

Concentration-Compactness Principle for Generalized Trudinger Inequalities

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Abstract. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain and let $\alpha < n - 1$. We prove the Concentration-Compactness Principle for the embedding of the Orlicz-Sobolev space $W_0^1 L^n \log^\alpha L(\Omega)$ into the Orlicz space with the Young function $\exp(t^{\frac{n}{n-1-\alpha}}) - 1$.

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1. Introduction

Throughout the paper Ω denotes a bounded domain in \mathbb{R}^n , $n \geq 2$. It is well-known that the Sobolev space $W_0^{1,p}(\Omega)$, $1 \leq p < n$, is continuously embedded into $L^{\frac{pn}{n-p}}(\Omega)$. For $p > n$ we know that each function from $W_0^{1,p}(\Omega)$ is bounded, i.e., it belongs to $L^\infty(\Omega)$, but this is not true for the limiting case $p = n$. For $p = n$ there is a famous result by Trudinger [18] (see also Yudovič [21]) which implies that the first-order Sobolev space $W_0^{1,n}(\Omega)$ is continuously embedded to the Orlicz space $L^\Phi(\Omega)$ with the Young function of the exponential type $\Phi(t) = \exp(t^{\frac{n}{n-1}}) - 1$, $t > 0$.

It is a general fact that embeddings are usually not compact in the limiting cases. For example embedding of the Sobolev space $W_0^{1,p}(\Omega)$ into $L^{\frac{pn}{n-p}}(\Omega)$ for $1 \leq p < n$ or into $L^\Phi(\Omega)$ for $p = n$ are not compact. However there is the

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amazing Concentration-Compactness Principle (see [19] and references given there for history and applications) that some substitute for compactness is still available for many embeddings. This principle is usually telling us that from each bounded sequence we can either select a subsequence that converges in the target space or we can select a subsequence that has very special behavior. For example it concentrates around one point and in some sense converges to the Dirac mass at this point or after suitable translations it concentrates around one point. This observation is very useful and can be used in many problems connected with the Calculus of Variations (see, e.g., [9, 10, 14, 19]).

The aim of this paper is to prove the Concentration-Compactness Principle for embeddings that generalize the Trudinger embedding. For $\alpha < n - 1$ set

$$\gamma = \frac{n}{n - 1 - \alpha}, \quad B = 1 - \frac{\alpha}{n - 1} \quad \text{and} \quad K_{n,\alpha} = B^{\frac{1}{B}} n \omega_{n-1}^{\frac{\gamma}{n-1}},$$

here ω_{n-1} denotes the surface area of the unit sphere. The space $W_0 L^n \log^\alpha L(\Omega)$ of the Sobolev type (see Preliminaries for the definition) is continuously embedded into the Orlicz space with the Young function $\exp(t^\gamma) - 1$. These results are due to Fusco, Lions, Sbordone [11] for $\alpha < 0$ and Edmunds, Gurka, Opic [6, 7] in general. In [6] the space $W L^n \log^\alpha L$ is modeled as a set of functions with Bessel potential in the generalized Lorentz Zygmund space and the results are much more general than those we mention here.

In this paper we consider differentiable Young functions Φ such that

$$\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t^n \log^\alpha(t)} = 1 \tag{1}$$

with $\alpha < n - 1$. In the critical case $K = K_{n,\alpha}$ we usually also require existence of $t_\Phi > 1$ and $a \in (0, \min(1, \frac{1}{\gamma}))$ such that

$$\Phi(t) \geq t^n \log^\alpha(t) \left(1 + \log^{-a}(t)\right) \quad \text{for } t \geq t_\Phi. \tag{2}$$

The main result of this paper is the following theorem saying:

Suppose that we have a normalized sequence $\{u_k\}_{k=1}^\infty$ of functions from the Orlicz-Sobolev space $W_0 L^\Phi(\Omega) := W_0^1 L^\Phi(\Omega)$. Then we can select a weakly convergent subsequence (satisfying (3)) so that, either this subsequence concentrates around one point $x_0 \in \bar{\Omega}$ and we can find its subsequence $\{u_{k_i}\}_{i=1}^\infty$ such that $\exp(K|u_{k_i}|^\gamma) - 1$ converges to a multiple of the Dirac mass at x_0 (see (i)), or $\exp(K|u_k|^\gamma)$ are uniformly bounded in $L^{1+\delta}(\Omega)$ and, by the reflexivity of $L^{1+\delta}(\Omega)$, these exponentials converge in $L^1(\Omega)$ (see (ii)).

Theorem 1.1. *Let $n \geq 2$, $\alpha < n - 1$ and let Φ be a Young function satisfying (1) and (2). Let $\{u_k\}_{k=1}^\infty \subset W_0 L^\Phi(\Omega)$ satisfy $\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leq 1$. Further suppose that*

$$u_k \rightharpoonup u \text{ in } W_0 L^\Phi(\Omega), \quad u_k \rightarrow u \text{ a.e. in } \Omega \quad \text{and} \quad \Phi(|\nabla u_k|) \xrightarrow{*} \mu \text{ in } \mathcal{M}(\bar{\Omega}). \tag{3}$$

- (i) If $u=0$ and $\mu = \delta_{x_0}$ for some $x_0 \in \bar{\Omega}$, then the sequence $\{\exp(K_{n,\alpha}|u_k|^\gamma)\}_{k=1}^\infty$ is relatively compact with respect to the weak* convergence in $\mathcal{M}(\bar{\Omega})$ and the limits of convergent subsequences belong to $\{\mathcal{L}_n|_\Omega + c\delta_{x_0} : c \geq 0\}$.
- (ii) Otherwise there is $\delta > 0$ such that $\exp(K_{n,\alpha}(1 + \delta)|u_k|^\gamma)$ is bounded in $L^1(\Omega)$ and

$$\exp(K_{n,\alpha}|u_k|^\gamma) \xrightarrow{k \rightarrow \infty} \exp(K_{n,\alpha}|u|^\gamma) \quad \text{in } L^1(\Omega).$$

Notice that this result cannot be valid with a constant $K > K_{n,\alpha}$ because similarly to Moser’s result [15] the integral from $\exp(K|u|^\gamma)$ can be made arbitrarily large if $K > K_{n,\alpha}$ (see remarks after Theorem 2.3 and [13]). In the case $K < K_{n,\alpha}$ the situation is much simpler and we have just the compactness as an easy corollary of the Moser-type result.

Corollary 1.2. *Let $n \geq 2$, $\alpha < n - 1$, $K < K_{n,\alpha}$ and let Φ be a Young function satisfying (1). Let $\{u_k\}_{k=1}^\infty \subset W_0L^\Phi(\Omega)$ satisfy $\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leq 1$. Further suppose that $u_k \rightarrow u$ a.e. in Ω . Then*

$$\exp(K|u_k|^\gamma) \xrightarrow{k \rightarrow \infty} \exp(K|u|^\gamma) \quad \text{in } L^1(\Omega).$$

For the proof of our main theorem we use the method inspired by Lions [14] and Carleson, Chang [2] but we have to include several new ideas. We need to use the results and techniques from [13] to show the boundedness of $\exp(K_{n,\alpha}|u|^\gamma)$. Moreover we extend these estimates to show that the critical sequence of functions converges to 0 (see Lemma 3.6).

At the end let us mention some possible applications of our results and open problems. First it was shown by Carleson and Chang [2] (see also [10]) that the extremal constant in Moser’s inequality is actually attained by some function. Using our Concentration-Compactness result it is not difficult to prove the following version of such a result for functional with the sub-critical growth:

Theorem 1.3. *Let $n \geq 2$, $\alpha < n - 1$ and let Φ be a Young function satisfying (1). Suppose that the function $F : \mathbb{R} \mapsto \mathbb{R}$ is even and continuous. Further suppose that either*

$$\lim_{t \rightarrow \infty} \frac{F(t)}{\exp(K|t|^\gamma)} = 0 \quad \text{for some } K < K_{n,\alpha} \tag{4}$$

or Φ satisfies the additional condition (2) and

$$\lim_{t \rightarrow \infty} \frac{F(t)}{\exp(K_{n,\alpha}|t|^\gamma)} = 0. \tag{5}$$

Then the functional $\Lambda_F(u) = \int_\Omega F(u(x)) dx$ attains its maximum on the set $\{u \in W_0L^\Phi(\Omega) : \|\Phi(|\nabla u|)\|_{L^1(\Omega)} \leq 1\}$.

We would like to know if it is possible to obtain the generalization of Theorem 1.3 also for functionals with the critical growth, i.e., if the limit in (5) is not zero but one. To obtain this result it would be necessary to have a version of some technical estimates from [2] also for the generalized Trudinger inequalities.

Another application of the Concentration-Compactness Alternative for Trudinger inequalities in dimension two can be found in de Figueiredo, Miyagaki, Ruf [9] (see also [16] for the higher dimensional version). It is shown there that the functional $\Gamma_F(u) = \int_{\Omega} (|\nabla u(x)|^2 - F(x, u(x))) dx$ has a non-zero critical point and thus there is a nontrivial solution of the equation

$$-\Delta u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Here F is a primitive of f , these functions satisfy some additional technical conditions and $f(x, u)$ behaves like e^{Ku^2} for u big. By [4] it is possible to obtain an analogue of above result showing that there are non-zero critical points of the functional $\Gamma_{F,\Phi} = \int_{\Omega} (\Phi(|\nabla u(x)|) - F(u(x))) dx$. Here Φ satisfies (1) and $F(u)$ behaves like e^{Ku^γ} for u big. Let us note that the proof in [4] uses a different version of the Concentration-Compactness Principle (see [4, proof of Lemma 5.2]).

2. Preliminaries

The n -dimensional Lebesgue measure is denoted by \mathcal{L}_n . Further, $\mathcal{L}_n|_{\Omega}$ is its restriction to Ω , i.e., $\mathcal{L}_n|_{\Omega}(A) = \mathcal{L}_n(A \cap \Omega)$ for every measurable set $A \subset \mathbb{R}^n$. If u is a measurable function on Ω , then by $u = 0$ (or $u \neq 0$) we mean that u is equal (or not equal) to the zero function a.e. on Ω . Sometimes we abbreviate the integral with respect to the Lebesgue measure $\int f(x) dx$ to $\int f$ if there is no danger of confusion to the reader.

By $\mathcal{M}(A)$ we denote the set of all Radon measures on a compact set A . We write that $\mu_j \xrightarrow{*} \mu$ in $\mathcal{M}(A)$ if $\int_A \psi d\mu_j \rightarrow \int_A \psi d\mu$ for every $\psi \in C(A)$. It is well known that each sequence bounded in $L^1(A)$ contains a subsequence converging weakly* in $\mathcal{M}(A)$.

By $B(x_0, R)$ we denote an open Euclidean ball in \mathbb{R}^n centered at x_0 with the radius $R > 0$. If $x_0 = 0$ we simply write $B(R)$.

By C we denote a generic positive constant which may depend on n , α , $\mathcal{L}_n(\Omega)$ and Φ . This constant may vary from expression to expression as usual.

Young functions and Orlicz spaces. A function $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a Young function if Φ is increasing, convex, $\Phi(0) = 0$ and $\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = \infty$.

Denote by $L^{\Phi}(A, d\mu)$ the Orlicz space corresponding to a Young function Φ on a set A with a measure μ . If $\mu = \mathcal{L}_n$ we simply write $L^{\Phi}(A)$. The space

$L^\Phi(A, d\mu)$ is equipped with the norm

$$\|f\|_{L^\Phi(A, d\mu)} = \inf \left\{ \lambda > 0 : \int_A \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq \Phi(1) \right\}. \tag{6}$$

This is slightly different from the usual Luxemburg definition where we have $\int_A \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1$. We use (6) to have the Hölder’s inequality (7) with a sharp constant.

Given a differentiable Young function Φ we can define the generalized inverse function to $\phi(u) = \Phi'(u)$ by

$$\psi(s) = \inf\{u : \phi(u) > s\} \quad \text{for } s > 0$$

and further we define the associated Young function Ψ by

$$\Psi(t) = \int_0^t \psi(s) ds \quad \text{for } t \geq 0.$$

The dual space to $L^\Phi(A, d\mu)$ can be identified as the Orlicz space $L^\Psi(A, d\mu)$.

If we have $\Phi(1) + \Psi(1) = 1$ then the following generalization of Hölder’s inequality is valid (see [17, p. 58] for the proof)

$$\int_A |f(y)g(y)| d\mu(y) \leq \|f\|_{L^\Phi(A, d\mu)} \|g\|_{L^\Psi(A, d\mu)}. \tag{7}$$

We use this inequality for a measurable set $A \subset \mathbb{R}$ and the measure $d\mu(y) = \omega_{n-1}y^{n-1}dy$.

If our Young function Φ satisfies (2), in a standard way we can prove that there is a Young function $\Phi_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

- Φ_1' is continuous and increasing on $(0, \infty)$,
 - $\Phi_1(t) = \frac{1}{n}t^n$ for $t \in [0, 1]$,
 - there is a $G > t_\Phi$ such that for every $t \geq G$ we have
- $$\Phi_1(t) = \frac{1}{n}t^n \log^\alpha(t) \left(1 + \log^{-a}(t)\right) \leq \frac{1}{n}\Phi(t). \tag{8}$$

Denote by Ψ the Young function associated to the function Φ_1 . Clearly $\Psi(t) = \frac{n-1}{n}t^{\frac{n}{n-1}}$ for $t \in [0, 1]$. Hence $\Phi_1(1) + \Psi(1) = 1$. Therefore (Φ_1, Ψ) is a normalized complementary Young pair and we can use inequality (7).

We need the following estimate from [13, Lemma 4.4].

Lemma 2.1. *Assume that the Young function Φ satisfies (2). Then there are $t_0 \in (0, 1)$ and $b \in (a, \min\{1, \frac{1}{\gamma}\})$ such that for $0 < t \leq t_0$ we have*

$$\left\| \frac{1}{y^{n-1}} \right\|_{L^\Psi((t, R), \omega_{n-1}y^{n-1}dy)} \leq \left(\frac{\omega_{n-1}}{B}\right)^{\frac{n-1}{n}} \log^{\frac{1}{\gamma}}\left(\frac{1}{t}\right) \left(1 - \log^{-b}\left(\frac{1}{t}\right)\right). \tag{9}$$

For an introduction to Orlicz spaces see, e.g., [17].

Δ_2 -condition. We say that the Young function Φ satisfies the Δ_2 -condition, if there are $t_\Delta \geq 0$ and $C_\Delta > 1$ such that $\Phi(2t) \leq C_\Delta \Phi(t)$ whenever $t \geq t_\Delta$. It is easy to see that if Φ satisfies the Δ_2 -condition for one fixed $t_\Delta > 0$ then it satisfies this condition with arbitrary $\tilde{t}_\Delta > 0$ with a different constant $\tilde{C}_\Delta > 1$. From the Δ_2 -condition it is not difficult to deduce that for any $\eta > 0$ we can find $\varepsilon > 0$ so that

$$\Phi((1 + \varepsilon)t) \leq (1 + \eta)\Phi(t), \quad t \geq t_\Delta. \tag{10}$$

It is not difficult to check the Δ_2 -condition for our Young functions satisfying (1). Therefore one easily proves

$$\|f\|_{L^\Phi(A, d\mu)} < 1 \iff \int_A \Phi(|f|) d\mu(x) < \Phi(1), \tag{11}$$

and

$$\|f_j\|_{L^\Phi(A, d\mu)} \xrightarrow{j \rightarrow \infty} 0 \iff \int_A \Phi(|f_j|) d\mu(x) \xrightarrow{j \rightarrow \infty} 0. \tag{12}$$

Orlicz-Sobolev spaces. Let A be a nonempty open set in \mathbb{R}^n and let Φ be a Young function. In this subsection we consider Orlicz spaces only with the Lebesgue measure. We define the Orlicz-Sobolev space $WL^\Phi(A)$ as the set

$$WL^\Phi(A) := \{u : u, |\nabla u| \in L^\Phi(A)\}$$

equipped with the norm $\|u\|_{WL^\Phi(A)} := \|u\|_{L^\Phi(A)} + \|\nabla u\|_{L^\Phi(A)}$, where ∇u is the gradient of u and we use its Euclidean norm in \mathbb{R}^n .

We put $W_0L^\Phi(A)$ for the closure of $C_0^\infty(A)$ in $WL^\Phi(A)$. For this space we prefer to use throughout the paper the equivalent norm (see [12, Corollary 5.8]) $\|u\|_{W_0L^\Phi(A)} := \|\nabla u\|_{L^\Phi(A)}$. The space $W_0L^\Phi(A)$ is a reflexive Banach space and it is compactly embedded into $L^\Phi(A)$ (see [8]).

We write that $f_k \rightharpoonup f$ in $W_0L^\Phi(A)$, if

$$\int_A \frac{\partial f_k}{\partial x_i} g \, dx \rightarrow \int_A \frac{\partial f}{\partial x_i} g \, dx \quad \text{for every } g \in L^\Psi(A) \text{ and } i \in \{1, \dots, n\}.$$

Non-increasing rearrangement. The non-increasing rearrangement f^* of a measurable function f on Ω is

$$f^*(t) = \inf \left\{ s > 0 : \mathcal{L}_n(\{x \in \Omega : |f(x)| > s\}) \leq t \right\}, \quad t > 0.$$

We also define the non-increasing radially symmetric rearrangement $f^\#$ by

$$f^\#(x) = f^* \left(\frac{\omega_{n-1}}{n} |x|^n \right) \quad \text{for } x \in B(R), \quad \mathcal{L}_n(B(R)) = \mathcal{L}_n(\Omega).$$

For an introduction to these rearrangements see , e.g., [18]. We need the fact that for every Young function Φ and for every measurable function $f : \Omega \rightarrow \mathbb{R}$ we have

$$\int_{\Omega} \Phi(|f(x)|) dx = \int_{B(R)} \Phi(|f^{\#}(x)|) dx = \int_0^{\mathcal{L}_n(\Omega)} \Phi(|f^*(y)|) dy.$$

We also use the Polya-Szegö principle (see, e.g., Talenti [18] for the proof).

Theorem 2.2. *Let Ω be an open bounded set and let $R > 0$ be such that $\mathcal{L}_n(B(R)) = \mathcal{L}_n(\Omega)$. Let Φ be a Young function. Suppose that the function $f : \Omega \rightarrow \mathbb{R}$ is Lipschitz continuous and supported in Ω . Then f^* is locally absolutely continuous and*

$$\int_{\Omega} \Phi(|\nabla f(x)|) dx \geq \int_{B(R)} \Phi(|\nabla f^{\#}(x)|) dx.$$

On embeddings into exponential spaces. The following theorem from [13] generalizes the famous result of Moser [15].

Theorem 2.3. *Let $\alpha < n - 1$ and let Φ be a Young function that satisfies (1). Suppose that $f \in W_0L^{\Phi}(\Omega)$ and $\|\Phi(|\nabla f|)\|_{L^1(\Omega)} \leq 1$.*

- (i) *If $K < K_{n,\alpha}$, then $\|\exp(K|f(x)|^{\gamma})\|_{L^1(\Omega)} \leq C_K$.*
- (ii) *If $K = K_{n,\alpha}$ and (2) is satisfied, then $\|\exp(K|f(x)|^{\gamma})\|_{L^1(\Omega)} \leq C_K$.*

The constant C_K always depends on $n, \alpha, \mathcal{L}_n(\Omega), K$ and Φ only.

Analogously to Moser’s result the norm in the exponential space can be made arbitrary large if $K > K_{n,\alpha}$. For detailed discussion about the limiting case $K = K_{n,\alpha}$ see [13].

In the proof of Theorem 1.1 we apply Theorem 2.3 to handle $\{u_k\}_{k=k_0+1}^{\infty}$, where $k_0 \in \mathbb{N}$ is sufficiently large. For dealing with $\{u_k\}_{k=1}^{k_0}$ we need the following lemma from [6, Remarks 3.11(iv)].

Lemma 2.4. *Let $n \geq 2, \alpha < n - 1, K \geq 0$ and let $f \in W_0L^{\Phi}(\Omega)$. Then $\exp(K|f(x)|^{\gamma}) \in L^1(\Omega)$.*

Tools from Measure Theory. We have

Lemma 2.5. *Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of measurable functions and let $u_k \rightarrow u$ a.e. in Ω . Suppose that there are $\alpha, \beta, \tau, C_1 > 0$ such that $\|\exp(\alpha(1+\beta)|u_k|^{\tau})\|_{L^1(\Omega)} < C_1$ for all $k \in \mathbb{N}$. Let F be an even continuous function such that*

$$\sup_{t \in (t_0, \infty)} \frac{|F(t)|}{\exp(\alpha|t|^{\tau})} < \infty \quad \text{for some } t_0 > 0.$$

Then $F(u_k) \xrightarrow{k \rightarrow \infty} F(u)$, in particular, $\exp(\alpha|u_k|^{\tau}) \xrightarrow{k \rightarrow \infty} \exp(\alpha|u|^{\tau})$ in the $L^1(\Omega)$ -norm.

Proof. As $\exp(\alpha|t|^\tau) \geq 1$ on \mathbb{R} , from the assumptions on F we obtain $L > 0$ such that

$$|F(t)| \leq L \exp(\alpha|t|^\tau) \quad \text{for every } t \in \mathbb{R}. \tag{13}$$

By Fatou’s lemma we have $\exp(\alpha|u|^\tau) \in L^1(\Omega)$, and thus for every $\varepsilon > 0$ there is $\delta > 0$ so that

$$\int_A \exp(\alpha|u|^\tau) < \frac{\varepsilon}{L} \quad \text{provided } \mathcal{L}_n(A) < \delta. \tag{14}$$

Next since $u \in L^1(\Omega)$ we find $M_1 > 0$ such that

$$\mathcal{L}_n(\{x \in \Omega : |u(x)| > M_1\}) < \delta. \tag{15}$$

Fix $M \geq M_1$ large enough so that $\frac{C_1}{\exp(\alpha\beta M^\tau)} < \frac{\varepsilon}{L}$. We have by (13) – (15)

$$\int_{\{|u| \geq M\}} |F(u)| \leq L \int_{\{|u| \geq M\}} \exp(\alpha|u|^\tau) < L \frac{\varepsilon}{L} = \varepsilon$$

and similarly we use the integrability of $\exp(\alpha(1 + \beta)|u_k|^\tau)$

$$\begin{aligned} \int_{\{|u_k| \geq M\}} |F(u_k)| &\leq L \int_{\{|u_k| \geq M\}} \exp(\alpha|u_k|^\tau) \\ &= L \int_{\{|u_k| \geq M\}} \frac{\exp(\alpha(1 + \beta)|u_k|^\tau)}{\exp(\alpha\beta|u_k|^\tau)} \\ &\leq L \frac{C_1}{\exp(\alpha\beta M^\tau)} \\ &< \varepsilon. \end{aligned}$$

Finally, the assumption $u_k \rightarrow u$ a.e. in Ω and the continuity of F imply $g_k := F(u_k)\chi_{\{|u_k| < M\}} - F(u)\chi_{\{|u| < M\}} \xrightarrow{k \rightarrow \infty} 0$ a.e. in Ω . Moreover

$$|g_k(x)| \leq L \exp(\alpha|u_k|^\tau)\chi_{\{|u_k| < M\}} + L \exp(\alpha|u|^\tau)\chi_{\{|u| < M\}} \leq 2L \exp(\alpha M^\tau)$$

where the last term is a $L^1(\Omega)$ -function. Hence, for $k \in \mathbb{N}$ large enough, the Lebesgue Dominated Convergence Theorem gives

$$\int_\Omega |F(u_k) - F(u)| \leq \int_{\{|u_k| \geq M\}} |F(u_k)| + \int_{\{|u| \geq M\}} |F(u)| + \int_\Omega |g_k| < 3\varepsilon$$

and the result follows. □

3. Concentration-Compactness

Proof of Corollary 1.2. Since $K < K_{n,\alpha}$, we can find $\delta > 0$ such that $\tilde{K} := (1 + \delta)K < K_{n,\alpha}$. Now, assumptions of Lemma 2.5 are satisfied with $\alpha = K$, $\beta = \delta$, $\tau = \gamma$ and $C_1 = C_{\tilde{K}}$, where $C_{\tilde{K}} < \infty$ comes from Theorem 2.3(i). Therefore we can use Lemma 2.5 to conclude the proof. □

In the proof of Theorem 1.1 we distinguish three cases. These cases are studied separately in Propositions 3.3 – 3.5 bellow.

Case 1. In this subsection we prove the Compactness in the case $u = 0$ and $\mu \neq \delta_{x_0}$.

Lemma 3.1. *Let $n \geq 2$, $\alpha < n - 1$ and let Φ be a Young function satisfying (1). Let $\{u_k\}_{k=1}^\infty \subset W_0L^\Phi(\Omega)$ satisfy $\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leq 1$. Suppose that*

$$u_k \rightharpoonup 0 \text{ in } W_0L^\Phi(\Omega) \quad \text{and} \quad \Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\bar{\Omega}).$$

Let $F, N \subset \bar{\Omega}$ be compact sets such that $F \cap N = \emptyset$ and $\mu(N) > 0$. Then there is $\delta > 0$ such that

$$\|\exp(K_{n,\alpha}(1 + \delta)|u_k|^\gamma)\|_{L^1(F)} \text{ is bounded.} \tag{16}$$

Proof. First let us briefly outline the idea of the proof. Since $\mu(N) > 0$ we obtain that $\int_N \Phi(|\nabla u_k|)$ cannot be small for k big enough and thus we can find $\delta > 0$ such that $\|\Phi((1 + 2\delta)|\nabla u_k|)\|_{L^1(F)} \leq 1$. Then, using Theorem 2.3 for some modification of the function $(1 + 2\delta)u_k$ we obtain (16).

We use $\Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\bar{\Omega})$ for a test function $\psi \equiv 1$ to obtain

$$1 \geq \int_{\bar{\Omega}} \Phi(|\nabla u_k|) = \int_{\bar{\Omega}} \psi \Phi(|\nabla u_k|) \xrightarrow{k \rightarrow \infty} \int_{\bar{\Omega}} \psi d\mu = \mu(\bar{\Omega}). \tag{17}$$

Set $\sigma = \frac{1}{5}\mu(N)$ and recall that C_Δ, t_Δ are the constants from the Δ_2 -condition (i.e. $\Phi(2t) \leq C_\Delta \Phi(t)$, $t \geq t_\Delta$). For $\tau > 0$ denote $G_\tau = \{x \in \mathbb{R}^n : \text{dist}(x, F) > \tau\}$. Clearly, we can find $0 < a < b < \text{dist}(F, N)$ such that

$$\mu(G_a \setminus G_b) \leq \frac{\sigma}{2C_\Delta^2} \quad \text{and} \quad \mathcal{L}_n(G_a \setminus G_b) < \frac{\sigma}{C_\Delta^2 \Phi(t_\Delta)}. \tag{18}$$

Set $M_1 = \bar{\Omega} \setminus G_a$ and $M_2 = \bar{\Omega} \setminus G_b$. We observe $F \subset M_1 \subset M_2$ and $M_2 \cap N = \emptyset$.

If $\psi \in C(\bar{\Omega})$ is such that $0 \leq \psi \leq 1$, $\psi \equiv 0$ on N and $\psi \equiv 1$ on M_2 then

$$\int_{M_2} \Phi(|\nabla u_k|) \leq \int_{\bar{\Omega}} \psi \Phi(|\nabla u_k|) \xrightarrow{k \rightarrow \infty} \int_{\bar{\Omega}} \psi d\mu \leq 1 - \mu(N) = 1 - 5\sigma.$$

Hence there is $k_1 \in \mathbb{N}$ such that

$$\int_{M_2} \Phi(|\nabla u_k|) \leq 1 - 4\sigma \quad \text{for } k > k_1. \tag{19}$$

Using (18) the same way as above we can find $k_2 > k_1$ such that

$$\int_{M_2 \setminus M_1} \Phi(|\nabla u_k|) \leq \frac{\sigma}{C_\Delta^2} \quad \text{for } k > k_2. \tag{20}$$

We claim that there is $\delta \in (0, \frac{1}{2})$ such that

$$\int_{M_2} \Phi((1 + 2\delta)|\nabla u_k|) \leq 1 - 3\sigma \quad \text{for } k > k_1. \tag{21}$$

Indeed, we can find $t_\Delta > 0$ so that $\Phi(2t_\Delta) \leq \frac{\sigma}{2\mathcal{L}_n(M_2)}$ and set $\eta = \frac{1-(3+\frac{1}{2})\sigma}{1-4\sigma}$. Then there is $\varepsilon \in (0, 1)$ so that (10) holds on $[t_\Delta, \infty)$ (see Preliminaries). Thus setting $\delta = \frac{\varepsilon}{2}$ we can use (19) to obtain

$$\begin{aligned} \int_{M_2} \Phi((1+2\delta)|\nabla u_k|) &= \int_{M_2 \cap \{|\nabla u_k| \geq t_\Delta\}} \Phi((1+\varepsilon)|\nabla u_k|) + \int_{M_2 \cap \{|\nabla u_k| < t_\Delta\}} \Phi((1+\varepsilon)t_\Delta) \\ &\leq 1 - \left(3 + \frac{1}{2}\right)\sigma + \frac{1}{2}\sigma \\ &= 1 - 3\sigma \end{aligned}$$

and (21) is proved.

Now we can define v_k . Take $\psi \in C^1(\bar{\Omega})$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ on M_1 and $\psi \equiv 0$ on $\bar{\Omega} \setminus \text{Int}(M_2)$. Set $v_k = (1+2\delta)\psi u_k$. Our aim is to apply Theorem 2.3 to v_k , thus we need to prove that there is $k_3 > k_2$ such that

$$I := \int_{\Omega} \Phi(|\nabla v_k|) \leq 1 \quad \text{for } k > k_3. \tag{22}$$

We have $I = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= \int_{M_1} \Phi(|\nabla v_k|) = \int_{M_1} \Phi((1+2\delta)|\nabla u_k|) \leq 1 - 3\sigma \quad \text{for } k > k_1 \text{ by (21),} \\ I_2 &= \int_{\Omega \setminus M_2} \Phi(|\nabla v_k|) = \int_{\Omega \setminus M_2} \Phi(0) = 0, \\ I_3 &= \int_{M_2 \setminus M_1} \Phi(|\nabla v_k|). \end{aligned}$$

Set $P = \max_{x \in \bar{\Omega}} |\nabla \psi(x)|$. From $\delta \in (0, \frac{1}{2})$ we have on $M_2 \setminus M_1$

$$\Phi(|\nabla v_k|) \leq \Phi((1+2\delta)\psi|\nabla u_k| + (1+2\delta)|u_k||\nabla \psi|) \leq \Phi(2|\nabla u_k| + 2P|u_k|). \tag{23}$$

It is convenient for us to decompose $M_2 \setminus M_1$ into three sets

$$\begin{aligned} A_k^1 &= \{x \in M_2 \setminus M_1 : t_\Delta \geq |\nabla u_k(x)|, t_\Delta \geq P|u_k(x)|\}, \\ A_k^2 &= \{x \in M_2 \setminus M_1 : |\nabla u_k(x)| \geq t_\Delta, |\nabla u_k(x)| \geq P|u_k(x)|\}, \\ A_k^3 &= \{x \in M_2 \setminus M_1 : P|u_k(x)| \geq t_\Delta, P|u_k(x)| \geq |\nabla u_k(x)|\}. \end{aligned}$$

As $M_2 \setminus M_1 = A_k^1 \cup A_k^2 \cup A_k^3$, we have

$$I_3 = \int_{M_2 \setminus M_1} \Phi(|\nabla v_k|) \leq \int_{A_k^1} \dots + \int_{A_k^2} \dots + \int_{A_k^3} \dots$$

First, by (18) and (23) we have

$$\int_{A_k^1} \Phi(|\nabla v_k|) \leq \int_{A_k^1} \Phi(4t_\Delta) \leq C_\Delta^2 \Phi(t_\Delta) \mathcal{L}_n(G_a \setminus G_b) \leq C_\Delta^2 \Phi(t_\Delta) \frac{\sigma}{C_\Delta^2 \Phi(t_\Delta)} = \sigma. \tag{24}$$

Second, (20) and (23) imply

$$\int_{A_k^2} \Phi(|\nabla v_k|) \leq \int_{A_k^2} \Phi(4|\nabla u_k|) \leq C_\Delta^2 \int_{M_2 \setminus M_1} \Phi(|\nabla u_k|) \leq C_\Delta^2 \frac{\sigma}{C_\Delta^2} = \sigma. \tag{25}$$

Third, by the compact embedding of $W_0L^\Phi(\Omega)$ into $L^\Phi(\Omega)$ we see, that the weak convergence $u_k \rightharpoonup 0$ in $W_0L^\Phi(\Omega)$ implies $u_k \rightarrow 0$ in $L^\Phi(\Omega)$. Then, using (12), we find $k_3 > k_2$ such that for $k > k_3$ we have

$$\int_{A_k^3} \Phi(|\nabla v_k|) \leq \int_{A_k^3} \Phi(4P|u_k|) < \sigma. \tag{26}$$

Estimates (24), (25) and (26) imply $I_3 < 3\sigma$ and (22) follows.

Therefore $v_k \in W_0L^\Phi(\Omega)$ and $\|\Phi(|\nabla v_k|)\|_{L^1(\Omega)} \leq 1$. Thus using Theorem 2.3(i) with $K = (\frac{1+\delta}{1+2\delta})^\gamma K_{n,\alpha}$ and the fact that $v_k = (1 + 2\delta)u_k$ on F we obtain for $k > k_3$

$$\begin{aligned} \|\exp(K_{n,\alpha}(1 + \delta)^\gamma |u_k|^\gamma)\|_{L^1(F)} &= \|\exp(K(1 + 2\delta)^\gamma |u_k|^\gamma)\|_{L^1(F)} \\ &\leq \|\exp(K|v_k|^\gamma)\|_{L^1(\Omega)} \\ &\leq C_K. \end{aligned}$$

Moreover, for every fixed $k < k_3$ there is C_k such that $\|\exp(K_{n,\alpha}(1+\delta)^\gamma |u_k|^\gamma)\|_{L^1(F)} \leq C_k$ by Lemma 2.4. Hence we obtain (16) for $\tilde{\delta} = (1 + \delta)^\gamma - 1$ with the bound $\max(C_1, \dots, C_{k_3}, C_K)$. \square

Remark 3.2. It can be easily seen that if we have $\mu(\bar{\Omega}) < 1$, then there is a simplified version of the above proof giving us $\delta > 0$ such that the following norm $\|\exp(K_{n,\alpha}(1 + \delta)|u_k|^\gamma)\|_{L^1(\Omega)}$ is bounded.

Proposition 3.3. *Let $n \geq 2$, $\alpha < n - 1$ and let Φ be a Young function satisfying (1). Let $\{u_k\}_{k=1}^\infty \subset W_0L^\Phi(\Omega)$ satisfy $\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leq 1$. Further suppose that*

$$u_k \rightharpoonup 0 \text{ in } W_0L^\Phi(\Omega), \quad u_k \rightarrow 0 \text{ a.e. in } \Omega \quad \text{and} \quad \Phi(|\nabla u_k|) \xrightarrow{*} \mu \text{ in } \mathcal{M}(\bar{\Omega}),$$

where μ is not a Dirac mass at one point. Then there is $\delta > 0$ such that

$$\exp(K_{n,\alpha}(1 + \delta)|u_k|^\gamma) \quad \text{is bounded in } L^1(\Omega)$$

and

$$\exp(K_{n,\alpha}|u_k|^\gamma) \xrightarrow{k \rightarrow \infty} \exp(K_{n,\alpha}|u|^\gamma) \quad \text{in } L^1(\Omega).$$

Proof. As $\mu(\bar{\Omega}) \leq 1$ (see (17)), we distinguish two cases. If $\mu(\bar{\Omega}) < 1$, then the first assertion follows from Remark 3.2. Now, let $\mu(\Omega) = 1$. As μ is not a Dirac mass at one point, there is $N_1 \subset \bar{\Omega}$ compact such that $\mu(N_1) \in (0, 1)$. We denote

$G = \mathbb{R}^n \setminus N_1$ and $G_\tau = \{x \in \mathbb{R}^n : \text{dist}(x, N_1) > \tau\}$ for $\tau > 0$. Considering μ as a Radon measure on \mathbb{R}^n supported in $\bar{\Omega}$ we obtain $\lim_{\tau \rightarrow 0^+} \mu(G_\tau) = \mu(G) = 1 - \mu(N_1) \in (0, 1)$. Therefore there is $\tau > 0$ such that $0 < \mu(G_{2\tau}) \leq \mu(G_\tau) < 1$. Set $F_1 = \bar{\Omega} \setminus G_\tau$, $F_2 = \bar{\Omega} \cap \bar{G}_\tau$ and $N_2 = \bar{\Omega} \cap \bar{G}_{2\tau}$. Clearly F_1, F_2, N_1, N_2 are compact sets, $F_1 \cup F_2 = \bar{\Omega}$. Moreover $\mu(N_2) \geq \mu(G_{2\tau}) > 0$, $N_2 \cap F_1 = \emptyset$ and $N_1 \cap F_2 = \emptyset$.

Applying Lemma 3.1 to $F = F_1$ and $N = N_2$ we obtain that there is $\delta_1 > 0$ such that $\|\exp(K_{n,\alpha}(1 + \delta_1)|u_k|^\gamma)\|_{L^1(F_1)}$ is bounded. If $F = F_2$ and $N = N_1$ then Lemma 3.1 gives us $\delta_2 > 0$ such that $\|\exp(K_{n,\alpha}(1 + \delta_2)|u_k|^\gamma)\|_{L^1(F_2)}$ is bounded. From $F_1 \cup F_2 = \bar{\Omega}$ we conclude that $\|\exp(K_{n,\alpha}(1 + \delta)|u_k|^\gamma)\|_{L^1(\Omega)}$ is bounded for $\delta = \min(\delta_1, \delta_2)$.

Finally, we apply Lemma 2.5 to prove the last assertion. □

Case 2. In this subsection we prove the concentration in the case $u = 0$ and $\mu = \delta_{x_0}$.

Proposition 3.4. *Let $n \geq 2$, $\alpha < n - 1$ and let Φ be a Young function satisfying (1). Let $\{u_k\}_{k=1}^\infty \subset W_0L^\Phi(\Omega)$ satisfy $\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leq 1$. Further suppose that*

$$u_k \rightharpoonup 0 \text{ in } W_0L^\Phi(\Omega), \quad u_k \rightarrow 0 \text{ a.e. in } \Omega \quad \text{and} \quad \Phi(|\nabla u_k|) \xrightarrow{*} \delta_{x_0} \text{ in } \mathcal{M}(\bar{\Omega}),$$

where $x_0 \in \bar{\Omega}$.

- (i) *If $\int_\Omega (\exp(K_{n,\alpha}|u_k|^\gamma) - 1) \xrightarrow{k \rightarrow \infty} c \in [0, \infty)$, then $\exp(K_{n,\alpha}|u_k|^\gamma) - 1 \xrightarrow{*} c\delta_{x_0}$ in $\mathcal{M}(\bar{\Omega})$.*
- (ii) *Moreover, if Φ satisfies (2), then the sequence $\{\exp(K_{n,\alpha}|u_k|^\gamma) - 1\}_{k=1}^\infty$ is relatively compact with respect to the weak* convergence in $\mathcal{M}(\bar{\Omega})$ and the limits of convergent subsequences belong to $\{c\delta_{x_0} : c \in [0, C_K - \mathcal{L}_n(\Omega)]\}$ ($C_K \geq \mathcal{L}_n(\Omega)$ comes from Theorem 2.3 (ii)).*

Proof. Let us prove (i). First, we claim that

$$\eta > 0 \implies \int_{\Omega \setminus B(x_0, \eta)} \exp(K_{n,\alpha}|u_k|^\gamma) - 1 \xrightarrow{k \rightarrow \infty} 0. \tag{27}$$

From Lemma 3.1 for $N = \overline{B(x_0, \frac{\eta}{2})}$ we obtain that $\int_{\Omega \setminus B(x_0, \eta)} \exp(K_{n,\alpha}(1 + \delta)|u_k|^\gamma)$ is bounded for some $\delta > 0$ and thus we may use Lemma 2.5 to obtain (27).

Further we observe that (27) and assumption $\int_\Omega \exp(K_{n,\alpha}|u_k|^\gamma) - 1 \rightarrow c$ imply

$$\eta > 0 \implies \int_{B(x_0, \eta)} \exp(K_{n,\alpha}|u_k|^\gamma) - 1 \xrightarrow{k \rightarrow \infty} c. \tag{28}$$

Fix arbitrary test function $\psi \in C(\bar{\Omega})$ and let $\varepsilon > 0$. Then there is $\eta > 0$ such that

$$|\psi(x) - \psi(x_0)| < \frac{\varepsilon}{2 \max(c, 1)} \quad \text{whenever } |x - x_0| < \eta. \tag{29}$$

We have

$$\begin{aligned}
 I &:= \left| \int_{\bar{\Omega}} \psi \, d(c\delta_{x_0}) - \int_{\Omega} \psi(\exp(K_{n,\alpha}|u_k|^\gamma) - 1) \right| \\
 &= \left| c\psi(x_0) - \int_{\Omega} \psi(\exp(K_{n,\alpha}|u_k|^\gamma) - 1) \right| \\
 &\leq \int_{\Omega \setminus B(x_0, \eta)} |\psi|(\exp(K_{n,\alpha}|u_k|^\gamma) - 1) + \int_{B(x_0, \eta)} |\psi - \psi(x_0)|(\exp(K_{n,\alpha}|u_k|^\gamma) - 1) \\
 &\quad + |\psi(x_0)| \cdot \left| c - \int_{B(x_0, \eta)} (\exp(K_{n,\alpha}|u_k|^\gamma) - 1) \right| \\
 &= I_1 + I_2 + I_3.
 \end{aligned}$$

By (27) and $\sup_{\Omega} |\psi| < \infty$ we see that there is $k_1 \in \mathbb{N}$ such that $I_1 < \varepsilon$ for $k > k_1$. Further, using (28) and (29) we obtain

$$\begin{aligned}
 I_2 &= \int_{B(x_0, \eta)} |\psi - \psi(x_0)|(\exp(K_{n,\alpha}|u_k|^\gamma) - 1) \\
 &\leq \frac{\varepsilon}{2 \max(c, 1)} \int_{B(x_0, \eta)} \exp(K_{n,\alpha}|u_k|^\gamma) - 1 \xrightarrow{k \rightarrow \infty} \frac{\varepsilon}{2} \frac{c}{\max(c, 1)}.
 \end{aligned}$$

Therefore we can find $k_2 > k_1$ such that $I_2 < \varepsilon$ for $k > k_2$. Finally, from (28) and $|\psi(x_0)| < \infty$ we obtain $k_3 > k_2$ such that $I_3 < \varepsilon$ for $k > k_3$. Hence we have $I < 3\varepsilon$ for k large and the first assertion is proved.

Let us prove the second assertion. We apply Theorem 2.3 to obtain

$$\|\exp(K_{n,\alpha}|u_k|^\gamma) - 1\|_{L^1(\Omega)} \leq C_K - \mathcal{L}_n(\Omega). \tag{30}$$

Now, we use the fact that every set bounded in the $L^1(\Omega)$ -norm is relatively compact in $\mathcal{M}(\bar{\Omega})$ with respect to the weak*-convergence. Further, suppose that $\{v_k\}_{k=1}^\infty \subset \{u_k\}_{k=1}^\infty$ is such that $\exp(K_{n,\alpha}|v_k|^\gamma) - 1 \xrightarrow{*} \nu$ in $\mathcal{M}(\bar{\Omega})$. Choosing the test function $\psi \equiv 1$ we obtain

$$\int_{\Omega} (\exp(K_{n,\alpha}|v_k|^\gamma) - 1) = \int_{\bar{\Omega}} \psi(\exp(K_{n,\alpha}|v_k|^\gamma) - 1) \xrightarrow{k \rightarrow \infty} \int_{\bar{\Omega}} \psi \, d\nu = \nu(\bar{\Omega}).$$

Thus the sequence $\{v_k\}_{k=1}^\infty$ satisfies the assumptions of the first part of our proposition with $c = \nu(\bar{\Omega}) \in [0, C_K - \mathcal{L}_n(\Omega)]$ (for the upper estimate of c we use (30)), thus the first assertion concludes the proof. \square

Case 3. In this subsection we prove the compactness for $u \neq 0$.

Proposition 3.5. *Let $n \geq 2$, $\alpha < n - 1$ and let Φ be a Young function satisfying (1) and (2). Assume that $\{u_k\}_{k=1}^\infty \subset W_0L^\Phi(\Omega)$ are such that*

$$\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leq 1, \quad u_k \rightharpoonup u \text{ in } W_0L^\Phi(\Omega) \quad \text{and} \quad u \neq 0.$$

Then there is $\delta > 0$ such that

$$\|\exp(K_{n,\alpha}(1 + \delta)|u_k|^\gamma)\|_{L^1(\Omega)} \text{ is bounded}$$

and

$$\exp(K_{n,\alpha}|u_k|^\gamma) \rightarrow \exp(K_{n,\alpha}|u|^\gamma) \text{ in } L^1(\Omega).$$

The key ingredient of the proof of Proposition 3.5 is the following lemma telling us that if the sequence $\{u_k\}_{k=1}^\infty$ satisfies condition (31) (which is what we do not want in Proposition 3.5), then we actually have $u = 0$.

Lemma 3.6. *Let $n \geq 2$, $\alpha < n - 1$ and let Φ be a Young function satisfying (1) and (2). Let $R > 0$ and let $\{g_k\}_{k=1}^\infty$ be non-increasing locally absolutely continuous functions on $[0, R]$ satisfying $g_k(R) = 0$. Set $u_k(x) = g_k(|x|)$, for $x \in B(R)$, $k \in \mathbb{N}$ and assume that $\|\Phi(|\nabla u_k|)\|_{L^1(B(R))} \leq 1$. If*

$$\lim_{k \rightarrow \infty} \|\exp(K_{n,\alpha}(1 + \delta)|u_k|^\gamma)\|_{L^1(B(R))} = \infty \text{ for every } \delta > 0, \tag{31}$$

then $u_k \xrightarrow{k \rightarrow \infty} 0$ uniformly on $B(R) \setminus B(r)$ for every $r \in (0, R)$.

Proof. First let us prove

$$\int_{B(R) \setminus B(r)} \Phi(|\nabla u_k|) \xrightarrow{k \rightarrow \infty} 0 \text{ for every } r \in (0, R). \tag{32}$$

If (32) is not true then passing to a subsequence we can find $\tau > 0$ and $r_0 \in (0, R)$ such that $\int_{B(R) \setminus B(r_0)} \Phi(|\nabla u_k|) \geq \tau$ for all $k \in \mathbb{N}$ and thus

$$\int_{B(r_0)} \Phi(|\nabla u_k|) \leq 1 - \tau \text{ for all } k \in \mathbb{N}. \tag{33}$$

Put $d\mu(y) = \omega_{n-1}y^{n-1}dy$ and let Φ_1 be the Young function from (8). Fix $t \in (0, r_0)$ and for every $k \in \mathbb{N}$ set

$$A_k = \{y \in (t, r_0) : |g'_k(y)| > G\}, \quad \tilde{A}_k = \{y \in (r_0, R) : |g'_k(y)| > G\}$$

(recall that the constant G comes from (8)). From (8) and (33) we obtain

$$\begin{aligned} \int_{A_k} \Phi_1(|g'_k(y)|)\omega_{n-1}y^{n-1} dy &\leq \frac{\omega_{n-1}}{n} \int_{A_k} \Phi(|g'_k(y)|)y^{n-1} dy \\ &\leq \frac{\omega_{n-1}}{n} \int_0^{r_0} \Phi(|g'_k(y)|)y^{n-1} dy \\ &= \frac{1}{n} \int_{B(r_0)} \Phi(|\nabla u_k(x)|) dx \\ &\leq \frac{1}{n}(1 - \tau) \\ &= (1 - \tau)\Phi_1(1). \end{aligned} \tag{34}$$

Thus (11) gives $\tilde{\tau} > 0$ such that

$$\|g'_k(y)\|_{L^{\Phi_1}(A_k, d\mu)} \leq 1 - 2\tilde{\tau}, \quad k \in \mathbb{N}. \tag{35}$$

The same way we obtain from $\|\Phi(|\nabla u_k|)\|_{L^1(B(R))} \leq 1$ that

$$\|g'_k(y)\|_{L^{\Phi_1}(\tilde{A}_k, d\mu)} \leq 1, \quad k \in \mathbb{N}. \tag{36}$$

Hence Hölder's inequality gives

$$\begin{aligned} g_k(t) &\leq \int_t^R |g'_k(y)| dy \\ &= \int_{(r_0, R) \setminus \tilde{A}_k} \dots + \int_{\tilde{A}_k} \dots + \int_{(t, r_0) \setminus A_k} \dots + \int_{A_k} \dots \\ &\leq GR + \int_{\tilde{A}_k} |g'_k(y)| \frac{1}{\omega_{n-1} y^{n-1}} d\mu(y) + Gr_0 + \int_{A_k} |g'_k(y)| \frac{1}{\omega_{n-1} y^{n-1}} d\mu(y) \\ &\leq C + \frac{1}{\omega_{n-1}} \|g'_k(y)\|_{L^{\Phi_1}(\tilde{A}_k, d\mu)} \left\| \frac{1}{y^{n-1}} \right\|_{L^\Psi((r_0, R), d\mu)} \\ &\quad + \frac{1}{\omega_{n-1}} \|g'_k(y)\|_{L^{\Phi_1}(A_k, d\mu)} \left\| \frac{1}{y^{n-1}} \right\|_{L^\Psi((t, R), d\mu)}. \end{aligned}$$

Therefore Lemma 2.1, (35), (36) and $\left\| \frac{1}{y^{n-1}} \right\|_{L^\Psi((r_0, R), d\mu)} \leq C \left(\frac{1}{y^{n-1}} \right)$ is bounded on $[r_0, R]$) imply

$$g_k(t) \leq C + \frac{1}{\omega_{n-1}} (1 - 2\tilde{\tau}) \left(\frac{\omega_{n-1}}{B} \right)^{\frac{n-1}{n}} \log^{\frac{1}{\gamma}} \left(\frac{1}{t} \right) \quad \text{for all } t \in (0, \min(t_0, r_0)), \quad k \in \mathbb{N}.$$

Therefore there is $t_1 \in (0, \min(t_0, r_0))$ such that

$$g_k(t) \leq \frac{1}{\omega_{n-1}} (1 - \tilde{\tau}) \left(\frac{\omega_{n-1}}{B} \right)^{\frac{n-1}{n}} \log^{\frac{1}{\gamma}} \left(\frac{1}{t} \right) \quad \text{for every } t \in (0, t_1), \quad k \in \mathbb{N}. \tag{37}$$

Finally pick $\delta_0 > 0$ small enough so that

$$\eta := (1 + \delta_0)^\gamma (1 - \tilde{\tau})^\gamma < 1 \tag{38}$$

and let us show that we have a contradiction with (31). Indeed, (37), (38), $\eta < 1$ and $K_{n,\alpha} \left(\frac{1}{\omega_{n-1}} \left(\frac{\omega_{n-1}}{B} \right)^{\frac{n-1}{n}} \right)^\gamma = n$ imply

$$\begin{aligned} &\int_{B(R)} \exp(K_{n,\alpha}(1 + \delta_0)|u_k(x)|^\gamma) dx \\ &= \omega_{n-1} \int_0^R \exp(K_{n,\alpha}(1 + \delta_0)|g_k(y)|^\gamma) y^{n-1} dy \\ &\leq C \int_{t_1}^R \exp(C|g_k(t_1)|^\gamma) y^{n-1} dy + C \int_0^{t_1} \exp\left(\eta n \log\left(\frac{1}{y}\right)\right) y^{n-1} dy \\ &\leq C + C \int_0^{t_1} y^{n-1-\eta n} dy \\ &\leq C. \end{aligned}$$

Hence we have a contradiction with (31) and thus (32) is proved.

Now, fix $r \in (0, R)$ and let us check the uniform convergence. From (12) and (32) we obtain $\|\nabla u_k\|_{L^\Phi(B(R)\setminus B(r))} \rightarrow 0$. Hence $\|\nabla u_k\|_{L^1(B(R)\setminus B(r))} \rightarrow 0$ and thus the radial symmetry of u_k , $u_k|_{\partial B(R)} = 0$ and the monotonicity with respect to $|x|$ imply the uniform convergence. \square

Proof of Proposition 3.5. We prove Proposition 3.5 by contradiction. Suppose that $\sup_k \|\exp(K_{n,\alpha}(1+\delta)|u_k|^\gamma)\|_{L^1(\Omega)} = \infty$ for every $\delta > 0$. Recall that for a fixed $k \in \mathbb{N}$ and $\delta > 0$ we have $\|\exp(K_{n,\alpha}(1+\delta)|u_k|^\gamma)\|_{L^1(\Omega)} < \infty$ by Lemma 2.4. Thus passing to a subsequence, we can suppose that

$$\|\exp(K_{n,\alpha}(1+\delta)|u_k|^\gamma)\|_{L^1(\Omega)} \xrightarrow{k \rightarrow \infty} \infty \quad \text{for every } \delta > 0. \tag{39}$$

By a standard symmetrization argument (use Theorem 2.2 and the density of C_0^∞ -functions) we may assume that Ω is a ball, u_k, u are continuous, spherically symmetric, non-negative, non-increasing with respect to $|x|$ and locally absolutely continuous. Indeed, since $u_k \rightharpoonup u$ in $W_0L^\Phi(\Omega)$ we have $u_k \rightarrow u$ in $L^1(\Omega)$ and thus $u_k^\# \rightarrow u^\#$ in $L^1(\Omega)$. Moreover $\|\Phi(|\nabla u_k^\#|)\|_{L^1(\Omega)} \leq 1$ by Theorem 2.2. It follows that the rearranged sequence contains a subsequence that converges weakly in $W_0L^\Phi(\Omega)$ to a non-zero function. Since $u_k^\# \rightarrow u^\#$ in L^1 , it is easy to see that this limit function must be $u^\#$. The subsequence again satisfies (39) and hence assumptions of Lemma 3.6 are satisfied for this new subsequence which we again denote as u_k . We obtain that u_k converge uniformly to the zero function on $B(R) \setminus B(r)$ for every $r \in (0, R)$. This implies $u = 0$ a.e. and we have a contradiction with $u \neq 0$.

The last assertion of the Proposition 3.5 follows from Lemma 2.5. \square

Proof of Theorem 1.1. Theorem 1.1 follows from the Propositions 3.3 – 3.5. \square

4. Norm attaining functionals

In this section we apply the Concentration-Compactness Principle to the functionals with the sub-critical growth.

Proof of Theorem 1.3. Put

$$S := \sup \{ \Lambda_F(u) : u \in W_0L^\Phi(\Omega), \|\Phi(|\nabla u|)\|_{L^1(\Omega)} \leq 1 \}.$$

If $S = \mathcal{L}_n(\Omega)F(0)$, then the proof is trivial, because for $u = 0$ we have $\Lambda_F(u) = \mathcal{L}_n(\Omega)F(0)$. Otherwise there is a sequence $\{u_k\}_{k=1}^\infty \subset \{u \in W_0L^\Phi(\Omega) : \|\Phi(|\nabla u|)\|_{L^1(\Omega)} \leq 1\}$ such that $\Lambda_F(u_k) \xrightarrow{k \rightarrow \infty} S$. We can further suppose that

$$u_k \rightharpoonup u \text{ in } W_0L^\Phi(\Omega), \quad u_k \rightarrow u \text{ a.e. in } \Omega \quad \text{and} \quad \Phi(|\nabla u_k|) \xrightarrow{*} \mu \text{ in } \mathcal{M}(\bar{\Omega}),$$

otherwise we pass to a subsequence (note that $W_0L^\Phi(\Omega)$ is reflexive). Obviously we have the estimate $\|\Phi(|\nabla u|)\|_{L^1(\Omega)} \leq 1$ and thus all we need to show is $\Lambda_F(u) = S$.

If (4) is satisfied, then we find $\delta > 0$ such that $\tilde{K} := (1 + \delta)K < K_{n,\alpha}$. Now, we can use Lemma 2.5 (with $\alpha = K$, $\beta = \delta$, $\tau = \gamma$ and $C_1 = C_{\tilde{K}}$, where $C_{\tilde{K}} < \infty$ comes from Theorem 2.3(i)) to conclude the proof.

The rest of the proof is devoted to the case when (5) is satisfied. By Theorem 1.1 we have either

$$\exp(K_{n,\alpha}(1 + \delta)|u_k|^\gamma) \text{ is bounded in } L^1(\Omega)$$

or

$$u = 0 \text{ and } \Phi(|\nabla u_k|) \xrightarrow{*} \delta_{x_0} \text{ in } \mathcal{M}(\bar{\Omega}).$$

In the first case we easily conclude the proof using Lemma 2.5 as above.

Now, it is enough to prove that in the second case we have

$$\lim_{k \rightarrow \infty} \Lambda_F(u_k) = \mathcal{L}_n(\Omega)F(0). \tag{40}$$

Fix $\varepsilon > 0$. As $\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leq 1$, we can use Theorem 2.3(ii) to obtain $C_2 > 0$ such that

$$\int_{\Omega} \exp\left(K_{n,\alpha}|u_k|^\gamma\right) \leq C_2. \tag{41}$$

Next, by (5), there is $t_0 > 0$ such that

$$|F(t)| \leq \frac{\varepsilon}{C_2} \exp\left(K_{n,\alpha}|t|^\gamma\right) \text{ for } |t| \geq t_0. \tag{42}$$

Now, we have

$$\begin{aligned} |\Lambda_F(u_k) - \mathcal{L}_n(\Omega)F(0)| &\leq \int_{\Omega} |F(u_k) - F(0)| \\ &\leq \int_{\Omega} \left|F(u_k)\chi_{\{|u_k| \leq t_0\}} - F(0)\right| + \int_{\Omega} |F(u_k)|\chi_{\{|u_k| > t_0\}} \\ &= I_1 + I_2. \end{aligned}$$

Since F is continuous and $u_k \rightarrow 0$ a.e. in Ω , by the Lebesgue Dominated Convergence Theorem we obtain $I_1 \rightarrow 0$. By (41) and (42) we see that $I_2 \leq \varepsilon$. We have proved (40) and we are done. \square

5. Concluding remarks

(i) The original statement of the Concentration-Compactness Principle for the space $W_0^{1,n}(\Omega)$ (see [14, Theorem I.6]) can be misunderstood in the sense that it might seem that if $\{u_k\}_{k=1}^\infty \subset W_0^{1,n}(\Omega)$ satisfy

$$\|\nabla u_k\|_{L^n(\Omega)} \leq 1, \quad u_k \rightharpoonup u \text{ in } W_0^{1,n}(\Omega), \quad \text{and} \quad |\nabla u_k|^n \xrightarrow{*} \delta_{x_0} \text{ in } \mathcal{M}(\bar{\Omega}),$$

then there is just one $c \geq 0$ such that $\exp(K_{n,0}|u_k|^{\frac{n}{n-1}}) - 1 \overset{*}{\rightharpoonup} c\delta_{x_0}$ in $\mathcal{M}(\bar{\Omega})$. Our next proposition says that for a concentrating sequence the constant c is not unique in general.

Proposition 5.1. *Let $n \geq 2$, $\alpha < n - 1$, $K \geq 0$ and let Φ be a Young function satisfying (1) and let $K \leq K_{n,\alpha}$. Suppose that there is a sequence $\{u_k\}_{k=1}^\infty \subset W_0L^\Phi(\Omega)$ such that*

$$\begin{aligned} &\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leq 1, \quad u_k \rightharpoonup 0 \text{ in } W_0L^\Phi(\Omega), \quad u_k \rightarrow 0 \text{ a.e. in } \Omega, \\ &\Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \delta_{x_0} \text{ in } \mathcal{M}(\bar{\Omega}) \quad \text{and} \quad \{\exp(K|u_k|^\gamma) - 1\}_{k=1}^\infty \overset{*}{\rightharpoonup} c\delta_{x_0} \text{ in } \mathcal{M}(\bar{\Omega}) \end{aligned}$$

with $c > 0$. Then for every $d \in [0, c]$ there is a sequence $\{v_k\}_{k=1}^\infty \subset W_0L^\Phi(\Omega)$ such that

$$\begin{aligned} &\|\Phi(|\nabla v_k|)\|_{L^1(\Omega)} \leq 1, \quad v_k \rightharpoonup 0 \text{ in } W_0L^\Phi(\Omega), \quad v_k \rightarrow 0 \text{ a.e. in } \Omega, \\ &\Phi(|\nabla v_k|) \overset{*}{\rightharpoonup} \delta_{x_0} \text{ in } \mathcal{M}(\bar{\Omega}) \quad \text{and} \quad \{\exp(K|v_k|^\gamma) - 1\}_{k=1}^\infty \overset{*}{\rightharpoonup} d\delta_{x_0} \text{ in } \mathcal{M}(\bar{\Omega}). \end{aligned}$$

Proof. If $d = c$, we set $v_k = u_k$ and we are done. Thus suppose $0 \leq d < c$. We can also suppose that $x_0 = 0 \in \bar{\Omega}$. Applying Theorem 2.2 and the density of C_0^∞ -functions in $W_0L^\Phi(\Omega)$ we can further suppose that $\Omega = B(R)$, $R > 0$, u_k are radially symmetric, continuous, non-negative, non-increasing with respect to $|x|$ and locally absolutely continuous. This means that for every $k \in \mathbb{N}$ there is a bounded continuous non-increasing non-negative function $g_k : [0, R] \mapsto [0, \infty)$, such that $g_k(R) = 0$, g_k is differentiable a.e. in $[0, R]$ and $u_k(x) = g_k(|x|)$. Our functions v_k will be defined as a suitable modification of u_k .

Clearly $c_k := \int_{B(R)} (\exp(K|u_k|^\gamma) - 1) \overset{k \rightarrow \infty}{\rightarrow} c$. We can suppose that $c_k > d$, otherwise we pass to a subsequence. Fix $k \in \mathbb{N}$. The function

$$\psi(t) = \int_{B(R)} (\exp(K|\min(t, u_k)|^\gamma) - 1)$$

is continuous on $[0, \infty)$ and satisfies $\psi(0) = 0$ and $\psi(u_k(0)) = c_k > d$. Thus there is $0 \leq t_0 < u_k(0) = g_k(0)$ such that

$$\psi(t_0) = \int_{B(R)} \exp(K|\min(t_0, u_k)|^\gamma) - 1 = d. \tag{43}$$

As g_k is continuous, there is $a_0 \in (0, R]$ such that $g_k(a_0) = t_0$. Find $L \in \mathbb{N}$ large enough so that

$$\tau := \frac{g_k(0) - t_0}{2L} \leq \frac{1}{k}. \tag{44}$$

The continuity of g_k implies that there are $0 = a_{2L} < a_{2L-1} < \dots < a_1 < a_0$ such that $g_k(a_j) = j\tau + t_0$, $j = 0, \dots, 2L$. Let us define

$$h_k(t) = \begin{cases} g_k(t) - 2j\tau, & t \in [a_{2j+1}, a_{2j}], \quad j = 0, \dots, L - 1, \\ 2(j + 1)\tau - g_k(t) + 2t_0, & t \in [a_{2j+2}, a_{2j+1}], \quad j = 0, \dots, L - 1. \end{cases}$$

It is easy to check that this function is continuous and satisfies

$$0 \leq h_k \leq g_k, \quad h_k = g_k \text{ on } [a_0, R], \quad h_k \in [t_0, t_0 + \tau] \text{ on } [0, a_0] \quad (45)$$

and $|h'_k| = |g'_k|$ on $[0, R] \setminus \bigcup_{j=1}^{2L} a_j$. Set $v_k(x) = h_k(|x|)$ for $x \in \bar{B}(R)$. From the properties of h_k we obtain

$$0 \leq v_k \leq u_k, \quad v_k \rightarrow 0 \text{ a.e. in } B(R), \quad \Phi(|\nabla v_k|) = \Phi(|\nabla u_k|) \text{ a.e. in } B(R),$$

$$v_k \in W_0L^\Phi(B(R)), \quad \|\Phi(|\nabla v_k|)\|_{L^1(B(R))} \leq 1, \quad |\Phi(|\nabla v_k|)| \xrightarrow{*} \delta_0 \text{ in } \mathcal{M}(\bar{B}(R)).$$

Furthermore, we can suppose that $v_k \rightarrow 0$ in $W_0L^\Phi(B(R))$. Indeed, as the norm $\|\Phi(|\nabla v_k|)\|_{L^1(B(R))}$ is bounded, passing to a subsequence we can suppose that $v_k \rightarrow v$ in $W_0L^\Phi(B(R))$ for some $v \in W_0L^\Phi(B(R))$. Hence $v_k \rightarrow v$ in $L^1(B(R))$. Thus passing to a subsequence again we can suppose that $v_k \rightarrow v$ a.e. in $B(R)$ and hence $v = 0$ (recall that $u_k \rightarrow 0$ a.e.).

Finally from (43) – (45), $v_k(x) = h_k(|x|)$ and the continuity of the function \exp we see that $\int_{B(R)} (\exp(K|v_k|^\gamma) - 1) \rightarrow d$. Thus Proposition 3.4 concludes the proof. \square

(ii) In view of Corollary 1.2, one can ask whether there actually exist concentrating sequences satisfying the condition $\exp(K_{n,\alpha}|u_k|^\gamma) - 1 \xrightarrow{*} c\delta_{x_0}$, with $c > 0$. An explicit so called Moser sequence is known for the case $x_0 = 0$, $n = 2$ and $\alpha = 0$ and it is defined by

$$u_k(x) = \begin{cases} \frac{1}{2\pi} \log^{\frac{1}{2}}(k), & 0 \leq |x| \leq \frac{1}{k}, \\ \frac{1}{2\pi} \frac{\log(\frac{1}{|x|})}{\log^{\frac{1}{2}}(k)}, & \frac{1}{k} \leq |x| \leq 1, \\ 0, & |x| \geq 1. \end{cases}$$

(iii) The technical condition (2) was used in our proofs mainly for the application of Theorem 2.3(ii). If it is possible to prove the analogue of Theorem 2.3(ii) under weaker assumptions then we believe that it is possible to obtain our results under this weaker assumptions as well.

(iv) It is possible to use similar methods to obtain the Concentration-Compactness Principle also for embedding into multiple exponential spaces (see forthcoming paper [3]). In these results it is necessary to use [5] instead of Theorem 2.3.

(v) Even though it was essential for us to work with the norm given by (6) (this is the norm giving the strong Hölder’s inequality), the statements of our theorems with assumptions $\|\Phi(|\nabla u_k|)\|_{L^1(\Omega)} \leq 1$ rather correspond to the standard Luxemburg norm (see Preliminaries) because the above assumption reads that the Luxemburg norms of u_k are bounded by 1. If the bound of the Luxemburg norm was not 1 but say $C > 0$ then one can easily see that our assertions would still hold but with the critical parameter $K = \frac{K_{n,\alpha}}{C}$.

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