Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 31 (2012), 55–74
DOI: 10.4171/ZAA/1448

On Equivalent Conditions for the General Weighted Hardy Type Inequality in Space $L^{p(\cdot)}$

Farman I. Mamedov and Yusuf Zeren

Abstract. We study the Hardy type two-weighted inequality for the multidimensional Hardy operator in the norms of generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. In this way we prove equivalent conditions for $L^{p(\cdot)} \to L^{q(\cdot)}$ boundedness of Hardy operator in the case of exponents $q(0) \geq p(0)$, $q(\infty) \geq p(\infty)$. We also prove that the condition for such inequality to hold coincides with condition for validity of two weighted Hardy inequalities with constant exponents, if we require the exponents to be regular near zero and at infinity.

Keywords. Hardy operator, Hardy inequality, variable exponents, weighted inequality.

Mathematics Subject Classification (2010). Primary 42B26, secondary 46E30

1. Introduction

In this paper, we derive the Hardy type two-weighted inequality,

$$\left\| v^{\frac{1}{q(x)}} Hf(x) \right\|_{q(\cdot)} \leq C \left\| \omega^{\frac{1}{p(x)}} f(x) \right\|_{p(\cdot)}, \quad Hf(x) = \int_{\{t \in \mathbb{R}^n : |t| \leq |x|\}} f(t)dt. \tag{1.1}$$

in norms of generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$.

We prove that the validity of inequality (1.1) coincides with the validity of two ordinary weighted Hardy inequalities with constant exponent. In fact, we prove equivalent conditions for the inequality (1.1) to hold when $q(0) \geq p(0)$, $q(\infty) \geq p(\infty)$. To characterize the behavior of exponents, in our considerations we use a modified log conditions near zero and infinity.

F. I. Mamedov: Institute Mathematics and Mechanics of N.A.S., Republic of Azerbaican; farman-m@mail.ru
Y. Zeren: Department of Mathematics, Yıldız Technical University, Republic of Turkey; yusufzeren@hotmail.com

With a recent introduction of the ideas and approaches of Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ to the theory of integral operators and to the theory of non-Newtonian systems (see survey, e.g. [2,3,5,9,20]) it has become possible to consider Hardy type inequalities with a variable Lebesgue exponent. Recently there have been quite a number of papers discussing Hardy inequality in norms of $L^{p(\cdot)}(\mathbb{R}^n)$ spaces (see [1,4,6,7,10,12–15,19,21,22]). For constant exponents the Hardy inequality is a classical topic (see, e.g. [11,16,23]). It follows from the classical results that for $1 < p \le q < \infty$ the case reduces to a one-parameter problem on maximum and to the convergence of an improper integral for $0 < q < p$, $1 < p < \infty$. Note that different criteria are available for characterizing of the same inequality (see [17,18,23]). These criterions have proved their usefulness in our investigation.

2. Auxiliary statements, definitions and notation

Denote $p^+ = \sup_{x\in\mathbb{R}^n} p(x)$, $p^- = \inf_{x\in\mathbb{R}^n} p(x)$ for measurable functions $p : \mathbb{R}^n \to \mathbb{R}$. Denote by χ_E characteristic function of set $E \subset \mathbb{R}^n$. By P we denote the set of measurable functions $p(x)$ defined on $\mathbb{R}^n$ which satisfy the condition $0 < p^- \le p(x) \le p^+ < \infty$. By $L^{p(\cdot)}(\mathbb{R}^n)$ we denote the Banach space of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\}, \quad p^- \ge 1,$$

where the modular $I_p(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx$. For basic properties of the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ we refer to [8].

The weight functions $v(x)$, $\omega(x)$ are assumed to be measurable and having nonnegative finite values almost everywhere in $\mathbb{R}^n$. For weight functions we assume the following conditions:

$$\sigma = \omega(x)^{-\frac{1}{p(x)-1}} \in L^1(B(0,a)), \quad v(x) \in L^1\big(\mathbb{R}^n \backslash B(0,a)\big)$$

for any $a > 0$. We use the notation

$$V(x) = \int_{|y|>|x|} v(y)dy; \quad W(x) = \int_{|y|<|x|} \sigma(y)dy.$$

It is clear that the function $V(x)$ and $W(x)$ are radial, i.e., depend only on $|x|$. We denote them by $\widetilde{V}(|x|)$ and $\widetilde{W}(|x|)$ correspondingly, where the upper symbol $\tilde{}$ denotes a function of one variable. For these functions we suppose the conditions

$$\widetilde{V}(0) = \infty, \ \widetilde{W}(\infty) = \infty. \tag{2.1}$$

Let $0 < m < 1$ be such that $\widetilde{W}(m) < 1$, $\widetilde{V}(m) > 1$. Let also $M > 1$ be such that $\widetilde{W}(M) > 1$, $\widetilde{V}(M) < 1$. Denote by Λ_0 the class of functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the condition:

$$\exists f(0) \in \mathbb{R} : \sup_{|x|<m} |f(x) - f(0)| \ln \frac{1}{\widetilde{W}(|x|)} \le C_1, \tag{2.2}$$

and by Λ_∞ - the condition:

$$\exists f(\infty) \in \mathbb{R} : \sup_{|x|>M} |f(x) - f(\infty)| \ln \frac{1}{\widetilde{V}(|x|)} \le C_2. \tag{2.3}$$

We denote by C the positive constants $C, C_3, C_4, C_5, \ldots$ which may depend only on the constants n, m, M, p^+, p^-, q^+, q^-, C_1, C_2, $\widetilde{V}(m)$, $\widetilde{V}(M)$, $\widetilde{W}(m)$, $\widetilde{W}(M)$.

We write $g(x) \sim f(x)$ if there exist constants C_3 and C_4 such that $C_3 f(x) \le g(x) \le C_4 f(x)$.

We use the following technical lemma.

Lemma 2.1. *Let $s \in P$ and $s \in \Lambda_0$. Then*

$$e^{-c_1} W(x)^{s(0)} \le W(x)^{s(x)} \le e^{c_1} W(x)^{s(0)}, \quad |x| \le m. \tag{2.4}$$

Let $s \in P$ and $s \in \Lambda_\infty$. Then

$$e^{-c_2} V(x)^{s(\infty)} \le V(x)^{s(x)} \le e^{c_2} V(x)^{s(\infty)}, \quad |x| \ge M. \tag{2.5}$$

Proof. To prove Lemma 2.1, for example (2.4), it suffice to rewrite the inequality (2.4) in the form $e^{-c_1} \le W(x)^{s(x)-s(0)} \le e^{c_1}$, hence

$$-C_1 \le [s(x) - s(0)] \ln W(x) \le C_{1,}$$

which coincides with the condition (2.2). □

When proving our results, we use the following known Hardy inequality results for constant exponents [22, 23].

Theorem 2.2. *Let $0 \le a < b \le \infty$, $1 < p_1 \le q_1 < \infty$ and $v_1, \omega_1 : \mathbb{R}^n \to [0, \infty)$ be positive measurable functions so that*

$$\sigma_1 = \omega_1^{-\frac{1}{p-1}} \in L^1(B(0,t)); \quad v_1 \in L^1\big(B(0,b) \backslash B(0,t)\big)$$

for any $a < t < b$. Let $f_1(x) \ge 0$ be any measurable function. Then the following conditions are equivalent:

(1) *There is a positive constant* $C>0$ *such that the inequality*

$$\begin{aligned}&\left(\int_{a<|x|<b}\left(\int_{a<|y|<|x|} f_1(y)dy\right)^{q_1} v_1(x)dx\right)^{\frac{1}{q_1}}\\&\leq C\left(\int_{a<|x|<b} f_1(x)^{p_1}\omega_1(x)dx\right)^{\frac{1}{p_1}}\end{aligned} \tag{2.6}$$

holds.

(2) *The condition*

$$A_M(p_1,q_1)=\sup_{a<t<b}\left(\int_{t<|x|<b} v_1(x)dx\right)^{\frac{1}{q_1}}\left(\int_{a<|x|<t}\omega(x)^{\frac{-1}{p_1-1}}dx\right)^{\frac{1}{p_1'}}<\infty \tag{2.7}$$

holds.

(3) *The condition*

$$\begin{aligned}&A_{PS}(p_1,q_1)\\&=\sup_{a<t<b}\left(\int_{a<|x|<t} v_1(x)\left(\int_{a<|y|<|x|}\sigma_1(y)dy\right)^{q_1}dx\right)^{\frac{1}{q_1}}\left(\int_{a<|x|<t}\sigma_1(x)dx\right)^{-\frac{1}{p_1}}\\&<\infty\end{aligned} \tag{2.8}$$

holds.

(4) *The condition*

$$\begin{aligned}&A_{MK}(p_1,q_1)\\&=\sup_{a<t<b}\left(\int_{t<|x|<b}\left(\int_{|x|<|y|<b} v_1(y)dy\right)^{p_1'}\sigma_1(x)dx\right)^{\frac{1}{p_1'}}\left(\int_{t<|y|<b} v_1(y)dy\right)^{-\frac{1}{q_1'}}\\&<\infty\end{aligned} \tag{2.9}$$

holds.

Moreover, the best constant $C>0$ *in* (2.6) *is estimated as*

$$C\sim A_M(p_1,q_1)\sim A_{PS}(p_1,q_1)\sim A_{MK}(p_1,q_1). \tag{2.10}$$

3. Main result

Main result of the paper is the following statement.

Theorem 3.1. . *Let* $p,q\in\Lambda_0\cap\Lambda_\infty\cap P$. *Let* $f(x)\geq 0$ *be a measurable function and suppose that*

$$p^->1,\ q(0)\geq p(0),\ q(\infty)\geq p(\infty).$$

Then the following conditions are equivalent:

(1) *There is a positive constant C which depends only n, m, M, p^+, p^-, q^+, q^-, C_1, C_2, $\widetilde{V}(m)$, $\widetilde{V}(M)$, $\widetilde{W}(m)$, $\widetilde{W}(M)$ such that the weighted norm inequality*

$$\left\| v^{\frac{1}{q(x)}} Hf(x) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C \left\| \omega^{\frac{1}{p(x)}} f(x) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$

holds for any f.

(2) *There exist positive constants C_3 and C_4 such that both the weighted inequality*

$$\left(\int_{B(0,m)} \left(\int_{|y|<|x|} f(y)dy \right)^{q(0)} v(x)dx \right)^{\frac{1}{q(0)}} \leq C_3 \left(\int_{B(0,m)} f(x)^{p(0)} \sigma^{1-p(0)} dx \right)^{\frac{1}{p(0)}}$$

and

$$\left(\int_{|x|>M} \left(\int_{M<|y|<|x|} f(y)dy \right)^{q(0)} v(x)dx \right)^{\frac{1}{q(0)}} \leq C_4 \left(\int_{|x|>M} f(x)^{p(0)} \sigma^{1-p(0)} dx \right)^{\frac{1}{p(0)}}$$

hold for any f.

(3) *Both the condition*

$$\sup_{0<t<m} \widetilde{V}(t)^{\frac{1}{q(0)}} \cdot \widetilde{W}(t)^{\frac{1}{p'(0)}} \leq C_5 < \infty \tag{3.1}$$

and

$$\sup_{t>M} \widetilde{V}(t)^{\frac{1}{q(\infty)}} \cdot \widetilde{W}(t)^{\frac{1}{p'(\infty)}} \leq C_6 < \infty. \tag{3.2}$$

hold, where the positive constants C_5 and C_6 do not depend on t.

(4) *The condition*

$$\widetilde{V}(x)^{\frac{1}{q(x)}} \cdot \widetilde{W}(x)^{\frac{1}{p'(x)}} \leq C_7 < \infty; \quad x \in \mathbb{R}^n.$$

holds, where the positive constant C_7 does not depends on x.

3.1. Proof of (3) $\Rightarrow$ (1). Let $f(x) \geq 0$ be an arbitrary function satisfying the condition $\left\| f\omega_{p(\cdot)}^{\frac{1}{p}} \right\|_{p(\cdot)} \leq 1$. Then

$$I_{p(\cdot)} \left(f\omega^{\frac{1}{p}} \right) \leq 1. \tag{3.3}$$

To prove (3) $\Rightarrow$ (1) we have to establish the inequality $I_{q(\cdot)} \left(v^{\frac{1}{q}} Hf \right) \leq C$. Proof of this inequality takes three steps.

Step 1. Estimation near the origin.
Let $x \in B(0,m)$ be an arbitrary fixed point. Denote $p_x^- = \inf_{t\in B(0,|x|)} p(t)$. We write the Hardy operator as sum of two summand with "largest" and "smallest" parts:

$$\int_{B(0,|x|)} f(t)dt = \int_{B(0,|x|)} f\chi_{\frac{f(t)}{\sigma(t)}\geq 1}dt + \int_{B(0,|x|)} f\chi_{\frac{f(t)}{\sigma(t)}<1}dt. \tag{3.4}$$

Estimation of "largest" part. Using (3.3) and Hölder inequality, we have

$$\begin{aligned}\int_{B(0,|x|)} f\chi_{\frac{f}{\sigma}\geq 1}dt &= \int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)\chi_{\frac{f}{\sigma}\geq 1}\sigma(t)dt \\ &\leq \int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)^{\frac{p(t)}{p_x^-}} \sigma dt \\ &\leq \left(\int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)^{p(t)} \sigma dt\right)^{\frac{1}{p_x^-}} \left(\int_{B(0,|x|)} \sigma dt\right)^{\frac{1}{p_x^-}} \\ &\leq \left(\int_{B(0,|x|)} \sigma dt\right)^{\frac{1}{p_x^-}}\end{aligned} \tag{3.5}$$

Using (3.5) we have that

$$\begin{aligned}\left(\int_{B(0,|x|)} f\chi_{\frac{f}{\sigma}\geq 1}dt\right)^{q(x)} &\leq \left(\int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)^{\frac{p(t)}{p_x^-}} \chi_{\frac{f}{\sigma}\geq 1}\sigma dt\right)^{q(x)} \\ &= W(x)^{\frac{q(x)}{(p_x^-)'}} \left(W(x)^{\frac{-1}{(p_x^-)'}} \int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)^{\frac{p(t)}{p_x^-}} \chi_{\frac{f}{\sigma}\geq 1}\sigma dt\right)^{q(x)} \\ &\leq W(x)^{\frac{q(x)-q_x^-}{(p_x^-)'}} \left(\int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)^{\frac{p(t)}{p_x^-}} \sigma dt\right)^{q_x^-}\end{aligned}$$

where we have used that the term in second parentheses in the right-hand side is less then one, because we can pass to the small exponent q_x^-. Hence

$$\left(\int_{B(0,|x|)} f\chi_{\frac{f}{\sigma}\geq 1}dt\right)^{q(x)} \leq W(x)^{\frac{q(x)-q_x^-}{(p_x^-)'}} \left(\int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)^{\frac{p(t)}{p_x^-}} \sigma dt\right)^{q_x^-}.$$

Applying Hölder inequality, we have

$$\left(\int_{B(0,|x|)} f\chi_{\frac{f}{\sigma}\geq 1}dt\right)^{q(x)} \leq W(x)^{q_x^- + \frac{q(x)-q_x^-}{(p_x^-)'}} \left(\frac{1}{W(x)}\int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)^{\frac{p}{p^-}} \sigma dt\right)^{\frac{p^- q_x^-}{p_x^-}} \tag{3.6}$$

and hence

$$\int_{B(0,|x|)} v(x) \left(\int_{B(0,|x|)} f\chi_{\frac{f}{\sigma}\geq 1}(t)dt \right)^{q(x)} dx$$
$$\leq \int_{B(0,|x|)} W(x)^{\frac{q(x)-q_x^-}{(p_x^-)'}} W(x)^{\frac{p_x^- - p^-}{p_x^-} q_x^-} \left(\frac{1}{W(x)} \int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)^{\frac{p}{p^-}} \sigma dt \right)^{\frac{p^- q_x^-}{p_x^-}} v(x)\, dx.$$

For $x \in B(0,m)$ we have $W(x)^{\frac{q(x)-q_x^-}{p_x^-}} \leq 1$. According to Lemma 2.1 and conditions $p, q \in \Lambda_0$,

$$W(x)^{\frac{q_x^-(p_x^- - p^-)}{p_x^-}} \leq C_5 W(x)^{\frac{q(0)(p(0)-p^-)}{p(0)}}; \quad x \in B(0,m). \tag{3.7}$$

To prove this inequality, we apply Lemma 2.1 and verify $\frac{q_x^-(p_x^- - p^-)}{p_x^-} \in \Lambda_0$. The last belonging easily follows from the conditions $p, q \in \Lambda_0$ and the equality $\frac{q_x^-(p_x^- - p^-)}{p_x^-} - \frac{q(0)(p(0)-p^-)}{p(0)} = \frac{1}{p_x^- p(0)}\big[p(0)p_x^-\left(q_x^- - q(0)\right) + q(0)p^-\left(p_x^- - p(0)\right) + p^- p(0)\left(q(0) - q_x^-\right)\big]$. Hence, using (3.7), we conclude that

$$\begin{aligned} &\int_{B(0,m)} v(x) \left(\int_{B(0,|x|)} f\chi_{\frac{f}{\sigma})\geq 1}(t)dt \right)^{q(x)} dx \\ &\leq C_5 \int_{B(0,m)} W(x)^{\frac{q(0)(p(0)-p^-)}{p(0)}} \left(\int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)^{\frac{p(t)}{p^-}} \sigma dt \right)^{\frac{p^- q_x^-}{p_x^-}} v(x)\, dx. \end{aligned} \tag{3.8}$$

Note that, by Hölder inequality and (3.3) we have

$$\int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)^{\frac{p(t)}{p^-}} \sigma dt \leq \left(\int_{B(0,|x|)} f(t)^{p(t)} \omega dt \right)^{\frac{1}{p^-}} W(x)^{\frac{1}{(p^-)'}} \leq W(x)^{\frac{1}{(p^-)'}} \tag{3.9}$$

To continue the estimate (3.8), we shall use inequality (3.9). Denote

$$g(x) = W(x)^{-\frac{1}{(p^-)'}} \int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)^{\frac{p(t)}{p^-}} \sigma dt$$

By using of (3.9) we have $0 \leq g(x) \leq 1$. Hence

$$\begin{aligned} &\int_{B(0,m)} v(x) \left(\int_{B(0,|x|)} f\chi_{\frac{f}{\sigma}\geq 1}(t)dt \right)^{q(x)} dx \\ &\leq C_5 \int_{B(0,m)} W(x)^{\frac{q(0)(p(0)-p^-)}{p(0)}} g^{\frac{p^- q_x^-}{p_x^-}} W(x)^{\frac{p^- q_x^-}{p_x^-} \frac{1}{(p^-)'}} v(x) dx. \end{aligned} \tag{3.10}$$

Using $p, q \in \Lambda_0$ it is easy to verify $\frac{q_x^-}{p_x^-} \in \Lambda_0$. Then according to Lemma 2.1 we have

$$W(x)^{\frac{p^- q_x^-}{p_x^-} \frac{1}{(p^-)'}} \leq C_6 W(x)^{\frac{(p^- -1)q(0)}{p(0)}}; \quad x \in B(0,m). \tag{3.11}$$

By (3.11) the right-hand side of (3.10) does not exceed $C_6(I_1 + I_2)$ where

$$I_1 := \int_{\{x \in B(0,m): g(x) \geq W(x)\}} W(x)^{\frac{q(0)}{p(0)}} g^{\frac{p^- q_x^-}{p_x^-}} v(x) dx$$
$$I_2 := \int_{\{x \in B(0,m): g(x) < W(x)\}} W(x)^{\frac{q(0)}{p(0)}} g^{\frac{p^- q_x^-}{p_x^-}} v(x) dx.$$

Using (3.1) and $\widetilde{W}(m) < 1$, for the integral I_2 we have

$$\begin{aligned} I_2 &\leq \int_{B(0,m)} W(x)^{\frac{q(0)}{p(0)} + \frac{p^- q_x^-}{p_x^-}} v(x) dx \\ &\leq C_7 \int_{B(0,m)} V(x)^{\frac{-p(0)}{q(0)}\left(\frac{q(0)}{p(0)} + \frac{p^- q^-}{p^+}\right)} v(x) dx \\ &= -C_7 \int_0^m \widetilde{V}(t)^{-\left(1 + \frac{p(0)}{q(0)} \frac{q^-}{p^+}\right)} d\widetilde{V}(t) \\ &= C_7 \left(\frac{p(0)}{q(0)} \frac{q^-}{p^+}\right)^{-1} \widetilde{V}(m)^{-\frac{p(0)}{q(0)} \frac{q^-}{p^+}} \\ &= C_8. \end{aligned} \tag{3.12}$$

For the integral I_1 by partitioning of integration region, we have

$$\begin{aligned} I_1 = &\int_{\left\{x \in B(0,m): g(x) \geq W(x), \frac{q_x^-}{p_x^-} \geq \frac{q(0)}{p(0)}\right\}} W(x)^{\frac{q(0)}{p'(0)}} g^{\left(\frac{q_x^-}{p_x^-} - \frac{q(0)}{p(0)}\right) p^-} v(x) dx \\ &+ \int_{\left\{x \in B(0,m): g(x) \geq W(x), \frac{q_x^-}{p_x^-} < \frac{q(0)}{p(0)}\right\}} W(x)^{\frac{q(0)}{p'(0)}} g^{\left(\frac{q_x^-}{p_x^-} - \frac{q(0)}{p(0)}\right) p^-} v(x) dx \\ := &I_{11} + I_{12}. \end{aligned} \tag{3.13}$$

As we noted above, the exponent $\frac{q_x^-}{p_x^-} \in \Lambda_0$. For I_{11} we have the estimates

$$\begin{aligned} I_{11} &\leq \int_{B(0,m)} W(x)^{\frac{q(0)}{p'(0)}} g^{\frac{p^- q(0)}{p(0)} p^-} v(x) dx \\ &\leq \int_{B(0,m)} W(x)^{\frac{q(0)(p(0)-p^-)}{p(0)}} \left(\int_{B(0,|x|)} \left(\frac{f}{\sigma}\right)^{\frac{p(t)}{p^-}} \sigma dt \right)^{\frac{p^- q(0)}{p(0)}} v(x) dx. \end{aligned} \tag{3.14}$$

Let us now apply Theorem 2.2 with condition (2.8), assuming $a = 0$, $b = m$ and $q_1 = \frac{q(0)p^-}{p(0)}$, $p_1 = p^-$, $v_1 = W(x)^{\frac{q(0)(p(0)-p^-)}{p(0)}} v(x)$, $\omega_1 = \sigma(x)^{1-p^-}$, $f_1 = \left(\frac{f}{\sigma}\right)^{\frac{p}{p^-}} \sigma$. Then by (3.14) we obtain the estimate

$$I_{11} \leq C_9 \left(A_{PS}(p_1, q_1) \int_{B(0,|x|)} f^p \omega dt \right)^{\frac{q(0)}{p(0)}} \leq C_{10} A_{PS}(p_1, q_1)^{\frac{q(0)}{p(0)}} \tag{3.15}$$

Let us now make sure that the value $A_{PS}(p_1, q_1)$ is finite, i.e., verify the fulfillment of condition (2.8):

$$\begin{aligned} &\left(\int_{|y| \leq t} v_1(y) W_1(y)^{q_1} dy \right)^{\frac{1}{q_1}} \widetilde{W}_1(t)^{-\frac{1}{p_1}} \\ &= \left(\int_{|y| \leq t} W(y)^{\frac{q(0)(p(0)-p^-)}{p(0)} + \frac{q(0)p^-}{p(0)}} v(y) dy \right)^{\frac{p(0)}{q(0)p^-}} \widetilde{W}(t)^{-\frac{1}{p^-}} \\ &= \left[\left(\int_{|y| \leq t} v(y) W(y)^{q(0)} dy \right)^{\frac{1}{q(0)}} \widetilde{W}_1(t)^{-\frac{1}{p(0)}} \right]^{\frac{p(0)}{p^-}} \\ &\leq C_{11} \left[\sup_{0<t<m} \widetilde{V}(t)^{\frac{1}{q(0)}} \cdot \widetilde{W}(t)^{\frac{1}{q'(0)}} \right]^{\frac{p(0)}{p^-}} \\ &\leq C_{12}, \quad 0 < t < m, \end{aligned} \tag{3.16}$$

i.e., $A_{PS}(p_1, q_1) \leq C_{13}$, where $W_1(y) = \int_{|x|<|y|} \omega_1(x)^{-\frac{1}{p_1 - 1}} dx$ and the last inequalities follow from (2.10), (3.1). Since condition (2.8) is fulfilled, the above given passing from (3.14) to (3.15) is legitimate. According to Lemma 2.1 and the conditions $0 \leq g(x) \leq 1$, $g(x) > W(x)$; $x \in B(0, m)$ for the second integral term at the right-hand side of the inequality (3.13) we have the estimate

$$g(x)^{\frac{q_x^-}{p_x^-} - \frac{q(0)}{p(0)}} \leq W(x)^{\frac{C_1}{\ln \frac{1}{W(x)}}} \leq e^{C_1}.$$

By using of (3.14) and (3.15) we receive the estimates for I_{12} also:

$$I_{12} \leq \int_{B(0,m)} W(x)^{\frac{q(0)}{p'(0)}} g^{\frac{p^- q(0)}{p(0)}} v(x) dx \leq C_{14} \int_{B(0,m)} f(t)^{p(t)} \omega dx \leq C_{14}. \tag{3.17}$$

Hence

$$I_1 \leq C_{10} C_{12}^{\frac{p(0)}{\sigma(0)}} + C_{14}. \tag{3.18}$$

Using estimates (3.12) and (3.18), we get

$$\int_{B(0,m)} \left(\int_{B(0,|x|)} f \chi_{\frac{f}{\sigma} \geq 1} dt \right)^{q(x)} v(x) dx \leq C_{15}. \tag{3.19}$$

Estimation of "smallest" part. According to the Theorem 2.2 with condition (2.7) assuming $q_1 = q(0)$, $p_1 = p(0)$, $v_1 = v(x)$, $\omega_1 = \sigma(x)^{1-p(0)}$, $f_1 = \sigma(x)$ we obtain

$$\begin{aligned}
&\int_{B(0,m)} \left(\int_{B(0,|x|)} f\chi_{\frac{f}{\sigma}<1} dt \right)^{q(x)} v(x)dx \\
&\leq \int_{B(0,m)} \left(\int_{B(0,|x|)} \sigma dt \right)^{q(x)} v(x)dx \\
&= \int_{B(0,m)} W(x)^{q(x)} v(x)dx \quad (\text{by } q \in \Lambda_0) \\
&\leq C_{16} \int_{B(0,m)} \left(\int_{B(0,|x|)} \sigma dt \right)^{q(0)} v(x)dx \quad (\text{by Theorem 2.1}) \\
&\leq C_{17} \left(\int_{B(0,m)} \sigma dx \right)^{\frac{q(0)}{p(0)}} \\
&= C_{18}.
\end{aligned} \tag{3.20}$$

To verify condition (2.7) of Theorem 2.2 we use (3.1). Estimates (3.19) and (3.20) complete the estimation near zero:

$$\int_{B(0,m)} \left(\int_{B(0,|x|)} f dt \right)^{q(x)} v(x)dx \leq C_{19} = C_{15} + C_{18}. \tag{3.21}$$

Step 2. Estimation near the infinity.
Again writing the Hardy operator as sum of two summand with "largest" and "smallest" parts, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus B(0,M)} \left(\int_{B(0,|x|)\setminus B(0,M)} f dt \right)^{q(x)} v(x)dx \\
&\leq 2^{q^+-1} \int_{\mathbb{R}^n \setminus B(0,M)} \left(\int_{B(0,|x|)\setminus B(0,M)} f\chi_{\frac{f}{\sigma}\geq 1} dt \right)^{q(x)} v(x)dx \\
&+ 2^{q^+-1} \int_{\mathbb{R}^n \setminus B(0,M)} \left(\int_{B(0,|x|)\setminus B(0,M)} f\chi_{\frac{f}{\sigma}<1} dt \right)^{q(x)} v(x)dx \\
&=: 2^{q^+-1}(i_1 + i_2).
\end{aligned}$$

Estimation of "largest" part. Taking assumption (3.3) into account, we find that

$$\begin{aligned} i_1 &= \int_{\mathbb{R}^n\setminus B(0,M)} \left(\int_{B(0,|x|)\setminus B(0,M)} \left(\frac{f}{\sigma}\right) \chi_{\frac{f}{\sigma}\geq 1} \sigma dt \right)^{q(x)} v(x)dx \\ &\leq \int_{\mathbb{R}^n\setminus B(0,M)} \left(\int_{B(0,|x|)\setminus B(0,M)} \left(\frac{f}{\sigma}\right)^{p(t)} \chi_{\frac{f}{\sigma}\geq 1} \sigma dt \right)^{q(x)} v(x)dx \\ &\leq C_{20} \int_{\mathbb{R}^n\setminus B(0,M)} v(x)dx \\ &= C_{21}. \end{aligned} \tag{3.22}$$

Estimation of "smallest" part. Let us now estimate i_2. We denote

$$F(x) = \int_{B(0,|x|)\setminus B(0,M)} f(t)\chi_{\frac{f}{\sigma}<1} dt \quad \text{and} \quad G(x) = \frac{F(x)}{W(x)}.$$

It is clear that, $F(x) \leq \int_{B(0,|x|)} \sigma dt = W(x)$, hence $0 \leq G(x) \leq 1$; $x \in \mathbb{R}^n \setminus B(0,M)$. We have the following elementary estimates, derived by partitioning of integration region:

$$\begin{aligned} i_2 &= \int_{\mathbb{R}^n\setminus B(0,M)} F(x)^{q(x)} v(x)dx \\ &= \int_{\mathbb{R}^n\setminus B(0,M)} G^{q(x)} W^{q(x)} v(x)dx \\ &= \int_{\left\{|x|>M: G(x)<\frac{1}{W(x)}\right\}} G^{q(x)} W^{q(x)} v(x)dx \\ &\quad + \int_{\left\{|x|>M: G(x)\geq\frac{1}{W(x)}\right\}} G^{q(x)} W^{q(x)} v(x)dx \\ &\leq \int_{\mathbb{R}^n\setminus B(0,M)} v(x)dx + \int_{\left\{|x|>M: G(x)>\frac{1}{W(x)}\right\}} G^{q(\infty)} G^{q(x)-q(\infty)} W^{q(x)} v(x)dx \\ &\leq \widetilde{V}(M) + \int_{\left\{|x|>M: G(x)>\frac{1}{W(x)}\right\}\cap\{x: q(x)>q(\infty)\}} G^{q(\infty)} G^{q(x)-q(\infty)} W^{q(x)} v(x)dx \\ &\quad + \int_{\left\{|x|>M: G(x)>\frac{1}{W(x)}\right\}\cap\{q(x)\leq q(\infty)\}} G^{q(\infty)} G^{q(x)-q(\infty)} W^{q(x)} v(x)dx. \end{aligned} \tag{3.23}$$

Note that for $s \in \Lambda_\infty$ according to Lemma 2.1 and (3.2) using $\widetilde{W}(M) > 1$ for $|x| > M$, we have $W^{s(x)} \leq W^{s(\infty)} W^{s(x)-s(\infty)} \leq W^{s(\infty)} W^{|s(x)-s(\infty)|} \leq W^{s(\infty)} W^{\frac{C_2}{\ln \frac{1}{V(x)}}}$ and hence

$$W^{s(x)} \leq W^{s(\infty)} \left(\frac{C_{22}}{V(x)}\right)^{\frac{C_2 p'(\infty)}{q(\infty)\ln\frac{1}{V(x)}}} = C_{23} e^{\frac{C_2 p'(\infty)}{q(\infty)}} W^{s(\infty)}. \tag{3.24}$$

The same inequality for $W^{s(\infty)}$ gives backward inequality: $W^{s(\infty)} \leq W^{s(x)}W^{s(\infty)-s(x)} \leq W^{s(x)}W^{|s(x)-s(\infty)|} \leq W^{s(x)}W^{\frac{C_2}{\ln \frac{1}{V(x)}}}$ which implies

$$W^{s(\infty)} \leq W^{s(x)} \left(\frac{C_{22}}{V(x)} \right)^{\frac{C_2 p'(\infty)}{q(\infty) \ln \frac{1}{V(x)}}} = C_{23} e^{\frac{C_2 p'(\infty)}{q(\infty)}} W^{s(x)}. \tag{3.25}$$

Using (3.24) with exponent in the integral term over the set

$$\left\{ x : |x| > N,\, G(x) \geq \frac{1}{W(x)},\, q(x) \leq q(\infty) \right\}$$

of the last inequality (3.23), we have

$$\begin{aligned}
i_2 \leq & \widetilde{V}(M) + \int_{\left\{|x|>M : G(x) > \frac{1}{W(x)}\right\} \cap \{q(x) > q(\infty)\}} G^{q(\infty)} G^{q(x)-q(\infty)} W^{q(x)} v(x)dx \\
& + C_{24} \int_{\mathbb{R}^n \setminus B(0,M)} G^{q(\infty)} W^{q(\infty)} v(x) dx \quad (\text{by } 0 < G(x) \leq 1) \\
\leq & \widetilde{V}(M) + \int_{\mathbb{R}^n \setminus B(0,M)} G^{q(\infty)} W^{q(x)} v(x) dx + \int_{\mathbb{R}^n \setminus B(0,M)} G^{q(\infty)} W^{q(\infty)} v(x) dx \\
\leq & \widetilde{V}(M) + \int_{\mathbb{R}^n \setminus B(0,M)} F^{q(\infty)} W^{q(x)-q(\infty)} v(x) dx + \int_{\mathbb{R}^n \setminus B(0,M)} F^{q(\infty)} v(x) dx \\
& (\text{by } (3.24) \text{ with } s = q) \\
\leq & \widetilde{V}(N) + C_{25} \int_{\mathbb{R}^n \setminus B(0,M)} F^{q(\infty)} v(x) dx.
\end{aligned} \tag{3.26}$$

Let us apply Theorem 2.2 with condition (2.7) to the last summand, assuming $q_1 = q(\infty)$; $p_1 = p(\infty)$; $v_1 = v$; $\omega_1 = \omega^{\frac{p(\infty)-1}{p(x)-1}}$; $f_1 = f\chi_{\frac{f}{\sigma}<1}$. Then we have

$$\int_{\mathbb{R}^n \setminus B(0,M)} F^{q(\infty)} v(x) dx \leq C_{26} \left(\int_{\mathbb{R}^n \setminus B(0,M)} f_1^{p(\infty)} \omega^{\frac{p(\infty)-1}{p(x)-1}} dx \right)^{\frac{q(\infty)}{p(\infty)}}.$$

The condition (2.7) follows from (3.2). Hence

$$i_2 \leq C_{26} \left(\int_{\mathbb{R}^n \setminus B(0,M)} f^{p(\infty)} \omega^{\frac{p(\infty)-1}{p(x)-1}} \chi_{\frac{f}{\sigma}<1} dx \right)^{\frac{q(\infty)}{p(\infty)}} + \widetilde{V}(M). \tag{3.27}$$

By virtue of (3.24) with $s := p$ and partitioning of the integration region we have the estimates:

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B(0,M)} f^{p(\infty)}\omega^{\frac{p(\infty)-1}{p(x)-1}}\chi_{\frac{f}{\sigma}<1}dx \\
&= \int_{\mathbb{R}^n\setminus B(0,M)} \left(\frac{f}{\sigma}\right)^{p(\infty)} \sigma\chi_{\frac{f}{\sigma}<1}dx \\
&= \int_{\left\{|x|>M:\frac{f}{\sigma}<W^{\frac{-2}{\sigma(\infty)}}\right\}} \left(\frac{f}{\sigma}\right)^{p(\infty)} \sigma\chi_{\frac{f}{\sigma}<1}dx \\
&\quad + \int_{\left\{|x|>M:\frac{f}{\sigma}\geq W^{\frac{-2}{\sigma(\infty)}}\right\}} \left(\frac{f}{\sigma}\right)^{p(\infty)} \sigma\chi_{\frac{f}{\sigma}<1}dx \\
&\leq \int_{\mathbb{R}^n\setminus B(0,M)} W^{-2}\sigma dx \\
&\quad + \int_{\left\{|x|>M:p(x)<p(\infty),\ \frac{f}{\sigma}>W^{\frac{-2}{\sigma(\infty)}}\right\}} (f\sigma)^{p(x)} \left(\frac{f}{\sigma}\right)^{p(\infty)-p(x)} \sigma\chi_{\frac{f}{\sigma}<1}dx \\
&\quad + \int_{\left\{|x|>M:p(x)\geq p(\infty),\ \frac{f}{\sigma}>W^{\frac{-2}{p(\infty)}}\right\}} \left(\frac{f}{\sigma}\right)^{p(x)} \left(\frac{f}{\sigma}\right)^{p(\infty)-p(x)} \sigma\chi_{\frac{f}{\sigma}<1}dx \\
&\leq \frac{1}{\widetilde{V}(M)} + \int_{\{|x|>M:p(x)<p(\infty)\}} \left(\frac{f}{\sigma}\right)^{p(x)} \sigma dx \\
&\quad + \int_{\{|x|>M:p(x)\geq p(\infty)\}} W^{2(p(x)-p(\infty))} \left(\frac{f}{\sigma}\right)^{p(x)} \sigma dx \\
&\leq \frac{1}{\widetilde{V}(M)} + C_{28}\int_{\mathbb{R}^n\setminus B(0,M)} \sigma^{1-p(x)} f^{p(x)} dx + 1 \\
&= C_{28} + 1 + \frac{1}{\widetilde{V}(M)} \\
&= C_{29}.
\end{aligned} \tag{3.28}$$

Again we use (3.24) with $s = q$ in this calculation. From estimates (3.22), (3.26), (3.27) and the inequality (3.28) we obtain

$$\int_{\mathbb{R}^n\setminus B(0,M)} \left(\int_{B(0,|x|)\setminus B(0,M)} f(t)dt\right)^{q(x)} v(x)dx \leq C_{30}. \tag{3.29}$$

Step 3. Estimation in the middle.
Since $I_{p'(\cdot)}\big(\omega^{-\frac{1}{p}}\chi_{B(0,M)}\big) = \int_{B(0,M)} \omega^{-\frac{1}{p-1}}dt =: \widetilde{W}(M) \leq C_{31}$ by virtue of (3.3) and Hölder inequality, for the $p(\cdot)$-norms we have

$$\int_{B(0,M)} f(t)dt \leq C_0 \left\| f\omega^{\frac{1}{p}}\chi_{B(0,M)}\right\|_{p(\cdot)} \left\|\omega^{-\frac{1}{p}}\chi_{B(0,M)}\right\|_{p'(\cdot)} \leq C_{32}. \tag{3.30}$$

Hence

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B(0,M)}\left(\int_{B(0,M)}f(t)dt\right)^{q(x)}v(x)dx\\&\le\left(1+\left(\int_{B(0,M)}f(t)dt\right)^{q^+}\right)\int_{\mathbb{R}^n\setminus B(0,M)}v(x)dx.\end{aligned}\tag{3.31}$$

Using (3.30) and (3.31) we have

$$\int_{\mathbb{R}^n\setminus B(0,M)}\left(\int_{B(0,N)}f(t)dt\right)^{q(x)}v(x)dx\le\left(1+C_{29}^{q^+}\right)\widetilde{V}(N)=C_{30}.\tag{3.32}$$

Further, the elementary inequality $(a+b)^p\le 2^{p-1}\left(a^p+b^p\right)$ with $a\ge 0$, $b\ge 0$, $p\ge 1$ and using the inequalities (3.21), (3.29), (3.32) give us following estimates:

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(\int_{B(0,|x|)}f(t)dt\right)^{q(t)}v(x)dx\\&\le\int_{B(0,m)}\left(\int_{B(0,|x|)}f(t)dt\right)^{q(x)}v(x)dx+\int_{B(0,M)\setminus B(0,m)}\left(\int_{B(0,|x|)}f(t)dt\right)^{q(x)}v(x)dx\\&\quad+2^{q^+-1}\int_{\mathbb{R}^n\setminus B(0,M)}\left(\int_{B(0,M)}f(t)dt\right)^{q(x)}v(x)dx\\&\quad+2^{q^+-1}\int_{\mathbb{R}^n\setminus B(0,M)}\left(\int_{B(0,|x|)\setminus B(0,M)}f(t)dt\right)^{q(x)}v(x)dx\\&\le C_{34}.\end{aligned}\tag{3.33}$$

The part (3) $\Rightarrow$ (1) of Theorem 3.1 is proved.

3.2. Proof of (1) $\Rightarrow$ (3). Now, we shall construct an auxiliary function. Using it in the inequality (1.1), necessary conditions will be derived. Let $t\in(0,m)$ be fixed. We denote

$$f_t(x)=\left(\int_{B(0,t)}\sigma ds\right)^{-\frac{1}{p(0)}}\sigma(x)\chi_{B(0,t)}(x).\tag{3.34}$$

For $|x|\le t$ by using $\widetilde{W}(m)<1$, $p\in\Lambda_0$ we have

$$\begin{aligned}\widetilde{W}(t)^{p(x)}&=\widetilde{W}(t)^{p(0)}\widetilde{W}(t)^{p(x)-p(0)}\\&\ge\widetilde{W}(t)^{p(0)}\widetilde{W}(t)^{\frac{C_1}{\ln\frac{1}{W(x)}}}\quad\text{(since } W(x)\le\widetilde{W}(t)<1)\\&\ge\widetilde{W}(t)^{p(0)}\widetilde{W}(t)^{\frac{C_1}{\ln\frac{1}{\widetilde{W}(t)}}}\\&=e^{-C_1}\widetilde{W}(t)^{p(0)}.\end{aligned}\tag{3.35}$$

By the same way using $q \in \Lambda_0$ we have $\widetilde{W}(t)^{q(x)} = \widetilde{W}(t)^{q(0)} \widetilde{W}(t)^{q(x)-q(0)} \le \widetilde{W}(t)^{q(0)} \widetilde{W}(t)^{-\frac{C_1}{\ln \frac{1}{W(x)}}}$ and

$$\widetilde{W}(t)^{q(x)} \le \widetilde{W}(t)^{q(0)} \widetilde{W}(t)^{\frac{C_1}{\ln \frac{1}{\widetilde{W}(t)}}} = e^{C_1} \widetilde{W}(t)^{q(0)}. \tag{3.36}$$

Hence, using (3.35), we have

$$I_{p(\cdot)}\left(\omega^{\frac{1}{p}} f_t\right) = \int_{B(0,t)} \frac{\sigma(x)dx}{\widetilde{W}(t)^{\frac{p(x)}{p(0)}}} \le C_{35}. \tag{3.37}$$

Let (1.1) holds for the function f_t. From (3.37) we derive $\left\| \omega^{\frac{1}{p}} f_t \right\|_{p(\cdot)} \le C_{36}$. Then (1.1) implies $\left\| v^{\frac{1}{q}} H\left(f_t\right) \right\|_{q(\cdot)} \le C_{37}$. Hence, $I_{q(\cdot);B(0,m)}\left(v^{\frac{1}{q}} H f_t\right) \le C_{38}$ and

$$H f_t(x) \le \int_{|y| \le |x|} \chi_{B(0,m)} f_t(y) dy = \frac{W(x)}{\widetilde{W}(t)^{\frac{1}{p(0)}}} \chi_{B(0,t)}(x) + \widetilde{W}(t)^{\frac{1}{p'(0)}} \chi_{B(0,m) \setminus B(0,t)}(t).$$

According to Lemma 2.1, for $q \in \Lambda_0$ we have $W(x)^{q(x)} \sim W(x)^{q(0)}$. Hence using (3.36) we have

$$\begin{aligned} I_{q(\cdot)}\left(v^{\frac{1}{q}} \chi_{B(0,m)} H f_t\right) &= \int_{B(0,m)} v(y) \left(H f_t\right)^{q(y)} dy \\ &\ge \int_{B(0,m)} v(x) \left(H f_t\right)^{q(x)} dx \\ &\ge \int_{B(0,t)} v(x) \widetilde{W}(t)^{-\frac{q(x)}{p(0)}} W(x)^{q(x)} dx \quad \text{(by (3.36))} \\ &\ge C_{39} \widetilde{W}(t)^{-\frac{q(x)}{p(0)}} \int_{B(0,t)} v(x) W(x)^{q(0)} dx. \end{aligned}$$

Hence

$$\left(\int_{B(0,t)} v(x) W(x)^{q(0)} dx \right)^{\frac{1}{q(0)}} \widetilde{W}(t)^{-\frac{1}{p(0)}} \le C_{40}, \quad 0 \le t \le m. \tag{3.38}$$

According to Theorem 2.2, conditions (3.38) and (2.8) lead to the inequality

$$\left(\int_{B(0,m)} (Hf)^{q(0)} v dx \right)^{\frac{1}{q(0)}} \le C_{41} \left(\int_{B(0,m)} f(x)^{p(0)} \sigma^{1-p(0)} dx \right)^{\frac{1}{p(0)}}. \tag{3.39}$$

Inequality (3.39) generates the condition

$$\left(\int_{B(0,m) \setminus B(0,t)} v(x) dx \right)^{\frac{1}{q(0)}} \widetilde{W}(t)^{\frac{1}{p'(0)}} \le C_{42}, \quad t \in (0,m). \tag{3.40}$$

Now, let us show that (3.40) implies (3.1). For $|x| < \frac{m}{2}$ we have the inequality

$$\int_{|y|\le|x|<m} v(y)dy \ge \left(1 - \frac{\widetilde{V}(m)}{\widetilde{V}\left(\frac{m}{2}\right)}\right)\widetilde{V}(|x|). \tag{3.41}$$

To prove last inequality it suffices to use the elementary inequality $\widetilde{V}(m) \le \frac{\widetilde{V}(m)}{\widetilde{V}(\frac{m}{2})}\widetilde{V}(|x|)$ for $|x| < \frac{m}{2}$ in the equality $\widetilde{V}(m) + \int_{|y|\le|x|<m} v(y)dy = \widetilde{V}(|x|)$. Hence using (3.41) in (3.40) for any $t \in \left(0, \frac{m}{2}\right)$ we have

$$\widetilde{V}(t)\widetilde{W}(t)^{\frac{q(0)}{p'(0)}} \le C_{43} \tag{3.42}$$

which easily implies (3.1). The condition (3.1) has been proved.

Proof of (1) $\Rightarrow$ (3.2). By duality (see, e.g. [8]) in the spaces $L^{p(\cdot)}$, $\left(L^{p(\cdot)}\right)^* = L^{p'(\cdot)}$ and

$$\int_{\mathbb{R}^n} Hf(x)\cdot g(x)dx = \int_{\mathbb{R}^n} H^*g(x)\cdot f(x)dx.$$

Here $H^*g(x) = \int_{|y|>|x|} g(y)dy$. Hence (1.1) equivalently to the inequality

$$\left\|\sigma^{\frac{1}{p'(x)}} H^*f(x)\right\|_{p'(\cdot)} \le C_{44}\left\|v^{\frac{1-q'(x)}{q'(x)}} f(x)\right\|_{q'(\cdot)} \tag{3.43}$$

We have the inequality (3.43) with any test function $f(x) \ge 0$.

Fix any $t > M$ and put $f_t(x) = \widetilde{V}(t)^{-\frac{1}{q'(\infty)}} v(x)\chi_{|x|>t}(x)$ in (3.43). Using $q \in \Lambda_\infty$ we have $\frac{1}{q'} \in \Lambda_\infty$. Hence using $\widetilde{V}(M) < 1$ and $\frac{1}{q'} \in \Lambda_\infty$ for any $|x| > t$ we have $\widetilde{V}(t)^{-\frac{1}{q'(\infty)}} = \widetilde{V}(t)^{-\frac{1}{q'(x)}}\widetilde{V}(t)^{\frac{1}{q'(x)}-\frac{1}{q'(\infty)}} \le \widetilde{V}(t)^{-\frac{1}{q'(x)}}\widetilde{V}(t)^{-\frac{C_2}{\ln\frac{1}{V(x)}}} \le \widetilde{V}(t)^{-\frac{1}{q'(x)}}\widetilde{V}(t)^{\frac{C_2}{\ln\frac{1}{\widetilde{V}(t)}}} = e^{C_2}\widetilde{W}(t)^{-\frac{1}{q'(x)}}$, i.e.,

$$\widetilde{W}(t)^{-\frac{1}{q'(\infty)}} \le e^{C_2}\widetilde{W}(t)^{-\frac{1}{q'(x)}}; \quad |x| > t. \tag{3.44}$$

By the same way, using $p' \in \Lambda_\infty$ and $\widetilde{V}(M) < 1$ we have $\widetilde{V}(t)^{-p'(x)} = \widetilde{V}(t)^{-p'(\infty)}\widetilde{V}(t)^{p'(\infty)-p'(x)} \ge \widetilde{V}(t)^{-p'(\infty)}\widetilde{V}(t)^{\frac{C_2}{\ln\frac{1}{V(x)}}}$ and

$$\widetilde{V}(t)^{-p'(x)} \ge \widetilde{V}(t)^{-p'(\infty)}\widetilde{V}(t)^{\frac{C_2}{\ln\frac{1}{\widetilde{V}(x)}}} = e^{-C_2}\widetilde{V}(t)^{-p'(\infty)}. \tag{3.45}$$

By using of (3.44) we have

$$I_{q'(\cdot)}\left(v^{\frac{1-q'}{q'}} f_t\right) = \int_{|x|>t}\widetilde{V}(t)^{-\frac{q'(x)}{q'(\infty)}} v dx \le e^{C_2}\int_{|x|>t}\widetilde{V}(t)^{-1}v(x)dx \le e^{C_2}. \tag{3.46}$$

Testing inequality (3.43) with the function $f_t(x)$ and using (3.46) we have $I_{p'(\cdot)}\left(\sigma^{\frac{1}{p'}}H^*(f)_t\right) \le C_{45}$. Then using (3.45) we have

$$C_{45} \ge \int_{|x|>t} \sigma(x)\left(\int_{|y|>|x|} v(y)V(t)^{\frac{-1}{q'(\infty)}}dy\right)^{p'(x)} dx \ge e^{-C_2}\int_{|x|>t} V(x)^{p'(x)}\widetilde{V}(t)^{\frac{-p'(\infty)}{q'(\infty)}}\sigma(x)dx.$$

By using $p' \in \Lambda_\infty$ and Lemma 2.1 from here we infer

$$C_{45} \ge \widetilde{V}(t)^{-\frac{p'(\infty)}{q'(\infty)}} \int_{|x|>t} V(x)^{p'(\infty)}\sigma(x)dx.$$

Hence $\left(\int_{|x|>t} V(x)^{p'(\infty)}\sigma(x)dx\right)^{\frac{1}{p'(\infty)}} \widetilde{V}(t)^{-\frac{1}{q'(\infty)}} \le C_{46}$. According to Theorem 2.2 this condition implies (3.2). The implication (1) $\Leftrightarrow$ (3) has been proved.

3.3. Completion of the proof of Theorem 3.1. *Implication* (3) $\Leftrightarrow$ (2). It follows by simple applications of Theorem 2.2 with condition (2.7).

Proof of implication (4) $\Leftrightarrow$ (3). To prove this implication it suffices to prove that

$$V(x)^{\frac{1}{q(x)}} \sim V(x)^{\frac{1}{q(0)}}; \quad \text{for } x \in B(0,m) \tag{3.47}$$

and

$$W(x)^{\frac{1}{p'(x)}} \sim W(x)^{\frac{1}{p'(\infty)}}; \quad \text{for } |x| > M. \tag{3.48}$$

Indeed, let us demonstrate (4) $\Rightarrow$ (3.1). Denote by q^+, q^-, p^+, p^- the relevant values of exponents p, q over the set $B(0,m)$. Using $\widetilde{V}(m) > 1$ and $\widetilde{W}(m) < 1$ we have $C_{47} \ge V(x)^{\frac{1}{q(x)}}W(x)^{\frac{1}{p'(x)}} \ge V(x)^{\frac{1}{q^+}}W(x)^{\frac{1}{(p^+)'}}$ for $x \in B(0,m)$. Hence,

$$V(x) \le C_{48}W(x)^{-\frac{q^+}{(p^+)'}}; \quad x \in B(0,m). \tag{3.49}$$

Using $p, q \in \Lambda_0$ we have $\frac{1}{q} \in \Lambda_0$. By using (3.49) and $\widetilde{V}(m) > 1$, $\frac{1}{q} \in \Lambda_0$ we have

$$\begin{aligned} V(x)^{\frac{1}{q(x)}} &= V(x)^{\frac{1}{q(0)}}V(x)^{\frac{1}{q(x)}-\frac{1}{q(0)}} \\ &\le V(x)^{\frac{1}{q(0)}}V(x)^{\frac{C_1}{\ln\frac{1}{W(x)}}} \\ &\le C_{49}V(x)^{\frac{1}{q(0)}}W(x)^{-\frac{q^+C_1}{(p^+)'\ln\frac{1}{W(x)}}} \\ &= C_{49}e^{\frac{q^+C_1}{(p^+)'}}V(x)^{\frac{1}{q(0)}}. \end{aligned} \tag{3.50}$$

The same inequality holds for the term $V(x)^{\frac{1}{q(0)}}$:

$$\begin{aligned} V(x)^{\frac{1}{q(0)}} &= V(x)^{\frac{1}{q(x)}} V(x)^{\frac{1}{q(0)}-\frac{1}{q(x)}} \\ &\le V(x)^{\frac{1}{q(x)}} V(x)^{\frac{C_1}{\ln \frac{1}{W(x)}}} \\ &\le C_{50} V(x)^{\frac{1}{q(x)}} W(x)^{-\frac{q^+ C_1}{(p^+)' \ln \frac{1}{W(x)}}} \\ &= C_{50} e^{\frac{q^+ C_1}{(p^+)'}} V(x)^{\frac{1}{q(x)}}. \end{aligned} \tag{3.51}$$

Hence $V(x)^{\frac{1}{q(x)}} \sim V(x)^{\frac{1}{q(0)}}$ for $x \in B(0,m)$.

According to Lemma 2.1 and the conditions (2.2) for p we have $W(x)^{\frac{1}{p'(x)}} \sim W(x)^{\frac{1}{p'(\infty)}}$ for $x \in B(0,m)$. Hence for $x \in B(0,m)$ we have

$$V(x)^{\frac{1}{q(x)}} W(x)^{\frac{1}{p'(x)}} \sim V(x)^{\frac{1}{q(0)}} \; W(x)^{\frac{1}{p'(0)}} \tag{3.52}$$

and (4) $\Rightarrow$ (3.1) has been proved.

Let us prove the same equivalence at the infinity, i.e., prove that (4) $\Rightarrow$ (3.2). We need to prove $W(x)^{\frac{1}{p'(x)}} \sim W(x)^{\frac{1}{p'(\infty)}}$ for $|x| > M$. Denote by q^+, q^-, p^+, p^- the relevant values of exponents p, q over the set $\mathbb{R}^n \backslash B(0,M)$. Using $\widetilde{V}(M) < 1$ and $\widetilde{W}(M) > 1$ we have $C_{51} \ge V(x)^{\frac{1}{q(x)}} W(x)^{\frac{1}{p'(x)}} \ge V(x)^{\frac{1}{q^-}} W(x)^{\frac{1}{(p^-)'}}$ for $x \in B(0,m)$. Hence,

$$W(x) \le C_{52} V(x)^{-\frac{(p^-)'}{q^-}}; \quad |x| > M. \tag{3.53}$$

Now, using $q \in \Lambda_\infty$ we have $\frac{1}{q} \in \Lambda_\infty$. By using (3.53), $\widetilde{W}(M) > 1$ and $\frac{1}{p'} \in \Lambda_\infty$ we have for $|x| > M$:

$$\begin{aligned} W(x)^{\frac{1}{p'(x)}} &= W(x)^{\frac{1}{p'(\infty)}} W(x)^{\frac{1}{p'(x)}-\frac{1}{p'(\infty)}} \\ &\le W(x)^{\frac{1}{p'(\infty)}} W(x)^{\frac{C_2}{\ln \frac{1}{V(x)}}} \\ &\le C_{53} W(x)^{\frac{1}{p'(\infty)}} V(x)^{-\frac{(p^-)' C_2}{q^- \ln \frac{1}{V(x)}}} \\ &= C_{53}' e^{\frac{(p^-)' C_2}{q^-}} W(x)^{\frac{1}{p'(\infty)}} \end{aligned} \tag{3.54}$$

The same inequality holds for the term $W(x)^{\frac{1}{p'(\infty)}}$ for $|x| > M$:

$$\begin{aligned} W(x)^{\frac{1}{p'(\infty)}} &= W(x)^{\frac{1}{p'(x)}} W(x)^{\frac{1}{p'(\infty)}-\frac{1}{p'(x)}} \\ &\le W(x)^{\frac{1}{p'(x)}} W(x)^{\frac{C_2}{\ln \frac{1}{V(x)}}} \\ &\le C_{54} W(x)^{\frac{1}{p'(x)}} V(x)^{-\frac{(p^-)' C_2}{q^- \ln \frac{1}{V(x)}}} \\ &= C_{54} e^{\frac{(p^-)' C_2}{q^-}} W(x)^{\frac{1}{p'(x)}} \end{aligned} \tag{3.55}$$

According to Lemma 2.1 and the conditions (2.3) for q we have the implication $V(x)^{\frac{1}{q(x)}} \sim V(x)^{\frac{1}{q(\infty)}}$ for $|x| > M$. Hence for $|x| > M$

$$V(x)^{\frac{1}{q(x)}} W(x)^{\frac{1}{p'(x)}} \sim V(x)^{\frac{1}{q(0)}}\ W(x)^{\frac{1}{p'(0)}} \tag{3.56}$$

and (4) $\Rightarrow$ (3.2) has been proved. Hence (3.52) and (3.56) gives the implication (4) $\Rightarrow$ (3).

By the same way as (4) $\Rightarrow$ (3) we can easily show that (3) $\Rightarrow$ (4) and Theorem 3.1 has been proved.

Remark 3.2. Applying Theorem 3.1 in the case of weights

$$v(x) = |x|^{q(x)\left(\beta(x) - \frac{n}{p(x)} - \frac{n}{q(x)}\right)}; \quad \omega = |x|^{p(x)\beta(x)}$$

we can improve the condition $q(x) \geq p(x)$ in the paper [12] to $q(0) \geq p(0)$, $q(\infty) \geq p(\infty)$.

Acknowledgement. We are grateful to the referee for careful reading of the manuscript and useful comments.

References

[1] Canestro, M. I. A. and Salvador, P. O., Weighted weak type inequalities with variable exponents for Hardy and maximal operators. *Proc. Japon. Acad. Ser. A* 82 (2006), 126 – 130.

[2] Cruz-Uribe, D., Fiorenza, A., Martell, J. M. and Pérez, C., The boundedness of classical operators on variable L^p-spaces. *Ann. Acad. Sci. Fenn. Math.* 31 (2006), 239 – 264.

[3] Diening, L., Harjulehto, P., Hästö, P., Růžička, M.: *Lebesgue and Sobolev Spaces with Variable Exponents.* Lect. Notes Math. 2017. Berlin: Springer 2011.

[4] Diening, L. and Samko, S., Hardy inequality in variable exponent Lebesgue spaces. *Fract. Calc. Appl. Anal.* 10 (2007), 1 – 17.

[5] Diening, L., Hästö, P. and Nekvinda, A., Open problems in variable exponent Lebesgue and Sobolev spaces. In: *FSDONA 04. Proceedings* (Milovy; eds.: P. Drábek et al.). Prague: Academy of Sciences of the Czech Republic 2005, pp. 38 – 58.

[6] Edmunds, D. E., Kokilashvili, V., Meskhi, A., On the boundedness and compactness of the weighted Hardy operators in $L^{p(x)}$ spaces. *Georgian Math. J.* 12 (2005)(1), 27 – 44.

[7] Harjulehto, P., Hästö, P. and Koskenoja, M., Hardy's inequality in variable exponent Sobolev spaces. *Georgian Math. J.* 12 (2005)(3), 431 – 442.

[8] Kovacik, O. and Rákosník, J., On spaces $L^{p(x)}$ and $W^{1,p(x)}$. *Czechoslovak Math. J.* 41(116) (1991), 592 – 618.

[9] Kokilashvili, V. and Samko, S. G., Weighted boundedness of the maximal, singular and potential operators in variable exponent spaces. In: *Analytic Methods of Analysis and Differential Equations* (Minsk 2006; eds.: A. A. Kilbas et al.). Cambridge: Cambridge Scientific Publishers 2008, pp. 139 – 164.

[10] Kokilashvili, V. and Samko, S. G., Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Rev. Mat. Iberoamericana* 20 (2004)(2), 145 – 156.

[11] Kufner, A., Maligranda, L. and Persson, L. E., *The Hardy Inequality. About Its History and Some Related Results.* Plzeň: Vydavatelský Servis 2007.

[12] Mamedov, F. I., Harman, A., On weighted inequality of Hardy type in $L^{p(x)}$. *J. Math. Anal. Appl.* 353 (2009)(2), 521 – 530.

[13] Mamedov, F. I. and Harman, A., On a Hardy type general weighted inequality in spaces $L^{p(x)}$. *Integr. Equ. Oper. Theory* 66 (2010), 565 – 592.

[14] Mashiyev, R., Çekiç, B., Mamedov, F. I. and Ogras, S., Hardy's inequality in power-type weighted $L^{p(x)}(0,\infty)$. *J. Math. Anal. Appl.* 334 (2007)(1), 289 – 298.

[15] Mamedov, F. I. and Harman, A., On boundedness of weighted Hardy operator in $L^{p(x)}$ and regularity condition. *J. Ineq. Appl.* 2010, Art. ID 837951, 14 pp.

[16] Maz'ya, V. G., *Sobolev Spaces.* Berlin: Springer 1985.

[17] Okpoti, C. A., Persson, L.-E. and Sinnamon, G., Equivalence theorem for some integral conditions with general measures related to Hardy's inequality I. *J. Math. Anal. Appl.* 326 (2007), 398 – 413.

[18] Okpoti, C. A., Persson, L.-E. and Sinnamon, G., Equivalence theorem for some integral conditions with general measures related to Hardy's inequality II. *J. Math. Anal. Appl.* 337 (2008), 219 – 230.

[19] Rafeiro, H. and Samko, S., Hardy type inequality in variable Lebesgue spaces. *Ann. Acad. Sci. Fen. Mathematica* 34 (2009), 279 – 289.

[20] Růžička, M., *Electrorheological Fluids: Modeling and Mathematical Theory.* Lect. Notes Math. 1748. Berlin: Springer 2000.

[21] Samko, S. G., Hardy-Littlewood-Stein-Weiss inequality in the Lebesgue spaces with variable exponent. *Fract. Calc. Appl. Anal.* 6 (2003)(4), 421 – 440.

[22] Samko, S. G., Hardy inequality in the generalized Lebesgue spaces. *Fract. Calc. Appl. Anal.* 4 (2003), 355 – 362.

[23] Wedestig, A., Weighted inequalities of Hardy-type and their limiting inequalities. Doctoral Thesis. Dep. Math. Lulea University 2003.

Received March 10, 2010; revised October 8, 2010