + w_{,tr}^2(0) + w_{,t}^2(0) r^{-2}\right) r^{2\mu} dr \Bigg].$$

Proof. Differentiating (1.9) with respect to t, multiplying the result by $w_{,tt} r^{2\mu}$, integrating over B_R and using boundary conditions either (1.11) or (1.12) we obtain $\frac{1}{2}\frac{d}{dt}\int_0^R w_{,tt}^2 r^{2\mu} dr + \int_0^R \left(\frac{\partial^2 \psi}{\partial \vartheta^2}\vartheta_{,t} + \mu_0 w_{,rtt}\right) r^2 \left(r^{2\mu-2} w_{,tt}\right)_{,r} dr + \int_0^R \frac{\partial^2 \psi}{\partial \eta^2}\eta_{,t}\eta_{,tt} r^{2\mu} dr + \mu_0 \int_0^R w_{,tt}^2 r^{2\mu-2} dr = 0$. Performing calculations it follows

$$\frac{1}{2}\frac{d}{dt}\int_0^R w_{,tt}^2 r^{2\mu} dr + \int_0^R \frac{\partial^2 \psi}{\partial \vartheta^2}\vartheta_{,t}\left(\vartheta_{,tt} r^{2\mu} + (2\mu-2) w_{,tt} r^{2\mu-1}\right) dr$$

$$+ \mu_0 \int_0^R \left(w_{,rtt}^2 r^{2\mu} + (2\mu-2) w_{,rtt} w_{,tt} r^{2\mu-1}\right) dr$$

$$+ \int_0^R \frac{\partial^2 \psi}{\partial \eta^2}\eta_{,t}\eta_{,tt} r^{2\mu} dr + \mu_0 \int_0^R w_{,tt}^2 r^{2\mu-2} dr$$

$$= 0.$$

Continuing, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_0^R w_{,tt}^2 r^{2\mu}dr + \frac{1}{2}\int\limits_0^R \frac{\partial^2\psi}{\partial\vartheta^2}\frac{\partial}{\partial t}w_{,tr}^2 r^{2\mu}dr + \frac{1}{2}\int\limits_0^R \frac{\partial^2\psi}{\partial\eta^2}\frac{\partial}{\partial t}\eta_{,t}^2 r^{2\mu}dr\\
&+\mu_0\int\limits_0^R w_{,rtt}^2 r^{2\mu}dr + \mu_0\int\limits_0^R w_{,tt}^2 r^{2\mu-2}dr\\
=&-(2\mu-2)\mu_0\int\limits_0^R w_{,rtt}w_{,tt}r^{2\mu-1}dr - (2\mu-2)\int\limits_0^R \frac{\partial^2\psi}{\partial\vartheta^2}w_{,rt}w_{,tt}r^{2\mu-1}dr.
\end{aligned}$$

Continuing, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_0^R \left(w_{,tt}^2 + \frac{\partial^2\psi_1}{\partial\vartheta^2}w_{,tr}^2 + \frac{\partial^2\psi_2}{\partial\eta^2}\eta_{,t}^2\right)r^{2\mu}dr\\
&+\mu_0\int\limits_0^R w_{,rtt}^2 r^{2\mu}dr + \mu_0\int\limits_0^R w_{,tt}^2 r^{2\mu-2}dr\\
=&-\frac{1}{2}\int\limits_0^R \frac{\partial^3\psi_1}{\partial\vartheta^3}\vartheta_{,t}^3 r^{2\mu}dr - \frac{1}{2}\int\limits_0^R \frac{\partial^3\psi_2}{\partial\eta^3}\eta_{,t}^3 r^{2\mu}dr\\
&-(2\mu-2)\mu_0\int\limits_0^R w_{,rtt}w_{,tt}r^{2\mu-1}dr - (2\mu-2)\int\limits_0^R \frac{\partial^2\psi_1}{\partial\vartheta^2}w_{,rt}w_{,tt}r^{2\mu-1}dr.
\end{aligned}\tag{3.19}$$

Now we estimate the particular terms from the r.h.s. of (3.19). The third term on the r.h.s. of (3.19) equals

$$\begin{aligned}
&-(\mu-1)\mu_0\int\limits_0^R \frac{\partial}{\partial r}w_{,tt}^2 r^{2\mu-1}dr\\
=&-(\mu-1)\mu_0\int\limits_0^R \frac{\partial}{\partial r}(w_{,tt}^2 r^{2\mu-1})dr + (\mu-1)(2\mu-1)\mu_0\int\limits_0^R w_{,tt}^2 r^{2\mu-2}dr,
\end{aligned}$$

where the first integral equals

$$-(\mu-1)\mu_0\left(w_{,tt}^2(R)R^{2\mu-1} - \lim_{r\to 0} w_{,tt}^2(r)r^{2\mu-1}\right) \equiv I_1.$$

For $\mu<1$ we have

$$I_1 \le (1-\mu)\mu_0 w_{,tt}^2(R)R^{2\mu-1} \equiv I_2,$$

but for $\mu > 1$ it follows that

$$I_1 \le (\mu - 1)\mu_0 \lim_{r\to 0} w_{,tt}^2(r) r^{2\mu-1} \equiv I_3.$$

Looking for such solutions to problems (1.9), (1.10) with either (1.11) or (1.12) that the last integral on the l.h.s. of (3.19) is finite we obtain that $I_3 = 0$.

Applying the Hölder and the Young inequalities to the last term on the r.h.s. of (3.19) we see that it is bounded by

$$\varepsilon_1 \int_0^R w_{,tt}^2 r^{2\mu-2} dr + \frac{4(1-\mu)^2}{\varepsilon_1} \int_0^R \left| \frac{\partial^2 \psi_1}{\partial \vartheta^2} \right|^2 w_{,rt}^2 r^{2\mu} dr.$$

Hence, in view of (3.15), the second integral is estimated by $\frac{4(1-\mu)^2}{\varepsilon_1} \beta_1^2 \int_0^R w_{,rt}^2 r^{2\mu} dr$. In view of (3.15) and in the case $\mu < 1$, from (3.19), we obtain the inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_0^R (w_{,tt}^2 + \psi_{1,\vartheta\vartheta} w_{,tr}^2 + \psi_{2,\eta\eta}\eta_{,t}^2) r^{2\mu} dr + \mu_0 \int_0^R w_{,rtt}^2 r^{2\mu} dr \\ &+ \mu_0(1 + (1-\mu)(2\mu-1) - \varepsilon_1) \int_0^R w_{,tt}^2 r^{2\mu-2} dr \\ &\le c\left(\int_0^R |\vartheta_{,t}|^3 r^{2\mu} dr + \int_0^R |\eta_{,t}|^3 r^{2\mu} dr \right) + c w_{,tt}^2(R,t) + c \int_0^R w_{,rt}^2 r^{2\mu} dr. \end{aligned} \tag{3.20}$$

To guarantee that the coefficient near the last integral on the l.h.s. is positive we need $1 + (1-\mu)(2\mu-1) - \varepsilon_1 > 0$ which implies that

$$\frac{3 - \sqrt{9 - 8\varepsilon_1}}{4} < \mu < \frac{3 + \sqrt{9 - 8\varepsilon_1}}{4} \tag{3.21}$$

Since ε_1 can be chosen arbitrary small we see that (3.21) holds for $\mu \in \left(0, \frac{3}{2}\right)$.

Let us consider the case $\mu > 1$. Then condition (3.21) is too restrictive. To relax the condition we consider the last two terms on the l.h.s. of (3.19) together. Applying the Hardy inequality (see Notation)

$$\left(\mu - \frac{1}{2}\right)^2 \int_0^R u^2 r^{2\mu-2} dr \le \int_0^R u_{,r}^2 r^{2\mu} dr$$

for functions vanishing for $r > R$, we estimate the last two terms on the l.h.s. of (3.19) from below by $\mu_0\left[1 + (1-\mu)(2\mu-1) + \left(\frac{1}{2} - \mu\right)^2 - \varepsilon_1\right] \int_0^R w_{,tt}^2 r^{2\mu-2} dr$. To get any estimate from (3.19) we need

$$1 + (1-\mu)(2\mu-1) + \left(\frac{1}{2} - \mu\right)^2 - \varepsilon_1 > 0 \;\Rightarrow\; \mu^2 - 2\mu + \varepsilon_1 - \frac{1}{4} < 0,$$

so

$$1 < \mu < \frac{2+\sqrt{5-4\varepsilon_1}}{2} < \frac{2+\sqrt{5}}{2}. \tag{3.22}$$

Then from (3.19) we obtain the inequality for $\mu > 1$ and the Dirichlet problem

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^R (w_{,tt}^2+\psi_{1,\vartheta\vartheta}w_{,rt}^2+\psi_{2,\eta\eta}\eta_{,t}^2)r^{2\mu}dr+\mu_0\int_0^R w_{,rtt}^2 r^{2\mu}dr+\mu_0\int_0^R w_{,tt}^2 r^{2\mu}dr \\ &\le c\bigg(\int_0^R |\vartheta_{,t}|^3 r^{2\mu}dr+\int_0^R |\eta_{,t}|^3 r^{2\mu}dr\bigg)+c\int_0^R w_{,rt}^2 r^{2\mu}dr. \end{aligned} \tag{3.23}$$

In the case of the Neumann problem (1.9)–(1.11) we apply the extension theorem to estimate the last but one term on the r.h.s. of (3.20) by

$$\varepsilon_2\int_{\frac{R}{2}}^R w_{,rtt}^2 dr + c\left(\frac{1}{\varepsilon_2}\right)\int_{\frac{R}{2}}^R w_{,tt}^2 dr \le \varepsilon_2 c(R)\int_0^R w_{,rtt}^2 r^{2\mu}dr + c\left(\frac{1}{\varepsilon_2},R\right)\int_0^R w_{,tt}^2 r^{2\mu}dr.$$

Then for sufficiently small ε_2 we obtain from (3.20) the inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^R (w_{,tt}^2+\psi_{1,\vartheta\vartheta}w_{,tr}^2+\psi_{2,\eta\eta}\eta_{,t}^2)r^{2\mu}dr \\ &+\mu_0\int_0^R w_{,rtt}^2 r^{2\mu}dr+\mu_0\big(1+(1-\mu)(2\mu-1)-\varepsilon_1\big)\int_0^R w_{,tt}^2 r^{2\mu-2}dr \\ &\le c\bigg(\int_0^R |\vartheta_{,t}|^3 r^{2\mu}dr+\int_0^R |\eta_{,t}|^3 r^{2\mu}dr\bigg)+c(R)\int_0^R w_{,tt}^2 r^{2\mu}dr+c\int_0^R w_{,rt}^2 r^{2\mu}dr. \end{aligned} \tag{3.24}$$

Integrating (3.23) and (3.24) with respect to time and applying the Gronwall lemma we obtain (3.17) and (3.18). This concludes the proof. □

To estimate the first integral on the r.h.s. of (3.17) and (3.18) we use the Pego transformation

$$p(r,t)=\int_0^r w_t(r',t)r'^2dr', \quad q(r,t)=\mu_0 w_{,r}(r,t)r^2-p(r,t). \tag{3.25}$$

Next we have

Lemma 3.4. *Assume w is sufficiently regular. Assume that w, $w_{,r}$ vanish at $r=0$. Then functions p and q are solutions to the problems*

$$p_{,t}-\mu_0 p_{,rr}=\frac{\partial\psi}{\partial w_{,r}}r^2-\int\limits_0^r\left(\frac{\partial\psi}{\partial\eta}+\mu_0\frac{w_{,t}}{r}\right)rdr-2\mu_0 w_{,t}r \tag{3.26}$$

$$p|_{t=0}=\int\limits_0^r w_1(r)r^2dr\equiv p_0 \tag{3.27}$$

$$p(R,t)=\int\limits_0^R w_{,t}(r,t)r^2dr,\quad |p(R,t)|\le R^{\frac{3}{2}}c_0^{\frac{1}{2}} \tag{3.28}$$

$$p_{,r}|_{r=R}=w_{,t}(R,t)R^2 \tag{3.29}$$

and

$$q_{,t}=-\frac{\partial\psi}{\partial w_{,r}}r^2+\int\limits_0^r r\left(\frac{\partial\psi}{\partial\eta}+\mu_0\frac{w_{,t}}{r}\right)dr \tag{3.30}$$

$$q|_{t=0}=\mu_0 w_{0,r}r^2-\int\limits_0^r w_1(r)r^2dr\equiv q_0. \tag{3.31}$$

Proof. First we find an equation for p. We calculate

$$\begin{aligned}
p_{,t}-\mu_0 p_{,rr}&=\int\limits_0^r w_{,tt}r^2dr-\mu_0\Big(\int\limits_0^r w_{,t}r^2dr\Big)_{,rr}\\
&=\int\limits_0^r\left\{\frac{1}{r^2}\left[\left(\frac{\partial\psi}{\partial w_{,r}}+\mu_0 w_{,rt}\right)r^2\right]_{,r}-\frac{1}{r}\left(\frac{\partial\psi}{\partial\eta}+\mu_0\frac{w_{,t}}{r}\right)\right\}r^2dr-\mu_0(w_{,t}r^2)_{,r}\\
&=\frac{\partial\psi}{\partial w_{,r}}r^2\Big|_{r=0}^{r=r}+\mu_0 w_{,rt}r^2\Big|_{r=0}^{r=r}-\int\limits_0^r\left(\frac{\partial\psi}{\partial\eta}+\mu_0\frac{w_{,t}}{r}\right)rdr-\mu_0 w_{,rt}r^2-2\mu_0 w_{,t}r.
\end{aligned}$$

Now we examine the behaviour of the first two terms from the r.h.s. First we examine the second term. From (3.7) we have

$$\int\limits_0^R w_{,rt}^2 r^{2\mu}dr<\infty\quad\text{for a.a. } t\in(0,T)\quad\text{and}\quad \mu\in\left(0,\frac{3}{2}\right).$$

Hence $w_{,rt}\sim r^\alpha$ with $\alpha>-\left(\frac{1}{2}+\mu\right)$. Then we see that $w_{,rt}r^2|_{r=0}=0$.

In view of the assumptions of the lemma we have $\frac{\partial\psi}{\partial w_{,r}}r^2\big|_{r=0} = 0$. Then the equation for p takes the form (3.26). From (3.25) and (1.10) we have (3.27). Finally, from (3.25) it follows $p(R,t) = \int_0^R w_{,t}(r,t)r^2dr$, so

$$|p(R,t)| \le \int\limits_0^R |w_{,t}(r,t)|r^2dr \le R\int\limits_0^R |w_{,t}(r,t)r|dr \le R^{\frac{3}{2}}\bigg(\int\limits_0^R w_{,t}^2r^2dr\bigg)^{\frac{1}{2}} \le R^{\frac{3}{2}}c_0^{\frac{1}{2}}.$$

Hence (3.28) is proved.

It seems that the Neumann boundary condition for the parabolic equation (3.26) is more convenient than the Dirichlet boundary condition (3.28). Therefore, we formulate it in the form (3.29).

From the definition of q and the assumptions of the lemma we have

$$\begin{aligned}
q_{,t} &= \mu_0 w_{,rt}r^2 - p_{,t} = \mu_0 w_{,rt}r^2 - \int\limits_0^r w_{,tt}r^2dr \\
&= \mu_0 w_{,rt}r^2 - \int\limits_0^r \bigg[\bigg(\frac{\partial\psi}{\partial w_{,r}} + \mu_0 w_{,rt}\bigg)r^2\bigg]_{,r} dr + \int\limits_0^r r\bigg(\frac{\partial\psi}{\partial\eta} + \mu_0\frac{w_{,t}}{r}\bigg)dr \\
&= -\frac{\partial\psi}{\partial w_{,r}}r^2\bigg|_{r=0}^{r=r} + \mu_0 w_{,rt}r^2\bigg|_{r=0} + \int\limits_0^r r\bigg(\frac{\partial\psi}{\partial\eta} + \mu_0\frac{w_{,t}}{r}\bigg)dr \\
&= -\frac{\partial\psi}{\partial w_{,r}}r^2 + \int\limits_0^r r\bigg(\frac{\partial\psi}{\partial\eta} + \mu_0\frac{w_{,t}}{r}\bigg)dr.
\end{aligned} \tag{3.32}$$

Finally we calculate

$$q|_{t=0} = \mu_0 w_{0,r}r^2 - p|_{t=0} = \mu_0 w_{0,r}r^2 - \int\limits_0^r w_1(r)r^2dr \equiv q_0. \tag{3.33}$$

From (3.32) and (3.33) we obtain (3.30) and (3.31), respectively. This concludes the proof. □

Lemma 3.5. *Let the assumptions of Lemma* 3.2 *be satisfied. Let* $\nu < \frac{1}{\sigma} + \frac{1}{2}$, $\sigma > 1$, $\delta > 0$ *but arbitrary small,*

$$B_{R,R_0} = \{r \in B_R :\ R_0 < r\}.$$

Then solutions to problem (3.26), (3.27), (3.29) *satisfy*

$$\begin{aligned}
\|p\|_{V^{2,1}_{\sigma,-\nu}(B_R^T)} \le{}& c\|w_{,r}r^2\|_{L_{\sigma,-\nu}(B_R^T)} + cc_1 + c\|w_1\|_{B^{1-\frac{2}{\sigma}}_{\sigma,-\nu}(B_R)} \\
&+ \varepsilon_1\|w_{,t}\|_{W_\sigma^{1+\delta,\frac{1}{2}+\frac{\delta}{2}}(B^T_{R,R_0})} + c\left(\frac{1}{\varepsilon_1}\right)\|w_{,t}\|_{L_2(B^T_{R,R_0})}
\end{aligned} \tag{3.34}$$

and solutions to problem (3.30), (3.31) *are bounded by*

$$\|q_{,t}\|_{L_{\sigma,-\nu}(B_R^T)} \le c\|w_{,r}r^2\|_{L_{\sigma,-\nu}(B_R^T)} + cc_1, \tag{3.35}$$

where c_1 *is introduced in* (3.7).

Proof. For solutions to problem (3.26), (3.27), (3.29) we have (see Lemma 2.1)

$$\begin{aligned} \|p\|_{V^{2,1}_{\sigma,-\nu}(B_R^T)} \le c\bigg(&\|w_{,r}r^2\|_{L_{\sigma,-\nu}(B_R^T)} + \bigg\|\int_0^r (|w|+|w_{,t}|)dr\bigg\|_{L_{\sigma,-\nu}(B_R^T)} \\ &+\|w_{,t}r\|_{L_{\sigma,-\nu}(B_R^T)} + \|p_0\|_{W^{2-\frac{2}{\sigma}}_{\sigma,-\nu}(B_R)} + \|w_{,t}(R,t)\|_{W_\sigma^{\frac{1}{2}-\frac{1}{2\sigma}}(0,T)}\bigg). \end{aligned} \tag{3.36}$$

The first norm on the r.h.s. of (3.36) equals

$$\bigg(\int_{B_R^T} |w_{,r}|^\sigma r^{(2-\nu)\sigma} drdt\bigg)^{\frac{1}{\sigma}}. \tag{3.37}$$

By the Hölder inequality the second integral on the r.h.s. of (3.36) is estimated by

$$\bigg(\int_{B_R^T}\bigg|\bigg(\int_0^r r^{-2\nu_1}dr\bigg)^{\frac{1}{2}}\bigg(\int_0^r (w^2+w_{,t}^2)r^{2\nu_1}dr\bigg)^{\frac{1}{2}}\bigg|^\sigma r^{-\sigma\nu}drdt\bigg)^{\frac{1}{\sigma}} \le cc_1, \tag{3.38}$$

where the last inequality holds in virtue of Lemma 3.2 and under assumption $\nu_1 + \nu < \frac{1}{\sigma} + \frac{1}{2}$, where ν_1 can be chosen arbitrary small.

We express the third integral on the r.h.s. of (3.36) in the form

$$\bigg(\int_{B_R^T} |w_{,t}|^\sigma r^{(1-\nu)\sigma} drdt\bigg)^{\frac{1}{\sigma}} \equiv J_1.$$

Assuming $\nu < 1$, setting $1-\nu = \mu > 0$ and recalling the imbedding (2.2) and Lemma 3.2 we obtain

$$J_1 \le c\sup_t \int_{B_R} |(wr^\mu)_{,t}|^2 dr + c\int_0^T\int_{B_R} |\nabla(w_{,t}r^\mu)|^2 drdt \le cc_1^2,$$

where $\sigma \le 6$.

Using (3.27) the last but one term on the r.h.s. of (3.36) equals

$$
\begin{aligned}
&\|p_0\|_{W^{2-\frac{2}{\sigma}}_{\sigma,-\nu}(B_R)}\\
&=\bigg\|\int_0^r w_1(r')r'^2dr'\bigg\|_{B^{2-\frac{2}{\sigma}}_{\sigma,-\nu}(B_R)}\\
&=\bigg(\int_0^R\bigg|\int_0^r w_1(r')r'^2dr'\bigg|^\sigma r^{-\sigma\nu}dr\bigg)^{\frac{1}{\sigma}}\\
&\quad+\bigg(\int_0^R\int_0^R\frac{\big|r_1^{-\nu}\partial_{r_1}\int_0^{r_1}w_1(r')r'^2dr'-r_2^{-\nu}\partial_{r_2}\int_0^{r_2}w_1(r')r'^2dr'\big|^\sigma}{|r_1-r_2|^{1+\sigma(1-\frac{2}{\sigma})}}dr_1dr_2\bigg)^{\frac{1}{\sigma}}\\
&\equiv I_1+I_2,
\end{aligned}
$$

where $\sigma>2$. First we examine

$$
\begin{aligned}
I_2&=\bigg(\int_0^R\int_0^R\frac{|w_1(r_1)r_1^{2-\nu}-w_1(r_2)r_2^{2-\nu}|^\sigma}{|r_1-r_2|^{\sigma-1}}dr_1dr_2\bigg)^{\frac{1}{\sigma}}\\
&\le\bigg(\int_0^R\int_0^R\frac{|w_1(r_1)r_1^{-\nu}-w_1(r_2)r_2^{-\nu}|^\sigma}{|r_1-r_2|^{\sigma-1}}dr_1dr_2\bigg)^{\frac{1}{\sigma}}\\
&\quad+\bigg(\int_0^R\int_0^R\frac{|w_1(r_2)r_2^{-\nu}(r_1^2-r_2^2)|^\sigma}{|r_1-r_2|^{\sigma-1}}dr_1dr_2\bigg)^{\frac{1}{\sigma}}\\
&\le\|w_1\|_{\overset{\bullet}{B}{}^{1-\frac{1}{\sigma}}_{\sigma,-\nu}(B_R)}+c\|w_1\|_{L_{\sigma,-\nu}(B_R)}.
\end{aligned}
$$

By the Hölder inequality we have

$$
\begin{aligned}
I_1&\le\bigg(\int_0^R\bigg[\bigg(\int_0^r r'^{(2+\nu)\sigma'}dr'\bigg)^{\frac{1}{\sigma'}}\bigg(\int_0^r|w_1(r')r'^{(1-\nu)}|^\sigma dr'\bigg)^{\frac{1}{\sigma}}\bigg]^\sigma r^{-\sigma\nu}dr\bigg)^{\frac{1}{\sigma}}\\
&\le c\|w_1\|_{L_{\sigma,-\nu}(B_R)}\bigg(\int_0^R r^{(2+\nu)\sigma'+1}r^{-\sigma\nu}dr\bigg)^{\frac{1}{\sigma}}\equiv I_1',
\end{aligned}
$$

where $\sigma'=\frac{\sigma}{\sigma-1}$. Performing calculations, we have $I_1'\le c\|w_1\|_{L_{\sigma,-\nu}(B_R)}$ for $\sigma>1$. By the inverse trace theorem the last term on the r.h.s. of (3.36) is estimated by

$$
\|w_{,t}(R,\cdot)\|_{W_\sigma^{\frac{1}{2}-\frac{1}{2\sigma}}(0,T)}\le\varepsilon\|w_{,t}\|_{W_\sigma^{1+\delta,\frac{1}{2}+\frac{\delta}{2}}(B^T_{R,R_0})}+c\left(\frac{1}{\varepsilon}\right)\|w_{,t}\|_{L_2(B^R_{R,R_0})},\tag{3.39}
$$

where $\delta > 0$ but arbitrary small and $B_{R,R_0} = \{r \in B_R : R_0 < r < R\}$. Using the above estimates in the r.h.s. of (3.36) implies (3.34). Finally we calculate

$$\|q_{,t}\|_{L_{\sigma,-\nu}(B_R^T)} \le c\bigg(\|w_{,r}r^2\|_{L_{\sigma,-\nu}(B_R^T)} + \bigg\|\int\limits_0^r (|w| + |w_{,t}|)dr\bigg\|_{L_{\sigma,-\nu}(B_R^T)}\bigg)$$

so the r.h.s. is bounded by expressions from (3.37) and (3.38). This implies (3.35) and concludes the proof. □

Corollary 3.6. *Let the assumptions of Lemma 3.5 be satisfied. Then* (3.25) *and* (3.34), (3.35) *imply*

$$\begin{aligned}\|w_{,rt}r^2\|_{L_{\sigma,-\nu}(B_R^T)} &\le c\big(\|q_{,t}\|_{L_{\sigma,-\nu}(B_R^T)} + \|p_{,t}\|_{L_{\sigma,-\nu}(B_R^T)}\big)\\ &\le c\|w_{,r}r^2\|_{L_{\sigma,-\nu}(B_R^T)} + \varepsilon_1\|w_{,t}\|_{W_\sigma^{1+\delta,\frac12+\frac\delta2}(B_{R,R_0}^T)}\\ &\quad + c\left(\frac{1}{\varepsilon_1}\right)\|w_{,t}\|_{L_2(B_{R,R_0}^T)} + cc_3,\end{aligned} \tag{3.40}$$

where $c_3 = c_1 + \|w_1\|_{B_{\sigma,-\nu}^{1-\frac{2}{\sigma}}(B_R)}$. *Employing* (3.7) *to the last but one term on the r.h.s. of* (3.40) *yields*

$$\|w_{,rt}r^2\|_{L_{\sigma,-\nu}(B_R^T)} \le c\|w_{,r}r^2\|_{L_{\sigma,-\nu}(B_R^T)} + \varepsilon_1\|w_{,t}\|_{W_\sigma^{1+\delta,\frac12+\frac\delta2}(B_{R,R_0}^T)} + c\left(\frac{1}{\varepsilon_1}\right)c_3, \tag{3.41}$$

where $\nu < \frac12 + \frac1\sigma$.

Remark 3.7. In the case of the Dirichlet boundary condition (1.12), the terms with ε_1 in (3.41) vanish. Hence, in this case, (3.41) is replaced by

$$\|w_{,rt}r^2\|_{L_{\sigma,-\nu}(B_R^T)} \le c\|w_{,r}r^2\|_{L_{\sigma,-\nu}(B_R^T)} + cc_3. \tag{3.42}$$

Now we estimate the first integral on the r.h.s. of (3.18). Let ν_0 be such that $2\mu = 3\nu_0$. Then the integral equals

$$\int\limits_0^t\int\limits_0^R \left(|w_{,rt}|^3 + \left|\frac{w_{,t}}{r}\right|^3\right) r^{3\nu_0}drdt \le c\int\limits_0^t\int\limits_0^R |w_{,rt}|^3 r^{3\nu_0}drdt, \tag{3.43}$$

where the Hardy inequality was used.

We estimate (3.43) by using (3.42) with $\sigma = 3$, $\nu = 2 - \nu_0$. Since $\nu < \frac56$ we have that $\nu_0 > \frac76$ and $\mu = \frac32\nu_0 > \frac74$. Therefore from (3.18), (3.42), (3.43) and in the case of the Dirichlet condition (1.12) we obtain the inequality for $\mu > \frac{7}{4}$

$$\begin{aligned}
&\int_0^R (w_{,tt}^2 + w_{,tr}^2 + w_{,t}^2 r^{-2}) r^{2\mu} dr + \mu_0 \int_0^t \int_0^R (w_{,rtt}^2 + w_{,tt}^2 r^{-2}) r^{2\mu} dr dt' \\
&\le c \int_0^t \int_0^R |w_{,r}|^3 r^{2\mu} dr dt + c c_3 + \int_0^R (w_{,tt}^2(0) + w_{,tr}^2(0) + w_{,t}^2(0) r^{-2}) r^{2\mu} dr \qquad (3.44) \\
&\equiv c \int_0^t \int_0^R |w_{,r}|^3 r^{2\mu} dr dt + c c_4.
\end{aligned}$$

From now we are going to obtain such inequalities that the first integral on the r.h.s. of (3.44) could be absorbed. Then we obtain an estimate. For this purpose we need to prove a series of lemmas.

Lemma 3.8. *Let the assumptions of Lemma 3.2 be satisfied for $\mu = \nu$, $\nu \in \left(0, \frac{3}{2}\right)$. Assume that*

$$w_{,rtt} \in L_{2,\mu}((0,R)\times(0,T)), \quad w_1 \in H_\mu^1(0,R), \quad \mu = 1+\nu, \quad \nu \in \left(0, \frac{3}{2}\right). \qquad (3.45)$$

Then the following inequality holds

$$\begin{aligned}
&\int_0^t \int_0^R w_{,tt}^2 r^{2\mu} dr dt' + \frac{\mu_0}{2} \int_0^R (w_{,rt}^2 + w_{,t}^2 r^{-2}) r^{2\mu} dr \\
&\le \varepsilon_1 \int_0^t \int_0^R w_{,rtt}^2 r^{2\mu} dr dt' + c\left(\frac{1}{\varepsilon_1}\right) c_1^2 + \frac{\mu_0}{2} \int_0^R (w_{,rt}^2(0) + w_{,t}^2(0) r^{-2}) r^{2\mu} dr \qquad (3.46) \\
&\equiv \varepsilon_1 \int_0^t \int_0^R w_{,rtt}^2 r^{2\mu} dr dt' + c\left(\frac{1}{\varepsilon_1}\right) c_5^2,
\end{aligned}$$

where $\mu = 1 + \nu$, $\nu \in \left(0, \frac{3}{2}\right)$ and $\varepsilon_1 \in (0,1)$.

Proof. Multiplying (1.9) by $w_{,tt} r^{2\mu}$ and integrating over B_R yields

$$\begin{aligned}
&\int_0^R w_{,tt}^2 r^{2\mu} dr + \int_0^R \left(\frac{\partial \psi}{\partial \vartheta} + \mu_0 w_{,rt}\right) w_{,rtt} r^{2\mu} dr \\
&+ (2\mu - 2) \int_0^R \left(\frac{\partial \psi}{\partial \vartheta} + \mu_0 w_{,rt}\right) w_{,tt} r^{2\mu-1} dr + \int_0^R \frac{1}{r} \left(\frac{\partial \psi}{\partial \eta} + \mu_0 \frac{w_t}{r}\right) w_{,tt} r^{2\mu} dr \qquad (3.47) \\
&= 0.
\end{aligned}$$

Using that $\left|\frac{\partial\psi}{\partial\vartheta}\right| \le c|w_{,r}|$, $\left|\frac{\partial\psi}{\partial\eta}\right| \le c|\eta|$ we obtain from (3.47) the inequality

$$\begin{aligned}
&\int_0^R w_{,tt}^2 r^{2\mu}dr + \frac{\mu_0}{2}\frac{d}{dt}\int_0^R w_{,rt}^2 r^{2\mu}dr + \frac{\mu_0}{2}\frac{d}{dt}\int_0^R w_{,t}^2 r^{2\mu-2}dr \\
&\le \varepsilon_1\int_0^R w_{,rtt}^2 r^{2\mu}dr + c\Big(\frac{1}{\varepsilon_1}\Big)\int_0^R w_{,r}^2 r^{2\mu}dr + \varepsilon_2\int_0^R w_{,tt}^2 r^{2\mu}dr \\
&\quad + c\Big(\frac{1}{\varepsilon_2}\Big)\int_0^R (w_{,r}^2 + w_{,rt}^2) r^{2\mu-2}dr + \varepsilon_3\int_0^R w_{,tt}^2 r^{2\mu}dr + c\Big(\frac{1}{\varepsilon_3}\Big)\int_0^R \left|\frac{w}{r}\right|^2 r^{2\mu-2}dr.
\end{aligned}$$

Choosing ε_2 and ε_3 sufficiently small we get

$$\begin{aligned}
&\int_0^R w_{,tt}^2 r^{2\mu}dr + \frac{\mu_0}{2}\frac{d}{dt}\int_0^R (w_{,rt}^2 + w_{,t}^2 r^{-2}) r^{2\mu}dr \\
&\le \varepsilon_1\int_0^R w_{,rtt}^2 r^{2\mu}dr + c\Big(\frac{1}{\varepsilon_1}\Big)\int_0^R w_{,r}^2 r^{2\mu}dr + c\int_0^R \Big(w_{,rt}^2 + w_{,r}^2 + \frac{w^2}{r^2}\Big) r^{2\mu-2}dr.
\end{aligned} \tag{3.48}$$

Integrating (3.48) with respect to time, assuming that $\mu = 1+\nu$, $\nu \in \left(0, \frac{3}{2}\right)$ and using (3.7) for $\mu = \nu$ we obtain (3.46). This concludes the proof. □

Lemma 3.9. *Let the assumptions of Lemma* 3.2 *be satisfied. Let us assume that*

$$\begin{aligned}
&\frac{\partial^2\psi}{\partial\vartheta^2} \ge \alpha_1, \quad \left|\frac{\partial\psi}{\partial\vartheta}\right| \le c|\vartheta|, \quad \left|\frac{\partial\psi}{\partial\eta}\right| \le c|\eta|, \\
&w_{,tt} \in L_{2,\mu}((0,R)\times(0,T)), \quad w_{0,rr} \in L_{2,\mu}(0,R), \\
&\mu = \nu + 1, \quad \nu \in \Big(0, \frac{3}{2}\Big).
\end{aligned} \tag{3.49}$$

Then the following inequality for solutions to problem (1.9), (1.10), (1.12) *is valid*

$$\begin{aligned}
&\int_0^t\int_0^R w_{,rr}^2 r^{2\mu}drdt' + \frac{\mu_0}{2}\int_0^R w_{,rr}^2 r^{2\mu}dr \\
&\le c\int_0^t\int_0^R w_{,tt}^2 r^{2\mu}drdt' + cc_1^2 + \frac{\mu_0}{2}\int_0^R w_{,rr}^2(0) r^{2\mu}dr \\
&\equiv c\int_0^t\int_0^R w_{,tt}^2 r^{2\mu}drdt' + cc_6^2,
\end{aligned} \tag{3.50}$$

where $\mu = \nu+1$, $\nu \in \left(0, \frac{3}{2}\right)$ *and* c_1 *is the constant from* (3.7).

Proof. Multiplying (1.9) by $w_{,rr}r^{2\mu}$ and integrating over B_R we obtain

$$\int_0^R w_{,tt}w_{,rr}r^{2\mu}dr = \int_0^R \frac{1}{r^2}\left[\left(\frac{\partial\psi}{\partial\vartheta}+\mu_0 w_{,rt}\right)r^2\right]_{,r} w_{,rr}r^{2\mu}dr - \int_0^R \left(\frac{1}{r}\frac{\partial\psi}{\partial\eta}+\mu_0\frac{w_{,t}}{r}\right)w_{,rr}r^{2\mu}dr.$$

Continuing, we have

$$\begin{aligned}&\int_0^R \left(\frac{\partial^2\psi}{\partial\vartheta^2}w_{,rr}^2+\mu_0 w_{,rrt}w_{,rr}\right)r^{2\mu}dr\\ &=\int_0^R w_{,tt}w_{,rr}r^{2\mu}dr+\int_0^R\left(\frac{\partial\psi}{\partial\vartheta}+\mu_0 w_{,rt}\right)w_{,rr}r^{2\mu-1}dr+\int_0^R\left(\frac{\partial\psi}{\partial\eta}+\mu w_{,t}\right)w_{,rr}r^{2\mu-1}dr.\end{aligned}$$

Using that $\frac{\partial^2\psi}{\partial\vartheta^2}\ge\alpha_1$, $\left|\frac{\partial\psi}{\partial\vartheta}\right|\le c|\vartheta|$, $\left|\frac{\partial\psi}{\partial\eta}\right|\le c|\eta|$ we get

$$\begin{aligned}&\alpha_1\int_0^R w_{,rr}^2r^{2\mu}dr+\frac{\mu_0}{2}\frac{d}{dt}\int_0^R w_{,rr}^2r^{2\mu}dr\\ &\le\varepsilon_1\int_0^R w_{,rr}^2r^{2\mu}dr+c\left(\frac{1}{\varepsilon_1}\right)\int_0^R w_{,tt}^2r^{2\mu}dr+c\int_0^R(w_{,r}^2+w_{,rt}^2+w_{,t}^2+w^2r^{-2})r^{2\mu-2}dr.\end{aligned}$$

Integrating the result with respect to time, assuming that ε_1 is sufficiently small, using that $\mu=1+\nu$, $\nu\in\left(0,\frac{3}{2}\right)$ and employing (3.7) with $\mu=\nu$ we obtain (3.50). This concludes the proof. $\square$

From (3.46) and (3.50) we derive the inequality

$$\begin{aligned}&\int_0^t\int_0^R(w_{,tt}^2+w_{,rr}^2)r^{2\mu}drdt'+\frac{\mu_0}{2}\int_0^R(w_{,rt}^2+w_{,rr}^2+w_{,t}^2r^{-2})r^{2\mu}dr\\ &\le\varepsilon_1\int_0^t\int_0^R w_{,rtt}^2r^{2\mu}drdt'+c\left(\frac{1}{\varepsilon_1}\right)c_7^2,\end{aligned}\tag{3.51}$$

where $\mu=1+\nu$, $\nu\in\left(0,\frac{3}{2}\right)$ and $c_7=c_5+c_6$. From (3.18) and (3.51) for sufficiently small ε_1 we obtain

$$\begin{aligned}&\int_0^R(w_{,tt}^2+w_{,rt}^2+w_{,rr}^2+w_{,t}^2r^{-2})r^{2\mu}dr+\int_0^t\int_0^R(w_{,rtt}^2+w_{,rr}^2+w_{,tt}^2r^{-2})r^{2\mu}drdt'\\ &\le c(t)\int_0^t\int_0^R(|\vartheta_{,t}|^3+|\eta_{,t}|^3)r^{2\mu}drdt'+cc_8^2,\end{aligned}\tag{3.52}$$

where $\mu = 1+\nu$, $\nu \in \left(0, \frac{3}{2}\right)$ and $c_8^2 = c_7^2 + \int_0^R (w_{,tt}^2(0) + w_{,tr}^2(0) + w_{,t}^2(0)r^{-2})r^{2\mu}dr$.

Now we estimate the integral on the r.h.s. of (3.52). By the Hardy inequality we have

$$\int_0^T\int_0^R |w_{,t}|^3 r^{2\mu-3}drdt' \le c\int_0^t\int_0^R |w_{,rt}|^3 r^{2\mu}drdt'.$$

Hence the integral on the r.h.s. of (3.52) is bounded by $c\int_0^t\int_0^R |w_{,rt}|^3 r^{2\mu}drdt'$. Inequality (3.41) in the case of the Dirichlet problem implies

$$\int_0^t\int_0^R |w_{,rt}|^3 r^{6-3\nu}drdt' \le c\int_0^t\int_0^R |w_{,r}|^3 r^{6-3\nu}drdt' + cc_3^3, \tag{3.53}$$

where $\nu < \frac{5}{6}$. Since $w_{,r}(r,t) = \int_0^t w_{,rt'}(r,t')dt' + w_{,r}(r,0)$, from (3.53) we obtain the inequality

$$\int_0^t\int_0^R |w_{,rt'}|^3 r^{6-3\nu}drdt' \le c\int_0^t\int_0^R\int_0^{t'} |w_{,rt''}(t'')|^3 dt'' r^{6-3\nu}drdt' + cc_9^3(t), \tag{3.54}$$

where $c_0(t)$ is an increasing function of t. Hence the Gronwall inequality implies

$$\int_0^t\int_0^R |w_{,rt'}|^3 r^{6-3\nu}drdt' \le cc_{10}^3(t), \tag{3.55}$$

where $c_{10}(t)$ is an increasing function of t. Summarizing the above considerations yields

Lemma 3.10. *Let us assume that*

$$\psi(\vartheta,\eta) = \psi_1(\vartheta)+\psi_2(\eta),\ \frac{\partial^2\psi_1}{\partial\vartheta^2} \ge \alpha_1,\ \frac{\partial^2\psi_2}{\partial\eta^2} \ge \alpha_2,\ \left|\frac{\partial\psi_1}{\partial\vartheta}\right| \le \beta_1|\vartheta|,\ \left|\frac{\partial\psi_2}{\partial\eta}\right| \le \beta_2|\eta|,$$

where α_1, α_2, β_1, β_2 are positive constants, $w_0 \in H_\mu^2(0,R)$, $w_1 \in H_\mu^2(0,R)$, $\mu \in \left(1, 1+\frac{5}{6}\right)$. Then, for solutions of the Dirichlet problem (1.9), (1.10), (1.12), *the following a priori estimate holds*

$$\begin{aligned}
&\int\limits_0^R (w_{,tt}^2 + w_{,rt}^2 + w_{,rr}^2 + w_{,t}^2 r^{-2}) r^{2\mu} dr \\
&+ \int\limits_0^t \int\limits_0^R (w_{,rtt}^2 + w_{,rr}^2 + w_{,tt}^2 r^{-2}) r^{2\mu} dr dt' \\
&\le c\bigg(\|w_0\|_{H_\mu^1(0,R)}^2 + \|w_{0,rr}\|_{L_{2,\mu}(0,R)}^2 + \|w_1\|_{L_{2,\mu}(0,R)}^2 \\
&+ \int\limits_0^R (w_{,tt}^2(0) + w_{,rt}^2(0) + w_{,t}^2(0) r^{-2}) r^{2\mu} dr \bigg),
\end{aligned} \tag{3.56}$$

where $\mu \in \left(1, 1+\frac{5}{6}\right)$.

Now we derive some local properties of solutions to problem (1.9), (1.10), (1.12).

Lemma 3.11. *Let the assumptions of Lemma* 3.2 *be satisfied. Then the following estimates hold*

$$|w(r,t)| \le \frac{R^{\frac{1}{2}-\mu}}{\sqrt{1-2\mu}} c_1, \quad \mu \in \left(0, \frac{1}{2}\right), \tag{3.57}$$

so $w \in B(B_R \times (0,T))$.

$$\|w_{,t}\|_{L_2(0,T;C^{\frac{1}{2}-\mu/2}(B_R))} \le cc_1, \qquad \mu \in \left(0, \frac{1}{2}\right), \tag{3.58}$$

$$\|w\|_{B(0,T;C^{\frac{1}{2}-\mu/2}(B_R))} \le cc_1, \qquad \mu \in \left(0, \frac{1}{2}\right), \tag{3.59}$$

$$\|w\|_{L_\beta(0,R;C^{\frac{\beta-1}{\beta}}(0,T))} \le cc_1, \quad 1 < \beta < \frac{2}{2\mu+1}, \quad \mu \in \left(0, \frac{1}{2}\right). \tag{3.60}$$

Proof. From (3.7) we have $|w(r,t)| = \left|\int_R^r w_{,r}(r,t)dr\right| = \left|\int_R^r r^{-\mu} w_{,r} r^\mu dr\right| \le \left(\int_0^R r^{-2\mu} dr\right)^{\frac{1}{2}} \left(\int_0^R w_{,r}^2 r^{2\mu} dr\right)^{\frac{1}{2}} \le \frac{R^{\frac{1}{2}-\mu}}{\sqrt{1-2\mu}} c_1$, $\mu \in (0, \frac{1}{2})$. Hence (3.57) holds.

Next we calculate $w_{,t}(r',t) - w_{,t}(r'',t) = \int_{r''}^{r'} w_{,tr} dr$. Hence

$$|w_{,t}(r',t) - w_{,t}(r'',t)| \le \frac{1}{1-2\mu} \left|(r')^{1-2\mu} - (r'')^{1-2\mu}\right|^{\frac{1}{2}} \left| \int\limits_{r''}^{r'} w_{,rt}^2 r^{2\mu} dr \right|^{\frac{1}{2}}.$$

From the above inequality it follows

$$\frac{|w_{,t}(r',t)-w_{,t}(r'',t)|^2}{|r'-r''|^{1-2\mu}} \le \frac{1}{(1-2\mu)^2}\bigg|\int\limits_{r''}^{r'} w_{,rt}^2 r^{2\mu}dr\bigg|.$$

Taking supremum with respect to $r', r'' \in [0,R]$ and integrating the result with respect to time yields

$$\int\limits_0^T dt \sup_{r',r''\in[0,R]} \frac{|w_{,t}(r',t)-w_{,t}(r'',t)|^2}{|r'-r''|^{1-2\mu}} \le \frac{1}{(1-2\mu)^2}\int\limits_0^T dt\int\limits_0^R w_{,rt}^2 r^{2\mu}dr \le \frac{1}{(1-2\mu)^2}c_1^2.$$

Since $w_{,t}(r,t)=\int_R^r w_{,rt}(r,t)dr$ we obtain

$$|w_{,t}(r,t)| = \bigg|\int\limits_R^r r^{-\mu}w_{,rt}r^{\mu}dr\bigg| \le \bigg(\frac{R^{1-2\mu}}{1-2\mu}\bigg)^{\frac{1}{2}}\bigg(\int\limits_0^R w_{,rt}^2 r^{2\mu}dr\bigg)^{\frac{1}{2}}.$$

Taking the L_2 norm with respect to time yields

$$\int\limits_0^T w_{,t}^2(r,t)dt \le \frac{R^{1-2\mu}}{1-2\mu}\int\limits_0^T dt\int\limits_0^R w_{,rt}^2 r^{2\mu}dr \le \frac{R^{1-2\mu}}{1-2\mu}c_1^2.$$

Hence (3.58) is proved.

To show (3.60) we consider

$$|w(r,t')-w(r,t'')| = \bigg|\int\limits_{t''}^{t'} w_{,t}dt\bigg| = \bigg|\int\limits_{t''}^{t'} r^{-\mu}w_{,t}r^{\mu}dt\bigg| \le |t'-t''|^{\frac{1}{\beta_1}}\bigg|\int\limits_{t''}^{t'}|r^{-\mu}w_{,t}r^{\mu}|^{\beta_2}dt\bigg|^{\frac{1}{\beta_2}},$$

where $\frac{1}{\beta_1}+\frac{1}{\beta_2}=1$, $\beta_2<2$. Taking the $L_{\beta_2}(0,R)$ norm of the above inequality yields

$$\bigg(\int\limits_0^R |w(r,t')-w(r,t'')|^{\beta_2}dr\bigg)^{\frac{1}{\beta_2}} \le |t'-t''|^{\frac{1}{\beta_1}}\bigg(\int\limits_0^R dr\bigg|\int\limits_{t''}^{t'}|r^{-\mu}w_{,t}r^{\mu}|^{\beta_2}dt\bigg|\bigg)^{\frac{1}{\beta_2}}$$

Continuing,

$$\begin{aligned}&\sup_{t',t''\in[0,T]}\bigg(\int\limits_0^R \frac{|w(r,t')-w(r,t'')|^{\beta_2}}{|t'-t''|^{\frac{\beta_2}{\beta_1}}}dr\bigg)^{\frac{1}{\beta_2}}\\ &\le \bigg(\int\limits_0^T dt\int\limits_0^R |r^{-\mu}w_{,t}r^{\mu}|^{\beta_2}dr\bigg)^{\frac{1}{\beta_2}}\\ &\le \bigg(\int\limits_0^T dt\bigg(\int\limits_0^R dr r^{-\mu\beta_2\gamma_1}\bigg)^{\frac{1}{\gamma_1}}\bigg(\int\limits_0^R dr|w_{,t}r^{\mu}|^{\beta_2\gamma_2}\bigg)^{\frac{1}{\gamma_2}}\bigg)^{\frac{1}{\beta_2}},\end{aligned}$$

where $\frac{1}{\gamma_1} + \frac{1}{\gamma_2} = 1$. We set $\beta_2\gamma_2 = 2$ and we need that $\mu\beta_2\gamma_1 < 1$. Therefore $\gamma_2 = \frac{2}{\beta_2}$ and $\gamma_1 = \frac{2}{2-\beta_2}$. Hence $\mu\beta_2\frac{2}{2-\beta_2} < 1$ and $1 < \beta_2 < \frac{2}{2\mu+1}$ and $\mu \in \left(0, \frac{1}{2}\right)$. By (3.7) estimate (3.60) follows.

Finally, we show (3.59). Since $w(r',t) - w(r'',t) = \int_{r''}^{r'} w_r dr$ we obtain

$$\frac{|w(r',t) - w(r'',t)|^2}{|r' - r''|^{1-2\mu}} \le \int_0^R w_{,r}^2 r^{2\mu} dr \le c_1^2,$$

where $0 < \mu < \frac{1}{2}$ and the last inequality follows from (3.7). Therefore (3.59) is shown and Lemma 3.11 is proved. This concludes the proof. □

Finally, we derive some properties of the Pego functions p, q (see (3.25)).

Remark 3.12. From (3.25) it follows that $\frac{p_{,r}}{r^2} = w_{,t}$. Then (3.7) implies

$$\sup_t \int_0^R p_{,r}^2 r^{2(\mu-2)} dr \le cc_1, \qquad \int_0^T \int_0^R p_{,r}^2 r^{2(\mu-3)} dr \le cc_1, \tag{3.61}$$

where $\mu \in (0, \frac{3}{2})$.

Since $p(0,t) = 0$ we have

$$\left|\frac{p(r)}{r^\nu}\right| \le \int_0^r \left|\partial_r \frac{p}{r^\nu}\right| dr \le r^{\frac{1}{2}} \left(\int_0^r (p_{,r}^2 r^{-2\nu} + p^2 r^{-2\nu-2}) dr \right)^{\frac{1}{2}}$$

Setting $\nu = 2-\mu$ and using the Hardy inequality $\int p^2 r^{2(\mu-3)} dr \le c \int p_{,r}^2 r^{2(\mu-2)} dr$ and (3.7) it follows

$$\frac{|p(r,t)|}{r^{\frac{5}{2}-\mu}} \le cc_1.$$

Remark 3.13. We have to emphasize that the final Lemma 3.10 does not hold for solutions to the Neumann initial boundary value problem (1.9)–(1.11). This follows from the fact that the regularity described by the l.h.s. of (3.56) is not enough to absorb the first norm on the r.h.s. of (3.39). To absorb the norm we need an estimate for

$$\int_0^T \int_0^R w_{,rrt}^2 r^{2\mu} dr dt.$$

This, however, needs many additional estimates.

4. Existence

We prove the existence of solutions to problem (1.9), (1.10), (1.12) by the Faedo-Galerkin method (cf. [6, 12]).

We take the basis $\{\varphi_k(r)\}$ in $W_2^1(B_R)$, such that $\varphi_k(R) = 0$. We assume additionally that $(\varphi_k, \varphi_l)_{L_2(B_R)} = \delta_{lk}$, where $(\cdot, \cdot)_{L_2(B_R)}$ is the scalar product in $L_2(B_R)$ and δ_{lk} is the Kronecker delta. Moreover, we assume that there exist constants $c_k < \infty$, $k \in \mathbb{N} \cup \{0\}$ such that

$$\|\varphi_k, \varphi_{k,r}\|_{L_\infty(B_R)} \le c_k.$$

We are looking for the approximate solution $w^N(r,t)$ in the form

$$w^N(r,t) = \sum_{k=1}^{N} c_k^N(t)\varphi_k(r).$$

Then $c_k^N(t)$ are solutions to the following system of ordinary differential equations

$$\begin{aligned} &\int\limits_{B_R} w^N_{,tt}\varphi_k r^2 dr + \int\limits_{B_R} [a_1(r, w^N, w^N_{,r}) + \mu_0 w^N_{,rt}]\varphi_{k,r} r^2 dr \\ &\quad + \int\limits_{B_R} \left[a_2(r, w^N, w^N_{,r}) + \mu_0 \frac{w^N_{,t}}{r}\right]\varphi_k r dr \\ &= 0, \end{aligned} \tag{4.1}$$

where we introduced the notation

$$a_1 = \psi_{,w_{,r}}, \quad a_2 = \psi_{,\eta}.$$

Repeating the proof of Lemma 3.2 we obtain the following estimate for the approximate solution

$$\int\limits_{B_R} \left[\frac{1}{2}|w^N_{,t}|^2 + \psi\left(w^N_{,r}, \frac{w^N}{r}\right)\right] r^{2\mu} dr + \mu_0 \int\limits_0^t dt' \int\limits_{B_R} \left[|w^N_{,rt}|^2 + \left|\frac{w^N_{,t}}{r}\right|^2\right] r^{2\mu} dr \le c_1^2. \tag{4.2}$$

In view of the growth condition (3.15) estimate (4.2) implies

$$\begin{aligned} &\int\limits_{B_R} \left[\frac{1}{2}|w^N_{,t}|^2 + c_1|w^N_{,r}|^2 + c_2\left|\frac{w^N}{r}\right|^2\right] r^{2\mu} dr \\ &\quad + \mu_0 \int\limits_0^t dt' \int\limits_{B_R} \left[|w^N_{,rt}|^2 r^{2\mu} + |w^N_{,t}|^2 r^{2\mu-2}\right] dr \\ &\le c_1^2, \end{aligned} \tag{4.3}$$

where $\mu \in \left(0, \frac{3}{2}\right)$. From (4.3) we have

$$w^N \in L^1_\infty(0,T;L_{2,\mu}(B_R)) \cap L_\infty(0,T;H^1_\mu(B_R)) \cap L^1_2(0,T;H^1_\mu(B_R)) \equiv \mathcal{M}(\Omega^T). \quad (4.4)$$

In view of (4.4) we have that w^N weakly-star converges in $L^1_\infty(0,T;L_{2,\mu}(B_R)) \cap L_\infty(0,T;H^1_\mu(B_R))$ and weakly in $L^1_2(0,T;H^1_\mu(B_R))$ to some $w \in \mathcal{M}(\Omega^T)$.

To prove the existence of weak solutions to problem (1.9)–(1.11) we recall that the Faedo-Galerkin approximations satisfy the following integral identities

$$\begin{aligned}
&-\int_0^T dt \int_{B_R} w_t^N \varphi_{tt} r^{2\mu} dr + \int_{B_R} w_t^N \varphi_t r^{2\mu} |dr|_{t=0}^{t=T} \\
&+ \int_0^T dt \int_{B_0 R} [a_1(r, w^N, w^N_{,r}) + \mu_0 w^N_{,rt}] \varphi_{,rt} r^{2\mu} dr \\
&+ \int_0^T dt \int_{B_R} \left[a_2(r, w^N, w^N_{,r}) + \mu_0 \frac{w^N_{,t}}{r^2}\right] \varphi_{,t} r^{2\mu} dr \\
&= 0,
\end{aligned} \quad (4.5)$$

which holds for any function $\varphi \in P_N$, where $P_N = \{\varphi: \ \varphi = \sum_{k=1}^N d_k(t)\varphi_k(r)\}$. Hence $w^N \in P_N$.

We assume additionally that

$$a_2(r, w, w_{,r}) = a_2'(r, w) w_{,r} + a_2''(r, w). \quad (4.6)$$

Since

$$\frac{d}{dt} \int_{B_R} \psi(r, w, w_{,r}) r^{2\mu} dr = \int_{B_R} [a_1(r, w, w_{,r}) w_{,rt} + a_2(r, w, w_{,r}) w_t] r^{2\mu} dr$$

we have that

$$a_1 = \psi_{,w_{,r}}, \quad a_2 = \psi_{,\frac{w}{r}} \ .$$

To pass to the limit in the integral identity (4.5) we assume the monotonicity condition

$$\int_{B_R^T} [a_1(r, w^N, w^N_{,r}) - a_1(r, w^N, \eta_{,r})](w^N_{,rt} - \eta_{,rt}) r^{2\mu} dr dt + f\left(\|w^N - \eta\|_{\mathcal{M}(\Omega^T)}\right) \geq 0, \quad (4.7)$$

where $f(\tau)$ is a continuous function for $\tau \geq 0$ and satisfying $\lim_{\varepsilon \to 0} \varepsilon^{-1} f(\varepsilon\tau) = 0$ for any $\tau > 0$. The condition (4.7) is called the monotonicity condition.

Condition (4.7) is a restriction on the considered viscoelasticity system because $a_1 = \psi_{,w_{,r}}$ and ψ determines function F (see (1.8)) which partially generate the stress tensor σ (see (1.4)). Condition (4.7) can be satisfied in the case of linear function a_1 with respect to the last argument and sufficiently nonlinear function f. Moreover, we have to emphasize that the L_∞ norms in $\mathcal{M}(\Omega^T)$ can be replaced by the norm sup in view of estimate (4.3). We hope that condition (4.7) holds for more general a_1.

We need the monotonicity condition because passing to the limit in (4.5) for any function $\varphi \in P_N$ we obtain in view of (4.4) the identity

$$\begin{aligned} &-\int_0^T dt \int_{B_R} w_{,t}\varphi_{,tt} r^{2\mu} dr + \int_{B_R} w_{,t}\varphi_{,t} r^{2\mu} dr\big|_{t=0}^{t=T} \\ &+\int_0^T dt \int_{B_R} (A + \mu_0 w_{,rt})\varphi_{,rt} r^{2\mu} dr + \int_0^T dt \int_{B_R} \left(a_2(r, w, w_{,r}) + \mu_0 \frac{w_{,t}}{r^2} \right) \varphi_{,t} r^{2\mu} dr \\ &= 0, \end{aligned} \tag{4.8}$$

where we used (4.6) and $A = \lim_{N\to\infty} a_1(r, w^N, w^N_{,r})$. Replacing φ by a sequence $\varphi^{N'} \in P_{N'}$ we can pass with $N' \to \infty$, so we obtain that (4.8) holds for any $\varphi \in \bigcup_{k=1}^\infty P_k$.

To show that $A = a_1(r, w, w_{,r})$ we use the monotonicity condition. Expressing (4.5) with $\varphi = w^N - \eta$, $\eta \in P_N$, yields

$$\begin{aligned} &\int_{B_R} |w^N_{,t}|^2 r^{2\mu} dr\big|_{t=0}^{t=T} + \int_{B_R^T} w^N_{,t} \eta_{,tt} r^{2\mu} dr dt - \int_{B_R} w^N_{,t} \eta_{,t} r^{2\mu} dr\big|_{t=0}^{t=T} \\ &+ \int_{B_R^T} [a_1(r, w^N, w^N_{,r}) + \mu_0 w^N_{,rt}](w^N_{,rt} - \eta_{,rt}) r^{2\mu} dr dt \\ &+ \int_{B_R^T} \left(a_2(r, w^N, w^N_{,r}) + \mu_0 \frac{w^N_{,t}}{r^2} \right) (w^N_{,t} - \eta_{,t}) r^{2\mu} dr dt \\ &= 0. \end{aligned} \tag{4.9}$$

Eliminating the term $\int_{B_R^T} a_1(r, w^N, w^N_{,r})(w^N_{,rt} - \eta_{,rt}) r^{2\mu} dr dt$ in (4.7) by employing

(4.9) we obtain (4.7) in the form

$$
\begin{aligned}
&-\int\limits_{B_R^T} a_1(r,w^N,\eta_{,r})(w^N_{,rt}-\eta_{,rt})r^{2\mu}drdt-\int\limits_{B_R}|w^N_{,t}|^2r^{2\mu}dr|_{t=0}^{t=T}\\
&-\int\limits_{B_R^T} w^N_{,t}\eta_{,tt}r^{2\mu}drdt+\int\limits_{B_R} w^N_{,t}\eta_{,t}r^{2\mu}dr|_{t=0}^{t=T}-\mu_0\int\limits_{B_R^T} w^N_{,rt}(w^N_{,rt}-\eta_{,rt})r^{2\mu}drdt\\
&-\int\limits_{B_R^T}\left[a_2(r,w^N,w^N_{,r})+\mu_0\frac{w^N_{,t}}{r^2}\right](w^N_{,t}-\eta_{,t})r^{2\mu}drdt+f\big(\|w^N-\eta\|_{\mathcal{M}(\Omega^T)}\big)\\
&\geq 0.
\end{aligned}
\tag{4.10}
$$

Since w^N weak-star converges in $\mathcal{M}(\Omega^T)$ to $w\in\mathcal{M}(\Omega^T)$ we can pass to the limit in (4.10). Hence we get

$$
\begin{aligned}
&-\int\limits_{B_R^T} a_1(r,w,\eta_{,r})(w_{,rt}-\eta_{,rt})r^{2\mu}drdt-\int\limits_{B_R}|w_{,t}|^2r^{2\mu}dr|_{t=0}^{t=T}-\int\limits_{B_R^T} w_{,t}\eta_{,tt}r^{2\mu}dr\\
&+\int\limits_{B_R} w_{,t}\eta_{,t}r^{2\mu}dr|_{t=0}^{t=T}-\mu_0\int\limits_{B_R^T} w_{,rt}(w_{,rt}-\eta_{,rt})r^{2\mu}drdt\\
&+\int\limits_{B_R^T}\left[a_2(r,w,w_{,r})+\mu_0\frac{w_{,t}}{r^2}\right](w_{,t}-\eta_{,t})r^{2\mu}drdt+f\big(\|w-\eta\|_{\mathcal{M}(\Omega^T)}\big)\\
&\geq 0.
\end{aligned}
\tag{4.11}
$$

Replacing, in (4.8), φ by $w-\eta$ with the help of explanation from [12, Chapter 5, Section 6 between formulas (6.60) and (6.61)] and comparing the result with (4.11) yields

$$
\int\limits_{B_R^T}(A(r,t)-a_2(r,w,w_{,r}))(w_{,rt}-\eta_{,rt})r^{2\mu}drdt+f\big(\|w-\eta\|_{\mathcal{M}(\Omega^T)}\big)\geq 0. \tag{4.12}
$$

Setting $\eta=w-\varepsilon\zeta(r,t)$, where $\zeta(r,t)$ is a smooth function and repeating the considerations from [12, Chapter 5, Section 6 and between formulas (6.61) and (6.62)] we obtain that

$$
A(r,t)=a_2(r,w,w_{,r}).
$$

Hence, we have proved the result

Lemma 4.1. *Let us assume that* $w_1\in L_{2,\mu}(B_R)$, $w_0\in H^1_\mu(B_R)$. *Then there exists a weak solution to problem* (1.1), (1.10), (1.12) *in the space described by* (4.4) *satisfying estimate* (3.7).

Proof of the Main Theorem. Let w be a weak solution to problem (1.9), (1.10), (1.12). We show a higher regularity of the weak solution by deriving better estimates.

Hence using the classical techniques of increasing regularity of weak solutions and repeating the considerations from the proof of Lemma 3.11 we conclude the proof. □

5. Concluding remarks

Using the method presented in this paper, we can extend our considerations to the initial boundary value problem for non-linear symmetric thermoviscoelasticity in the domain Ω_R, which is the ball with radius $R > 0$. It will be done in our future paper.

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