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Equilibrium State of a Pendant Drop with Inter-phase Layer

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Abstract. We consider equilibrium state of axisymmetrical pendant drop taking into account intermediate layer. We show that the contact angle between drop's surface and horizontal plane differs from the classical one and depends on the width of the layer and on the radii of the contact circle. When width of the layer is equal to zero our representation of the contact angle reduces to the classical one.
We prove the existence of equilibrium forms of the axisymmetrical drops on the basis of the variational principle.

Keywords. Intermediate layer, mean curvature, Gauß curvature, contact angle, variational principle, generalized derivative, Steiner symmetrization

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1. Formulation of the problem

Let us consider equilibrium state of an axisymmetrical drop pending from the horizontal plane P. Let us denote through S the drop's surface and through S^* its projection onto the plane P. Let Σ be the circle of the contact of the two surfaces S and S^*.

We suppose that the line L generating the surface S is a rectifiable curve whose length is equal to l. We introduce Cartesian coordinates (x, y) in the meridian section of the drop orienting the axis x along the line perpendicular to the plane P and we denote through $w = w(s) = (x(s), y(s))$, $0 \leq s \leq l$, the natural parameterization of the curve L.

In the book [4], the different equilibrium states of the type just described were investigated on the basis of variational principles for energy functional. We will follow the same way but we take into account the intermediate layer between liquid and gas phases.

In the paper [1], on the basis of Gibbs theory, a new condition generalizing the well known Laplace condition was proposed. In the general case, this

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condition has a complicated form depending on the variable width of the intermediate layer, variable potentials C_1, C_2 and potential Φ of the external forces. But when the width of the layer and potentials C_1, C_2 are constant and there are no external forces this formula reduces to the following form

$$J \cdot \gamma + 2K \cdot C_1 = P_2 - P_1. \tag{1}$$

Here, in the denotations of the paper [1], γ is the coefficient of surface tension, J is equal to double mean curvature of the equilibrium surface, K denotes the Gauß curvature, and the difference $P_2 - P_1$ denotes the difference between the pressures in the different phases. When the quantities γ, C_1, C_2 are not constant the condition (1) can also be considered as the approximation of the abovementioned general condition.

In the paper [8], the similar condition of equilibrium was proposed. We will represent here in short terms the simple arguments of the authors of this paper. Let us suppose that, at the moment $\tau = 0$, two phases which we denote as L (liquid) and V (vapor) are divided by the surface A_0.

We will suppose also that, as the result of phase interchange, two new surfaces appear. The boundary of the liquid phase we will denote as A_1 and the boundary of the vapor phase we will denote as A_2.

Let us suppose that A_1 moves into the liquid phase with the velocity ω_1 and A_2 moves into the vapor phase with the velocity ω_2 along the normal $\bar{n}$ of the surface A_0 at the point the deformation whose vicinity we consider. The velocities ω_1, ω_2 tend to zero when time τ tends to infinity so that the following limit exists

$$\lim_{\tau \to \infty} \int_0^\tau (\omega_1 + \omega_2) \cdot \mathrm{d}t.$$

These deformations of the surfaces A_1, A_2 are described by the differentials $\mathrm{d}A_1$, $\mathrm{d}A_2$.

Let $\bar{f} = \bar{f}(u, v)$ be a local representation of the surface A_0 then the local representations of the surfaces A_1, A_2 can be written in the following forms

$$A_1: \ \bar{f}(u, v) - \bar{n} \int_0^\tau \omega_1 \cdot \mathrm{d}t, \quad A_2: \ \bar{f}(u, v) + \bar{n} \int_0^\tau \omega_2 \cdot \mathrm{d}t.$$

Let us "freeze" the boundary A_2 then the relative position of the boundary A_1 to the boundary A_2 can be described by the expression $\bar{f}(u,v) - \bar{n} \cdot \int_0^\tau (\omega_1 + \omega_2) \cdot \mathrm{d}t$.

The following calculations in [8] are based on the thermodynamic definition of the capillary pressure Π

$$\Pi := P_L - P_V = \sigma \frac{\mathrm{d}(A_1 - A_2)}{\mathrm{d}V_{V,L}}. \tag{2}$$

The difference $\mathrm{d}(A_1 - A_2)$ characterizes the variation of the element of the surface A_0 due to the phase interchange and the differential $\mathrm{d}V_{V,L}$ characterizes the variation of the volume of vapor phase (this formula can be written also for the differential of the volume characterizing the variation of the liquid phase, in this case we must put the signal minus in the right part of the equation (2)). Now on application of the Olinde Rodrigues theorem we get that

$$dA_1 - dA_2 = \left[(k_1+k_2)+k_1k_2\int_0^\tau(\omega_1+\omega_2)\cdot \mathrm{d}t\right]\int_0^\tau(\omega_1+\omega_2)\cdot \mathrm{d}t\times\left|\bar{f}_u\times\bar{f}_v\right|\cdot \mathrm{d}u\mathrm{d}v \quad (3)$$

where k_1 and k_2 are the principal curvatures of the surface A_0.

The variation of the volume due to the variation of the element of the surface over the period $[0,\tau]$ can be written in the following form

$$dV_{L,P} = \int_0^\tau (\omega_1+\omega_2)\cdot \mathrm{d}t \times \left|\bar{f}_u\times\bar{f}_v\right|\cdot \mathrm{d}u\mathrm{d}v. \quad (4)$$

Passing to the limit in the formulas (3), (4) and substituting the resulting variations of elements of area and volume into the formula (2) we arrive at the following representation

$$P_L - P_V = \sigma 2H + Kl_p\sigma. \quad (5)$$

Here l_p is equal to $\lim_{\tau\to\infty}\int_0^\tau(\omega_1+\omega_2)\cdot \mathrm{d}t$, H is the mean curvature of the surface A_0, and K is equal to its Gauß curvature.

The formulas (1) and (5) have similar structure. In the sequel we will use the formula (5) in our calculations because it contains explicitly the width of the intermediate layer.

It is necessary to indicate that there are other formulas describing the generalization of the Laplace condition and bearing nonlinear character in order of H (see [2, 6]). Hypothesis adopted in the last paper lead to the disappearance of the linear term of H.

In spite of the critics of the approach realized in the paper [2] by the authors of [1], it seems that, on assuming that the mass distributions of the different phases are quasi independent on each sphere [5], the term including H^2 instead of H may appear.

In what follows we will study the simple version of the general theory described by the equation (5).

We procure the surface S describing the liquid drop pendant from the plane P. Of course the surface S does not coincide with the surface A_0 determined by the condition (5) for the constant λ. But as the width of the intermediate layer is small we suppose that the condition (5) is satisfied on the boundary of the liquid phase for some constant λ^*. Thus in fact we study two phased model. We take into account the intermediate layer by introducing the term $\sigma\cdot l_p\cdot K$ into classical Laplace condition.

Now we proceed with the formulation of the variational principle which will help us to prove the existence of the surfaces satisfying the generalized Laplace condition (5). As it was already said we use the same variational technique that was exposed in the book [4] but now it is necessary for us to include the energy needed for the formation of intermediate layer. It means that the energy functional under its variation must yield not only the mean curvature but the Gauß curvature as well.

Let $\mathbf{S}$ be the area of the surface S, Γ be the function describing the gravitational potential, ρ be fluid's density, β be coefficient of the relative adhesion, W be drop's interior part, and V be the volume of the domain W. Now let us consider the functional F represented as follows

$$F(S) = \sigma\left(\mathbf{S} + l_p\Xi - \beta\int_{S^*} dS + \lambda V + \sigma^{-1}\cdot\iiint_W \Gamma\rho \mathrm{d}V\right). \tag{6}$$

In the formula (6) $\Xi(S)$ is the functional of the type

$$\Xi(S) = 2\pi\int_0^l f\left(\dot{y}\right)\cdot \mathrm{d}s, \quad \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}s}\ , \tag{7}$$

and the function f has the following representation

$$f(\dot{y}) = \frac{\sqrt{1-\dot{y}^2}}{2}\left\{E_0 - \int_0^{\dot{y}}\left(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2}\right)\left(1-\sigma^2\right)^{-\frac{3}{2}}\mathrm{d}\sigma\right\}. \tag{8}$$

It is easy to verify that the function f is a solution of the second order ordinary differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\tau^2}\cdot\sqrt{1-\tau^2} - \frac{\mathrm{d}f}{\mathrm{d}\tau}\frac{\tau}{\sqrt{1-\tau^2}} + f\frac{1}{\sqrt{1-\tau^2}} = -1. \tag{9}$$

The general solution of the equation (9) depends on two independent constants. One of them was chosen to be $-\frac{\pi}{2}$ in order to guarantee the convergence of the integral from the formula (7), the other one, E_0, can be chosen arbitrary.

All the terms in the representation of the functional F for the exception of the term Ξ correspond to the classical case [4]. Later we shall show that the variation of the functional Ξ is conjugated to the rest of the terms in such a way that the Gauß curvature will appear in the Euler necessary condition. Thus the term Ξ may be considered as the amount of the energy needed for the formation of intermediate layer.

Variational problem. Let M be the class of surfaces S described earlier.

For given values of the width l_p of intermediate layer and of the contact angle β and for a given volume V of the domain W (or for a given value of the constant λ) it is necessary to find a surface $S_e \in M$ and a constant λ such that

$$F(S_e) = \inf\left\{F(S), S\in M\right\}.$$

It is clear that in the case when the constant λ is given the volume V will be defined by the extremal element.

In what follows we omit the symbol denoting the extremal surface. The problem just formulated differs from the classical one of the same type [4] only by the term Ξ in the energy functional.

2. Necessary conditions of the extremum

In this section we prove the following lemma.

Lemma 2.1. *Let S be a solution of the variational problem formulated in Section* 1*. Let us suppose that the function $y = y(s)$ from the natural parameterization of its generating line L is twice continuously differentiable over $(0, l)$ and continuously differentiable over $[0, l]$.*

Then the mean and Gauß curvature of the surface S satisfy the equation

$$2H + l_p K = \lambda^*, \quad \lambda^* = \lambda + \frac{1}{\sigma}\Gamma\rho. \tag{10}$$

The contact angle γ and width l_p of the intermediate layer satisfy the following system of equations

$$\lambda = \frac{1}{\pi r^2}\left[2\pi r\cos\gamma + \frac{l_p \pi r^2}{2}\sin^2\gamma - \kappa V\right], \tag{11}$$

$$\cos\gamma - \frac{l_p}{r}\left[\gamma - \frac{\sin 2\gamma}{2}\right] = \beta. \tag{12}$$

The constant r in the equalities (11), (12) *is the radius of the domain S^*. The constant κ is equal to*

$$\kappa = \frac{\rho g}{\sigma}.$$

The equations (10)–(12) *coincide with the known equations from* [4] *when l_p is equal to zero.*

Proof. Let us denote as $[0, x_A]$ the projection of the domain W over x-axis. Let $\bar{T}$ denotes the vector tangential to the curve L and vector $\bar{N}$ the normal vector of it. We suppose both of these vectors to be normalized. We denote as $\bar{\varsigma}$ virtual displacement of the surface S

$$\bar{\varsigma} = \varepsilon\left[\xi(x, y)\bar{N} + \eta(x, y)\bar{T}\right] + o(\varepsilon), \quad \varepsilon \to 0.$$

We will select the functions $\eta = \eta(x) = \eta\left(x, y(x)\right), \ \xi = \xi(x) = \xi\left(x, y(x)\right),$ $x \in [0, x_A]$ in two different ways.

In the first case we will suppose that the supporters of these functions are contained in the interior part of the segment $[0, x_A]$ in a vicinity of an interior point under consideration. In the second case we will suppose that they are symmetrical in order to the point $x = 0$. We suppose also that the functions ξ, η are continuously differentiable ones.

In both cases we will select the displacements of different orders in variable ε in the representation of $\bar{\varsigma}$ to be along the axis x. Then the displacement $\bar{\varsigma}$ can be characterized by the function y^*

$$|\bar{\varsigma}| = y^*(x) = \varepsilon t(x) + o(\varepsilon), \quad \varepsilon \to 0.$$

Let us now calculate the variations of the different terms of the functional F. In the correspondence with the case under consideration we will denote these variations by one of the symbols 1 or 2. Let us start with the calculation of $\delta_1 S$,

$$\begin{aligned}
\delta_1 S &= 2\pi \int_0^{x_A} y_1(s_1) \cdot \mathrm{d}s_1 - 2\pi \int_0^{x_A} y(s) \cdot \mathrm{d}s \\
&= 2\pi \int_0^{x_A} (y(x) + \varepsilon t) \sqrt{1 + y'^2(x) + 2\varepsilon y' \cdot t'(x) + o(\varepsilon)} \cdot \mathrm{d}x \\
&\quad - 2\pi \int_0^{x_A} y(x) \sqrt{1 + y'^2(x)} \cdot \mathrm{d}x + o(\varepsilon) \\
&= 2\pi\varepsilon \int_0^{x_A} y y' t' \frac{\mathrm{d}x}{\sqrt{1 + y'^2}} + 2\pi\varepsilon \int_0^{x_A} t(x) \sqrt{1 + y'^2(x)} \cdot \mathrm{d}x + o(\varepsilon) \\
&= -2\pi\varepsilon \int_0^{l} \frac{\mathrm{d}\,(y\dot{y}\dot{x})}{\mathrm{d}s} t\,(x(s)) \cdot \mathrm{d}s \\
&\quad + 2\pi\varepsilon \int_0^{x_A} t(x) \sqrt{1 + y'^2(x)} \mathrm{d}x + o(\varepsilon) \\
&= -2\pi\varepsilon \int_0^{x_A} \left(\frac{y y'}{\sqrt{1 + y'^2}} \right)' t(x) \cdot \mathrm{d}x \\
&\quad + 2\pi\varepsilon \int_0^{x_A} t(x) \sqrt{1 + y'^2(x)} \cdot \mathrm{d}x + o(\varepsilon) \\
&= -2\pi\varepsilon \int_0^{x_A} 2H(x) y(x) t(x) \cdot \mathrm{d}x + o(\varepsilon), \quad \varepsilon \to 0.
\end{aligned} \tag{13}$$

It is clear that

$$\delta_1 \left(\pi \int_0^{x_A} y^2(x) \cdot \mathrm{d}x \right) = 2\pi\varepsilon \int_0^{x_A} y(x) t(x) \cdot \mathrm{d}x + o(\varepsilon), \quad \varepsilon \to 0. \tag{14}$$

Let us now calculate $\delta_1 \Xi$. When calculating $\delta_1 S$ we have already seen that

$$\frac{\mathrm{d}s}{\mathrm{d}s_1} = 1 - \varepsilon \dot{y} \dot{t}(s) + o\,(\varepsilon)\,.$$

Taking the last equation into account we will get

$$\begin{aligned}\delta_1\Xi &= 2\pi\int_0^{x_A} f\left(\frac{\mathrm{d}}{\mathrm{d}s_1}\left(y+\varepsilon t+o(\varepsilon)\right)\right)\sqrt{1+y'^2+2\varepsilon y't'+o(\varepsilon)}\mathrm{d}x\\
&\quad-2\pi\int_0^{x_A} f\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)\sqrt{1+y'^2}\cdot\mathrm{d}x\\
&= 2\pi\int_0^{l_1}\left\{f\left[\left(\dot y+\varepsilon\dot t+o(\varepsilon)\right)\left(1-\varepsilon\dot y\dot t\right)\right]-f\left(\dot y\right)\right\}\cdot\mathrm{d}s_1\\
&\quad+2\pi\int_0^{l_1} f\left(\dot y\right)\cdot\mathrm{d}s_1-2\pi\int_0^{l} f\left(\dot y\right)\cdot\mathrm{d}s\\
&= 2\pi\varepsilon\int_0^{l}\left[f_{\dot y}\dot x^2+f\left(\dot y\right)\dot y\right]\dot t(s)\cdot\mathrm{d}s+o(\varepsilon),\quad \varepsilon\to 0.\end{aligned}\tag{15}$$

The function f satisfies the equation (9). Thus integrating the last expression in (15) by parts and taking into account the equation (9) we arrive at the expression

$$\delta_1\Xi = 2\pi\varepsilon\int_0^{x_A}\ddot y(x)t(x)\cdot\mathrm{d}x+o(\varepsilon) = -2\pi\varepsilon\int_0^{x_A}K(x)y(x)t(x)\cdot\mathrm{d}x+o(\varepsilon),\tag{16}$$

Now the variations of the last terms of the functional F can be easily calculated. It is clear that in the first case we get that

$$\delta_1\left(\int_{S^*}\mathrm{d}s\right)=0\tag{17}$$

and

$$\delta_1\left(\iiint_W\Gamma\rho\cdot\mathrm{d}v\right)=2\pi\varepsilon\int_0^{x_A}\Gamma\rho t(x)\cdot\mathrm{d}x+o(\varepsilon),\quad\varepsilon\to 0.\tag{18}$$

Taking into account the equalities (13)–(18) we finally obtain

$$\begin{aligned}\delta_1 F = \varepsilon\sigma\Bigg\{&-2\pi\int_0^{x_A}2H(x)y(x)t(x)\cdot\mathrm{d}x\\
&-2\pi\int_0^{x_A}K(x)y(x)t(x)\cdot\mathrm{d}x+2\pi\lambda\int_0^{x_A}y(x)t(x)\cdot\mathrm{d}x\\
&+\frac{1}{\sigma}2\pi\int_0^{x_A}\Gamma\rho t(x)\cdot\mathrm{d}x\Bigg\}+o(\varepsilon),\quad\varepsilon\to 0.\end{aligned}\tag{19}$$

As the function $t=t(x)$ in the representation (19) is an arbitrary continuously differentiable function we arrive at the following condition

$$2H\left(x,y(x)\right)+l_pK\left(x,y(x)\right)=\lambda+\frac{1}{\sigma}\Gamma\rho.\tag{20}$$

This condition is satisfied at all the points of the interval $(0, x_A)$. If we put the width of the intermediate layer equal to zero than we get the well known condition from [4]

$$2H\left(x, y(x)\right) = \lambda + \frac{1}{\sigma}\Gamma\rho.$$

Now we deduce a condition for the contact angle γ between S and P. To this end we calculate the variation of the functional F the supporters of ξ and η being in the neighborhood $[0, \delta]$ of zero. As in the first case we have

$$y_2(x) = y(x) + \varepsilon t(x) + o(\varepsilon), \quad \varepsilon \to 0. \tag{21}$$

It is necessary to mention here that the function t now satisfies the inequality $t(0) \neq 0$. Thus we have

$$\delta_2 S = 2\pi\varepsilon \int_0^{x_A} yy't' \frac{\mathrm{d}x}{\sqrt{1+y'^2}} + 2\pi\varepsilon \int_0^{x_A} t(x)\sqrt{1+y'^2}\,\mathrm{d}x + o(\varepsilon), \quad \varepsilon \to 0. \tag{22}$$

Integrating again the first integral by parts we get

$$\delta_2 S = -2\pi\varepsilon \left[\int_0^{\delta} 2H(x)y(x)t(x)dx + \frac{y(0)y'(0)}{\sqrt{1+y'^2(0)}}t(0)\right] + o(\varepsilon). \tag{23}$$

In the same way we arrive at the following expression

$$\begin{aligned} \delta_2 \Xi &= 2\pi\varepsilon \int_0^{l_\delta} \left[f_{\dot y}\dot x^2 + f\left(\dot y\right)\dot y\right] \dot t(s) \cdot \mathrm{d}s + o(\varepsilon) \\ &= 2\pi\varepsilon \int_0^{\delta} \ddot y(x)t(x) \cdot \mathrm{d}x + 2\pi\varepsilon \left[f_{\dot y}\dot x^2 + f\left(\dot y\right)\dot y\right] \dot t(0) + o(\varepsilon). \end{aligned} \tag{24}$$

Here l_δ is the length of the arc connecting the points $(0, r)$ and $(\delta, y(\delta))$.

The variations

$$\delta_2\left(\pi \int_0^{x_A} y^2(x) \cdot \mathrm{d}x\right), \quad \delta_2\left(\iiint_W \Gamma\rho \cdot \mathrm{d}v\right)$$

in the case under consideration are defined by the expressions similar to that of the formulas (14) and (17) corresponding to the first case

$$\delta_2\left(\pi \int_0^{x_A} y^2(x) \cdot \mathrm{d}x\right) = 2\pi\varepsilon \int_0^{\delta} y(x)t(x) \cdot \mathrm{d}x + o(\varepsilon), \tag{25}$$

$$\delta_2\left(\iiint_W \Gamma\rho \cdot \mathrm{d}v\right) = 2\pi\varepsilon \int_0^{\delta} \Gamma\rho y(x)t(x) \cdot \mathrm{d}x + o(\varepsilon), \quad \varepsilon \to 0. \tag{26}$$

Now we easily get that

$$\delta_2\left(\int_{S^*} \mathrm{d}S\right) = 2\pi\varepsilon t(0)r + o(\varepsilon), \quad \varepsilon \to 0. \tag{27}$$

Using the equations (22)–(24) and (25)–(27) and arbitrariness of δ we arrive at the following condition for the contact angle γ

$$-r\cos\gamma t(0) + l_p\left[f_\tau\left(\cos\gamma\right)\sin^2\gamma + f\left(\cos\gamma\right)\right]t(0) + \beta r t(0) = 0. \tag{28}$$

Let us make some rearrangements in the equality (28). To this end we use the equation (9) written in the following form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[f_\tau\left(1-\tau^2\right) + f\left(\tau\right)\tau\right] = -\sqrt{1-\tau^2}. \tag{29}$$

It follows from the equation (29) that

$$f_\tau(\tau)\left(1-\tau^2\right) + f(\tau)\tau = \frac{\arccos\tau}{2} - \frac{\tau\sqrt{1-\tau^2}}{2}. \tag{30}$$

From the equations (28), (30) we now obtain

$$\cos\gamma - \frac{l_p}{r}\left[\gamma - \frac{\sin 2\gamma}{2}\right] = \beta. \tag{31}$$

We get classical condition from (31) if we put l_p equal to zero [4].

Now let us calculate the multiplier λ. To this end we integrate the equation (20) over the disk S^* and thus get

$$2\pi\int_0^r 2Hy\cdot\mathrm{d}y - 2\pi\int_0^r \ddot{y}\cdot\mathrm{d}y = \pi r^2 + \frac{\rho g}{\sigma}V. \tag{32}$$

It can be easily proved that $\dot{x}(y)|_{y=0} = 0$. The last condition implies that

$$\int_0^r \ddot{y}\cdot\mathrm{d}y = \int_0^L \ddot{y}\dot{y}\cdot\mathrm{d}s = \frac{\cos^2\gamma - 1}{2}. \tag{33}$$

It is known [4] that

$$\int_0^r\int_0^{2\pi} 2Hy\cdot\mathrm{d}y\mathrm{d}\varphi = \int_\Sigma \cos\gamma\cdot\mathrm{d}s = 2\pi r\cos\gamma. \tag{34}$$

It follows from the equations (32)–(34) that

$$2\pi\left[r\cos\gamma - l_p\frac{\cos^2\gamma - 1}{2}\right] = \pi r^2\lambda + \kappa V, \tag{35}$$

where $\kappa = (\rho g)/\sigma$. We now have that the multiplier λ is equal to

$$\lambda = \frac{1}{\pi r^2}\left[2\pi r\cos\gamma + \frac{l_p\pi r^2}{2}\sin^2\gamma - \kappa V\right]. \tag{36}$$

We again obtain the classical condition (see [4]) when l_p is equal to zero. □

3. Existence of the solution of variational problem

Here we prove the existence of the solution of the variational problem with the properties described in the Lemma 2.1.

Theorem 3.1. *Let $N = N(t)$, $f_0 = f_0(\sigma)$ be the functions defined as follows*

$$N = N(t) = \frac{\sqrt{2}}{2} \frac{1-t}{\sqrt{1+t^2}},$$

$$f_0(\sigma) = -\left(\arcsin \sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2}\right)\left(1-\sigma^2\right)^{-\frac{3}{2}}, \quad \sigma \in (0,1).$$

Let c_0 be the following constant

$$c_0 = \sup\left\{3\left(1-t^2\right)\frac{f_0\left(N(t)\right)}{2\left(1+t^2\right)^{\frac{5}{2}}} - \sqrt{2}(1+t)^3\frac{f'\left(N(t)\right)}{4(1+t)^3},\ t \in (-1,1)\right\}.$$

Let us suppose that the following inequality takes place

$$2 - c_0 l_p > 0. \tag{37}$$

Then there exist a solution of the variational problem from the Section 1.

Proof. We can select the constant E_0 in such a way that the values of the functional F should be bounded from below.

Let $\{S_n\}$ be minimal sequence for the problem under consideration. The values of the functional F do not increase under Steiner symmetrization of the domains [3]. It means that the lines L_n generating the surfaces S_n may be considered as monotone ones. It implies that these lines are the graphs of the monotone functions

$$y_n = y_n(x),\ x \in \left[0, x_A^n\right], \quad y_n\left(x_A^n\right) = 0, \quad y_n\left(0\right) = r_n.$$

As the sequences $\{x_A^n\}$, $\{r_n\}$ are evidently upper-bounded we arrive at the conclusion that the sequence $\{y_n\}$ of the functions y_n is compact in the sense of uniform convergence [7].

Let $y_e = y_e(x)$ be the limit of a convergent subsequence of $\{y_n\}$, and S_e be a surface of rotation generated by the graph of the function $y_e = y_e(x)$. On the assumption defined by the inequality (37) the functional F is semi-continuous from below on the set M of admissible surfaces [3]. It means that $F(S_e) = \inf\left\{F(S), S \in M\right\}$. □

Now we can prove the main theorem.

Theorem 3.2. *Let $y = y(s) = y(x(s))$, $s \in [0, l]$, be the function whose graph is given by the line L generating the extremal surface S. Than this function is twice differentiable over $(0, l)$ and continuously differentiable over $[0, l]$ and the condition (20) is satisfied with the constant λ defined by the expression (36). Besides the contact angle γ and width l_p of the intermediate layer satisfy the equation (35). The following equality also takes place*

$$\dot{y}(x_A) = 1. \tag{38}$$

Proof. We can write the variation $\delta_1 F$ corresponding to the variation of $y = y(s)$ in the neighborhood of the point $(x, y) \in S$ in the following form

$$\begin{aligned} \delta_1 F = 2\pi\varepsilon \int_0^l y\dot{y}\dot{t} \cdot \mathrm{d}s + 2\pi l_p \varepsilon \int_0^l \left[f_{\dot{y}}\dot{x}^2 + f(\dot{y})\dot{y}\right] \dot{t} \cdot \mathrm{d}s \\ + 2\pi\varepsilon\lambda \int_0^l y\dot{x}t \cdot \mathrm{d}s + 2\pi\varepsilon \int_0^l t \cdot \mathrm{d}s + \frac{2\pi}{\sigma} \int_0^l \Gamma\rho y t\dot{x} \cdot \mathrm{d}s + o(\varepsilon), \end{aligned} \tag{39}$$

$\varepsilon \to 0$. We get now from the equation (39) the following necessary condition for the surface S to be extremal

$$\int_0^l \left\{y\dot{y} + l_p \left[f_{\dot{y}}\dot{x}^2 + f(\dot{y})\dot{y}\right]\right\} \dot{t} \cdot \mathrm{d}s + \int_0^l (\lambda y\dot{x} + 1 + \kappa y x\dot{x})\, t \cdot \mathrm{d}s = 0. \tag{40}$$

The equation (40) means that the function $y\dot{y} + l_p \left[f_{\dot{y}}\dot{x}^2 + f(\dot{y})\dot{y}\right]$ has the generalized derivative equal to the following function

$$\lambda y\dot{x} + \kappa y x\dot{x} + 1.$$

It means that in the neighborhood of the point (x_0, y_0), $x_0 = x(s_0)$, $y_0 = y(s_0)$ where the derivative $\dot{y}(s_0)$ exists the following representation takes place

$$y\dot{y} + l_p \left[f_{\dot{y}}\dot{x}^2 + f(\dot{y})\dot{y}\right] = \int_{s_0}^s (\lambda y\dot{x} + \kappa y x\dot{x} + 1) \cdot \mathrm{d}s + y_0\dot{y}_0. \tag{41}$$

We denote by $\Psi = \Psi(s)$ the integral in the right part of the expression (41) and by $\Phi = \Phi(s, t)$ the expression

$$\Phi(s, \tau) = y(s_0)\tau + \dot{y}(s_0)(s - s_0)\tau + o(s - s_0) - l_p \int_{\tau_0}^{\tau} \sqrt{1 - \sigma^2} \cdot \mathrm{d}\sigma - \Psi(s) - y_0\dot{y}_0.$$

Let us now consider the following equation

$$\Phi(s, \tau) = 0. \tag{42}$$

It is clear that the function $\dot{y} = \dot{y}(s)$ satisfies the equation $\Phi(s, \dot{y}(s)) = 0$ for almost all the values of the parameter s.

We will use the equation (42) in order to prove that the function $\dot{y} = \dot{y}(s)$ is continuous over the interval $[0, l]$.

We see that

$$\frac{\partial \Phi}{\partial t}(s_0, t_0) = y_0 - l_p\sqrt{1 - \tau_0^2} > 0, \quad y_0 > l_p. \tag{43}$$

Since the derivative $\frac{\partial \Phi}{\partial s}(s_0, t_0)$ is positive over the interval $(0, l^*)$, $y(l^*) = l_p$, the equation (42) can be solved locally in the class of continuous functions for almost all the values of the natural parameter varying in the interval $(0, l^*)$. It means that the function $\dot{y} = \dot{y}(s)$ is continuous in the neighborhoods of almost all the points $s_0 \in (0, l^*)$. It implies that the second derivative $\ddot{y}(s_0)$ exists almost everywhere on the interval $(0, l^*)$ (see [3]). Using the equation (40) we arrive at the following equality

$$\ddot{y} \cdot \left(1 - \frac{l_p}{y}\right) = \frac{\dot{x}}{y} + \lambda + \kappa \dot{x} \tag{44}$$

for almost all the points of the interval $(0, l^*)$.

Taking into account the equation (41) we get that the function $\ddot{y} = \ddot{y}(s)$ is bounded over $(0, l^*) \subset [0, l]$ and that it is also continuous in the neighborhoods of the points $s_0 \in (0, l^*)$ where the derivatives $\dot{y}(s_0)$, $\ddot{y}(s_0)$ exist.

On the basis of the properties of the functions $\dot{y} = \dot{y}(s)$, $\ddot{y} = \ddot{y}(s)$ we will evaluate now the dimensions of the neighborhoods of the points $s_0 \in (0, l^*)$ where the derivatives $\dot{y}(s_0)$, $\ddot{y}(s_0)$ exist. We are going to prove that these neighborhoods overlap which will imply that the functions $\dot{y} = \dot{y}(s)$, $\ddot{y} = \ddot{y}(s)$ are continuous over the interval $(0, l^*)$.

Let $s_0 \in (0, l^*)$ be a point where the derivative$\tau_0 = \dot{y}(s_0)$ exists. The point (s_0, τ_0) is a solution of the equation (42). In accordance with the condition (43) we get that the function $\Phi_0(t) = \Phi(s_0, t)$ is increasing on the line R. Thus for any $\varepsilon > 0$ we get

$$\Phi(s_0, \tau_0 - \varepsilon) < 0 < \Phi(s_0, \tau_0 + \varepsilon),$$

$$\begin{aligned} \Phi(s_0, \tau_0 \pm \varepsilon) &= y_0(\tau_0 \pm \varepsilon) - l_p \int_{\tau_0}^{\tau_0 \pm \varepsilon} \sqrt{1 - \sigma^2} \cdot \mathrm{d}\sigma - y_0 \tau_0 \\ &= \pm y_0 \varepsilon - l_p \int_{\tau_0}^{\tau_0 \pm \varepsilon} \sqrt{1 - \sigma^2} \cdot \mathrm{d}\sigma. \end{aligned}$$

Now let us consider $\Phi(s, t_0 \pm \varepsilon)$ in the neighborhood of the point $s_0 \in (0, l^*)$. We see that

$$\begin{aligned} \Phi(s, \tau_0 \pm \varepsilon) &= y_0 \cdot (\tau_0 \pm \varepsilon) + (\tau_0 \pm \varepsilon)(s - s_0)\dot{y}(s) \\ &\quad - l_p \int_{\tau_0}^{\tau_0 \pm \varepsilon} \sqrt{1 - \sigma^2} \cdot \mathrm{d}\sigma - \Psi(s) - y_0\tau_0 + o(s - s_0)(\tau_0 \pm \varepsilon) \\ &= \Phi(s_0, \tau_0 \pm \varepsilon) + \Psi(s) + (\tau_0 \pm \varepsilon)(s - s_0)\dot{y}(s) \\ &\quad + o(s - s_0)(\tau_0 \pm \varepsilon). \end{aligned} \tag{45}$$

Using the evident identity

$$y(s) - y(s_0) = \dot{y}(s_0)(s - s_0) + (s - s_0) \int_{s_0}^{\sigma} \ddot{y}(s) \cdot \mathrm{d}s, \quad s < \sigma < s_0,$$

we can now evaluate the term $o(s - s_0)$ from the equation (45) in the following way

$$|o(s - s_0)| < \sup \left\{ |\ddot{y}(s)| \, (s - s_0)^2, \quad s \in (0, l^*) \right\}. \tag{46}$$

The inequality (46) implies that we can select ε^* sufficiently small so that for any s_0, $0 < s_0 < l^*$, and any $(s, t) \in [s_0 - \varepsilon^*, s_0 + \varepsilon^*] \times [\tau_0 - \varepsilon, \tau_0 + \varepsilon]$ the following conditions take place

$$\Phi(s, \tau_0 + \varepsilon)\Phi(s_0, \tau_0 + \varepsilon) > 0, \tag{47}$$

$$\Phi(s, \tau_0 - \varepsilon)\Phi(s_0, \tau_0 - \varepsilon) > 0. \tag{48}$$

These conditions guarantee the solvability of the equation (42) in terms of $\tau = \tau(s)$ in the class of the continuous functions in ε^*-neighborhoods of almost all the points of the interval $(0, l^*)$. The ε^*-neighborhoods that we have constructed are overlapping. It means that the function $\dot{y} = \dot{y}(s)$ is continuous over the interval $(0, l^*)$.

Using the equation (44) we get that the function $\ddot{y} = \ddot{y}(s)$ is continuous and bounded over the interval $(0, l^*)$. It implies that the function $\dot{y} = \dot{y}(s)$ is continuous over the interval $[0, l^*]$.

Let us now consider the segment $[l^*, l]$. It is clear that at the point (s_0, τ_0) such that $\frac{\partial \Phi}{\partial t}(s_0, \tau_0) = 0$, $l > s_0 > l^*$, the value of τ_0 differs from zero and unity.

Let us now consider an arbitrary sequence $\{s_n\}$, $\lim_{n \to \infty} s_n = s_0$. Let $\{\dot{y}_m\}$ be a convergent subsequence of the bounded sequence $\{\dot{y}_n\}$, $\dot{y}_n = \dot{y}(s_n)$, and $\tau_1 = \lim_{n \to \infty} y_m$. Using the equation (31) we get

$$y_0 \left(\dot{y}_m - \tau_0 \right) - l_p \int_{\tau_0}^{\dot{y}_m} \sqrt{1 - \sigma^2} \cdot \mathrm{d}\sigma = \Phi(s) + o(s - s_0). \tag{49}$$

Passing to the limit in the expression (49) with s_m tending to s_0 we arrive at the condition

$$(\tau_1 - \tau_0) = \frac{1}{\sqrt{1 - \tau_0^2}} \int_{\tau_0}^{\tau_1} \sqrt{1 - \sigma^2} \cdot \mathrm{d}\sigma. \tag{50}$$

The equation (50) is valid only in the case when $\tau_1 = \tau_0$. Really the following equation

$$\tau_1 - \tau_0 = \frac{1}{\sqrt{1 - \tau_0^2}} \sqrt{1 - \tau_0^2} (\tau_1 - \tau_0) - \frac{\tau_*^2}{\sqrt{1 - \tau_*^2}} (\tau_1 - \tau_0)^2 \tag{51}$$

takes place with the point τ_* situated between τ_0 and τ_1. The equation (50) shows that the point τ_1 is not equal to zero. As τ_* differs from zero in this case we get a contradiction. Thus in fact $\tau_1 = \tau_0$ and we get the continuity of the function $\dot{y} = \dot{y}(s)$ at the points of the set $D \subset (l^*, l)$ consisting of the points of differentiability of the function $\dot{y} = \dot{y}(s)$.

Let us now consider a point $s_1 \in D^c = (l^*, l)\backslash D$. We can extend the function $\dot{y}$ at the point (x_1, y_1), $x_1 = x(s_1)$, $y_1 = y(s_1)$ as equal to $\sqrt{l_p^2 - y_1^2}$. The function $\dot{y}$ thus extended will be continuous over the interval (l^*, l). In order to prove this we need only to confirm that

$$\lim_{n\to\infty} \dot{y}(s_n) = \sqrt{l_p^2 - y_1^2}, \quad \lim_{n\to\infty} y(s_n) = y_1, \quad \frac{\partial \Phi}{\partial t}(s_n) \neq 0.$$

It is clear that it is sufficient to consider the case when $\lim_{n\to\infty} \frac{\partial \Phi}{\partial t}(s_n) = 0$. As the derivative $\frac{\partial \Phi}{\partial t}$ of the function Φ at the point (s_n) is equal to $y(s_n) - l_p\sqrt{1 - \dot{y}^2(s_n)}$, and the points $y(s_n)$ tend to $y(s_1)$ we get that the points $\dot{y}(s_n)$ tend to the point $\sqrt{l_p^2 - y_1^2}$. This means that the extended function is continuous over (l^*, l).

As in the case of the interval $(0, l^*)$ we now get that the function $\dot{y}$ is continuous over $[l^*, l]$.

Taking into account the proven continuity of the function $\dot{y}$ over the sets $[0, l^*]$ and $[l^*, l]$ we get the continuity of this function over $[0, l]$. From the equation (44) we now get that the function $\ddot{y}$ is continuous over $(0, l)$.

Let us now show that $\dot{x}(l) = 0$. Suppose that it is not so. Than we have that $\dot{y}(l) \neq 1$. It means that the function $x = x(y)$ is differentiable at the point zero and its derivative is different from zero. We get a contradiction as the function $x = x(y)$ achieves its maximum value there. It means that the condition (38) is satisfied.

Thus we get that the function $y = y(s)$ satisfies all the conditions of regularity of Lemma 2.1. It means that it is a solution of the variational problem which has all the properties declared in the formulation of Theorem 3.2. $\square$

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