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An Approximation Result in Generalized Anisotropic Sobolev Spaces and Applications

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Abstract. In this paper, we give an approximation result in some anisotropic Sobolev space. We also describe the action of some distributions in the dual and we mention two applications to some strongly nonlinear anisotropic elliptic boundary value problems.

Keywords. Approximation, anisotropic Sobolev space, segment property, strongly nonlinear elliptic problems

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1. Introduction

In this paper, we use Hedberg-type's approximation to prove existence of distributional solutions in an appropriate function space for some nonlinear anisotropic elliptic equations. A prototype example is

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big)+g(x,u)=f \quad \text{in } \Omega,$$

where Ω is an open subset of $\mathbb{R}^N$ and the exponents $p_i \geq 1$ for $i=0,\dots,N$. The nonlinear function $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is assumed to be Carathéodory, measurable in $x\in\Omega$ for all $\sigma\in\mathbb{R}$ and continuous in $\sigma\in\mathbb{R}$ for a.e. $x\in\Omega$. Furthermore,

$$g(x,\sigma)\,\sigma \geq 0, \quad \text{for all } \sigma\in\mathbb{R} \text{ and a.e. } x\in\mathbb{R}^N.$$

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Let $p_0 \geq 1$, we will use the following anisotropic Sobolev space

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in L^{p_0}(\Omega),\ \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega),\ i = 1, 2, \ldots, N \right\}.$$

Later to prove existence results, we use Hedberg-type's approximation in anisotropic spaces $W^{1,\vec{p}}(\Omega)$ and another of type $W^{1,\vec{p},\varepsilon}(\Omega)$ with $0 < \varepsilon < 1$ small enough (see the definition of $W^{1,\vec{p},\varepsilon}(\Omega)$ introduced in Subsection 5.2). We approximate our function $u \in W^{1,\vec{p}}(\Omega)$ by a sequence of functions u_n which belong to $W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω, dominated by u and has the same sign as u almost everywhere in Ω (see, e.g., [3,5,8,9] for more details). This approximation was used before for nonlinear elliptic equations (see, e.g., [5,16]). It characterizes also the action of some distributions in the dual (see, e.g., [5]).

Hedberg-type's approximation was introduced first by Brézis and Browder [5] in the context of classical isotropic Sobolev space (in $W^{1,p}(\Omega)$). Next it was used by Benkirane and Gossez [2] in Orlicz-Sobolev space with the help of the Riesz potential theory (see [13] for more details).

In our case, without referring to this theory, we simply construct this approximation in the generalized space $W^{1,\vec{p}}(\Omega)$.

The remaining part of this paper is organized as follows: In Section 2, we introduce some notations, functional spaces and a technical lemma. We construct in Section 3 Hedberg-type's approximation in the anisotropic space $W^{1,\vec{p}}(\Omega)$. Section 4 is devoted to the description of the action of some distributions in the dual, with a regularity condition on Ω, namely, we impose the segment property on the domain. The results in Sections 3 and 4 generalize the work of H. Brézis and F. Browder [5] to the anisotropic space $W^{1,\vec{p}}(\Omega)$. As application, we use Hedberg-type's approximation to prove the existence of solutions for some strongly nonlinear boundary value problems of the form:

$$Au + g(x, u) = f \quad \text{in } \Omega,$$

where the operator A is strongly nonlinear satisfying appropriate growth, coerciveness and monotonicity conditions. The nonlinear term g is of Carathéodory and has to fulfil essentially the sign condition $g(x, \sigma)\sigma \geq 0$, but we do not assume any growth conditions with respect to $|u|$.

2. Anisotropic Sobolev spaces and a technical lemma

We start by recalling the notion of anisotropic Sobolev spaces. These spaces were introduced and studied by Nikolskiĭ [11], Slobodeckiĭ [12], and Troisi [14], and later by Trudinger [15] in the framework of Orlicz spaces.

Let Ω be an open domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. Let $p_0, p_1, \dots, p_N$ be $N+1$ real numbers with $p_i \geq 1$, $i = 0, 1, \dots, N$. We denote

$$\vec{p} = \{p_i,\ i = 0, 1, 2, \dots, N\}, \qquad D^0 u = u$$
$$\underline{p} = \min\{p_i,\ i = 0, 1, 2, \dots, N\}, \quad D^i u = \frac{\partial u}{\partial x_i} \quad (i = 1, \dots, N).$$

With a slight abuse of the notation, we introduce the anisotropic Sobolev space

$$W^{1,\vec{p}}(\Omega) = \{u \in L^{p_0}(\Omega),\ D^i u \in L^{p_i}(\Omega),\ i = 1, 2, \dots, N\},$$

under the norm

$$\|u\|_{1,\vec{p}} = \sum_{i=0}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{1}$$

We define also $W_0^{1,\vec{p}}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to the norm (1). The dual of $W_0^{1,\vec{p}}(\Omega)$ is denoted by $W^{-1,\vec{p'}}(\Omega)$, where $\vec{p'} = \{p'_i,\ i = 0, 1, 2, \dots, N\}$, $p'_i = \frac{p_i}{p_i - 1}$ and $p_i > 1$.

Remark 2.1. Arguing as Adams [1], it can be easily seen that $W^{1,\vec{p}}(\Omega)$ is a separable Banach space and reflexive if $1 < p_i < \infty$ for all $i = 0, 1, 2, \dots, N$.

We will use later the following Sobolev embedding.

Lemma 2.2. *Let Ω be a bounded open set in $\mathbb{R}^N$. Then the following embeddings are compact:*

if $\underline{p} < N$ then $W_0^{1,\vec{p}}(\Omega) \to L^q(\Omega)$, $\quad \forall q \in [\underline{p}, p^*[$, where $\frac{1}{p^*} = \frac{1}{\underline{p}} - \frac{1}{N}$;

if $\underline{p} = N$ then $W_0^{1,\vec{p}}(\Omega) \to L^q(\Omega)$, $\quad \forall q \in [\underline{p}, +\infty[$;

if $\underline{p} > N$ then $W_0^{1,\vec{p}}(\Omega) \to L^\infty(\Omega) \cap C^k(\overline{\Omega})$, where $k = E\left(1 - \frac{N}{\underline{p}}\right)$.

Herein, $E(x) = n$ for $x \in [n, n+1)$, $n \in \mathbb{N}$. The proof of this lemma follows from the fact that $W^{1,\vec{p}}(\Omega) \subset W^{1,\underline{p}}(\Omega)$ and the classical embedding theorems of Sobolev spaces.

3. Approximation theorem

In this section, we construct Hedberg-type's approximation in the anisotropic Sobolev space $W^{1,\vec{p}}(\mathbb{R}^N)$.

Definition 3.1. The vector $\vec{p} = \{p_i,\ i = 0, 1, 2 \dots, N\}$ is admissible if the following condition is satisfied:

$$p_0 = \sup\{p_i,\ i = 0, 1, \dots, N\}.$$

Theorem 3.2. *If $\vec{p}$ is an admissible vector and $u \in W^{1,\vec{p}}(\mathbb{R}^N)$, then there exists a sequence u_n such that*

i) *$u_n \in W^{1,\vec{p}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ has a compact support in $\mathbb{R}^N$ for all n*
ii) *$|u_n(x)| \le |u(x)|$ and $u_n(x)\, u(x) \ge 0$ for almost every $x \in \mathbb{R}^N$*
iii) *$u_n \to u$ strongly in $W^{1,\vec{p}}(\mathbb{R}^N)$ as $n \to +\infty$.*

Proof. Case $\underline{p} > N$. Let $\xi, \xi_n \in \mathcal{D}(\mathbb{R}^N)$ such that $0 \le \xi \le 1$, $\xi = 1$ in the neighbourhood of 0 with $\mathrm{supp}(\xi) \subset B(0,1)$, and $\xi_n(x) = \xi(\frac{x}{n})$. Herein, $B(0,\rho)$ is the ball with centre 0 and radius ρ. Observe that from the definition of ξ we deduce that $\mathrm{supp}(\xi_n) \subset B(0,n)$. Next, we let $u_n = \xi_n\, u$. It is easy to see that

$$|u_n(x)| \le |u(x)| \quad \text{and} \quad u_n(x)\, u(x) \ge 0 \quad \text{for almost every } x \in \mathbb{R}^N.$$

From Lemma 2.2, we deduce that $u_n \in W^{1,\vec{p}}(B_n) \subset L^\infty(B_n)$, where $B_n = \mathrm{supp(u)} \cap B(0,n)$. By Leibniz formula, we get

$$D^i u_n(x) = D^i \xi_n(x)\, u(x) + \xi_n(x)\, D^i u(x) = \frac{1}{n} D^i \xi(\frac{x}{n})\, u(x) + \xi_n(x)\, D^i u(x)\,.$$

This implies that as $n \to +\infty$

$$D^i u_n(x) \to D^i u(x) \text{ a.e. in } B_n, \quad |D^i u_n(x)| \le c(|u(x)| + |D^i u(x)|),$$

for some constant $c > 0$. From the definition of $\vec{p}$ (recall that $\vec{p}$ is an admissible vector), we obtain $u \in L^{p_0}(B_n) \subset L^{p_i}(B_n)$ and $D^i u \in L^{p_i}(B_n)$. Finally, using the Lebesgue dominated convergence theorem, we deduce $u_n \to u$ strongly in $W^{1,\vec{p}}(\mathbb{R}^N)$ as $n \to +\infty$.

Case $\underline{p} \le N$. Let $u_n = \xi_n\, u$ as in the case $\underline{p} > N$. We will use the following function

$$\begin{cases} F \in \mathcal{D}(\mathbb{R})\ , \ F = 1 \text{ in the neighbourhood of } 0 \\ \text{with } \mathrm{supp}(F) \subset]-1,1[\text{ and } 0 \le F \le 1. \end{cases}$$

We define the function $u_{n,k}$ by $u_{n,k} = F\left(\frac{u_n}{k}\right) u_n$. Observe that $|u_{n,k}| \le k$ and $u_{n,k} \in L^\infty(\mathbb{R}^N)$. Moreover, $|u_{n,k}| < |u|$ and $u_{n,k} u \ge 0$ almost everywhere in $\mathbb{R}^N$, and $\mathrm{supp}(u_{n,k})$ is bounded in $\mathbb{R}^N$. Applying Leibniz formula, we get

$$D^i u_{n,k} = D^i F\Big(\frac{u_n}{k}\Big)\, u_n + F\Big(\frac{u_n}{k}\Big)\, D^i u_n = \frac{1}{k} F'\Big(\frac{u_n}{k}\Big) D^i u_n\, u_n + F\Big(\frac{u_n}{k}\Big)\, D^i u_n$$

It follows that

$$D^i u_{n,k} \to D^i u_n \text{ a.e. in } B_n \quad \text{as } k \to +\infty, \quad |D^i u_{n,k}| \le c |D^i u_n|,$$

for some constant $c > 0$. This implies that $u_{n,k} \to u_n$ strongly in $W^{1,\vec{p}}(\mathbb{R}^N)$ as $k \to +\infty$. Using this and since $u_n \to u$ strongly in $W^{1,\vec{p}}(\mathbb{R}^N)$ as $n \to +\infty$, then there exists a subsequence of $u_{n,k}$ that converges to u strongly in $W^{1,\vec{p}}(\mathbb{R}^N)$. This concludes the proof of theorem. □

Remark 3.3. Note that in the isotropic case ($p_i = p$ for $i = 0, 1, \ldots, N$), we find the result obtained by Brézis and Browder [5] in the classical Sobolev space $W^{1,p}(\mathbb{R}^N)$.

4. Action of some distributions in the dual

In this section we assume that Ω is an open set of $\mathbb{R}^N$. We let the function $J \in C_c^\infty(\mathbb{R}^N)$ that satisfies

(i) $J \geq 0$, $J = 0$ if $|x| \geq 1$

(ii) $\int_{\mathbb{R}^N} J(x)\, dx = 1$.

For $\varepsilon > 0$, define the mollifier J_ε (see, e.g., [1]) by $J_\varepsilon(x) = \varepsilon^{-N} J\left(\frac{x}{\varepsilon}\right)$. Clearly, we have

(i) $J_\varepsilon \geq 0$, and $J_\varepsilon = 0$ if $|x| \geq \varepsilon$

(ii) $\int_{\mathbb{R}^N} J_\varepsilon(x)\, dx = \int_{\mathbb{R}^N} J(x)\, dx = 1$.

So J_ε approximates the Dirac mass δ_0. We therefore expect the convolution

$$(J_\varepsilon * u)(x) = \int_{\mathbb{R}^N} J_\varepsilon(x-y) u(y)\, dy$$

to approximate functions $u \in W^{1,\vec{p}}(\Omega)$. We start with the following classical result, we refer to [1] for the proof in the isotropic case.

Lemma 4.1. *Let $u \in W^{1,\vec{p}}(\Omega)$ and Ω' an open set such that $\Omega' \subset\subset \Omega$. Then* $\lim_{\varepsilon \to 0^+} J_\varepsilon * u = u$ *in* $W^{1,\vec{p}}(\Omega')$.

We have the following theorem.

Theorem 4.2. *Let $\vec{p}$ an admissible vector, $T \in W^{-1,\vec{p}'}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)$ and $u \in W^{1,\vec{p}}(\mathbb{R}^N)$. Assume there exists $h \in L^1(\mathbb{R}^N)$ such that $T(x)\, u(x) \geq h(x)$ for almost every $x \in \mathbb{R}^N$. Then*

$$T\, u \in L^1(\mathbb{R}^N) \quad \textit{and} \quad \langle T, u \rangle = \int_{\mathbb{R}^N} T(x)\, u(x)\, dx.$$

Proof. From Theorem 3.2 we know that there exists a sequence u_n such that

$$\begin{cases} u_n \in W^{1,\vec{p}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ has a compact support in } \mathbb{R}^N \text{ for all } n, \\ |u_n(x)| \leq |u(x)| \text{ and } u_n(x)\, u(x) \geq 0 \text{ for almost every } x \in \mathbb{R}^N, \\ u_n \to u \text{ strongly in } W^{1,\vec{p}}(\mathbb{R}^N) \text{ as } n \to +\infty. \end{cases}$$

Observe that from the definition of J_ε, we obtain $\int_{\mathbb{R}^N} T\, J_\varepsilon * u_n\, dx = \langle T, J_\varepsilon * u_n \rangle$ and

$$\langle T, J_\varepsilon * u_n \rangle \to \langle T, u_n \rangle \quad \text{as } \varepsilon \to 0. \tag{2}$$

Moreover

$$\begin{cases} T\, J_\varepsilon * u_n \to T\, u_n \text{ a.e. in } \mathbb{R}^N, \\ |T\, J_\varepsilon * u_n| \leq |T\, u_n| \in L^1(\mathbb{R}^N). \end{cases}$$

By Lebesgue theorem, we get

$$\int_{\mathbb{R}^N} T\, J_\varepsilon * u_n\, dx \to \int_{\mathbb{R}^N} T\, u_n\, dx \quad \text{as } \varepsilon \to 0. \tag{3}$$

From (2) and (3), we deduce

$$\langle T, u_n \rangle = \int_{\mathbb{R}^N} T\, u_n \, dx. \tag{4}$$

Using $T\, u_n \to T\, u$ a.e. in $\mathbb{R}^N$ and $Tu_n \geq -|h|$, then from Fatou's lemma, we deduce that

$$\int_{\mathbb{R}^N} T\, u \, dx \leq \lim_{n \to +\infty} \int_{\mathbb{R}^N} T\, u_n \, dx = \lim_{n \to +\infty} \langle T, u_n \rangle = \langle T, u \rangle < +\infty. \tag{5}$$

This proves that $T\, u \in L^1(\mathbb{R}^N)$. Observe that

$$\langle T, u_n \rangle \to \langle T, u \rangle \quad \text{as } n \to +\infty, \tag{6}$$

and by the construction of u_n, we have

$$\begin{cases} T\, u_n \to T\, u \text{ almost everywhere in } \mathbb{R}^N, \\ |T\, u_n| \leq |T\, u| \in L^1(\mathbb{R}^N). \end{cases}$$

By Lebesgue theorem, it follows

$$\int_{\mathbb{R}^N} T\, u_n \, dx \to \int_{\mathbb{R}^N} T\, u \, dx \quad \text{as } n \to +\infty.$$

Finally, in view of (4), (5) and (6), we obtain $\langle T, u \rangle = \int_{\mathbb{R}^N} T\, u \, dx$. □

Remark 4.3. Note that in the isotropic case ($p_i = p$ for $i = 0, 1, \ldots, N$), we find the result obtained by Brézis and Browder [5] in the classical Sobolev space $W^{1,p}(\mathbb{R}^N)$.

To extend our result in Theorem 4.2 to an open subset Ω of $\mathbb{R}^N$, we need to assume an additional condition about the regularity on Ω (see, e.g., [2,3,5]). Recall that an open subset Ω of $\mathbb{R}^N$ is said to have the segment property if, given any $x \in \partial\Omega$, there exists an open set G_x in $\mathbb{R}^N$ with $x \in G_x$ and y_x of $\mathbb{R}^N \backslash \{0\}$ such that, if $z \in \overline{\Omega} \cap G_x$ and $t \in]0, 1[$, then $z + t y_x \in \Omega$. This property allows us by a translation to push the support of a function u in Ω.

We have the following lemma.

Lemma 4.4. *Let Ω be an open subset of $\mathbb{R}^N$ satisfying the segment property, $\vec{p}$ be an admissible vector and $u \in W_0^{1,\vec{p}}(\Omega)$. Then, there exists a sequence u_k of $\mathcal{D}(\Omega)$ such that:*

- *$u_k \to u$ strongly in $W_0^{1,\vec{p}}(\Omega)$.*
- *There exists a compact Z of $\overline{\Omega}$ such that $supp(u_k) \subset Z$ for all k.*
- *If $u \in L^\infty(\Omega)$, there exists a constant $C > 0$ such that $|u_k| \leq C$ for all k.*

Proof. Using the admissibility condition of the vector $\vec{p}$, we can prove this lemma by adapting the same technique in the proof of [1, Theorem 3.18], see also [2, 3] in the setting of Sobolev-Orlicz spaces.

Indeed, by Theorem 3.1, we can suppose that u has a bounded support, it follows that the set $K = \{x \in \Omega : u(x) \neq 0\}$ is bounded, and $\overline{K}$ is a compact subset of $\overline{\Omega}$.

Observe that $F = \overline{K} \setminus (\cup_{x\in\partial\Omega} U_x)$ is a compact subset of Ω, where $\cup_{x\in\partial\Omega} U_x$ are open sets determined by the segment property. Moreover, there exists an open set U_0 of $\mathbb{R}^N$ such that $F \subset\subset U_0 \subset\subset \Omega$. Since $\overline{K}$ is compact, and using the fact that Ω satisfies the segment property, there exists a finite number of open sets U_x denoted by $U_1, U_2, \ldots, U_k$ such that $\overline{K} \subset U_0 \cup U_1 \cup \cdots \cup U_k$. Note that there exist also open sets $\tilde{U_1}, \tilde{U_2}, \ldots, \tilde{U_k}$ such that $\tilde{U_j} \subset\subset U_j$ for $j = 0, 1, \ldots, k$, with $\overline{K} \subset \tilde{U_0} \cup \tilde{U_1} \cup \cdots \cup \tilde{U_k}$. Let $\psi = (\psi_j)_{0\leq j\leq k}$ a partition of the unity which correspond to $(U_j)_{0\leq j\leq k}$ ($\text{supp}(\psi_j) \subset \tilde{U_j} \subset\subset \Omega$). Next, set

$$u_j = \psi_j u \quad \text{for } j = 1, \ldots, k. \tag{7}$$

Clearly $u_j \in W^{1,\vec{p}}(\Omega)$. Now, let $1 \leq i \leq N$, then there exist constants $C_i^m, C > 0$ such that

$$D^i u_j(x) = \sum_{0\leq m\leq i} C_i^m D^m \psi_j(x) D^{i-m} u(x),$$

and

$$\int_\Omega |D^i u_j(x)|^{p_i}\, dx \leq C \sum_{0\leq m\leq i} \|D^m \psi_j\|_r^{p_i} \|D^{i-m} u\|_{p_{i-m}}^{p_i},$$

with $r = \left(\frac{1}{p_i} - \frac{1}{p_{i-m}}\right)^{-1}$. This implies that $u_j \in W^{1,\vec{p}}(\Omega)$ since $\vec{p}$ is an admissible vector.

In the last step we let $\varepsilon > 0$ and we propose to find $\phi_j \in C_c^\infty(\mathbb{R}^N)$ such that

$$\|u_j - \phi_j\|_{1,\vec{p}} < \frac{\varepsilon}{k+1}. \tag{8}$$

Therefore, by setting $v = \sum_{j=0}^{j=k} \phi_j$, we deduce $\|u - v\|_{1,\vec{p}} < \varepsilon$ from (7) and (8). Indeed, since $\text{supp}(u_0) \subset \tilde{U_0} \subset\subset \Omega$, then in view of Lemma 4.1, we deduce the function ϕ_0. For $1 \leq j \leq k$, we extend u_j by zero out of Ω, then $u_j \in W^{1,\vec{p}}(\mathbb{R}^N \setminus \Gamma)$, with $\Gamma = \tilde{U_j} \cap \partial\Omega$.

Now let y be the associated vector to U_j (segment property), and for t small enough put $\Gamma_t = \Gamma + ty$ and $u_{j,t}(x) = u_j(x - ty)$. We then have $\Gamma_t \subset \tilde{U_j}$ and $\Gamma_t \cap \overline{\Omega} = \emptyset$. Moreover, we easily have $u_{j,t} \to u_j$ strongly in $W^{1,\vec{p}}(\Omega)$ as $t \to 0$. Finally, from Lemma 4.1 and the fact that $\Omega \cap U_j \subset\subset \mathbb{R}^N - \Gamma_t$, we may define for δ small enough: $\phi_j = J_\delta * u_{j,t}$. □

The next theorem is an extension of Theorem 4.2 to an open subset Ω of $\mathbb{R}^N$.

Theorem 4.5. *Let Ω be an open subset of $\mathbb{R}^N$ satisfying the segment property, $\vec{p}$ be an admissible vector, $T \in W^{-1,\vec{p}'}(\Omega) \cap L^1_{loc}(\overline{\Omega})$ and $u \in W_0^{1,\vec{p}}(\Omega)$. Assume there exists a function $h \in L^1(\Omega)$ such that $T(x)\,u(x) \geq h(x)$ almost everywhere in Ω. Then*

$$T\,u \in L^1(\Omega) \quad \text{and} \quad \langle T, u\rangle = \int_\Omega T(x)\,u(x)\,dx.$$

Proof. From Theorem 3.2 (in the cases $\underline{p} < N$ and $\underline{p} \geq N$), there exists a sequence u_n with bounded support (not necessary compact) such that

$$\begin{cases} u_n \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega) \text{ with bounded support for all } n, \\ |u_n(x)| \leq |u(x)| \text{ and } u_n(x)\,u(x) \geq 0 \text{ for almost every } x \in \Omega, \\ u_n \to u \text{ strongly in } W^{1,\vec{p}}(\Omega) \text{ as } n \to +\infty. \end{cases}$$

We deduce from Lemma 4.4 that for u_n there exists a sequence $(u_{n,k})_k$ of $C_c^\infty(\Omega)$ such that

$$\begin{cases} u_{n,k} \to u_n \text{ strongly in } W_0^{1,\vec{p}}(\Omega), \\ \text{there exists a compact } Z \text{ of } \overline{\Omega} \text{ such that } \operatorname{supp}(u_{n,k}) \subset Z \text{ for all } k, \\ \text{there exists a constant } C > 0 \text{ such that } |u_{n,k}| \leq C \text{ for all } k. \end{cases}$$

Now we can write $\langle T, u_n\rangle = \int_\Omega T\,u_n\,dx$. Proceeding as the proof of Theorem 4.2, we conclude that $\langle T, u\rangle = \int_\Omega T\,u\,dx$. This completes the proof of the theorem. □

5. Applications

This section is devoted to the application of Theorem 4.5 to the study of two anisotropic elliptic equations.

5.1. Nonlinear anisotropic elliptic equation with the data in the dual. Let A be a nonlinear operator from $W_0^{1,\vec{p}}(\Omega)$ into the dual $W^{-1,\vec{p}'}(\Omega)$. For the study of the problem

$$Au + g(x,u) = f, \quad x \in \Omega, \tag{9}$$

we will impose the following conditions:

$$\begin{cases} A \text{ is bounded, pseudo-monotone and coercive, i.e.,} \\ \displaystyle\lim_{\|u\|_{1,\vec{p}} \to +\infty} \frac{\langle Au, u\rangle}{\|u\|_{1,\vec{p}}} = +\infty, \text{ with } p_i > 1 \text{ for all } i = 0, 1, \ldots, N. \end{cases} \tag{10}$$

The nonlinear function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be a Carathéodory function satisfying:

$$\begin{cases} \text{There exists } h_s \in L^1(\Omega) \text{ such that} \\ \sup_{|u| \leq s} |g(x,u)| \leq h_s(x) \text{ for all } s > 0 \text{ and for a.e. } x \in \Omega, \\ g(x,\sigma)\,\sigma \geq 0 \text{ for all } \sigma \in \mathbb{R} \text{ and for a.e. } x \in \Omega. \end{cases} \tag{11}$$

We have the following result.

Theorem 5.1. *Let Ω be an open subset of $\mathbb{R}^N$ satisfying the segment property. Assume that the conditions* (10) *and* (11) *hold and $\vec{p}$ is an admissible vector. Then for all $f \in W^{-1,\vec{p}'}(\Omega)$ there exists $u \in W_0^{1,\vec{p}}(\Omega)$ such that*

$$\begin{cases} g(x,u) \in L^1(\Omega) \quad \textit{and} \quad g(x,u)\,u \in L^1(\Omega), \\ \langle Au, v\rangle + \displaystyle\int_\Omega g(x,u)v\,dx = \langle f, v\rangle \quad \forall v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega) \textit{ and } v = u. \end{cases}$$

Proof. The proof of Theorem 5.1 is based on approximation problem. We prove first existence of solutions to the approximate problem of (9), deriving a priori estimates, and then passing to the limit in the approximate solutions using monotonicity and compactness arguments. Having proved existence to the approximate problem of (9), the goal is to send the regularization parameter to $+\infty$ in sequences of such solutions and use Theorem 4.2 to fabricate weak solutions of the original equation (9). We proceed by steps:

Existence of the approximate problem. For a.e. $x \in \Omega$ set

$$g_k(x,u) = T_k g(x,u), \quad b_k(u,v) = \int_\Omega g_k(x,u)v\,dx \quad \text{for all } u, v \in W_0^{1,\vec{p}}(\Omega),$$

where T_k is the usual truncation given by

$$T_k\xi = \begin{cases} \xi & \text{if } |\xi| \le k \\ \frac{k\xi}{|\xi|} & \text{if } |\xi| > k. \end{cases}$$

Define the following operator

$$G_k u : W_0^{1,\vec{p}}(\Omega) \to \mathbb{R}; \quad v \mapsto \int_\Omega g_k(x,u)v\,dx.$$

It is easy to see that $G_k : W_0^{1,\vec{p}}(\Omega) \to W^{-1,\vec{p}'}(\Omega)$ is well defined. Next, we show that $A + G_k$ is pseudo-monotone. Indeed, let $u_j \to u$ weakly in $W_0^{1,\vec{p}}(\Omega)$ such that

$$(A + G_k)u_j \to y \text{ weakly in } W^{-1,\vec{p}'}(\Omega) \text{ and } \lim_{j\to+\infty} \langle (A+G_k)u_j, u_j - u\rangle \le 0.$$

Then, there exists a subsequence still denoted by u_j such that $u_j \to u$ a.e. in Ω. Hence

$$\begin{cases} g_k(x,u_j) \to g_k(x,u) \text{ a.e. in } \Omega, \\ |g_k(x,u_j)| \le k. \end{cases}$$

By dominated convergence Lebesgue theorem, we get $g_k(x,u_j) \to g_k(x,u)$ in $L^{p_0'}(\Omega)$. Since $u_j \to u$ weakly in $L^{p_0}(\Omega)$, it follows that

$$\int_\Omega g_k(x,u_j)(u_j - u)\,dx \to 0.$$

Therefore, we obtain

$$Au_j \to y - G_k u \text{ weakly in } W^{-1,\vec{p}'}(\Omega) \text{ and } \lim_{j\to+\infty} \langle Au_j, u_j - u\rangle \le 0.$$

Since A is pseudo-monotone, we deduce $y = Au + G_k u$, which implies that $A + G_k$ is pseudo-monotone. On the other hand, from (10) and (11), we easily deduce that $A+G_k$ is bounded and coercive. The operator $A+G_k$ satisfies then Lerray-Lions classical conditions [10], so there exists $u_k \in W_0^{1,\vec{p}}(\Omega)$ solution of the problem $Au_k + g_k(x, u_k) = f$, or variationally

$$\langle Au_k, v\rangle + \int_\Omega g_k(x, u_k) v\, dx = \langle f, v\rangle \quad \text{for all } v \in W_0^{1,\vec{p}}(\Omega). \tag{12}$$

Existence of solution to the original problem. Proceeding as in the proof of [4, Theorem 3.2], one obtains

$$u_k \to u \text{ weakly in } W_0^{1,\vec{p}}(\Omega), \quad Au_k \to \chi \text{ weakly in } W^{-1,\vec{p}'}(\Omega).$$

Moreover, we have

$$g(x, u)u \in L^1(\Omega) \quad \text{and} \quad g_k(x, u_k) \to g(x, u) \text{ strongly in } L^1(\Omega),$$

as $k \to +\infty$. By passing to limit in (12), we obtain

$$\langle \chi, v\rangle + \int_\Omega g(x, u) v\, dx = \langle f, v\rangle \quad \text{for all } v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

It remains to prove that $\chi = Au$. Indeed, on one hand, letting $v = u_k$ in (12). In view of Fatou's lemma, we deduce that

$$\limsup_{k\to+\infty} \langle Au_k, u_k\rangle \le \langle f, u\rangle - \int_\Omega g(x, u) u\, dx.$$

On the other hand, we have $T = g(x, u) = f - \chi \in W^{-1,\vec{p}'}(\Omega) \cap L^1_{loc}(\overline{\Omega})$. Now, by applying Hedberg-type's approximation as in Theorem 4.2, we obtain $\int_\Omega g(x, u)u\, dx = \langle f - \chi, u\rangle$. Therefore $\limsup_{k\to+\infty}\langle Au_k, u_k\rangle \le \langle \chi, u\rangle$. Since A is pseudo-monotone, then $\chi = Au$. □

5.2. Nonlinear anisotropic elliptic equation with data close to $L^1(\Omega)$. In this subsection, we will use the following anisotropic space

$$W^{1,\vec{p},\varepsilon}(\Omega) = \left\{ u \in W^{1,1+\frac{1}{\varepsilon}}(\Omega),\ \frac{\partial u}{\partial x_i} \in L^{p_i},\ i = 1, \dots, N \right\},$$

under the norm

$$\|u\|_{1,\vec{p},\varepsilon} = \|u\|_{L^{1+\frac{1}{\varepsilon}}(\Omega)} + \sum_{i=1}^N \|D^i u\|_{L^{p_i}(\Omega)}.$$

The space $W_0^{1,\vec{p},\varepsilon}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{1,\vec{p},\varepsilon}(\Omega)$ with respect to its norm, i.e., $W_0^{1,\vec{p},\varepsilon}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{1,\vec{p},\varepsilon}(\Omega)}$. The dual of $W_0^{1,\vec{p},\varepsilon}(\Omega)$ is denoted by $W^{-1,\vec{p'},\varepsilon}(\Omega)$, where $\vec{p'} = \{p'_i,\ i = 0,1,\dots,N\}$, $p'_i = \frac{p_i}{p_i-1}$, $p_0 = 1 + \frac{1}{\varepsilon}$, and $p_i > 1$ for $i = 1,\dots,N$. Herein, ε is a positive number satisfying $0 < \varepsilon < 1$.

Remark 5.2. By taking $p_0 = 1 + \frac{1}{\varepsilon}$, note that for $0 < \varepsilon < 1$ small enough, $\vec{p}$ is an admissible vector.

We prove existence and regularity of distributional solutions in an appropriate function space for the nonlinear elliptic equation

$$\begin{cases} u \in W_0^{1,\vec{p},\varepsilon}(\Omega), \\ Au + g(x,u) = f \quad \text{in } \Omega, \end{cases} \tag{13}$$

where $f \in L^{1+\varepsilon}(\Omega)$. Herein, the operator A and the function g satisfy (10) and (11) respectively.

Theorem 5.3. *Let Ω be an open subset of $\mathbb{R}^N$ satisfying the segment property. Assume* (10) *and* (11) *hold. Then for all $f \in L^{1+\varepsilon}(\Omega)$, the problem* (13) *has at least one solution, i.e., there exists $u \in W_0^{1,\vec{p},\varepsilon}(\Omega)$ such that*

$$\begin{cases} g(x,u) \in L^1(\Omega) \quad \textit{and} \quad g(x,u)u \in L^1(\Omega), \\ \langle Au, v\rangle + \displaystyle\int_\Omega g(x,u)v\,dx = \langle f, v\rangle \quad \forall v \in W_0^{1,\vec{p},\varepsilon}(\Omega) \cap L^\infty(\Omega) \ \textit{and} \ v = u. \end{cases}$$

Remark 5.4. Note that the term $\langle f, v\rangle$ is well defined since v is in $W_0^{1,\vec{p},\varepsilon}(\Omega)$ and $f \in L^{1+\varepsilon}(\Omega)$.

Proof. Proceeding exactly as the proof of the first step of Theorem 5.1, we obtain for all $f_n \in L^\infty(\Omega)$, there exists $u_n \in W_0^{1,\vec{p},\varepsilon}(\Omega)$ such that

$$\begin{cases} g(x,u_n) \in L^1(\Omega) \quad \text{and} \quad g(x,u_n)u_n \in L^1(\Omega), \\ \langle Au_n, v\rangle + \displaystyle\int_\Omega g(x,u_n)v\,dx = \langle f_n, v\rangle \quad \forall v \in W_0^{1,\vec{p},\varepsilon}(\Omega) \cap L^\infty(\Omega) \text{ and } v = u_n, \end{cases} \tag{14}$$

where $|f_n| \le |f|$ a.e. in Ω, $f_n \in L^\infty(\Omega)$ and $f_n \to f$ strongly in $L^{1+\varepsilon}(\Omega)$ and a.e. in Ω.

Observe that $W_0^{1,\vec{p},\varepsilon}(\Omega)$ is reflexive (recall that $p_i > 1$ for all $i = 0,1,\dots,N$) and repeating the same steps in the proof of [4, Theorem 3.2], we have

$$\begin{aligned} u_n &\to u && \text{weakly in } W_0^{1,\vec{p},\varepsilon}(\Omega) \\ Au_n &\to \chi && \text{weakly in } W^{-1,\vec{p'},\varepsilon}(\Omega) \\ g(x,u_n) &\to g(x,u) && \text{strongly in } L^1(\Omega) \end{aligned}$$

as $n \to +\infty$. Therefore, we obtain

$$\langle \chi, v\rangle + \int_\Omega g(x,u)v\,dx = \langle f, v\rangle \quad \forall\, v \in W_0^{1,\vec{p},\varepsilon}(\Omega) \cap L^\infty(\Omega).$$

Moreover, we have $T = g(x,u) = f - \chi \in W^{-1,\vec{p}',\varepsilon}(\Omega) \cap L^1_{loc}(\overline{\Omega})$. Hence, using Hedberg-type's approximation (Theorem 4.2), we have $\int_\Omega g(x,u)u\,dx = \langle f - \chi, u\rangle$. Now, substituting $v = u_n$ in (14), we get from Fatou's lemma $\limsup_{n\to+\infty}\langle Au_n, u_n\rangle \le \langle \chi, u\rangle$. Consequently $\chi = Au$ since A is pseudo-monotone operator. This completes the proof of Theorem 5.2. $\square$

Remark 5.5. Note that $\left(L^{1+\frac{1}{\varepsilon}}(\Omega)\right)' = L^{1+\varepsilon}(\Omega)$. So that, f is considered in a dual space close to $L^1(\Omega)$.

Observe also that when $\varepsilon \to 0$, we deal with a problem with L^1-data and u will belong to $L^\infty(\Omega)$ which is true by regarding the definition of our anisotropic space. Note that in this direction, the classical anisotropic case requires the condition $\underline{p} > N$ in order to get $u \in L^\infty(\Omega)$.

Remark 5.6. Remark that in the case where $\underline{p} > N$, the existence results stated in Theorem 5.1 and 5.2 can be proved without assuming the segment property of Ω nor the admissibility condition of the vector $\vec{p}$. Let us point out that a work in this direction can be found in [4], where the authors have studied the existence of solutions for a general class of nonlinear elliptic equations of order m with $m\underline{p} > N$.

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