Duals of Optimal Spaces for the Hardy Averaging Operator

Aleš Nekvinda and Luboš Pick

Abstract. The Hardy averaging operator $Af(x) := \frac{1}{x} \int_0^x f(t)\, dt$ is known to map boundedly the 'source' space S^p of functions on $(0,1)$ with finite integral

$$\int_0^1 \operatorname*{ess\,sup}_{t \in (x,1)} \frac{1}{t} \int_0^t |f|^p dx$$

into the 'target' space T^p of functions on $(0,1)$ with finite integral

$$\int_0^1 \operatorname*{ess\,sup}_{t \in (x,1)} |f(t)|^p dx$$

whenever $1 < p < \infty$. Moreover, the spaces S^p and T^p are optimal within the fairly general context of all Banach lattices. We prove a duality relation between such spaces. We in fact work with certain (more general) weighted modifications of these spaces. We prove optimality results for the action of A on such spaces and point out some applications to the variable-exponent spaces. Our method of proof of the main duality result is based on certain discretization technique which leads to a discretized characterization of the optimal spaces.

Keywords. Hardy averaging operator, optimal target and domain spaces, associate spaces, discretization, Banach lattice, weights, weighted spaces, variable-exponent spaces

Mathematics Subject Classification (2000). 47G100

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1. Introduction

We consider the Hardy averaging operator A given at a function $f \in L^1_{loc}(0,1)$ and $x \in (0,1)$ by

$$Af(x) = \frac{1}{x} \int_0^x f(t)dt$$

and the one-dimensional Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{a<x<b} \frac{1}{b-a} \int_a^b |f(t)|dt.$$

The operators M and A are bounded on $L^p(0,1)$ whenever $1 < p \leq \infty$, and this result cannot be essentially improved within the context of Lebesgue spaces. However, an improvement is possible if we are willing to settle for other, more general function spaces and classes. In [9], the spaces S^p and T^p were constructed for $1 < p < \infty$, as the collections of all functions on $(0,1)$ having finite norms

$$\|f\|_{S_p} = \left(\int_0^1 \operatorname*{ess\,sup}_{t\in(x,1)} \left(\frac{1}{t} \int_0^t |f(s)|ds \right)^p dx \right)^{\frac{1}{p}}$$

and

$$\|f\|_{T_p} = \left(\int_0^1 \operatorname*{ess\,sup}_{t\in(x,1)} |f(t)|^p dx \right)^{\frac{1}{p}},$$

respectively. These spaces satisfy $T^p \hookrightarrow L^p \hookrightarrow S^p$ and $A : S^p \to T^p$ (here, as usual, $\hookrightarrow$ denotes a continuous embedding and $\to$ a boundedness of an operator). Moreover, S^p and T^p are *optimal* in the following sense; if X, Y are Banach function spaces such that $X \subsetneq T^p$ and $S^p \subsetneq Y$ then A is no longer bounded from S^p to X of from Y to T^p.

Thus, the spaces of type S^p or T^p have interesting applications, while, at the same time, they are relatively new (although it should be noted that spaces similar to T^p were considered in connection with different matters by Grosse-Erdmann in [6], where, among other results, also their discrete versions were introduced). It is therefore desired to study their intrinsic properties.

In this paper, we focus, among other things, on the duality relationship between spaces of type S^p and T^p. We will in fact work in a more general context, studying certain weighted versions of these spaces and also their variable-exponent likes. In our main result we characterize associate spaces of the weighted versions of the optimal target and source spaces for the operator A. Interestingly, we shall see that the two types of spaces are linked together by a certain duality relation. The proof is based on an elementary discretization method which leads to several other characterizations of the spaces in question. We also consider applications to variable-exponent spaces which have

been recently extensively studied because of the wide range of their application in mathematical physics.

We also note that the duality relation can be used in quite an obvious way to obtain corresponding optimality results for the associate operator $A'g(t) := \int_t^1 \frac{g(s)}{s}\, ds$. We omit the details.

The paper is organized as follows. Section 2 contains background material. In Section 3 we introduce weighted and variable-exponent versions of the spaces S^p and T^p; we also study the action of the operator A on these spaces and show their optimality. In Section 4 we carry out a discretization procedure that will be needed later in the proof of the main duality results. By doing that, we obtain an alternative characterization of the optimal spaces. Finally, in Section 5, we characterize the associate spaces of the optimal spaces and point out the link between them.

2. Preliminaries

Let $\mathcal{M}(0,1)$ denote the class of all Lebesgue-measurable functions on $(0,1)$. Denote by $|E|$ the Lebesgue measure of any measurable subset E of $(0,1)$ and by χ_E the characteristic function of E.

We recall the definition of Banach function spaces from [1]. We note that the terminology is not unique in the literature. In particular, we shall assume the so-called Fatou property, given below as the axiom (P3).

Definition 2.1. We say that a normed linear space $(X, \|\cdot\|_X)$ is a *Banach function space* (*BFS* for short) if the following five conditions are satisfied:

(P1) the norm $\|f\|_X$ is defined for all $f \in \mathcal{M}(0,1)$ and $f \in X$ if and only if $\|f\|_X < \infty$

(P2) $\|f\|_X = \|\,|f|\,\|_X$ for every $f \in \mathcal{M}(0,1)$

(P3) if $0 \le f_n \nearrow f$ a.e. in $(0,1)$, then $\|f_n\|_X \nearrow \|f\|_X$

(P4) $\chi_{(0,1)} \in X$

(P5) for every set $E \subset (0,1)$, there exists a positive constant C_E such that $\int_E |f(x)|dx \le C_E \|f\|_X$.

Note that the condition (P3) immediately yields the following property:

(P6) if $0 \le f \le g$, then $\|f\|_X \le \|g\|_X$.

If $(X, \|.\|_X)$ is a BFS, then its *associate space* X' is defined as the collection of all functions in $\mathcal{M}(0,1)$ having finite the norm

$$\|f\|_{X'} = \sup\left\{ \int_0^1 f(x)g(x)dx; \|g\|_X \le 1 \right\}.$$

Then $X'' = X$, and the Hölder inequality

$$\int_0^1 |f(x)g(x)|\, dx \leq \|f\|_X \|g\|_{X'}$$

holds.

Let X, Y be two Banach spaces (not necessarily Banach function spaces). We say that X is (*continuously*) *embedded* into Y, written $X \hookrightarrow Y$, if there is a positive constant $C > 0$ such that $\|f\|_Y \leq C\|f\|_X$ for all $f \in X$. It is known ([1, Theorem 1.8]) that, for a pair of if Banach function spaces X and Y, the set-inclusion $X \subset Y$ already implies (and therefore is equivalent to) the embedding $X \hookrightarrow Y$.

We denote by $\mathcal{B}$ the set of all functions $p(\cdot) \in \mathcal{M}(0,1)$ defined on $(0,1)$ such that $1 < \operatorname{ess\,inf} p(x) \leq \operatorname{ess\,sup} p(x) < \infty$. For $p(\cdot) \in \mathcal{B}$, we define its *conjugate function* $p'(\cdot)$ by $p'(x) = \frac{p(x)}{p(x)-1}$.

Definition 2.2. Given a function $p(\cdot) \in \mathcal{B}$ and $\alpha \in \mathbb{R}$, we define the functional

$$m_{p(\cdot),\alpha}(f) = \int_0^1 |f(x)|^{p(x)} x^\alpha dx,$$

the corresponding *Luxemburg norm*

$$\|f\|_{L_\alpha^{p(\cdot)}} = \inf\left\{\lambda > 0; m_{p(\cdot),\alpha}\left(\frac{f(x)}{\lambda}\right) \leq 1\right\}, \quad f \in \mathcal{M}(0,1),$$

and the corresponding *weighted variable-exponent Lebesgue space*

$$L_\alpha^{p(x)} = \{f \in \mathcal{M}(0,1); \|f\|_{L_\alpha^{p(\cdot)}} < \infty\}.$$

Under our assumptions on $p(\cdot)$, $m_{p(\cdot)}$ is a convex modular, and $L_\alpha^{p(\cdot)}$ is a Banach space under the Luxemburg norm. We write $L^{p(x)}$ instead of $L_0^{p(x)}$. By a standard technique, one can show that, for a given $p(\cdot) \in \mathcal{B}$, one has

$$\|f\|_{L_\alpha^{p(\cdot)}} < \infty \iff \int_0^1 |f(x)|^{p(x)} x^\alpha dx < \infty.$$

We also recall that the *Hölder inequality* for $L^{p(\cdot)}$ spaces reads as follows (see [8, Theorem 2.1]). Let $p(\cdot) \in \mathcal{B}$, then there exists a positive constant $r_{p(\cdot)}$ such that

$$\int_0^1 |f(x)g(x)|dx \leq r_{p(\cdot)} \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}.$$

3. The action of the operator A on function spaces

In the theory of variable-exponent Lebesgue spaces, the following notion is of great importance.

Definition 3.1. Let $p(\cdot) : [0,1] \to \mathbb{R}$. We say that $p(\cdot)$ is *weak-Lipschitz* if there is a $C > 0$ such that

$$|p(x) - p(y)| \le \frac{C}{\log\left(\frac{\mathrm{e}^2}{|x-y|}\right)} \tag{3.1}$$

for all $x, y \in [0,1]$, $0 < |x - y| < 1$.

We say that $p(\cdot)$ is *weak-Lipschitz at zero* if there exist constants $\delta \in (0,1)$ and $C > 0$ such that

$$|p(x) - p(0)| \le \frac{C}{\log\left(\frac{\mathrm{e}^2}{x}\right)} \tag{3.2}$$

for all $x \in (0, \delta)$.

It has been known for several years that the condition (3.1) plays a crucial role in connection with the action of integral operators on $L^{p(\cdot)}$ (see [10] for example). In this paper we introduce a weighted version of the *maximal operator*, defined for $\beta \in \mathbb{R}$ at a function $f \in \mathcal{M}(0,1)$ by

$$M^\beta f(x) := x^\beta \sup_{r>0} \frac{1}{2r} \int_{(x-r,x+r)\cap(0,1)} y^{-\beta} |f(y)| dy, \quad x \in (0,1).$$

We have the following result due to Kokilashvili and Samko ([7, Theorem A]): whenever $p(\cdot) \in \mathcal{B}$ is weak-Lipschitz and $\beta \in \mathbb{R}$, then

$$M^\beta : L^{p(x)} \to L^{p(x)} \quad \text{if and only if} \quad -\frac{1}{p(0)} < \beta < \frac{1}{p'(0)}. \tag{3.3}$$

We write Mf instead of $M^0 f$.

Our aim is to to characterize those spaces $L_\alpha^{p(x)}$ on which the maximal operator is bounded. To this end, we shall first prove a key lemma of a technical nature.

Lemma 3.2. *Let $p(x), q(x) \in \mathcal{B}$. Assume that there exist constants $0 < \delta < \frac{1}{\mathrm{e}}$ and $C > 0$ such that for all $x \in (0, \delta)$*

$$|p(x) - q(x)| \le \frac{C}{\log\left(\frac{\mathrm{e}^2}{x}\right)}. \tag{3.4}$$

Then there is a positive constant D such that, for every $x \in (0,1)$,

$$D^{-1} x^{\frac{1}{q(x)}} \le x^{\frac{1}{p(x)}} \le D x^{\frac{1}{q(x)}}.$$

Proof. By symmetry, it suffices to prove just $x^{\frac{1}{p(x)}} \leq Dx^{\frac{1}{q(x)}}$. Since $p(\cdot), q(\cdot) \in \mathcal{B}$, there is a $B \geq 1$ such that $B^{-1} \leq (e^2)^{\frac{1}{p(x)} - \frac{1}{q(x)}} \leq B$, $x \in [0, 1]$. Writing

$$x^{\frac{1}{p(x)} - \frac{1}{q(x)}} = (e^2)^{\frac{1}{p(x)} - \frac{1}{q(x)}} \left(\frac{x}{e^2}\right)^{\frac{1}{p(x)} - \frac{1}{q(x)}} \leq Be^{\left(\frac{1}{p(x)} - \frac{1}{q(x)}\right)\log\left(\frac{x}{e^2}\right)}$$

and denoting

$$L(x) := B\left(e^{(q(x)-p(x))\left(-\log\left(\frac{e^2}{x}\right)\right)}\right)^{\frac{1}{p(x)q(x)}},$$

we get $L(x) \leq B$ when $q(x) - p(x) \geq 0$ (since then $(q(x) - p(x))\left(-\log\left(\frac{e^2}{x}\right)\right) \leq 0$), while, when $q(x) - p(x) < 0$, by (3.4) we have

$$L(x) = B\left(e^{(p(x)-q(x))\log\left(\frac{e^2}{x}\right)}\right)^{\frac{1}{p(x)q(x)}} \leq B\left(e^C\right)^{\frac{1}{p(x)q(x)}} \leq Be^C := D,$$

proving the claim. □

Theorem 3.3. *Let $p(\cdot) \in \mathcal{B}$ be weak-Lipschitz. Then $M : L_\alpha^{p(x)} \to L_\alpha^{p(x)}$ if and only if $-1 < \alpha < p(0) - 1$.*

Proof. Let $f \in L_\alpha^{p(x)}(0, 1)$. Since

$$\left|1 - \frac{p(0)}{p(x)}\right| = p(0)\left|\frac{1}{p(0)} - \frac{1}{p(x)}\right| \leq C\frac{p(0)}{\log\left(\frac{e^2}{x}\right)},$$

it follows from Lemma 3.2 that x^α is comparable to $x^{\frac{\alpha p(x)}{p(0)}}$, whence

$$\int_0^1 |f(x)|^{p(x)} x^{\frac{\alpha p(x)}{p(0)}} dx < \infty,$$

in other words, $\int_0^1 |x^{\frac{\alpha}{p(0)}} f(x)|^{p(x)} dx < \infty$. Denoting $g(x) = x^{\frac{\alpha}{p(0)}} f(x)$, we have $g \in L^{p(x)}(0, 1)$. Our assumption $-1 < \alpha < p(0) - 1$ is equivalent to $-\frac{1}{p(0)} < \frac{\alpha}{p(0)} < \frac{1}{p'(0)}$, which, by (3.3), in turn implies $M^{\frac{\alpha}{p(0)}} g \in L^{p(x)}(0, 1)$, or

$$\int_0^1 Mf(x)^{p(x)} x^{\frac{\alpha p(x)}{p(0)}} dx < \infty.$$

By the above observation, this is equivalent to $\int_0^1 Mf(x)^{p(x)} x^\alpha dx < \infty$, as desired. This establishes the "if" part of the theorem. The "only if" part can be proved along the same line of argument (in the opposite direction). □

We will now introduce a new function space which will turn out to be the optimal range for the average operator. Given a function $f(x)$ on $(0, 1)$, set

$$\widetilde{f}(x) = \operatorname*{ess\,sup}_{t \in (x,1)} |f(t)|.$$

Definition 3.4. Let $f \in \mathcal{M}(0,1)$, $p(\cdot) \in \mathcal{B}$ and $\alpha \in \mathbb{R}$. We define the functional

$$m_{T_\alpha^{p(\cdot)}}(f) = \int_0^1 (\widetilde{f}(x))^{p(x)} x^\alpha dx = \int_0^1 \left(\operatorname*{ess\ sup}_{t \in (x,1)} |f(t)| \right)^{p(x)} x^\alpha dx,$$

the norm

$$\|f\|_{T_\alpha^{p(\cdot)}} = \inf \left\{ \lambda > 0; m_{T_\alpha^{p(\cdot)}} \left(\frac{f}{\lambda} \right) \leq 1 \right\},$$

and the corresponding space

$$T_\alpha^{p(\cdot)} = \{ f \in \mathcal{M}(0,1); \|f\|_{T_\alpha^{p(\cdot)}} < \infty \}.$$

In cases when $p(\cdot) \equiv p$ for some constant $p \in (1, \infty)$, we write T_α^p instead of $T_\alpha^{p(\cdot)}$.

By a routine technique one proves that the functional $m_{T_\alpha^{p(\cdot)}}$ is a convex modular and $T_\alpha^{p(\cdot)}$ is a Banach space with respect to the norm $\| \cdot \|_{T_\alpha^{p(\cdot)}}$.

Lemma 3.5. *Let $p(\cdot) \in \mathcal{B}$ be weak-Lipschitz at zero. Assume $\alpha < p(0) - 1$. Then $L_\alpha^{p(\cdot)} \hookrightarrow L^1$.*

Proof. Let $f \in L_\alpha^{p(\cdot)}$. Using the Hölder inequality, we obtain

$$\int_0^1 |f(x)| dx = \int_0^1 \left| f(x) x^{\frac{\alpha}{p(x)}} x^{-\frac{\alpha}{p(x)}} \right| dx \leq r_{p(\cdot)} \left\| f(x) x^{\frac{\alpha}{p(x)}} \right\|_{L^{p(\cdot)}} \left\| x^{-\frac{\alpha}{p(x)}} \right\|_{L^{p'(\cdot)}}.$$

Since $\int_0^1 \left| f(x) x^{\frac{\alpha}{p(x)}} \right|^{p(x)} dx = \int_0^1 |f(x)|^{p(x)} x^\alpha dx < \infty$, we have $\left\| f(x) x^{\frac{\alpha}{p(x)}} \right\|_{L^{p(\cdot)}} < \infty$. Furthermore, thanks to our assumption $\alpha < p(0) - 1$,

$$\int_0^1 x^{-\frac{\alpha}{p(0)-1}} dx < \infty. \tag{3.5}$$

As $p(\cdot)$ is weak-Lipschitz at zero, one has

$$\left| \frac{\alpha}{p(x) - 1} - \frac{\alpha}{p(0) - 1} \right| \leq \alpha \frac{|p(x) - p(0)|}{(p(x) - 1)(p(0) - 1)} \leq \frac{C}{\log \left(\frac{e^2}{x} \right)}.$$

Now, Lemma 3.2 and (3.5) yield $\int_0^1 \left(x^{-\frac{\alpha}{p(x)}} \right)^{p'(x)} dx = \int_0^1 x^{-\frac{\alpha}{p(x)-1}} dx < \infty$, hence also $\left\| x^{-\frac{\alpha}{p(x)}} \right\|_{L^{p'(\cdot)}} < \infty$, and the assertion follows. $\square$

Remark 3.6. When $p(\cdot) \in \mathcal{B}$ is weak-Lipschitz at zero and $-1 < \alpha < p(0) - 1$, then, actually, $T_\alpha^{p(\cdot)}$ is a BFS.

Indeed, we already noticed that $T_\alpha^{p(\cdot)}$ always satisfies the conditions (P1) and (P2). By a standard technique, one can prove (P3), which in turn implies (P6). Since $\int_0^1 x^\alpha dx < \infty$, (P4) holds. Finally, since $|f| \leq \widetilde{f}$, we get $T_\alpha^{p(\cdot)} \hookrightarrow L_\alpha^{p(\cdot)}$, hence (P5) follows from Lemma 3.5.

Theorem 3.7. *Let $p(\cdot), q(\cdot) \in \mathcal{B}$ be weak-Lipschitz at zero, $p(0) = q(0) := p$. Assume $-1 < \alpha < p - 1$. Then the norms in $T_\alpha^{p(\cdot)}$ and $T_\alpha^{q(\cdot)}$ are equivalent.*

Proof. Assume that $f \geq 0$ and $\int_0^1 (\widetilde{f}(x))^{p(x)} x^\alpha dx < \infty$. By Lemma 3.5, we know that $\int_0^1 \widetilde{f}(x) dx < \infty$. Since $\widetilde{f}$ is non-increasing on $(0, 1)$, we have $x\widetilde{f}(x) \leq \int_0^1 \widetilde{f}(x) dx =: K < \infty$, which gives

$$\widetilde{f}(x) \leq \frac{K}{x}, \quad x \in (0, 1). \tag{3.6}$$

Set $\mathbb{A}_1 = \{x \in [0, 1]; q(x) < p(x)\}$, $\mathbb{A}_2 = \{x \in [0, 1]; q(x) \geq p(x)\}$, then

$$\int_0^1 \widetilde{f}(x)^{q(x)} x^\alpha dx = \int_{\mathbb{A}_1} \widetilde{f}(x)^{q(x)} x^\alpha dx + \int_{\mathbb{A}_2} \widetilde{f}(x)^{q(x)} x^\alpha dx =: A_1 + A_2,$$

say. Clearly, $A_1 \leq \int_{\mathbb{A}_1} (1 + \widetilde{f}(x))^{q(x)} x^\alpha dx \leq \int_{\mathbb{A}_1} (1 + \widetilde{f}(x))^{p(x)} x^\alpha dx < \infty$. It remains to estimate A_2. By (3.6), we have

$$\begin{aligned}
A_2 &= \int_{\mathbb{A}_2} \widetilde{f}(x)^{p(x)} x^\alpha \widetilde{f}(x)^{q(x)-p(x)} dx \\
&\leq \int_{\mathbb{A}_2} \widetilde{f}(x)^{p(x)} x^\alpha \left(\frac{K}{x}\right)^{q(x)-p(x)} dx \\
&\leq C \int_{\mathbb{A}_2} \widetilde{f}(x)^{p(x)} x^\alpha \left(\frac{1}{x}\right)^{q(x)-p(x)} dx.
\end{aligned}$$

Since $p(0) = q(0)$, we have $|p(x) - q(x)| \leq |p(x) - p(0)| + |q(x) - q(0)|$. This, together with the weak-Lipschitz property at zero of p and q, yields (3.4) for x sufficiently close to zero. This enables us to use Lemma 3.2, and we arrive at

$$\left(\frac{1}{x}\right)^{q(x)-p(x)} = x^{p(x)-q(x)} = \left(x^{\frac{1}{q(x)} - \frac{1}{p(x)}}\right)^{p(x)q(x)} \leq D^{p(x)q(x)} \leq \widetilde{D},$$

in other words, $A_2 \leq C\widetilde{D} \int_{\mathbb{A}_2} \widetilde{f}(x)^{p(x)} x^\alpha dx < \infty$. Complemented with the fact that $T_\alpha^{p(\cdot)}$ and $T_\alpha^{q(\cdot)}$ are Banach function spaces by Remark 3.6, this finishes the proof. $\square$

An important consequence of the preceding theorem is the relation between spaces of type T and L.

Corollary 3.8. *Let $p(\cdot)$ be weak-Lipschitz at zero. Set $p = p(0)$ and assume $-1 < \alpha < p - 1$. Then $T_\alpha^p \hookrightarrow L_\alpha^{p(\cdot)}$.*

Proof. The assertion is readily seen from $T_\alpha^p \hookrightarrow T_\alpha^{p(\cdot)} \hookrightarrow L_\alpha^{p(\cdot)}$, in which the former embedding follows from Theorem 3.7 while the latter and the latter was observed in Remark 3.6. $\square$

We shall now investigate the role of $T_\alpha^{p(\cdot)}$ as the target space for the averaging operator.

Lemma 3.9. *Assume that $p(\cdot) \in \mathcal{B}$ is weak-Lipschitz and that $-1 < \alpha < p(0) - 1$. Then $A : L_\alpha^{p(\cdot)} \to T_\alpha^{p(\cdot)}$.*

Proof. Let f be defined on $(0,1)$. When appropriate, we consider its extension by zero outside $(0,1)$. Set $g(x) = \widetilde{A|f|}(x)$. In [9, (5.1)], the inequality

$$g(x) \le 4M(A|f|)(x)$$

was noted. By Theorem 3.3, M is bounded on $L_\alpha^{p(\cdot)}$, and, since $|Af| \le A|f| \le Mf$, so is A. Let $f \in L_\alpha^{p(\cdot)}$. Then $A|f| \in L_\alpha^{p(\cdot)}$ and, by continuity of M on $L_\alpha^{p(\cdot)}$, also $M(A|f|) \in L_\alpha^{p(\cdot)}$. Altogether, $g \in L_\alpha^{p(\cdot)}$, as desired. (We have shown of course only that $f \in L_\alpha^{p(\cdot)}$ implies $Af \in T_\alpha^{p(\cdot)}$, but one can obtain the corresponding boundedness of A just the same way as it is done for the identity operator, e.g., in [1, Theorem 1.8]). $\square$

Theorem 3.10. *Let $p(\cdot) \in \mathcal{B}$ be weak-Lipschitz at zero and set $p := p(0)$. Assume $-1 < \alpha < p - 1$. Then $A : L_\alpha^{p(\cdot)} \to T_\alpha^p$.*

Proof. Let C and δ be the constants from (3.2). Set $d := \operatorname{ess\,inf} p(x)$, $D := (d - p) \log \frac{\delta}{e^2}$ and

$$q(x) = \max\left(d, p - \frac{\max(C, D)}{\log\left(\frac{e^2}{x}\right)}\right).$$

Since $p(\cdot) \in \mathcal{B}$, we have $d > 1$, and so $q(\cdot) \in \mathcal{B}$.

We claim that $q(x) \le p(x)$. Assume first $D \ge C$. Then

$$q(x) = \max\left(d, p - \frac{D}{\log\left(\frac{e^2}{x}\right)}\right) \tag{3.7}$$

and $q(\delta) = d$. Since the function $x \mapsto p - \dfrac{D}{\log\left(\frac{e^2}{x}\right)}$ is nonincreasing in x, we have

$$
\begin{aligned}
q(x) &= \left(p - \frac{D}{\log\left(\frac{e^2}{x}\right)}\right)\chi_{(0,\delta)}(x) + d\,\chi_{(\delta,1)}(x) \\
&\le \left(p - \frac{C}{\log\left(\frac{e^2}{x}\right)}\right)\chi_{(0,\delta)}(x) + d\,\chi_{(\delta,1)}(x) \\
&\le p(x)\chi_{(0,\delta)}(x) + p(x)\chi_{(\delta,1)}(x) = p(x),
\end{aligned}
$$

where we used the estimate $p - \dfrac{C}{\log\left(\frac{e^2}{x}\right)} \le p(x)$, $x \in (0,\delta)$, which follows from the weak-Lipschitz property at zero.

Now assume $0 < D < C$ (note that then $d > p$). Denote $\lambda := \mathrm{e}^2\,\mathrm{e}^{\frac{C}{d-p}}$, then $p - \frac{C}{\log\left(\frac{\mathrm{e}^2}{\lambda}\right)} = d$. Thus, $q(\lambda) = d$. Since $\delta = \mathrm{e}^2\,\mathrm{e}^{\frac{D}{d-p}}$, we have $\lambda < \delta$, whence

$$q(x) = \left(p - \frac{C}{\log\left(\frac{\mathrm{e}^2}{x}\right)}\right)\chi_{(0,\lambda)}(x) + d\,\chi_{(\lambda,1)}(x) \le p(x)\chi_{(0,\lambda)}(x) + p(x)\chi_{(\lambda,1)}(x) = p(x),$$

proving our claim. Therefore, $L_\alpha^{p(\cdot)} \hookrightarrow L_\alpha^{q(\cdot)}$.

The case $D = 0$ or, equivalently, $d = p$, is analogous. Thus, assuming $f \ge 0$ and $\int_0^1 f(x)^{p(x)} x^\alpha dx < \infty$, we also have $\int_0^1 f(x)^{q(x)} x^\alpha dx < \infty$. Observe that, by [9, Lemmas 5.4 and 5.3], $q(\cdot)$ is weak-Lipschitz. This fact enables us to apply Lemma 3.9, and we get $\int_0^1 \widetilde{Af}(x)^{q(x)} x^\alpha dx < \infty$. Finally, we obtain, by Theorem 3.7, applied to $p(x) \equiv p$, $\int_0^1 \widetilde{Af}(x)^p x^\alpha dx < \infty$, which finishes the proof. $\qquad\square$

We shall now introduce another new space, $S_\alpha^{p(\cdot)}$, and study its basic functional properties. This space is in a counterpart of the space $S_\alpha^{p(\cdot)}$ in the sense that it serves as an optimal source space for the average operator A.

Definition 3.11. Let $f \in \mathcal{M}(0,1)$, $p(\cdot) \in \mathcal{B}$ and $\alpha \in \mathbb{R}$. We define the functional

$$m_{S_\alpha^{p(\cdot)}}(f) = \int_0^1 \widetilde{A|f|}(x)^{p(x)} x^\alpha dx = \int_0^1 \left(\operatorname*{ess\,sup}_{t \in (x,1)} \frac{1}{t}\int_0^t |f(s)| ds\right)^{p(x)} x^\alpha dx,$$

the norm

$$\|f\|_{S_\alpha^{p(\cdot)}} = \big\|A|f|\,\big\|_{T_\alpha^{p(\cdot)}},$$

and the corresponding space

$$S_\alpha^{p(\cdot)} = \{f \in \mathcal{M}(0,1); \|f\|_{S_\alpha^{p(\cdot)}} < \infty\}.$$

Then, again, $m_{S_\alpha^{p(\cdot)}}$ is a convex modular and $S_\alpha^{p(\cdot)}$ is a Banach space with respect to the norm $\|\cdot\|_{S_\alpha^{p(\cdot)}}$.

Theorem 3.12. *Let $p(\cdot) \in \mathcal{B}$ be weak-Lipschitz at zero. Set $p = p(0)$ and assume $-1 < \alpha < p - 1$. Then $L_\alpha^{p(\cdot)} \hookrightarrow S_\alpha^{p(\cdot)} \hookrightarrow L^1$.*

Proof. Let us first show that $L_\alpha^{p(\cdot)} \hookrightarrow S_\alpha^{p(\cdot)}$. Let $0 \le f \in L_\alpha^{p(\cdot)}$. By Theorems 3.10 and 3.7, we have $Af \in T_\alpha^{p(\cdot)}$, i.e., $\int_0^1 \widetilde{Af}(x)^p x^\alpha dx < \infty$, whence $f \in S_\alpha^{p(\cdot)}$.

Now, we will prove $S_\alpha^{p(\cdot)} \hookrightarrow L^1$. Assume that $f \ge 0$ and $f \notin L^1$, i.e., $\int_0^1 f(t) dt = \infty$. Then, for every $0 < x < 1$,

$$\widetilde{Af}(x) = \operatorname*{ess\,sup}_{y \in (x,1)} \frac{1}{y}\int_0^y f(t) dt \ge \lim_{y \to 1_-} \frac{1}{y}\int_0^y f(t) dt = \infty,$$

whence $f \notin S_\alpha^{p(\cdot)}$, and the assertion follows. $\qquad\square$

Lemma 3.13. *Let $p(\cdot)$ be weak-Lipschitz at zero. Set $p = p(0)$ and assume $-1 < \alpha < p - 1$. Then $S_\alpha^{p(\cdot)}$ is a Banach function space.*

Proof. It is not difficult to see that $S_\alpha^{p(\cdot)}$ satisfies the conditions (P1) and (P2). (P3), and so, (P6). Since $\int_0^1 x^\alpha dx < \infty$, (P4) holds. Finally, Theorem 3.12 yields (P5). □

We shall now concentrate on the *sharpness* of the embeddings $L_\alpha^{p(\cdot)} \hookrightarrow S_\alpha^{p(\cdot)}$ and $T_\alpha^p \hookrightarrow L_\alpha^{p(\cdot)}$. Indeed, given $p(\cdot)$, take $q(\cdot)$ from (3.7). Then $q(\cdot)$ is weak-Lipschitz at zero and $q(x) \leq p(x)$, just as in the proof of Theorem 3.10. Set

$$r(x) = \max\left(q(x) - \frac{C}{\log\left(\frac{e^2}{x}\right)}, \operatorname*{ess\,inf}_{x \in (0,1)} q(x)\right).$$

Clearly, $r(x) \leq q(x)$. Moreover, $r(x) < q(x)$ on an interval $(0, a)$ for some $a > 0$. Hence,

$$L_\alpha^{p(\cdot)} \hookrightarrow L_\alpha^{q(\cdot)} \overset{\hookrightarrow}{\neq} L_\alpha^{r(\cdot)} \hookrightarrow S_\alpha^{p(\cdot)}.$$

The sharpness of the embedding $T_\alpha^p \hookrightarrow L_\alpha^{p(\cdot)}$ can be shown in an analogous manner.

Remark 3.14. For every $1 < p$ and $-1 < \alpha < p - 1$, we have $T_\alpha^p \hookrightarrow S_\alpha^p$. Indeed, by Corollary 3.8 and Theorem 3.12 (applied to the constant function $p(\cdot) \equiv p$), we have $T_\alpha^p \hookrightarrow L_\alpha^p \hookrightarrow S_\alpha^p$.

So far we only know that the operator A is bounded on L_α^p as long as $p \in (1, \infty)$ and $-1 < \alpha < p - 1$. We shall now obtain a tighter result.

Theorem 3.15. *Let $\alpha \in \mathbb{R}$ and $1 < p < \infty$.*
  (i) *We have $A : S_\alpha^p \to T_\alpha^p$.*
 (ii) *If $r(\cdot), s(\cdot) \in \mathcal{B}$ are weak-Lipschitz at zero, $r(0) = s(0) =: p$ and $-1 < \alpha < p - 1$, then $A : L_\alpha^{r(\cdot)} \to L_\alpha^{s(\cdot)}$.*
(iii) *If $-1 < \alpha < p - 1$, then*

$$A : S_\alpha^p \to S_\alpha^p \tag{3.8}$$
$$A : T_\alpha^p \to T_\alpha^p. \tag{3.9}$$

Proof. The assertion (i) is an immediate consequence of the definitions of the spaces S_α^p and T_α^p, (ii) follows from (i), Corollary 3.8 and Theorem 3.12, and, finally, (iii) is an easy consequence of Remark 3.14 and (i). □

We shall now prove that the spaces S_α^p and T_α^p that appear in the embeddings (3.8) and (3.9) are sharp in a fairly general sense.

Theorem 3.16. *Let $1 < p$ and $-1 < \alpha < p - 1$. Assume that Z is a BFS and $Z \subsetneq T_\alpha^p$. Then $A : T_\alpha^p \not\hookrightarrow Z$.*

Proof. Take $0 \leq g \in T_\alpha^p \setminus Z$ and set $h(x) = \widetilde{g}(t)$. Then h is non-increasing, $h \geq g$ and $h \in T_\alpha^p$. Since Z is a BFS, we have $h \notin Z$. So, $h \in T_\alpha^p \setminus Z$. The monotonicity of h implies $Ah \geq h$, whence $Ah \notin Z$. By Theorem 3.15 (ii), we obtain $Ah \in T_\alpha^p$ and, consequently, $A : T_\alpha^p \not\to Z$. $\qquad\square$

Now we turn our attention to the optimality of S_α^p in (3.8).

Theorem 3.17. *Let $1 < p$ and $-1 < \alpha < p-1$. Assume that Z be a BFS such that $S_\alpha^p \subsetneq Z$. Then $A : Z \not\to S_\alpha^p$.*

Proof. Take $0 \leq f \in Z \setminus S_\alpha^p$. Since Z is a BFS, we have $f \in L^1$. Then

$$K := \operatorname*{ess\,sup}_{z \in (\frac{1}{e}, 1)} \left(\frac{1}{z} \int_0^z f \right)^p \leq \left(e \int_0^1 f \right)^p < \infty.$$

By the Fubini theorem,

$$
\begin{aligned}
L : &= \int_0^1 \left(\widetilde{A(\widetilde{Af})}(x) \right)^p x^\alpha dx \\
&= \int_0^1 \operatorname*{ess\,sup}_{y \in (x,1)} \left(\frac{1}{y} \int_0^y f(s) \log \frac{y}{s} ds \right)^p x^\alpha dx \\
&\geq \int_0^1 \operatorname*{ess\,sup}_{y \in (x,1)} \left(\frac{1}{y} \int_0^{\frac{y}{e}} f(s) \log \frac{y}{s} ds \right)^p x^\alpha dx \\
&\geq \int_0^1 \operatorname*{ess\,sup}_{y \in (x,1)} \left(\frac{1}{y} \int_0^{\frac{y}{e}} f(s) ds \right)^p x^\alpha dx \\
&= e^{-p} \int_0^1 \operatorname*{ess\,sup}_{y \in (x,1)} \left(\frac{e}{y} \int_0^{\frac{y}{e}} f(s) ds \right)^p x^\alpha dx \\
&= e^{-p} \int_0^1 \operatorname*{ess\,sup}_{z \in (\frac{x}{e}, \frac{1}{e})} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p x^\alpha dx.
\end{aligned}
$$

Fix $x \in (0,1)$. Denote $a := \operatorname*{ess\,sup}_{z \in (\frac{x}{e}, \frac{1}{e})} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p$. As $a \geq \max(a, K) - K$, we have

$$
\begin{aligned}
\operatorname*{ess\,sup}_{z \in (\frac{x}{e}, \frac{1}{e})} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p &\geq \max\left(\operatorname*{ess\,sup}_{z \in (\frac{x}{e}, \frac{1}{e})} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p, \operatorname*{ess\,sup}_{z \in (\frac{1}{e}, 1)} \left(\frac{1}{z} \int_0^z f \right)^p \right) - K \\
&= \operatorname*{ess\,sup}_{z \in (\frac{x}{e}, 1)} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p - K \\
&\geq \operatorname*{ess\,sup}_{z \in (x, 1)} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p - K.
\end{aligned}
$$

Finally, since $f \notin S_\alpha^p$

$$L \geq \mathrm{e}^{-p} \int_0^1 \operatorname*{ess\,sup}_{z \in (x,1)} \left(\frac{1}{z} \int_0^z f(s) ds \right)^p dx - \mathrm{e}^{-p} K = \infty.$$

This gives $Af \notin S_\alpha^p$ which finishes the proof. $\qquad\qquad\square$

4. Equivalent norms in T_α^p and S_α^p

We introduce in this section a family of norms equivalent either to the norm in T_α^p or to the norm in S_α^p. We shall now summarize definitions of all spaces considered in what follows.

Definition 4.1. Let $p \geq 1$, $\alpha \in \mathbb{R}$. We define the space X_α^p by

$$\|f\|_{X_\alpha^p} := \left(\sum_{n=0}^\infty 2^{-n(\alpha+1)} \operatorname*{ess\,sup}_{2^{-n-1} \leq t \leq 2^{-n}} |f(t)|^p \right)^{\frac{1}{p}},$$

and the space $(Y)_\alpha^p$ by

$$\|f\|_{Y_\alpha^p} := \left(\sum_{n=0}^\infty 2^{n(p-\alpha-1)} \left(\int_{2^{-n-1}}^{2^{-n}} |f(t)| dt \right)^p \right)^{\frac{1}{p}}.$$

Next, consider an $f \in \mathcal{M}(0,1)$, extended by zero elsewhere on $\mathbb{R}$ if necessary. We define the norm

$$\|f\|_{\mathbb{S}_\alpha^p} := \left(\int_0^1 \left(\int_{\frac{x}{2}}^{2x} t^{-1} |f(t)| dt \right)^p x^\alpha dx \right)^{\frac{1}{p}},$$

and the corresponding space $\mathbb{S}_\alpha^p$. Finally, we define the norm

$$\|f\|_{\widetilde{S}_\alpha^p} = \left(\int_0^1 \left(\int_x^1 t^{-1} |f(t)| dt \right)^p x^\alpha dx \right)^{\frac{1}{p}}$$

and denote by $\widetilde{S}_\alpha^p$ the corresponding space.

We are going to use discretization and anti-discretization methods in the spirit of [4, 6]. We will need three auxiliary lemmas, the first of which is just a special case of [4, Lemma 3.1 (i)], apparently first proved in [5].

Lemma 4.2. *Let $a > 1$, $1 \leq p < \infty$. Then there is a $C > 0$ such that the inequality*

$$\sum_{n=0}^\infty a^n \left(\sum_{k=n}^\infty b_k \right)^p \leq C \sum_{n=0}^\infty a^n b_n^p$$

holds for any sequence $b_k \geq 0$.

Using the duality between ℓ^p and $\ell^{p'}$ spaces we obtain a dual version of this inequality.

Lemma 4.3. *Let* $a > 1$, $1 \leq p < \infty$. *Then there is a* $D > 0$ *such that the inequality*

$$\sum_{n=0}^{\infty} a^{-n} \left(\sum_{k=0}^{n} c_k \right)^p \leq D \sum_{n=0}^{\infty} a^{-n} c_n^p$$

holds for any sequence $c_k \geq 0$.

The following assertion is a particular case of [4, Lemma 3.2 (ii)]. It also follows from Lemma 4.3 with $D = \frac{1}{1-\beta}$.

Lemma 4.4. *Let* $0 < \beta < 1$ *and* $a_n \geq 0$, $n \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{\infty} \beta^n \max_{0 \leq k \leq n} a_k \leq \frac{1}{1-\beta} \sum_{n=1}^{\infty} \beta^n a_n.$$

Observe that, denoting, for $\alpha > 0$, $B_\alpha := \frac{2^{\alpha+1}-1}{(\alpha+1)2^{\alpha+1}}$, we have

$$2^{n(\alpha+1)} \int_{2^{-n-1}}^{2^{-n}} x^\alpha dx = B_\alpha.$$

Proposition 4.5. *Let* $-1 < \alpha < p-1$. *Then* $X_\alpha^p = T_\alpha^p$ *with equivalent norms.*

Proof. We will first establish the inequality $\|f\|_{T_\alpha^p} \leq C\|f\|_{X_\alpha^p}$. Set $a_n := \operatorname{ess\,sup}_{2^{-n-1} \leq t \leq 2^{-n}} |f(t)|$. By Lemma 4.4,

$$\|f\|_{T_\alpha^p}^p = \int_0^1 \left(\widetilde{f}(x) \right)^p x^\alpha dx = \int_0^1 \left(\operatorname*{ess\,sup}_{x \leq t \leq 1} |f(t)| \right)^p x^\alpha dx = \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \operatorname*{ess\,sup}_{x \leq t \leq 1} |f(t)|^p x^\alpha dx$$

and hence

$$\|f\|_{T_\alpha^p}^p \leq \sum_{n=0}^{\infty} \operatorname*{ess\,sup}_{2^{-n-1} \leq t \leq 1} |f(t)|^p \int_{2^{-n-1}}^{2^{-n}} x^\alpha dx$$

$$= \sum_{n=0}^{\infty} \max_{0 \leq k \leq n} \operatorname*{ess\,sup}_{2^{-k-1} \leq t \leq 2^{-k}} |f(t)|^p \int_{2^{-n-1}}^{2^{-n}} x^\alpha dx$$

$$= B_\alpha \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \max_{0 \leq k \leq n} a_k^p$$

$$\leq \frac{B_\alpha}{1 - 2^{-(1+\alpha)}} \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} a_n^p$$

$$= \frac{1}{1+\alpha} \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \operatorname*{ess\,sup}_{2^{-n-1} \leq t \leq 2^{-n}} |f(t)|^p$$

$$= \frac{1}{1+\alpha} \|f\|_{X_\alpha^p}^p.$$

The converse inequality $\|f\|_{X_\alpha^p} \leq C\|f\|_{T_\alpha^p}$ follows from

$$
\begin{aligned}
\|f\|_{X_\alpha^p}^p &= \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \operatorname*{ess\,sup}_{2^{-n-1} \leq t \leq 2^{-n}} |f(t)|^p \\
&= 2^{\alpha+1} B_\alpha^{-1} \sum_{n=0}^{\infty} \int_{2^{-n-2}}^{2^{-n-1}} \operatorname*{ess\,sup}_{2^{-n-1} \leq t \leq 2^{-n}} |f(t)|^p x^\alpha dx \\
&\leq 2^{\alpha+1} B_\alpha^{-1} \sum_{n=0}^{\infty} \int_{2^{-n-2}}^{2^{-n-1}} \operatorname*{ess\,sup}_{x \leq t \leq 1} |f(t)|^p x^\alpha dx \\
&\leq 2^{\alpha+1} B_\alpha^{-1} \int_0^1 \operatorname*{ess\,sup}_{x \leq t \leq 1} |f(t)|^p x^\alpha dx \\
&= 2^{\alpha+1} B_\alpha^{-1} \|f\|_{T_\alpha^p}^p. \qquad \square
\end{aligned}
$$

Lemma 4.6. *Let $p > 1$ and $-1 < \alpha < p - 1$. Then $\mathbb{S}_\alpha^p \hookrightarrow L^1$.*

Proof. Let $f \geq 0$. Then we have, by Lemma 4.2 applied to $b_k = \int_{2^{-k-1}}^{2^{-k}} f(x)dx$,

$$
\begin{aligned}
\|f\|_{L^1}^p &= \left(\int_0^1 f(t)dt \right)^p \\
&\leq \sum_{n=0}^{\infty} 2^{n(p-\alpha-1)} \left(\int_0^{2^{-n}} f(t)dt \right)^p \\
&= \sum_{n=0}^{\infty} 2^{n(p-\alpha-1)} \left(\sum_{k=n}^{\infty} \int_{2^{-k-1}}^{2^{-k}} f(t)dt \right)^p \\
&\leq C \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(2^n \int_{2^{-n-1}}^{2^{-n}} f(t)dt \right)^p \\
&\leq C \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\int_{2^{-n-1}}^{2^{-n}} t^{-1}f(t)dt \right)^p \\
&= C \frac{2^{\alpha+1}(\alpha+1)}{2^{\alpha+1}-1} \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \left(\int_{2^{-n-1}}^{2^{-n}} t^{-1}f(t)dt \right)^p x^\alpha dx \\
&\leq \widetilde{C} \int_0^1 \left(\int_{\frac{x}{2}}^{2x} t^{-1}f(t)\,dt \right)^p x^\alpha dx \\
&= \widetilde{C} \|f\|_{\mathbb{S}_\alpha^p}^p. \qquad \square
\end{aligned}
$$

Lemma 4.7. *Assume $p > 1$ and $-1 < \alpha < p - 1$. Then $\mathbb{S}_\alpha^p$ is a BFS.*

Proof. It is not difficult to verify properties (P1), (P2) and (P3) from Definition 2.1. Due to Lemma 4.6 it suffices to verify that $f \equiv 1 \in \mathbb{S}_\alpha^p$. This follows from $\int_0^1 \left(\int_{\frac{x}{2}}^{2x} t^{-1}dt \right)^p x^\alpha dx = \int_0^1 (\log 4)^p x^\alpha dx < \infty$. $\qquad \square$

Lemma 4.8. *Assume $1 < p$ and $-1 < \alpha < p - 1$. Then the spaces $\mathbb{S}_\alpha^p$ and S_α^p coincide and their norms are equivalent.*

Proof. Let us prove first the embedding $\mathbb{S}_\alpha^p \hookrightarrow S_\alpha^p$. We have, for $f \geq 0$,

$$\|f\|_{S_\alpha^p}^p = \int_0^1 \operatorname*{ess\,sup}_{x<t<1} \left(\frac{1}{t} \int_0^t f(s)ds \right)^p x^\alpha dx$$

$$= \sum_{n=0}^\infty \int_{2^{-n-1}}^{2^{-n}} \operatorname*{ess\,sup}_{x<t<1} \left(\frac{1}{t} \int_0^t f(s)ds \right)^p x^\alpha dx$$

$$\leq \sum_{n=0}^\infty \int_{2^{-n-1}}^{2^{-n}} \operatorname*{ess\,sup}_{2^{-n-1}<t<1} \left(\frac{1}{t} \int_0^t f(s)ds \right)^p x^\alpha dx.$$

Denote $a_k = \operatorname*{ess\,sup}_{2^{-k-1}<t<2^{-k}} \frac{1}{t} \int_0^t f(s)ds$. Then, by Lemma 4.4

$$\|f\|_{S_\alpha^p}^p \leq \sum_{n=0}^\infty \left(\max_{0\leq k\leq n} a_k^p \right) \int_{2^{-n-1}}^{2^{-n}} x^\alpha dx$$

$$= \frac{2^{\alpha+1}-1}{2^{\alpha+1}(\alpha+1)} \sum_{n=0}^\infty 2^{-n(\alpha+1)} \max_{0\leq k\leq n} a_k^p$$

$$\leq \frac{2^{\alpha+1}-1}{2^{\alpha+1}(\alpha+1)} \frac{1}{1-2^{-(\alpha+1)}} \sum_{n=0}^\infty 2^{-n(\alpha+1)} a_n^p$$

$$= \frac{1}{\alpha+1} \sum_{n=0}^\infty 2^{-n(\alpha+1)} \left(\operatorname*{ess\,sup}_{2^{-n-1}<t<2^{-n}} \frac{1}{t} \int_0^t f(s)ds \right)^p$$

$$\leq \frac{2^p}{\alpha+1} \sum_{n=0}^\infty 2^{n(p-\alpha-1)} \left(\operatorname*{ess\,sup}_{2^{-n-1}<t<2^{-n}} \int_0^t f(s)ds \right)^p$$

$$\leq \frac{2^p}{\alpha+1} \sum_{n=0}^\infty 2^{n(p-\alpha-1)} \left(\int_0^{2^{-n}} f(s)ds \right)^p$$

$$= \frac{2^p}{\alpha+1} \sum_{n=0}^\infty 2^{n(p-\alpha-1)} \left(\sum_{k=n}^\infty \int_{2^{-k-1}}^{2^{-k}} f(s)ds \right)^p.$$

Denote $b_n = \int_{2^{-n-1}}^{2^{-n}} f(s)ds$. Then, by Lemma 4.2,

$$\|f\|_{S_\alpha^p}^p \leq \frac{2^p}{\alpha+1} \sum_{n=0}^\infty 2^{n(p-\alpha-1)} \left(\sum_{k=n}^\infty b_k \right)^p$$

$$\leq \frac{2^p}{\alpha+1} C \sum_{n=0}^\infty 2^{n(p-\alpha-1)} b_n^p$$

$$\leq \frac{2^p}{\alpha+1} C \sum_{n=0}^\infty 2^{-n(\alpha+1)} \left(\int_{2^{-n-1}}^{2^{-n}} s^{-1} f(s)ds \right)^p$$

$$= \frac{2^{p+\alpha+1}}{\alpha+1} C \frac{\alpha+1}{2^{\alpha+1}-1} \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \left(\int_{2^{-n-1}}^{2^{-n}} s^{-1} f(s) ds \right)^p x^\alpha dx$$

$$= \frac{2^{p+\alpha+1}}{2^{\alpha+1}-1} C \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \left(\int_{2^{-n-1}}^{2^{-n}} s^{-1+\frac{\alpha}{p}} f(s) ds \right)^p dx$$

$$\leq \widetilde{C} \int_0^1 \left(\int_{\frac{x}{2}}^{2x} s^{-1} f(s) ds \right)^p x^\alpha dx$$

$$= \widetilde{C} \| f \|_{\mathbb{S}_\alpha^p}^p .$$

We shall now prove the converse embedding $S_\alpha^p \hookrightarrow \mathbb{S}_\alpha^p$. Set $c_n = \int_{2^{-n-1}}^{2^{-n}} s^{-1} f(s) ds$. Note that, formally, $c_{-1} = 0$, whence,

$$\| f \|_{\mathbb{S}_\alpha^p}^p = \int_0^1 \left(\int_{\frac{x}{2}}^{2x} s^{-1} f(s) ds \right)^p x^\alpha dx$$

$$= \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \left(\int_{\frac{x}{2}}^{2x} s^{-1} f(s) ds \right)^p x^\alpha dx$$

$$\leq \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \left(\int_{2^{-n-2}}^{2^{-n+1}} s^{-1} f(s) ds \right)^p x^\alpha dx$$

$$= B_\alpha \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} (c_{n+1} + c_n + c_{n-1})^p$$

$$\leq B_\alpha 3^{p-1} \left(\sum_{n=0}^{\infty} 2^{-n(\alpha+1)} c_{n+1}^p + \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} c_n^p + \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} c_{n-1}^p \right)$$

$$= C_\alpha \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} c_n^p ,$$

where $C_\alpha := B_\alpha 3^{p-1} (2^{\alpha+1} + 1 + 2^{-\alpha-1})$. Thus,

$$\| f \|_{\mathbb{S}_\alpha^p}^p \leq C_\alpha \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\int_{2^{-n-1}}^{2^{-n}} s^{-1} f(s) ds \right)^p$$

$$\leq C_\alpha \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(2^{n+1} \int_{2^{-n-1}}^{2^{-n}} f(s) ds \right)^p$$

$$\leq 2^p C_\alpha \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\operatorname*{ess\,sup}_{2^{-n} < t < 1} \frac{1}{t} \int_0^t f(s) ds \right)^p$$

$$= 2^p \frac{C_\alpha}{B_\alpha} \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \left(\operatorname*{ess\,sup}_{2^{-n} < t < 1} \frac{1}{t} \int_0^t f(s) ds \right)^p x^\alpha dx$$

and

$$\|f\|_{\mathbb{S}_\alpha^p}^p \le 2^p \frac{C_\alpha}{B_\alpha} \sum_{n=0}^\infty \int_{2^{-n-1}}^{2^{-n}} \left(\underset{x<t<1}{\text{ess sup}} \frac{1}{t} \int_0^t f(s)ds \right)^p x^\alpha dx$$

$$= 2^p \frac{C_\alpha}{B_\alpha} \int_0^1 \left(\underset{x<t<1}{\text{ess sup}} \frac{1}{t} \int_0^t f(s)ds \right)^p x^\alpha dx$$

$$= 2^p \frac{C_\alpha}{B_\alpha} \|f\|_{S_\alpha^p}^p. \qquad \square$$

Proposition 4.9. *Let $-1 < \alpha < p-1$. The spaces Y_α^p and $\mathbb{S}_\alpha^p$ coincide and the norms are equivalent.*

Proof. If $x \in (2^{-n-1}, 2^{-n})$ then $(\frac{x}{2}, 2x) \supset (2^{-n-1}, 2^{-n})$. Thus, for any f,

$$\|f\|_{\mathbb{S}_\alpha^p}^p = \int_0^1 \left(\int_{\frac{x}{2}}^{2x} t^{-1}|f(t)|dt \right)^p x^\alpha dx$$

$$= \sum_{n=0}^\infty \int_{2^{-n-1}}^{2^{-n}} \left(\int_{\frac{x}{2}}^{2x} t^{-1}|f(t)|dt \right)^p x^\alpha dx$$

$$\ge \sum_{n=0}^\infty \int_{2^{-n-1}}^{2^{-n}} \left(\int_{2^{-n-1}}^{2^{-n}} t^{-1}|f(t)|dt \right)^p x^\alpha dx$$

$$\ge B_\alpha \sum_{n=0}^\infty 2^{n(p-\alpha-1)} \left(\int_{2^{-n-1}}^{2^{-n}} |f(t)|dt \right)^p$$

$$= B_\alpha \|f\|_{Y_\alpha^p}^p.$$

As for the converse inequality, we have

$$\|f\|_{\mathbb{S}_\alpha^p}^p \le \sum_{n=0}^\infty \int_{2^{-n-1}}^{2^{-n}} \left(\int_{2^{-n-2}}^{2^{-n+1}} t^{-1}|f(t)|dt \right)^p x^\alpha dx$$

$$\le 2^{2p} B_\alpha \sum_{n=0}^\infty 2^{n(p-\alpha-1)} \left(\int_{2^{-n-2}}^{2^{-n+1}} |f(t)|dt \right)^p.$$

Writing $\int_{2^{-n-2}}^{2^{-n+1}} \ldots = \int_{2^{-n-2}}^{2^{-n-1}} \ldots + \int_{2^{-n-1}}^{2^{-n}} \ldots + \int_{2^{-n}}^{2^{-n+1}} \ldots$, we get

$$\|f\|_{\mathbb{S}_\alpha^p}^p \le 3^{p-1} 2^{2p} B_\alpha \left[\sum_{n=0}^\infty 2^{n(p-\alpha-1)} \left(\int_{2^{-n-2}}^{2^{-n-1}} |f(t)|dt \right)^p \right.$$

$$\left. + \sum_{n=0}^\infty 2^{n(p-\alpha-1)} \left(\int_{2^{-n-1}}^{2^{-n}} |f(t)|dt \right)^p + \sum_{n=0}^\infty 2^{n(p-\alpha-1)} \left(\int_{2^{-n}}^{2^{-n+1}} |f(t)|dt \right)^p \right]$$

$$=: 3^{p-1} 2^{2p} B_\alpha \left[D_1 + D_2 + D_3 \right],$$

say. Now, D_2 is exactly $\|f\|_{Y_\alpha^p}^p$, while

$$D_1 = \sum_{n=0}^{\infty} 2^{n(p-\alpha-1)} \left(\int_{2^{-n-2}}^{2^{-n-1}} |f(t)|dt \right)^p$$

$$= 2^{-(p-\alpha-1)} \sum_{n=1}^{\infty} 2^{n(p-\alpha-1)} \left(\int_{2^{-n-1}}^{2^{-n}} |f(t)|dt \right)^p$$

$$\leq 2^{-(p-\alpha-1)} \sum_{n=0}^{\infty} 2^{n(p-\alpha-1)} \left(\int_{2^{-n-1}}^{2^{-n}} |f(t)|dt \right)^p$$

$$= 2^{-(p-\alpha-1)} D_2,$$

and one can show in an analogous manner that $D_3 = 2^{p-\alpha-1}D_2$. Altogether, this proves the claim. $\square$

Theorem 4.10. *Let $1 < p$ and $-1 < \alpha < p - 1$. Then the spaces $\widetilde{S}_\alpha^p$ and Y_α^p coincide and their norms are equivalent.*

Proof. Let us prove first $Y_\alpha^p \hookrightarrow \widetilde{S}_\alpha^p$. For a given function f, set $c_k = \int_{2^{-k-1}}^{2^{-n}} \frac{|f(s)|}{s}ds$. We can easily write

$$\|f\|_{\widetilde{S}_\alpha^p}^p = \int_0^1 \left(\int_x^1 \frac{|f(s)|}{s}ds \right)^p x^\alpha dx$$

$$= \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \left(\int_x^1 \frac{|f(s)|}{s}ds \right)^p x^\alpha dx$$

$$\leq \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} \left(\int_{2^{-n-1}}^1 \frac{|f(s)|}{s}ds \right)^p x^\alpha dx$$

$$= B_\alpha \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\sum_{k=0}^{n} \int_{2^{-k-1}}^{2^{-k}} \frac{|f(s)|}{s}ds \right)^p$$

$$= B_\alpha \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\sum_{k=0}^{n} c_k \right)^p.$$

By Lemma 4.3, we obtain

$$\|f\|_{\widetilde{S}_\alpha^p}^p \leq B_\alpha \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\sum_{k=0}^{n} c_k \right)^p$$

$$\leq B_\alpha D \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} c_n^p$$

$$\leq 2^p B_\alpha D \sum_{n=0}^{\infty} 2^{n(p-\alpha-1)} \left(\int_{2^{-n-1}}^{2^{-n}} |f(s)|ds \right)^p$$

$$= 2^p B_\alpha D \|f\|_{Y_\alpha^p}^p.$$

As for the converse embedding, we have

$$
\begin{aligned}
\|f\|_{Y_\alpha^p}^p &= \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(2^n \int_{2^{-n-1}}^{2^{-n}} |f(s)| ds \right)^p \\
&\leq \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\int_{2^{-n-1}}^{2^{-n}} \frac{|f(s)|}{s} ds \right)^p \\
&= \frac{2^{\alpha+1}}{B_\alpha} \sum_{n=0}^{\infty} \int_{2^{-n-2}}^{2^{-n-1}} \left(\int_{2^{-n-1}}^{2^{-n}} \frac{|f(s)|}{s} ds \right)^p x^\alpha dx \\
&\leq \frac{2^{\alpha+1}}{B_\alpha} \sum_{n=0}^{\infty} \int_{2^{-n-2}}^{2^{-n-1}} \left(\int_x^1 \frac{|f(s)|}{s} ds \right)^p x^\alpha dx \\
&\leq \frac{2^{\alpha+1}}{B_\alpha} \int_0^1 \left(\int_x^1 \frac{|f(s)|}{s} ds \right)^p x^\alpha dx \\
&= \frac{2^{\alpha+1}}{B_\alpha} \|f\|_{\widetilde{S}_\alpha^p},
\end{aligned}
$$

proving the claim of the theorem. $\qquad\square$

5. Duality of the optimal spaces

We shall now characterize the associate spaces of T_α^p and S_α^p.

Theorem 5.1. *Let $p > 1$ and $-1 < \alpha < p - 1$. Then $(X_\alpha^p)' = Y_{-\alpha(p'-1)}^{p'}$ with equivalent norms.*

Proof. By the Hölder inequality for series, we get

$$
\begin{aligned}
&\int_0^1 |f(x)g(x)| dx \\
&= \sum_{n=0}^{\infty} \int_{2^{-n-1}}^{2^{-n}} |f(x)g(x)| dx \\
&\leq \sum_{n=0}^{\infty} \left(2^{\frac{-n(\alpha+1)}{p}} \operatorname*{ess\,sup}_{2^{-n-1} \leq x \leq 2^{-n}} |f(x)| \right) \left(2^{\frac{n(\alpha+1)}{p}} \int_{2^{-n-1}}^{2^{-n}} |g(x)| dx \right) \\
&\leq \left(\sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \operatorname*{ess\,sup}_{2^{-n-1} \leq x \leq 2^{-n}} |f(x)|^p \right)^{\frac{1}{p}} \left(\sum_{n=0}^{\infty} 2^{\frac{n(\alpha+1)}{p-1}} \left(\int_{2^{-n-1}}^{2^{-n}} |g(x)| dx \right)^{p'} \right)^{\frac{1}{p'}} \\
&= \|f\|_{X_\alpha^p} \|g\|_{Y_{-\alpha(p'-1)}^{p'}},
\end{aligned}
$$

from which we get $\|g\|_{(X_\alpha^p)'} \leq \|g\|_{Y_{-\alpha(p'-1)}^{p'}}$.

For the converse inequality, take an arbitrary g, $\|g\|_{Y^{p'}_{-\alpha(p'-1)}} = 1$. We set $f_0(x) = \sum_{n=0}^{\infty} \left(2^{n(\alpha+1)} \int_{2^{-n-1}}^{2^{-n}} |g(t)|dt\right)^{\frac{p'}{p}} \chi_{(2^{-n-1}, 2^{-n})}(x)$ and $b_n := \int_{2^{-n-1}}^{2^{-n}} |g(t)|dt$. Then

$$
\begin{aligned}
\int_0^1 |f_0(x)g(x)|dx &= \int_0^1 \sum_{n=0}^{\infty} \left(2^{n(\alpha+1)}b_n\right)^{\frac{p'}{p}} \chi_{(2^{-n-1}, 2^{-n})}(x)|g(x)|dx \\
&= \sum_{n=0}^{\infty} 2^{n(\alpha+1)\frac{p'}{p}} b_n^{\frac{p'}{p}} \int_{2^{-n-1}}^{2^{-n}} |g(x)|dx \\
&= \sum_{n=0}^{\infty} 2^{n(\alpha+1)\frac{p'}{p}} b_n^{1+\frac{p'}{p}} \\
&= \sum_{n=0}^{\infty} 2^{n(\alpha+1)\frac{p'}{p}} \left(\int_{2^{-n-1}}^{2^{-n}} |g(t)|dt\right)^{p'} \\
&= \|g\|_{Y^{p'}_{-\alpha(p'-1)}}^{p'}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\|f_0\|_{X_\alpha^p}^p &= \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\operatorname*{ess\,sup}_{2^{-n-1} \leq t \leq 2^{-n}} |f(t)|\right)^p \\
&= \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\operatorname*{ess\,sup}_{2^{-n-1} \leq t \leq 2^{-n}} (2^{n(\alpha+1)}b_n)^{\frac{p'}{p}}\right)^p \\
&= \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} 2^{np'(\alpha+1)} b_n^{p'} \\
&= \sum_{n=0}^{\infty} 2^{n(\alpha+1)(p'-1)} \left(\int_{2^{-n-1}}^{2^{-n}} |g(t)|dt\right)^{p'} \\
&= \|g\|_{Y^{p'}_{-\alpha(p'-1)}}^{p'}
\end{aligned}
$$

Thus, $\|g\|_{(X_\alpha^p)'} = \sup_{\|f\|_{X_\alpha^p} \leq 1} \int_0^1 |f(x)g(x)|\,dx \geq \int_0^1 |f_0(x)g(x)|\,dx = \|g\|_{Y^{p'}_{-\alpha(p'-1)}}$, and the assertion follows. $\qquad\square$

We can thus formulate the following result.

Theorem 5.2. *Let* $-1 < \alpha < p-1$, $1 < p$. *Then the spaces* $\left(T_\alpha^p\right)'$ *and* $S^{p'}_{-\alpha(p'-1)}$ *coincide, and their norms are equivalent. Consequently, the spaces* $(S_\alpha^p)'$ *and* $T^{p'}_{-\alpha(p'-1)}$ *coincide, and their norms are equivalent.*

Remark 5.3. The statement of Theorem 5.2 is equivalent to the inequality

$$\int_0^1 |g(x)h(x)|\,dx \le C_h \left(\int_0^1 \left[\operatorname*{ess\,sup}_{x \le t \le 1} |g(x)| \right]^p x^\alpha\,dx \right)^{\frac{1}{p}},$$

in which $C_h \approx \left(\int_0^1 \left(\operatorname{ess\,sup}_{x \le t \le 1} \frac{1}{t} \int_0^t |h(y)|\,dy \right)^{p'} x^{-\alpha(p'-1)}\,dx \right)^{\frac{1}{p'}}$. This is a weighted reverse inequality for the supremum operator. We note that, for integral operators, such inequalities are studied, e.g., in [3]. Application of ideas developed there to supremum operators would give alternative proofs of duality results in the spirit of our Theorem 5.2.

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