

Continuity and Differentiability of Multivalued Superposition Operators with Atoms and Parameters I

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Abstract. For a given single- or multivalued function f and “atoms” S_i , let $S_f(\lambda, x)$ be the set of all measurable selections of the function $s \mapsto f(\lambda, s, x(s))$ which are constant on each S_i . Continuity and differentiability of such operators are studied in spaces of measurable functions containing ideal, Orlicz and L_p spaces with new results for the parameter-dependent case even for single-valued superposition operators without atoms. A motivation is to apply the results for variant of such maps S_f in Sobolev spaces in the second part of this article [Z. Anal. Anwend. 31 (2011) (to appear)].

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1. Introduction

The aim of this paper is to study continuity and differentiability of superposition (Nemytskij) operators in spaces of measurable functions. In the second part [26] of the article, we will apply these results to certain superposition type operators in Sobolev spaces. The main novelty of the first part is that we also include the case of parameter-dependent f (cf. Remark 1.1). Moreover, for both parts, we also include “atoms”. The latter is interesting mainly for multivalued f , and the motivation for it originates from obstacle problems for PDEs. As one of the simplest examples of the latter, consider on a domain S an equation like

$$-\Delta u(s) \in f(\lambda, s, u(s), \nabla u(s)) \quad \text{on } S, \quad u|_{\partial S} = 0, \quad (1)$$

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where f is either single-valued or also contains some “jumps”, e.g. for some real-valued functions g, h, u_0

$$f(\lambda, s, u, v) = \begin{cases} \{g(\lambda, s, u, v)\} & \text{if } u < u_0(\lambda, s, v) \\ [g(\lambda, s, u, v) - h(\lambda, s, u, v), g(\lambda, s, u, v)] & \text{if } u = u_0(\lambda, s, v) \\ \{g(\lambda, s, u, v) - h(\lambda, s, u, v)\} & \text{if } u > u_0(\lambda, s, v); \end{cases}$$

such jumps are an important tool for modeling unilateral obstacles, e.g. a source or sink working under some conditions (see [10, 11, 25] for more realistic such problems described by *systems* of equations). Now it can happen that on some disjoint subsets $S_i \subseteq S$ ($i \in I$) the obstacle does not act “pointwise” but only in an averaged sense, mathematically e.g. described by integrals like

$$\begin{aligned} -\Delta u(s) &\equiv \text{const} \in f(\lambda, s, \int_{S_i} u(t) dt, \int_{S_i} \nabla u(t) dt) \quad \text{on } S_i, \\ -\Delta u(s) &\in f(\lambda, s, u(s), \nabla u(s)) \quad \text{on } S \setminus \bigcup_{i \in I} S_i, \quad u|_{\partial\Omega} = 0, \end{aligned} \tag{2}$$

see [25]. Heuristically, on the obstacle S_i the “obstacle’s cause” $(u, \nabla u)$ is averaged on the right-hand side. Hence, it makes sense, heuristically, to require as in (2) that the “obstacle’s effect” $-\Delta u$ should (in the simplest case) be constant on S_i . In fact, the latter follows even automatically in similar problems from a natural weak formulation [9, 12].

Now the operator on the right-hand side of (2) can be described as the composition of differential and integral operators and of the (multivalued) operator

$$\begin{aligned} S_f(\lambda, u, v) &:= \{y : y \text{ measurable, } y(s) \in f(\lambda, s, u(s), v(s)) \text{ a.e.,} \\ &\quad \text{and } y|_{S_i} \text{ a.e. constant for every } i\}. \end{aligned}$$

We call S_f the superposition operator with parameter λ and *atoms* S_i . Mathematically, the meaning of the atoms is that we do not consider *arbitrary measurable* selections of $f(\lambda, \cdot, u(\cdot), v(\cdot))$, but only those selections which are measurable on the “reduced” measure space where we identify S_i as atoms of the measure space (recall that measurable functions are by definition a.e. constant on atoms of a measure space). Note, however, that we cannot easily reduce the study of S_f to the study of the “classical” superposition operator in the “reduced” measure space, since the measurability requirements in this “reduced” measure space would be too restrictive (i.e., also u, v and $f(\lambda, \cdot, u, v)$ would have to be constant on S_i). Indeed, the functions in e.g. (2) “live” naturally on the Lebesgue measure, and *only* in the definition of S_f we need to consider functions constant on S_i ; this will become particularly clear in the setting of Sobolev spaces explained in more detail in [26]. Therefore, the atoms S_i must really be treated as part of the operator S_f and not of the underlying measure space, i.e., the theory for the *operator* S_f has to take the atoms S_i into account.

The paper consists of two loosely related examinations: The study of the continuity properties of S_f in spaces of measurable functions (Section 3) and the study of differentiability properties of S_f in spaces of measurable functions (Section 4). Although we must treat atoms in all sections, the main technical difficulties concerning atoms will occur only in the next part [26] of the paper; in Sections 3 and 4 the main difficulty and novelty is the dependency on the parameter λ .

Since the continuity and differentiability of S_f in spaces of measurable functions is of independent interest (e.g. also for integral equations) we study both in the most natural framework in which this study can be done: For the continuity, this is the framework of ideal spaces (which is also the natural framework for integral equations), and for the differentiability, this is the setting of Orlicz spaces; both generalize L_p spaces and are introduced in the corresponding sections. Note that in the context of Sobolev spaces we have in view of the embedding theorems $(u, \nabla u) \in L_p \times L_q$ with $p \neq q$. Unfortunately, $L_p \times L_q$ is for $p \neq q$ neither an ideal space nor a (classical) Orlicz space, and so we are forced to deal with the more technical classes of *generalized* ideal spaces and *products* of Orlicz spaces in Section 3 and 4, respectively.

Roughly speaking, Section 3 can be considered as the generalization of the famous result of M. A. Krasnoselskij [14] that a superposition (Nemytskij) operator $F(x)(s) = f(s, x(s))$ generated by a Carathéodory function f is automatically continuous if it acts from L_p into L_q with $q < \infty$. However, for the multivalued case the situation is more complicated, since, contrary to what was claimed in [4] (see also [3, Theorem 8.2]), it was shown in [23] that the superposition operator is usually never upper semicontinuous. Nevertheless, it was shown in [23] that such operators are often upper semicontinuous in the uniform sense (see Section 2 for the terminology). Hence, the result in Section 3 is actually a generalization of the main result of [23] to the setting of superposition operators with atoms S_i and a parameter λ . Neither of these extensions (atoms or a parameter) can be reduced to the classical superposition operator $F(x)(s) = f(s, x(s))$ in any obvious manner (cf. Remark 1.1).

By “differentiability properties” we mean estimates like e.g.

$$\lim_{(u,v) \rightarrow 0} \sup_{\lambda \in \Lambda_0} \frac{\sup_{y \in S_f(\lambda, u, v)} \|y\|}{\|(u, v)\|} = 0, \tag{3}$$

or

$$\lim_{\substack{(\lambda, u, v) \rightarrow (\lambda_0, u, v) \\ (u, v) \neq (0, 0)}} \frac{\sup_{y \in S_f(\lambda, u, v)} \|y\|}{\|(u, v)\|} = 0, \tag{4}$$

where $\|\cdot\|$ denotes the norm of the respective considered spaces; the relation of such estimates with differentiability (in the single-valued case) is sketched at the beginning Section 4. Estimates like (3) or (4) are not only useful to prove

differentiability but also in connection with the study of bifurcation problems (also in the multivalued case). For instance, it follows from (3) that for a linear isomorphism J the operator $J - tS_f(\lambda, \cdot)$ has for $t \in [0, 1]$ no nontrivial zero in a neighborhood of 0, i.e., the inclusion $J(u, v) \in S_f(\lambda, u, v)$ has no bifurcation at 0, and e.g. the homotopy invariance of a corresponding degree theory can be used. In this context, the uniformness of (3) (or (4)) with respect to λ is often essential, as it implies that a corresponding neighborhood is independent of $\lambda \in \Lambda_0$ (or for λ close to λ_0 , respectively).

For the classical superposition operator without a parameter $F(x)(s) = f(s, x(s))$ sharp differentiability criteria are known in L_p and Orlicz spaces [1, 2] (see also [5]), but as remarked above, even to treat (1), we need this for products of such spaces. Our results in Section 4 extend the mentioned criteria to such products (and to the parameter-dependent and multivalued case). However, although these criteria are best possible, in a sense, they can hardly be verified in practice. Therefore, we derive from it some criteria which are (although mathematically weaker) rather simple to verify. For instance, we show that for a mapping from L_p into L_q with $q < p$ differentiability of f together with certain growth conditions already implies (3) (for $q \geq p$ there is no such result, since it is well-known that superposition operators are never differentiable in such cases unless they are affine, see e.g. [5, Theorem 3.12]). Various similar criteria can be found in literature (see the comments after Theorems 4.16), but rarely in connection with parameters (not to speak about multivalued f or atoms). The parameter-dependence is worth a separate remark.

Remark 1.1. It is a well-known trick that in some cases results about superposition operators with a parameter $F(\lambda, x)(s) = f(\lambda, s, x(s))$ can be reduced to results about classical superposition operators (without parameters) by considering λ as a function and applying the result for the corresponding auxiliary superposition operator $G(\lambda, x)(s) = f(\lambda(s), s, x(s))$ of a vector function (λ, x) . However, this trick cannot always be applied. For instance, in connection with differentiability, it is rather obvious that such a trick cannot directly lead to estimates like (3) where the limit should be uniform w.r.t. λ . Moreover, even for rather natural functions f , this trick sometimes cannot even be used to obtain the continuity of F . For instance, for $p, q \in (0, \infty)$, $\beta \in (0, \frac{p}{q})$, $\alpha \in (0, 1 - \frac{\beta q}{p})$, the function

$$f(\lambda, s, u) := |\lambda - s|^{-\alpha} |u|^{\beta-1} u$$

(in case $\beta \leq 1$ define $f(\lambda, s, 0) := 0$) satisfies by Jensen's inequality

$$|f(\lambda, s, u)| \leq \left(1 - \frac{\beta q}{p}\right) |\lambda - s|^{-(1 - \frac{\beta q}{p})^{-1} \alpha} + \frac{\beta q}{p} |u|^{\frac{p}{q}},$$

and so our results (Example 3.10 and Theorem 3.17) will imply that $F: \mathbb{R} \times L_p([0, 1]) \rightarrow L_q([0, 1])$ is continuous, in particular, continuous at 0. How-

ever, the mentioned auxiliary operator G is locally unbounded at 0 (and not even defined in a neighborhood of 0) on the space $L_r([0, 1]) \times L_p([0, 1])$, even in case $r = \infty$.

In view of Remark 1.1, the continuity and differentiability results obtained in Sections 3 and 4 are new for the parameter-dependent case (even for single-valued scalar superposition operators without atoms). Of course, the same remark holds (with similar examples as in Remark 1.1) for the setting of Sobolev spaces which is studied in the next part [26] of the paper.

2. General Notations

Throughout this paper, (S, Σ, μ) will denote a complete σ -finite measure space, and $S_i \in \Sigma$ ($i \in I$) will denote a fixed family of pairwise disjoint sets with $\mu(S_i) > 0$ ($i \in I$) which will play the role of atoms mentioned in the introduction. It is explicitly admissible that $I = \emptyset$, in which case the results of this paper deal with “ordinary” multivalued superposition operators.

Proposition 2.1. *I is at most countable.*

Proof. S is a union of countably many sets $E_n \in \Sigma$ with $\mu(E_n) < \infty$. Hence, I is the union of the countably many countable sets $I_n = \{i \in I : \mu(E_n \cap S_i) > 0\}$. \square

Section 3 deals also with spaces which are not normed, so let us fix some terminology. Let $(U, |\cdot|)$ be a quasi-pseudonormed space; here, “quasi” means that instead of the triangle inequality, we only assume

$$|x + y| \leq q \cdot (|x| + |y|)$$

with some finite constant q , and “pseudo” means that $|x|$ may be infinite and that $|x| = 0$ may hold also for $x \neq 0$.

Example 2.2. The space $U = L_p([0, 1])$ with $0 < p < 1$ is not normed but quasi-normed by

$$|x| := \left(\int_0^1 |x(s)|^p ds \right)^{\frac{1}{p}}.$$

(This is the reason, why we can include the case $p, q < 1$ in Remark 1.1.)

Of course, we understand U equipped with the induced uniform structure and corresponding topology. It is easy to see that with this topology, every quasi-normed space is a topological vector space, i.e., addition and scalar multiplication are continuous operations. However, $|\cdot|$ might be discontinuous (although this is usually not the case). In order to avoid pathological measurability problems, we will assume throughout that $|\cdot|$ is at least a Borel function.

We call a function $x: S \rightarrow U$ *measurable* if it can be approximated almost everywhere by a sequence of simple (assuming only finitely many values on measurable sets) functions; since we assume that $|\cdot|$ is a Borel function, also $|x(\cdot)|$ is measurable in this case. By $\mathcal{M}(S, U)$, we denote the space of measurable functions $x: S \rightarrow U$. Usually, we will tacitly identify functions $x, y \in \mathcal{M}(S, U)$ if $|x(s) - y(s)| = 0$ for almost all $s \in S$, i.e., we understand the elements of $\mathcal{M}(S, U)$ usually as corresponding equivalence classes.

By a multivalued function $F: X \multimap Y$, we mean a function from X into the powerset of Y . In contrast to usual practice, it will be convenient to allow that $F(x)$ is empty; instead, we define the notation

$$D(F) := \{x \in X : F(x) \neq \emptyset\}.$$

We call F *single-valued* if $F(x)$ contains at most one element for every $x \in X$; in this case, we will notationally not distinguish between F and the function $F: D(F) \rightarrow Y$ which is canonically induced by F , although this is of course a slight misuse of notation. As usual for multivalued functions, we will use the notation

$$F(X_0) := \bigcup_{x \in X_0} F(x) \quad (X_0 \subseteq X).$$

Moreover, we work with the small and large counter-images

$$\begin{aligned} F^-(M) &:= \{x \in X : F(x) \subseteq M\}, \\ F^+(M) &:= \{x \in X : F(x) \cap M \neq \emptyset\}. \end{aligned} \quad (M \subseteq Y) \quad (5)$$

Note that $F^-(M)$ contains by definition the complement of $D(F)$ and that $D(F) \supseteq F^+(M) \supseteq F^-(M) \cap D(F)$ (explaining the name “large counterimage” for $F^+(M)$).

In our case, the space Y will not only be a topological space but even carry a (quasi-)uniform structure, induced e.g. by a quasi-pseudonorm. Hence, for a set $M \subseteq Y$, we have two natural notions of neighborhoods: A “topological” neighborhood (i.e., a set containing an open set containing M) and a “uniform” neighborhood which is a set containing $U(M)$ for some element U of the quasi-uniform structure of Y , see e.g. [13] for the corresponding terminology.

The two kind of preimages and the topology/quasi-uniform structure on Y lead to four natural notions of continuity for multivalued maps.

Definition 2.3. Let $F: X \multimap Y$ be a multivalued function between a topological space X and a topological (quasi-uniform) space Y .

1. F is *upper semicontinuous at $x_0 \in X$ (in the uniform sense)* if for each topological (uniform) neighborhood $N \subseteq Y$ of $F(x_0)$ the set $F^-(N)$ is a neighborhood of x_0 .

2. F is *lower semicontinuous at* $x_0 \in X$ if for each $y \in F(x_0)$ and each topological neighborhood $N \subseteq Y$ of y the set $F^+(N)$ is a neighborhood of x_0 .
3. F is *lower semicontinuous at* $x_0 \in X$ *in the uniform sense* if for each element U of the quasi-uniform structure of Y there is a neighborhood $M \subseteq X$ of x_0 with $M \subseteq F^+(U(y))$ for each $y \in F(x_0)$.

What we call upper/lower semicontinuous at x_0 in the uniform sense is in literature sometimes called upper/lower semicontinuous at x_0 in the ε -sense [6] or (δ, ε) -upper/lower semicontinuous [8]. In the author's opinion the latter notions are unfortunately chosen, especially since they suggest a relation to some kind of metric, although it is actually the quasi-uniform structure of Y which is employed. We provide the short proof of the following result since it is perhaps slightly more general than what is well-known:

Proposition 2.4. *If F is upper semicontinuous at x_0 and Y is a quasi-uniform space, then F is upper semicontinuous at x_0 in the uniform sense; the converse holds if $F(x_0)$ is compact. If F is lower semicontinuous at x_0 in the uniform sense then it is lower semicontinuous at x_0 ; the converse holds if Y is a uniform space and $F(x_0)$ is precompact.*

Proof. The claim concerning upper semicontinuity follows from the subsequent Proposition 2.5. For the last claim, assume that F is lower semicontinuous at x_0 , $F(x_0)$ is precompact, and U is an element of the uniform structure of Y . Choosing an element V of the uniform structure of Y with $V \circ V^{-1} \subseteq U$, we find finitely many $y_1, \dots, y_n \in F(x_0)$ with $F(x_0) \subseteq \bigcup_{k=1}^n V(y_k)$. For each $k = 1, \dots, n$ there is a neighborhood $M_k \subseteq X$ of x_0 with $M_k \subseteq F^+(V(y_k))$. For each $y \in F(x_0)$ there is some k with $y \in V(y_k)$, hence $y_k \in V^{-1}(y)$, and so $M_k \subseteq F^+(V(y_k)) \subseteq F^+(U(y))$. Consequently, $M := M_1 \cap \dots \cap M_n$ satisfies $M \subseteq F^+(U(y))$ for every $y \in F(x_0)$. \square

We recall the following proof, observing that it requires really only a *quasi-uniform* structure.

Proposition 2.5. *Each uniform neighborhood of M in a quasi-uniform space is a topological neighborhood. The converse holds if M is compact.*

Proof. If N is a topological neighborhood of a compact set M , let \mathcal{O} denote the family of all open sets O with the property that there is some $x \in O$ and an element U of the uniform structure with $O \subseteq U(x)$ and $U^2(x) \subseteq N$. Then \mathcal{O} is an open cover of M . Let $O_1, \dots, O_n \in \mathcal{O}$ form a finite subcover, and choose corresponding $x_k \in O_k$ and U_k . Then $U := U_1 \cap \dots \cap U_n$ has the required property since for each $x \in M$ there is some k with $x \in O_k \subseteq U_k(x_k)$, and so $U(x) \subseteq U_k^2(x_k) \subseteq N$. \square

Unless we say otherwise, $(U, |\cdot|)$ and $(V, |\cdot|)$ are quasi-pseudo-normed spaces, and Λ is a topological space. Moreover, $f: \Lambda \times U \multimap V$ will usually denote a multivalued function with the property

$$|u_1 - u_2| = 0 \implies f(\lambda, s, u_1) = f(\lambda, s, u_2).$$

Then f induces a (multivalued and parameter-dependent) “superposition operator S_f with the atoms $(S_i)_{i \in I}$ ” in the following sense.

Definition 2.6. For $f: \Lambda \times U \multimap V$, let $S_f: \Lambda \times \mathcal{M}(S, U) \multimap \mathcal{M}(S, V)$ be defined as follows. For $\lambda \in \Lambda$ and $x \in \mathcal{M}(S, U)$, let $S_f(\lambda, x)$ denote the set of all $y \in \mathcal{M}(S, V)$ with the following two properties.

1. $y(s) \in f(\lambda, s, x(s))$ for almost all $s \in S$.
2. $y|_{S_i}$ is constant (almost everywhere) for every $i \in I$.

In particular, S_f depends on the atoms S_i ($i \in I$), although we do not mark this dependency explicitly in the notation.

3. Continuity in Generalized Ideal Spaces

As remarked in the introduction, it would not be sufficient for us to consider continuity only in L_p -spaces or, more general, ideal spaces. Instead, we have to deal with products of such spaces, and probably the most natural way to treat those is to work in the framework of “generalized ideal spaces” which were introduced in [23,24] exactly for this purpose. Unfortunately, this requires some terminology which we recall in the next section.

3.1. Generalized Ideal Spaces. For a measurable set $E \subseteq S$, we denote by χ_E the characteristic function of E , and by $P_E x(s) := \chi_E(s)x(s)$ the corresponding canonical projection in $\mathcal{M}(S, U)$. We are mainly interested in spaces where these projections are “bounded by 1” in the following sense.

Definition 3.1. Let $X \subseteq \mathcal{M}(S, U)$ be a nonempty subset, and $\|\cdot\|: X \rightarrow [0, \infty]$. We call $(X, \|\cdot\|)$ a (*quasi-pseudometric*) *projectable space* if the following holds.

1. $x, y \in X$ implies $x - y \in X$.
2. $\| -x \| = \| x \|$, and $\| 0 \| = 0$.
3. $\| x + y \| \leq q \cdot (\| x \| + \| y \|)$ for all $x, y \in X$.
4. $x \in X$ and $E \in \Sigma$ imply $P_E x \in X$ and $\| P_E x \| \leq \| x \|$.

We drop the additions “quasi” if we have $q = 1$, “pseudo” if we have $\|\cdot\|: X \rightarrow [0, \infty)$ and $\|x\| = 0 \implies x = 0$, “metric” (and write sometimes “normed” instead) if X is a (real) linear space and $\|\cdot\|$ is positively homogeneous.

We recall that $L_p([0, 1])$ with $0 < p < 1$ in Example 2.2 is not normed, but it is a quasinormed projectable space in the above sense.

The main reason to include even metric spaces in the definition is that we want to include $X = \mathcal{M}(S, U)$ with its usual quasi-pseudometric

$$\|x\|_{\mathcal{M}(S,U)} := \inf_{m>0} (m + \mu(\{s \in S : |x(s)| \geq m\})).$$

Note that if $\mu(S) = \infty$ then even in case $U = \mathbb{R}$, this is only a pseudometric (e.g. $\|1\| = \infty$), and the scalar multiplication is discontinuous. To avoid this inconvenience, we equip $\mathcal{M}(S, U)$ instead usually with the uniform structure of convergence in measure on sets of finite measure, i.e., with the quasi-pseudometric

$$\|x\|_{\mathcal{M}(S,U)} := \inf_{m>0} (m + \nu(\{s \in S : |x(s)| \geq m\})), \quad (6)$$

where ν is an equivalent normalized measure: The quantity (6) depends on the choice of ν of course, but the induced uniform structure does not, see [23] for details. If $(U, |\cdot|)$ is (quasi-)normed then $\mathcal{M}(S, U)$ is a (quasi-)metric projectable space and a topological vector space.

The most important projectable spaces are the ideal spaces:

Definition 3.2. A (quasi-pseudometric) projectable space X is *preideal* if the relations $x \in X$, $y \in \mathcal{M}(S, U)$, and $|y(s)| \leq |x(s)|$ a.e. imply that $y \in X$ and $\|y\| \leq \|x\|$. If X is also complete, we call it *ideal*.

Unfortunately, $L_p([0, 1]) \times L_q([0, 1])$, understood as a subset of $\mathcal{M}([0, 1], \mathbb{R}^2)$, is not an ideal space in case $p \neq q$, and so the class of ideal spaces is too narrow for our intentions. Therefore, we have to consider a more general class of spaces.

Definition 3.3. Let X be a (quasi-pseudometric) projectable space.

1. X is a (quasi-pseudometric) *generalized preideal space* if for each sequence $x_n \in X \cap L_\infty(S, U)$ which converges uniformly to 0 the following holds: For every set $E \in \Sigma$ with $\mu(E) > 0$ there is some $D \subseteq E$ in Σ with $\mu(D) > 0$ such that $\|P_D x_n\| \rightarrow 0$. If X is complete, we speak of a (quasi-pseudometric) *generalized ideal space*.
2. X is *embeddable* if for each sequence $x_n \in X \cap L_\infty(S, U)$ which converges uniformly to some $y \in L_\infty(S, U)$ and which converges in X (to a possibly different function) the following holds: For every set $E \in \Sigma$ with $\mu(E) > 0$ there is some $D \subseteq E$ in Σ with $\mu(D) > 0$ such that $P_D y \in X$.

Each (quasi-pseudonormed) preideal space is simultaneously generalized preideal and embeddable. Moreover, the product of (quasi-pseudonormed) embeddable generalized preideal spaces is also embeddable and generalized preideal, so that e.g. $L_p(S, U) \times L_q(S, V) \subseteq \mathcal{M}(S, U \times V)$ are embeddable generalized

ideal spaces. More general, if U is finite-dimensional, all ideal spaces and Orlicz spaces in the sense of [15, 16] are embeddable generalized ideal spaces. Proofs for all these claims (especially the last is not trivial) can be found in [23].

We will also need a convergence theorem of Vitali type. To this end, we define:

Definition 3.4. Let X be a (quasi-pseudonormed) projectable space. Then $M \subseteq X$ has *equicontinuous norm* if for each sequence $D_n \in \Sigma$

$$D_n \downarrow \emptyset \implies \lim_{n \rightarrow \infty} \sup_{x \in M} \|P_{D_n} x\| = 0.$$

The *regular part* X_0 of X is the set of all $x \in X$ for which $\{x\}$ has equicontinuous norm.

A trivial argument by contradiction implies that it suffices to consider countable subsets:

Lemma 3.5. $M \subseteq X$ has equicontinuous norm if and only if any sequence in M has equicontinuous norm.

Recall that we equipped $\mathcal{M}(S, U)$ with the uniform structure of convergence in measure on sets of finite measure (6). The term “embeddable” means that if X is complete, then it is (topologically and uniformly) embedded into $\mathcal{M}(S, U)$.

Proposition 3.6. Let $X \subseteq \mathcal{M}(S, U)$ be an embeddable quasi-pseudometric generalized ideal space and simultaneously be a topological vector space (for the induced topology). Then the identity map $id: X \rightarrow \mathcal{M}(S, U)$ is continuous (hence uniformly continuous by linearity) and thus each (uniform) neighborhood of a set $M \subseteq X$ in X is the restriction to X of a (uniform) neighborhood of M in $\mathcal{M}(S, U)$.

A generalized form of Vitali’s convergence theorem states that a certain converse holds for sets in the regular part under some hypothesis on equicontinuous norm [23, Corollary 3.2] (the relation to the classical Vitali theorem was explained in [23]).

Theorem 3.7 (Vitali). Let $X \subseteq \mathcal{M}(S, U)$ be a quasi-pseudometric generalized preideal space with regular part X_0 , and let $A \subseteq X$ have equicontinuous norm. If $M \subseteq X_0$ is such that each neighborhood of M in $\mathcal{M}(S, U)$ intersects A , then each neighborhood of M in X intersects A . An analogous statement holds for uniform neighborhoods if also M has equicontinuous norm.

3.2. Continuity in Generalized Ideal Spaces. Now we come to our main aim, the superposition operator S_f with atoms and parameters. In contrast to the superposition operator without atoms, it is not reasonable to assume that $D(S_f)$ has interior points, hence the well-known continuity results about superposition operators make no sense in our setting. In order to apply them anyway, it seems reasonable to consider besides S_f a superposition operator *without* atoms.

Definition 3.8. For $B: S \times U \dashrightarrow V$ satisfying

$$|u_1 - u_2| = 0 \implies B(s, u_1) = B(s, u_2),$$

we define $S_B^\circ: \mathcal{M}(S, U) \dashrightarrow \mathcal{M}(S, V)$ as follows. For $x \in \mathcal{M}(S, U)$, let $S_B^\circ(x)$ denote the set of all $y \in \mathcal{M}(S, V)$ such that $y(s) \in B(s, x(s))$ for almost all $s \in S$.

The main difference between the definition of S_B° and S_f is that the latter has atoms and depends on λ .

For superposition operators S_B° as above (without atoms and parameters) it is known that, roughly speaking, acting and Carathéodory type conditions already imply continuity in (generalized) ideal spaces. One cannot expect such a strong result with respect to parameters, since even in the autonomous single-valued case $f(\lambda, s, u) = a(\lambda, s)$, it is easy to give examples where $a(\cdot, s)$ is continuous for all $s \in [0, 1]$ and $a(\lambda, \cdot) \in Y = L_q([0, 1])$ for all $\lambda \in [0, 1]$ and $0 < q < \infty$ but $S_f(\lambda, x)$ is not continuous with respect to λ in Y ; consider for instance $a(\lambda, s) := \lambda^{-\frac{1}{q}} s^{\frac{1}{q}}$ ($\lambda > 0$), $a(0, \cdot) := \chi_{\{1\}}$, at $\lambda = 0$. The Vitali convergence theorem states that actually a necessary and sufficient condition for continuity of $a(\lambda, \cdot)$ in Y with respect to λ in such an example is that the family $a(\lambda, \cdot)$ has equicontinuous norm in Y .

Recall in this connection that Kransoselskij's famous acting condition states that the (classical) superposition operator generated by a (single-valued) function $f(\lambda, \cdot)$ acts from $X = L_p([0, 1])$ ($0 < p < \infty$) into Y if and only if there are functions $a_\lambda \in Y$ and constants b_λ with

$$f(\lambda, s, u) \leq a_\lambda(s) + b_\lambda |u|^{\frac{p}{q}}.$$

In the autonomous case discussed above, one can think of $b_\lambda = 0$, and therefore, it appears appropriate to assume for a continuity result that the family of functions a_λ has equicontinuous norm in Y (and that b_λ is bounded). We will discuss such a condition in a moment.

Note that the requirement that the family a_λ has equicontinuous norm is a much weaker requirement in the above growth assumption than the natural appearing hypothesis that a_λ and b_λ are both independent of λ : Even the autonomous superposition operator generated by $f(\lambda, s, u) = a(s - \lambda)$ with $\lambda \in [0, 1]$ and an essentially unbounded function $a \in L_q([-1, 1])$ would not

satisfy this requirement, although it is of course continuous in $Y = L_q([0, 1])$ and will be included in our weaker hypothesis.

The crucial condition for our continuity result will be formulated in terms of the following definition.

Definition 3.9. Let $D \subseteq \Lambda \times \mathcal{M}(S, U)$, $A \subseteq \mathcal{M}(S, V)$, $B: S \times U \multimap V$. For $E \in \Sigma$, we write $f \preceq A + B$ on D outside of E if for each $(\lambda, x) \in D$ we have $S_f(\lambda, x) \subseteq A + S_B^\circ(x)$ outside of E in the sense of equivalence classes, i.e., if for each $y \in S_f(\lambda, x)$ there is some $a \in A$ and some $b \in S_B^\circ(x)$ such that $y(s) = a(s) + b(s)$ for almost all $s \in S \setminus E$.

We point out that even for a “classical” single-valued superposition operator S_f without atoms but with a parameter, the function B will have to be chosen multivalued, in general. The following is the model example for the above definition:

Example 3.10. Let $X := L_{p_1}(S, U_1) \times \cdots \times L_{p_m}(S, U_m)$ and $Y := L_q(S, V)$ with $p_j \in (0, \infty]$, $0 < q < \infty$, and pseudonormed spaces U_j and V . For $D \subseteq \Lambda \times X$ and $E \in \Sigma$ assume that for any $(\lambda, x) \in D$, $x = (x_1, \dots, x_m)$, and any $y \in S_f(\lambda, x)$, we have for almost all $s \in S \setminus E$

$$|y(s)| \leq a_\lambda(s) + \sum_{\substack{j=1 \\ p_j \neq \infty}}^m b_j |x_j(s)|^{\frac{p_j}{q}},$$

where $a_\lambda \in L_q(S, \mathbb{R})$ (independent of x) and $b_j \in [0, \infty)$ is independent of $(\lambda, x) \in D$. (Note that for continuity in a point (λ_0, x_0) it suffices to consider neighborhoods of this point, hence one need only consider sets D which contain only couples (λ, x) with $\|x - x_0\|_X < \varepsilon$; in particular, the above inequality needs only be verified for functions x for which $\|x_j\|_{L_\infty} \leq \|x_{0,j}\|_{L_\infty} + \varepsilon$ for those components j with $p_j = \infty$.) Then $f \preceq A + B$ on D outside of E with

$$A := \{a_\lambda : \lambda \in \Lambda\}, \quad B(s, u_1, \dots, u_m) := \left\{ v \in V : |v| \leq \sum_{\substack{j=1 \\ p_j \neq \infty}}^m b_j |u_j|^{\frac{p_j}{q}} \right\},$$

and moreover, the superposition operator S_B° acts from X into Y , i.e., $X \subseteq (S_B^\circ)^-(Y)$ which will be the only requirement for B in our continuity result (actually, it is even more than what will be required). For A , we will require that it has equicontinuous norm in Y .

We will not make any assumptions on the continuity or measurability of B but only (implicitly) assumptions on the norm of the elements of $B(s, u)$. In this sense, the above example is typical: One should consider $B(s, u)$ as some

sort of “uniform bound” for the “norm” of the elements of $f(\lambda, s, u)$, allowing that the bound may be violated on null sets in a sense.

Our reason to introduce the exceptional set E is that for certain sets E the following hypothesis may be trivial to verify in some cases (at least for sets D which one will typically consider in connection with superposition operators with atoms).

Definition 3.11. Let $D \subseteq \Lambda \times \mathcal{M}(S, U)$, $(\lambda_0, x_0) \in D$, and $E \in \Sigma$. We say that S_f is *regular on E for (λ_0, x_0) on D* if for any sequence $(\lambda_n, x_n) \in D$ with $(\lambda_n, x_n) \rightarrow (\lambda_0, x_0)$ and any sequence $y_n \in S_f(\lambda_n, x_n)$ the set $\{P_E y_n : n\}$ has equicontinuous norm in Y .

Example 3.12. Suppose that E is the union of finitely many of the atoms S_{i_1}, \dots, S_{i_n} and that D is such that for any $(\lambda, x) \in D$ the function $x|_{S_{i_k}}$ is constant. If $Y = L_q(S, V)$ ($0 < q < \infty$) and $f(\cdot, s, \cdot)$ is uniformly bounded in a neighborhood of $(\lambda_0, x_0(s))$ for each $s \in E$, then S_f is regular on E for (λ_0, x_0) on D .

Theorem 3.13 (Continuity in $\mathcal{M}(S, V)$ implies continuity). *Let $X \subseteq \mathcal{M}(S, U)$ be an embeddable quasi-pseudometric generalized ideal space and simultaneously be a topological vector space (for the induced topology). Let $Y \subseteq \mathcal{M}(S, V)$ be a quasi-pseudometric generalized preideal space with regular part Y_0 .*

Let $S_f: \Lambda \times \mathcal{M}(S, U) \rightarrow \mathcal{M}(S, V)$ be a parameter-dependent superposition operator with atoms $(S_i)_{i \in I}$, and $(\lambda_0, x_0) \in D \subseteq \Lambda \times X$. Suppose that S_f is regular on E for (λ_0, x_0) on D and $f \preceq A + B$ on D outside of E where $A \subseteq Y$ has equicontinuous norm and x_0 is in the topology of X an inner point of $X \cap (S_B^\circ)^-(Y_0)$. Assume also that $\lambda_0 \in \Lambda$ has a countable neighborhood base.

Let $S_f: D \rightarrow \mathcal{M}(S, V)$ be upper or lower semicontinuous at (λ_0, x_0) (in the uniform sense). Then $S_f: D \rightarrow Y$ is upper or lower semicontinuous at (λ_0, x_0) (in the uniform sense), respectively.

An analogous result holds if we equip D in the hypothesis with the topology inherited from $\Lambda \times \mathcal{M}(S, U)$.

The hypothesis that x_0 is an inner point of $X \cap (S_B^\circ)^-(Y_0)$ means by Definition 3.8 and (5) that for every $x \in X$ in a neighborhood of x_0 all measurable functions y satisfying $y(s) \in B(s, x(s))$ a.e. belong to Y_0 . It is *not* necessary to verify whether such a function y exists.

Proof. The crucial tool of the proof is that, by [23, Corollary 4.1 and Proposition 5.1], we have for any sequence $x_n \in X$ with $\|x_n - x_0\|_X \rightarrow 0$ and any sequence $y_n \in S_B^\circ(x_n)$ that the set $\{y_n : n\}$ has equicontinuous norm in Y under our assumptions. It follows that S_f is regular on S for (λ_0, x_0) on D . Indeed, if $(\lambda_n, x_n) \in D$ converges to (λ_0, x_0) and $y_n \in S_f(\lambda_n, x_n)$, there are $a_n \in A$ and $\hat{y}_n \in S_B^\circ(x_n)$ with $y_n = P_E y_n + P_{S \setminus E} a_n + P_{S \setminus E} \hat{y}_n$. Since $\{P_E y_n, P_{S \setminus E} a_n, P_{S \setminus E} \hat{y}_n : n\}$ has equicontinuous norm it follows that also $\{y_n : n\}$ has equicontinuous norm.

As a side result we obtain for the particular choice $(\lambda_n, x_n) := (\lambda_0, x_0)$ by Lemma 3.5 that $M := S_f(\lambda_0, x_0)$ has equicontinuous norm.

Assume first by contradiction that $S_f: D \multimap Y$ is not upper semicontinuous at (λ_0, x_0) (in the uniform sense). Since λ_0 has a countable neighborhood base, we find a (uniform) neighborhood $N \subseteq Y$ of M and a sequence $(\lambda_n, x_n) \in D$ with $\lambda_n \rightarrow \lambda_0$, $\|x_n - x_0\|_X \rightarrow 0$, and $y_n \in S_f(\lambda_n, x_n) \setminus N$. We have shown that then the sets $A_0 := \{y_n : n\}$ and M have equicontinuous norm in Y . We conclude from Theorem 3.7 that there is a (uniform) neighborhood of M in $\mathcal{M}(S, V)$ which is disjoint from A_0 . This implies that $S_f: D \multimap \mathcal{M}(S, V)$ fails to be upper semicontinuous at (λ_0, x_0) (in the uniform sense). For the last claim, we use here Proposition 3.6.

Now assume that $S_f: D \multimap Y$ is not lower semicontinuous at (λ_0, x_0) (in the uniform sense). Then there are a sequence $(\lambda_n, x_n) \in X$ with $\lambda_n \rightarrow \lambda_0$, $\|x_n - x_0\|_X \rightarrow 0$, $\varepsilon > 0$, and a constant sequence $z_n := z \in M$ (resp. a sequence $z_n \in M$) such that $S_f(\lambda_n, x_n)$ contains no element y with $\|y - z_n\|_Y < \varepsilon$. However, if $S_f: D \multimap \mathcal{M}(S, V)$ is lower semicontinuous at (λ_0, x_0) (in the uniform sense), there are $y_n \in S_f(\lambda_n, x_n)$ with $y_n - z_n \rightarrow 0$ in $\mathcal{M}(S, V)$; for the last claim, we use here Proposition 3.6. Since we have shown that the sets $A_0 := \{y_n : n\}$ and $\{z_n : n\} \subseteq M$ have equicontinuous norm, we conclude from Theorem 3.7 that $\|y_n - z_n\|_Y \rightarrow 0$, a contradiction. \square

In order to apply Theorem 3.13, one has to verify some continuity of S_f in measure. It seems natural to expect that some Carathéodory type hypothesis suffices for the latter. For such an hypothesis, we need a notion of measurability of multivalued functions.

Definition 3.14. A multivalued function $G: S \multimap V$ is called *measurable* if $G^-(M) \in \Sigma$ for each open set $M \subseteq V$ and *weakly measurable* if $G^-(M) \in \Sigma$ for each closed set $M \subseteq V$. We call G (*weakly*) *measurable in the Bochner sense* if additionally G has essentially separable range, i.e., if there is a null set S_0 such that $G(S \setminus S_0)$ is separable.

Each measurable function is weakly measurable, and we have a Kuratowski-Ryll-Nardzewsky type selection theorem, i.e., any weakly measurable in the Bochner sense function $G: S \multimap V$ with nonempty complete values has a selection which is measurable in the sense that it can be approximated a.e. by simple functions [23, Section 6]. The following result is a straightforward extension of [23, Theorem 6.2]; only due to the presence of atoms an additional argument is required.

Lemma 3.15. *Let (λ_0, x_0) be such that $F_0(s) := f(\lambda, s, x_0(s))$ is compact and closed for almost all $s \in S$, and F_0 is measurable in the Bochner sense and that $f(\cdot, s, \cdot)$ is upper semicontinuous at $(\lambda_0, x_0(s))$ in the uniform sense for almost all $s \in S$. Assume that $\lambda_0 \in \Lambda$ has a countable neighborhood base.*

Then $S_f: \Lambda \times \mathcal{M}(S, U) \multimap \mathcal{M}(S, V)$ is upper semicontinuous at (λ_0, x_0) in the uniform sense, and hence the same holds if $\Lambda \times \mathcal{M}(S, U)$ is replaced by any subset D .

Proof. Otherwise, there are $\varepsilon > 0$, sequences $(\lambda_n, x_n) \in \Lambda \times \mathcal{M}(S, U)$ with $\lambda_n \rightarrow \lambda_0$ and $x_n \rightarrow x_0$ in measure on each set of finite measure, and sequences $y_n \in S_f(\lambda_n, x_n)$ such that for all $z \in S_f(\lambda_0, x_0)$ we have $\|y_n - z\|_{\mathcal{M}(S, V)} \geq \varepsilon$ for all n . In view of [17, (1.24)], we may assume that V is equipped with a pseudometric generating the same uniform structure; hence, by an extension of Riesz' theorem (see e.g. [22, Corollary 1.10]) we can assume, passing to a subsequence if necessary, that $x_n(s) \rightarrow x_0(s)$ for almost all $s \in S$. For almost all $s \in S$, since $f(\cdot, s, \cdot)$ is upper semicontinuous at $(\lambda_0, x_0(s))$ in the uniform sense, we obtain from $y_n(s) \in f(\lambda_n, s, x_n(s))$ that $F_0(s) \neq \emptyset$, and moreover, $\alpha_n(s) := \text{dist}(y_n(s), F_0(s)) \rightarrow 0$. Now repeating the arguments of the proof of [23, Theorem 6.2], we find measurable selections z_n of F_0 such that $y_n(s) - z_n(s) \rightarrow 0$ for almost all $s \in S$.

We use now that y_n is constant a.e. on S_i , say $y_n(s) = c_{n,i}$ for almost all $s \in S_i$. We can fix for each of the countably many $i \in I$ some $s_i \in S_i$ with $y_n(s_i) = c_{n,i}$, $y_n(s_i) - z_n(s_i) \rightarrow 0$ and such that $F_0(s_i)$ is compact. Since $\{z_n(s_i) : n\} \subseteq F_0(s_i)$, we can conclude that $(c_{n,i})_n$ has a convergent subsequence for every $i \in I$. Since I is countable, we find by a diagonal argument a subsequence n_k such that $(c_{n_k, i})_{n_k}$ converges for every $i \in I$. Passing to this subsequence, we can assume without loss of generality that $c_{n,i} \rightarrow c_i$ ($n \rightarrow \infty$) for every $i \in I$. In particular, $y_n(s) = c_{n,i} \rightarrow c_i$ for almost all $s \in S_i$, and so $y_n(s) - z_n(s) \rightarrow 0$ implies $z_n(s) \rightarrow c_i$ for almost all $s \in S_i$. Since $z_n(s) \in F_0(s)$, we thus have $c_i \in F_0(s)$ for almost all $s \in S_i$. We redefine now $z_n(s) := c_i$ for $s \in S_i$. Then $z_n(s) \in F_0(s)$ for almost all $s \in S$ and z_n is constant on each S_i . Hence, $z_n \in S_f(\lambda_0, x_0)$. By construction, we have $y_n(s) - z_n(s) \rightarrow 0$ for almost all $s \in S$. Indeed, for almost all $s \in S_i$, this follows from $y_n(s) = c_{n,i} \rightarrow c_i = z_n(s)$. Since $y_n(s) - z_n(s) \rightarrow 0$ for almost all $s \in S$, we obtain by [23, Proposition 2.2] that $\|y_n - z_n\|_{\mathcal{M}(S, V)} \rightarrow 0$ which is a contradiction. \square

Concerning lower semicontinuity, we have to suppose some assumptions on f and D concerning the atoms. Apart from that, the statement and proof of the following lemma are analogous to [23, Theorem 6.3].

Lemma 3.16. *Let $(\lambda_0, x_0) \in D \subseteq \Lambda \times \mathcal{M}(S, U)$ be such that for any $(\lambda, x) \in D$ the function $F(\lambda, x)(s) := f(\lambda, s, x(s))$ is measurable in the Bochner sense, assume a.e. nonempty compact closed values, and a.e. constant on each S_i ($i \in I$). Then:*

1. $D \subseteq D(S_f)$, i.e., $S_f(\lambda, x) \neq \emptyset$ for each $(\lambda, x) \in D$.

2. If λ_0 has a countable base of neighborhoods and if for almost all $s \in S$ the function $f(\cdot, s, \cdot)$ is lower semicontinuous at $(\lambda_0, x_0(s))$ (in the uniform sense) then $S_f: D \multimap \mathcal{M}(S, V)$ is lower semicontinuous at (λ_0, x_0) (in the uniform sense).

Proof. By the mentioned Kuratowski-Ryll-Nardzewsky theorem, there is a measurable selection of $F(\lambda, x)$; redefining this selection constant on each of the countably many atoms S_i (which is possible by hypothesis), we see $S_f(\lambda, x) \neq \emptyset$. For the second claim, we may assume as in the proof of Lemma 3.15 that V is pseudonormed. If S_f is not lower semicontinuous at (λ_0, x_0) (in the uniform sense), there are $\varepsilon > 0$, a constant (resp. not necessarily constant) sequence $z_n \in S_f(\lambda_0, x_0)$ and a sequence $(\lambda_n, x_n) \in D$ with $(\lambda_n, x_n) \rightarrow (\lambda_0, x_0)$ such that for each n and all $y \in S_f(\lambda_n, x_n)$ we have $\|y - z_n\|_{\mathcal{M}(S, V)} > 3\varepsilon$. Passing to a subsequence, we may assume that $x_n(s) \rightarrow x_0(s)$ for almost all $s \in S$. Repeating the arguments of the proof of [23, Theorem 6.3], we obtain from the lower semicontinuity of $f(\cdot, s, \cdot)$ (in the uniform sense) that there is a sequence y_n of measurable functions with $y_n(s) \in f(\lambda_n, s, x_0(s))$ and $|y_n(s) - z_n(s)| \leq \varepsilon$ for all $s \in E_n$ where $E_n \in \Sigma$ satisfy $E_n \uparrow S$ (up to a null set). By hypothesis, we can redefine y_n such that $y_n|_{S_i}$ is constant for each $i \in I$. Since $z_n \in S_f(\lambda_0, x_0)$ is constant on each $i \in I$, it remains true also for the redefined y_n that $|y_n(s) - z(s)| \leq \varepsilon$. Note that the redefined function satisfies $y_n \in S_f(\lambda_n, x_n)$. Now proceeding as in the proof of [23, Theorem 6.3], we obtain that $\|y_n - z_n\|_{\mathcal{M}(S, V)} < 3\varepsilon$ for all sufficiently large n which is a contradiction. \square

Combining Theorem 3.13 with Lemma 3.15 or Lemma 3.16, respectively, we obtain the following result:

Theorem 3.17 (Continuity for Functions of Carathéodory Type). *Let $X \subseteq \mathcal{M}(S, U)$ be an embeddable quasi-pseudometric generalized ideal space and simultaneously be a topological vector space (for the induced topology). Let $Y \subseteq \mathcal{M}(S, V)$ be a quasi-pseudometric generalized preideal space with regular part Y_0 .*

Let $S_f: \Lambda \times \mathcal{M}(S, U) \multimap \mathcal{M}(S, V)$ be a parameter-dependent superposition operator with atoms $(S_i)_{i \in I}$, and $(\lambda_0, x_0) \in D \subseteq \Lambda \times X$. Suppose that S_f is regular on E for (λ_0, x_0) on D and $f \preceq A + B$ on D outside of E such that A has equicontinuous norm in Y and for every $x \in X$ in a neighborhood of x_0 all measurable selections of the function $s \mapsto B(s, x(s))$ belong to Y_0 . Put $f_s(\lambda, u) := f(\lambda, s, u)$, and $F(\lambda, x)(s) := f(\lambda, s, x(s))$.

1. *Let the function $F(\lambda_0, x_0)$ be measurable in the Bochner sense and assume a.e. nonempty compact closed values. If λ_0 has a countable base of neighborhoods and for almost all $s \in S$, the function f_s is upper semicontinuous at $(\lambda_0, x_0(s))$ in the uniform sense, then $S_f: D \multimap Y$ is upper semicontinuous at (λ_0, x_0) in the uniform sense. If $F(\lambda_0, x_0)$ is constant on each S_i and assumes only nonempty values, then $S_f(\lambda_0, x_0) \neq \emptyset$.*

2. Suppose that for each $(\lambda, x) \in D$ the function $F(\lambda, x)$ is measurable in the Bochner sense, assumes a.e. nonempty compact values, and is constant on each S_i . Then $S_f: D \multimap Y$ assumes only nonempty values on D . If additionally λ_0 has a countable base of neighborhoods and for almost all $s \in S$, the function f_s is lower semicontinuous at $(\lambda_0, x_0(s))$ (in the uniform sense) then $S_f: D \multimap Y$ is lower semicontinuous at (λ_0, x_0) (in the uniform sense).

For the case $I = \emptyset$ (no atoms) and parameter-independent f , Theorem 3.17 was obtained in [23], and our proof followed that approach as far as possible. Special cases (for ideal spaces in \mathbb{R}^n) have been obtained in [4] and [3, Theorem 8.2], where however, also an analogous result for the upper semicontinuity (*not* in the uniform sense) was claimed, mistakenly: The example given in [23, Example 6.2] shows that S_f is practically never upper semicontinuous if f is not single-valued.

It should not be too surprising that for the claim $S_f(\lambda, x) \neq \emptyset$ we restrict ourselves to those (λ, x) for which $F(\lambda, x)$ is constant on the atoms. Also for lower semicontinuity, one cannot expect to drop this hypothesis, as can be seen even in the single-valued autonomous case:

Example 3.18. Let $S := [0, 2]$ with the Lebesgue measure, consider only one atom $I := \{1\}$, $S_1 := [0, 1]$, and $f(\lambda, s, u) := \{u\}$. Then $S_f(\lambda, x)$ is either $\{x\}$ or empty, depending on whether $x|_{[0,1]}$ is constant or not. In particular, $S_f: \Lambda \times L_p(S) \multimap L_q(S)$ fails to be lower semicontinuous at $(0, 0)$ for every $p, q \in (0, \infty]$.

Having this example in mind, it might appear rather surprising that we do *not* need any restriction for D or f concerning the atoms $(S_i)_i$ for the upper semicontinuity in the uniform sense in Theorem 3.17.

4. Differentiability in Products of Orlicz Spaces

For the rest of this paper, we will deal only with normed spaces; in particular, from now on, we assume that $(U, |\cdot|)$ is normed.

Moreover, we consider a much smaller class of function spaces than in the previous section, since even the class of ideal spaces is somewhat too large to obtain reasonable criteria for the differentiability of superposition operators. In fact, even for the classical superposition operator (single-valued, without atoms and parameters) there is not too much known about differentiability in ideal spaces (except e.g. [5, Theorem 2.14]). The largest subclass for which sharp criteria are known is the class of Orlicz spaces [5]. Therefore, we concentrate on that class of spaces. We recall the definition of that class.

Definition 4.1. A *Young function* is a function $\Phi: S \times \mathbb{R} \rightarrow [0, \infty]$ with the following two properties:

1. For any measurable function $x: S \rightarrow \mathbb{R}$ the superposition function $s \mapsto \Phi(s, x(s))$ is measurable.
2. For almost all $s \in S$, the function $\Phi(s, \cdot)$ is even, convex, and lower semicontinuous on \mathbb{R} with $\Phi(s, 0) = 0$. Moreover, Φ is not constant (i.e., only 0 or ∞) on $\mathbb{R} \setminus \{0\}$.

The *Orlicz space* $L_\Phi(S, U)$ generated by a Young function Φ is the set of all measurable functions $x: S \rightarrow U$ for which there is some $\lambda > 0$ with

$$\int_S \Phi(s, \lambda^{-1} |x(s)|) ds < \infty.$$

It is well-known that $L_\Phi(S, U)$ becomes a normed preideal space with the *Luxemburg norm*

$$\|x\|_{L_\Phi} := \inf \left\{ \lambda > 0 : \int_S \Phi(s, \lambda^{-1} |x(s)|) ds \leq 1 \right\}.$$

Moreover, the monotone convergence theorem implies that this space is perfect and thus an ideal space if U is a Banach space, see e.g. [21, Corollary 3.2.4].

Example 4.2. The spaces $L_p(S, U)$ ($1 \leq p \leq \infty$) are Orlicz spaces with the Young function $\tilde{\Phi}_p(s, u) = |u|^p$ (in case $p < \infty$) resp.

$$\tilde{\Phi}_\infty(s, u) := \begin{cases} 0 & \text{if } |u| \leq 1 \\ \infty & \text{if } |u| > 1. \end{cases}$$

By using the argument s of Φ , one sees that also L_p -spaces with weight-functions or $L_{p(\cdot)}$ spaces where $p(\cdot)$ varies with s are Orlicz spaces in the above sense.

Throughout this section, let U_1, \dots, U_m and V be real normed spaces, and $U := U_1 \times \dots \times U_m$. Let Φ_1, \dots, Φ_m and Ψ be Young functions, $X := L_{\Phi_1}(S, U_1) \times \dots \times L_{\Phi_m}(S, U_m)$, and $Y := L_\Psi(S, V)$. As in Section 2, we consider a given multivalued superposition operator S_f with a topological parameter space Λ and atoms $S_i \subseteq S$ ($i \in I$).

In the single-valued case, it is easy to guess how the derivative of S_f must look like. In order to formulate a corresponding result, we recall that in case of single-valued f and in the presence of atoms one cannot expect that the domain of definition of S_f contains an interior point. Hence, we call a map $F: D \rightarrow Y$ with $D \subseteq X$ *differentiable* at $x_0 \in D$ with a derivative $A: X \rightarrow Y$, if A is linear with

$$\lim_{\substack{h \rightarrow 0 \\ h \in D - x_0}} \frac{\|F(x_0 + h) - F(x_0) - Ah\|_Y}{\|h\|_X} = 0.$$

If x_0 is not an interior point of D , then A is not necessarily uniquely determined by this requirement. Nevertheless, A is uniquely determined on the set D_A° of all $h \in X$ with the property that there is a null sequence $t_n \in \mathbb{R} \setminus \{0\}$ with $x_0 + t_n h \in D$. By linearity, A is then of course even uniquely determined on $D_A := \text{span } D_A^\circ$.

Proposition 4.3. *Suppose that f is single-valued, $D \subseteq X$, and that the map $S_f(\lambda_0, \cdot): D \rightarrow Y$ is differentiable at x_0 with a derivative $A: X \rightarrow Y$, and that $f(\lambda_0, s, \cdot)$ is Gateaux differentiable at $x_0(s)$ for almost all $s \in S$ with derivative $f_u(\lambda_0, s, x_0(s)): U \rightarrow V$. Then we have for all $h \in D_A$ that*

$$Ah(s) = f_u(\lambda_0, s, x_0(s))h(s) \tag{7}$$

for almost all $s \in S$ and that (7) is constant on each S_i .

Proof. For $t \in \mathbb{R} \setminus \{0\}$ put $F_t(h) := t^{-1}(S_f(\lambda_0, x_0 + th) - S_f(\lambda_0, x_0)) - Ah$. Then $\|F_t(h)\|_{L_\Psi(S, V)} \rightarrow 0$ as $t \rightarrow 0$. Hence, if $h \in D_A^\circ$ and $t_n \neq 0$ is a null sequence with $x_0 + t_n h \in D_A$, we find by [21, Corollary 3.1.2] a subsequence n_k with $F_{t_{n_k}}(h)(s) \rightarrow 0$ for almost all $s \in S$. Since $F_{t_{n_k}}(h)(s) \rightarrow f_u(\lambda_0, s, x_0(s))h(s) - Ah(s)$ for almost all $s \in S$ and the first term in the definition of $F_t(h)$ is constant on each S_i , we obtain (7) and that Ah is constant on each S_i . Since this holds for every $h \in D_A^\circ$, it holds by linearity of both sides of (7) also for every $h \in D_A$. \square

Remark 4.4. If $I = \emptyset$ then at least in the scalar case (i.e., $U = V = \mathbb{R}$), one has a similar representation even without the hypothesis that $f(\lambda_0, s, \cdot)$ is Gateaux differentiable, if one assumes that x_0 is an interior point of $\{x : (\lambda_0, x) \in D\}$, see [5, Theorem 2.14].

Hence, in all natural situations, the derivative of $S_f(\lambda_0, \cdot)$ (if it exists) can be considered as a linear superposition operator $S_{f_u(\lambda_0, x_0)}$ with the same family of atoms. Thus, in order to prove that the derivative exists (and is given by $S_{f_u(\lambda_0, x_0)}$) one has to replace f by

$$\tilde{f}(\lambda_0, s, u) := f(\lambda_0, s, x_0(s) + u) - f(\lambda_0, s, x_0(s)) - f_u(\lambda_0, s, x_0(s))u \tag{8}$$

and has to verify that $\|S_{\tilde{f}}(\lambda_0, h)\|_Y = o(\|h\|_X)$. Summarizing, in order to discuss the differentiability of S_f , we can assume without loss of generality that $x_0 = 0$ and discuss only the case that the derivative is 0.

Similar considerations apply for higher derivatives of order $\alpha \in \mathbb{N}$ where it is for a similar reason in practical cases possible to consider only the case that all these derivatives are 0, and where by Peano's remainder term in the Taylor series the only difference is to replace $o(\|h\|_X)$ by $o(\|h\|_X^\alpha)$. For the case of scalar classical superposition operators (which in this respect is not different from ours), details can be found in [1, 2].

Summarizing our previous discussion: After perhaps replacing f by an auxiliary function, we are interested in showing that $\|S_f(\lambda, h)\| = o(\|h\|_X^\alpha)$ as $h \rightarrow 0$. This is what we study in this section now, observing also that it is useful (e.g. for bifurcation problems with λ as a bifurcation parameter) to know that such estimates holds uniformly w.r.t. λ (or at least as $(\lambda, h) \rightarrow (\lambda_0, 0)$). This question makes also sense in the multivalued case.

Given a multivalued map $f: \Lambda \times S \times U \multimap V$ and $\Lambda_0 \subseteq \Lambda$, we thus look for criteria such that

$$\sup_{\|x\|_X \leq r} \sup_{\lambda \in \Lambda_0} \sup_{y \in S_f(\lambda, x)} \|y\|_Y \leq c(r) \tag{9}$$

or

$$\limsup_{\substack{(\lambda, r) \rightarrow (\lambda_0, 0) \\ \lambda \in \Lambda, r > 0}} \sup_{\|x\|_X \leq r} \sup_{y \in S_f(\lambda, x)} \frac{\|y\|_Y}{c(r)} \leq 1 \tag{10}$$

holds where $c(r) \geq 0$ satisfies $\frac{c(r)}{r^\alpha} \rightarrow 0$ as $r \rightarrow 0$. Here, we use the conventions $\sup \emptyset := 0$, $\frac{0}{0} := 0$, $\frac{\infty}{c(r)} := \infty$, and put $\|y\|_Y := \infty$ if $y \notin Y$. For (10), we do not necessarily require that the accumulation point λ_0 belongs to Λ ; it may also belong to some larger space of which Λ is only a subspace. In the latter case, we will in a slight misuse of notation call a set $\Lambda_0 \subseteq \Lambda$ a neighborhood of λ_0 if it is an intersection of a neighborhood of λ_0 with Λ .

In our spaces, the problems (9) and (10) are theoretically almost completely answered by the following result. In order to quantify the estimates precisely, we assume that the Cartesian product space X is equipped with the max-norm.

For the rest of this section, we do not require that S be σ -finite; in particular, the number of atoms may also be uncountable.

Theorem 4.5. *Let $c, r \in (0, \infty)$ be given, and $\Lambda_0 \subseteq \Lambda$. Suppose that for each $\lambda \in \Lambda_0$, there are $C_\lambda \in [0, \infty)$ and functions $a_\lambda \in L_1(S, \mathbb{R})$ with $\|a_\lambda\|_{L_1} \leq C_\lambda$ such that for almost all $s \in S$ the estimate*

$$\sup_{v \in f(\lambda, s, u_1, \dots, u_m)} \Psi\left(s, \frac{C_\lambda + m}{c} |v|_V\right) \leq a_\lambda(s) + \sum_{j=1}^m \Phi_j\left(s, \frac{1}{r} |u_j|_{U_j}\right) \tag{11}$$

holds for all $(u_1, \dots, u_m) \in U$. Then (9) holds with $c(r) := c$.

Theorem 4.6. *Let $C \in [0, \infty)$, $\alpha \in (0, \infty)$. Suppose that for each $\varepsilon > 0$ there is some neighborhood Λ_0 of λ_0 (resp. suppose $\Lambda_0 \subseteq \Lambda$ is fixed) and some $R > 0$ such that for each $r \in (0, R)$ and each $\lambda \in \Lambda_0$ there are functions $a_{\lambda, r} \in L_1(S, \mathbb{R})$ with $\|a_{\lambda, r}\|_{L_1} \leq C$ such that for almost all $s \in S$ the estimate*

$$\sup_{v \in f(\lambda, s, u_1, \dots, u_m)} \Psi\left(s, \frac{1}{\varepsilon r^\alpha} |v|_V\right) \leq a_{\lambda, r}(s) + \sum_{j=1}^m \Phi_j\left(s, \frac{1}{r} |u_j|_{U_j}\right) \tag{12}$$

holds for all $(u_1, \dots, u_m) \in U$. Then (10) (or (9), respectively) holds with a function $c: (0, \infty) \rightarrow [0, \infty]$ satisfying $\frac{c(r)}{r^\alpha} \rightarrow 0$ as $r \rightarrow 0$.

Theorem 4.6 follows immediately from Theorem 4.5 by defining $c(r)$ as the left-hand side of (9) with Λ_0 corresponding to the choice $\varepsilon := r$. Indeed, Theorem 4.5 implies that for each $\varepsilon > 0$ there is some $R > 0$ with $c(r) \leq \varepsilon \frac{r^\alpha}{C+m}$ for all $r \in (0, R)$.

Remark 4.7. In contrast to Section 3, we do not consider an exceptional set $E \in \Sigma$ here on which the estimate (11) can be relaxed if one is interested only in

$$\lim_{r \rightarrow 0} \sup_{\substack{(\lambda, x) \in D \\ \|x\| \leq r, \lambda \in \Lambda_0}} \frac{\sup_{y \in S_f(\lambda, x)} \|y\|_Y}{r^\alpha} = 0 \quad (13)$$

or

$$\lim_{\substack{(\lambda, r) \rightarrow (\lambda_0, 0) \\ \lambda \in \Lambda, r > 0}} \sup_{\substack{(\lambda, x) \in D \\ \|x\|_X \leq r}} \frac{\sup_{y \in S_f(\lambda, x)} \|y\|_Y}{r^\alpha} = 0 \quad (14)$$

for some particular set $D \subseteq \Lambda \times X$ (where, e.g., all corresponding x are constant on certain subsets of E). Indeed, this would not make the result more general, since if one has other means to prove

$$\lim_{r \rightarrow 0} \sup_{\substack{(\lambda, x) \in D \\ \|x\| \leq r, \lambda \in \Lambda_0}} \frac{\sup_{y \in S_f(\lambda, x)} \|P_E y\|_Y}{r^\alpha} = 0$$

for such a set E , one obtains (13) by applying the above result on the measure space $S \setminus E$ and then using the triangle inequality for the sum $y = P_E y + P_{S \setminus E} y$; similarly for (14).

Proof of Theorem 4.5. Let $\|x\|_X \leq r$. Since we assume the max-norm on the product space X , this means that writing $x = (x_1, \dots, x_m)$ we have $x_j \in L_{\Phi_j}(S, U_j)$ and $\|x_j\|_{L_{\Phi_j}} \leq r$ for all $j \in \{1, \dots, m\}$. The definition of the Luxemburg norm thus implies

$$\int_S \Phi_j \left(s, \frac{|x_j(s)|_{U_j}}{r + \varepsilon} \right) ds \leq 1 \quad (j = 1, \dots, m) \quad (15)$$

for any $\varepsilon > 0$; letting $\varepsilon = \frac{1}{n}$ and using the monotone convergence theorem (recalling that $\Phi_j(s, \cdot)$ is monotone on $[0, \infty)$ and lower semicontinuous), we see that (15) holds also for $\varepsilon = 0$. Moreover, the convexity of $\Psi(s, \cdot)$ and $\Psi(s, 0) = 0$ imply

$$\Psi(s, t) = \Psi \left(s, \frac{(C_\lambda + m)t + 0}{C_\lambda + m} \right) \leq \frac{1}{C_\lambda + m} \Psi(s, (C_\lambda + m)t).$$

Hence, for any $y \in S_f(\lambda, x)$, the hypotheses imply

$$\begin{aligned} \int_S \Psi\left(s, \frac{1}{c} |y(s)|_V\right) ds &\leq \frac{1}{C_\lambda + m} \int_S \Psi\left(s, \frac{C_\lambda + m}{c} |y(s)|_V\right) ds \\ &\leq \frac{1}{C_\lambda + m} \left(\int_S a_\lambda(s) ds + \sum_{j=1}^m \int_S \Phi_j\left(s, \frac{|x_j(s)|_{U_j}}{r}\right) ds \right) \\ &\leq \frac{1}{C_\lambda + m} (C_\lambda + m) = 1, \end{aligned}$$

and so we have proved $y \in Y$, $\|y\|_Y \leq c$. \square

Since the proof of Theorem 4.5 (and thus of Theorem 4.6) is so strikingly straightforward, one might conjecture that much better results are available, but the lower estimate in [5, Theorem 4.3] (cf. also [2, Satz 2]) shows that this is not the case (up to possibly some multiplicative constant), at least if f is single-valued without parameters and atoms and $U = V = \mathbb{R}$.

In the setting of Lebesgue-Bochner spaces $X = L_{p_1}(S, U_1) \times \cdots \times L_{p_m}(S, U_m)$ and $Y = L_q(S, V)$ with $p_j \in [1, \infty]$, $q \in [1, \infty)$, the inequality (12) is easier to understand: This corresponds to the particular case when $\Phi_j := \tilde{\Phi}_{p_j}$ and $\Psi := \tilde{\Phi}_q$ where $\tilde{\Phi}_p$ are defined as in Example 4.2. Hence, (12) means

$$\sup_{v \in f(\lambda, s, u_1, \dots, u_m)} |v|^q \leq \varepsilon^q r^{\alpha q} a_{\lambda, r}(s) + \sum_{j=1}^m \varepsilon^q r^{\alpha q} \tilde{\Phi}_{p_j}\left(s, \frac{1}{r} |u_j|_{U_j}\right). \quad (16)$$

Note that Theorem 4.6 requires that this holds for all small $r > 0$. Apparently, this becomes very restrictive w.r.t. to the dependency on u_j in case $\alpha q > p_j$ (and also in case $\alpha q = p_j$) for some j .

For $\alpha = 1$ this is of course in accordance with Krasnoselskij's classical example: If the (classical) superposition operator is differentiable from $L_p([0, 1])$ into $L_q([0, 1])$ where $q > p$ or $q = p$ then this operator is automatically constant resp. affine, cf. e.g. [5, Theorem 3.12]. In fact, a similar result holds even for operators like $Fu(s) = f(s, \nabla u(s))$ from $W_0^{1,p}$ into L_q [20, Theorem 6.1].

Since we are not interested in such "degeneration" results here, we suppose that $\alpha q < p_j$ for all j . In this case, (16) can be reformulated such that the dependency on ε and r can be expressed in one variable δ . In the following formulation we also use Jensen's inequality to make the hypothesis look somewhat analogous to Example 3.10.

Theorem 4.8. *Let $X = L_{p_1}(S, U_1) \times \dots \times L_{p_m}(S, U_m)$ with $p_j \in [1, \infty]$, and $Y = L_q(S, V)$ with $q \in [1, \infty)$, and let $\alpha \in (0, \infty)$ be such that $\alpha q < p_j$ for all j . Let J_∞ and J_∞^c denote the set of all indices j such that $p_j = \infty$ or $p_j \neq \infty$, respectively. Suppose that for each $j \in J_\infty$ there are constants $\varepsilon_j \in (0, \infty)$ and $c_j \in [0, \infty)$ and that there is a monotone null sequence $\delta_n > 0$ with bounded $\frac{\delta_n}{\delta_{n+1}}$ such that for each $\lambda \in \Lambda_0$ (resp. $\lambda \in \Lambda$) there are functions $b_{\lambda,n} \in L_q(S, \mathbb{R})$ with $\|b_{\lambda,n}\|_{L_q} \rightarrow 0$ uniformly in $\lambda \in \Lambda_0$ as $n \rightarrow \infty$ (resp. as $(\lambda, n) \rightarrow (\lambda_0, \infty)$) such that for every index n and every λ the following holds for almost all $s \in S$:*

$$\sup_{v \in f(\lambda, s, u_1, \dots, u_m)} |v|_V \leq \delta_n^\alpha b_{\lambda,n}(s) + \sum_{j \in J_\infty^c} \delta_n^{\alpha - \frac{p_j}{q}} c_j |u_j|_{U_j}^{\frac{p_j}{q}} \tag{17}$$

for all $(u_1, \dots, u_m) \in U$ with $|u_j| < \varepsilon_j$ for all $j \in J_\infty$. Then (9) (resp. (10)) holds with a function $c: (0, \infty) \rightarrow [0, \infty]$ satisfying $\frac{c(r)}{r^\alpha} \rightarrow 0$ as $r \rightarrow 0$.

Proof. We use the above notation $\Phi_j := \tilde{\Phi}_{p_j}$ and $\Psi := \tilde{\Phi}_q$ with $\tilde{\Phi}_p$ as in Example 4.2 Without loss of generality, we assume $c_j > 0$. Given $\varepsilon > 0$ and $C := 1$, it suffices to show by Theorem 4.6 that (16) holds for all small $r > 0$ and all $\lambda \in \Lambda_0$ (Λ_0 a later specified neighborhood of λ_0) when we choose $a_{\lambda,r}$ with $\|a_{\lambda,r}\|_{L_1} \leq 1$ appropriately. For sufficiently small $r > 0$ there is some largest index $n = n(r)$ with $\delta_n \geq r$ and

$$\delta_n \geq r \left(\frac{\varepsilon}{c_j} \right)^{\frac{q}{\alpha q - p_j}} \quad (j \in J_\infty^c).$$

Then $n(r) \rightarrow \infty$ as $r \rightarrow 0$. Put $\delta(r) := \delta_{n(r)}$. Since $\frac{\delta_n}{\delta_{n+1}}$ is bounded, it follows from our choice of $n(r)$ (since the right-hand sides of the above formulas depend linearly on r) that $q(r) := \frac{\delta(r)}{r}$ is bounded as $r \rightarrow 0$. Hence, putting

$$a_{\lambda,r}(s) := \varepsilon^{-q} q(r)^{\alpha q} |b_{\lambda,n(r)}(s)|^q,$$

we thus have $\|a_{\lambda,r}\|_{L_1} \rightarrow 0$ uniformly with respect to $\lambda \in \Lambda_0$ as $r \rightarrow 0$ (resp. as $(\lambda, r) \rightarrow (\lambda_0, 0)$). In particular, there is some $R > 0$ (and some neighborhood Λ_0 of λ_0) with $\|a_{\lambda,r}\|_{L_1} \leq C = 1$ for all $r \in (0, R)$ and all $\lambda \in \Lambda_0$. We may also assume that $R \leq \varepsilon_j$ for all $j \in J_\infty$.

If $u = (u_1, \dots, u_m) \in U$ is such that $|u_j| \geq \varepsilon_j$ for some $j \in J_\infty$ then we have for all $r \in (0, R)$ that $\frac{|u_j|}{r} > 1$, and so (16) holds trivially, since the j -th term in the last sum is infinite. Otherwise, we can use (17), and so for each $v \in f(\lambda, s, u)$, we have by Jensen's inequality for every $r \in (0, R)$

$$\begin{aligned} |v|^q &\leq \delta(r)^{\alpha q} |b_{\lambda,n(r)}(s)|^q + \sum_{j \in J_\infty^c} \delta(r)^{\alpha q - p_j} c_j^q |u_j|_{U_j}^{p_j} \\ &\leq \varepsilon^q r^{\alpha q} a_{\lambda,r}(s) + \sum_{j \in J_\infty^c} \varepsilon^q r^{\alpha q - p_j} |u_j|_{U_j}^{p_j}. \end{aligned}$$

Hence, (16) holds in all cases for all $r \in (0, R)$. □

Theorem 4.8 is a generalization of [1, Theorem 4] (or of [5, Theorem 3.13]) and actually even in case $m = 1$ somewhat easier to verify: For $\lambda \in \Lambda$, one only has to determine countably many functions $b_{\lambda,n}$, needs to verify (17) only for small numbers $\delta_n > 0$, and also the constants c_j can be chosen conveniently for the verification. Note that the converse implication of [1, Theorem 4] implies that Theorem 4.8 is close to the best possible one can say.

In contrast to what is claimed in [1, Theorem 4], our result even holds if $\mu(S) = \infty$ (moreover, recall that we even do not suppose here that S be σ -finite). This is rather surprising, because, for instance, there is no absolutely continuous embedding of $L_q(\mathbb{R})$ into $L_p(\mathbb{R})$ for $p > q$, not even an embedding, in general, although one might expect from [5, Sections 3.6 and 4.6] that such an absolutely continuous embedding is necessary for the existence of nondegenerate differentiable superposition operators. But this is not the case. In fact, the following special case of Theorem 4.8 shows that there are a lot of superposition operators from e.g. $L_p(\mathbb{R})$ into $L_q(\mathbb{R})$ if $p > q$ which are differentiable at 0 although they are not degenerate in any sense, also not degenerate in the sense that their image near 0 does not have full support.

Theorem 4.9. *Let $X = L_{p_1}(S, U_1) \times \cdots \times L_{p_m}(S, U_m)$ with $p_j \in [1, \infty]$, and $Y = L_q(S, V)$ with $q \in [1, \infty)$, and let $\alpha \in (0, \infty)$ be such that $\alpha q < p_j$ for all j . Let J_∞ and J_∞^c denote the set of all indices j such that $p_j = \infty$ or $p_j \neq \infty$, respectively. Let $\Lambda_0 \subseteq \Lambda$. Suppose that there are $N \in \mathbb{N}$, $C \in (0, \infty)$, and $\varepsilon \in (0, \frac{p_j}{q} - \alpha)$ such that for each $j \in J_\infty$ there are $\varepsilon_j > 0$, and for each $j \in J_\infty^c$ and each $\lambda \in \Lambda_0$, there are N numbers $\beta_{\lambda,j,k} \in [\alpha + \varepsilon, \frac{p_j}{q})$ ($k = 1, \dots, N$) and corresponding measurable functions $a_{\lambda,j,k}: S \rightarrow \mathbb{R}$ with*

$$\int_S |a_{\lambda,j,k}(s)|^{\frac{p_j q}{p_j - \beta_{\lambda,j,k} q}} ds \leq C \quad (k = 1, \dots, N)$$

such that for almost all $s \in S$ the estimate

$$\sup_{v \in f(\lambda, s, u_1, \dots, u_m)} |v|_V \leq \sum_{j \in J_\infty^c} \sum_{k=1}^N a_{\lambda,j,k}(s) |u_j|_{U_j}^{\beta_{\lambda,j,k}} \tag{18}$$

holds for all $u = (u_1, \dots, u_m) \in U$ satisfying $|u_j| < \varepsilon_j$ for each $j \in J_\infty$. Then the hypotheses of Theorem 4.8 are satisfied, hence (9) holds with a function $c: (0, \infty) \rightarrow [0, \infty]$ satisfying $\frac{c(r)}{r^\alpha} \rightarrow 0$ as $r \rightarrow 0$.

Proof. For fixed $\lambda \in \Lambda_0$, the summand corresponding to j and k in (18) becomes in view of $p_j > \beta_{\lambda,j,k}q$ by Young's inequality for any $\delta > 0$

$$\begin{aligned} & \left(\delta^{\frac{(p_j - \alpha q)\beta_{\lambda,j,k}}{p_j}} a_{\lambda,j,k}(s) \right) \left(\delta^{-\frac{(p_j - \alpha q)\beta_{\lambda,j,k}}{p_j}} |u_j|_{U_j}^{\beta_{\lambda,j,k}} \right) \\ & \leq \frac{p_j - \beta_{\lambda,j,k}q}{p_j} \left(\delta^{\frac{(p_j - \alpha q)\beta_{\lambda,j,k}}{p_j}} |a_{\lambda,j,k}(s)| \right)^{\frac{p_j}{p_j - \beta_{\lambda,j,k}q}} \\ & \quad + \frac{\beta_{\lambda,j,k}q}{p_j} \left(\delta^{-\frac{(p_j - \alpha q)\beta_{\lambda,j,k}}{p_j}} |u_j|_{U_j}^{\beta_{\lambda,j,k}} \right)^{\frac{p_j}{\beta_{\lambda,j,k}q}} \\ & \leq \delta^{\frac{(p_j - \alpha q)\beta_{\lambda,j,k}}{p_j - \beta_{\lambda,j,k}q}} |a_{\lambda,j,k}(s)|^{\frac{p_j}{p_j - \beta_{\lambda,j,k}q}} + \delta^{\alpha - \frac{p_j}{q}} |u_j|_{U_j}^{\frac{p_j}{q}}. \end{aligned}$$

Now observe that the power of δ in the first summand is uniformly larger than α since $\beta_{\lambda,j,k} \geq \alpha + \varepsilon$. \square

It appears not so easy to include the two limit cases $\beta_{\lambda,j,k} \in \{\alpha, \frac{p_j}{q}\}$ in Theorem 4.9. The upper choice $\beta_{\lambda,j,k} = \frac{p_j}{q}$ will cause no problems if $\mu(S) < \infty$ and $a_{\lambda,j,k}$ are constant, as we will see. To allow the lower choice $\beta_{j,n} = \alpha$, one will at least have to assume that $f(\lambda, s, u)$ descends to 0 strictly faster than $|u|^\alpha$ in a sense. It seems that also this can be treated in a satisfactory way only if $\mu(S) < \infty$.

We will prepare now such a (sufficient) criterion. To this end, we introduce the notion of uniform convergence in measure: Suppose that for each $\lambda \in \Lambda$ there is a sequence of functions $y_{\lambda,n} : S \rightarrow [0, \infty]$. Then we write

$$\mu\text{-limsup}_{\substack{\lambda \in \Lambda_0 \\ n \rightarrow \infty}} y_{\lambda,n} = 0 \quad \text{or} \quad \mu\text{-limsup}_{(\lambda,n) \rightarrow (\lambda_0, \infty)} y_{\lambda,n} = 0 \quad (19)$$

if

$$\lim_{n \rightarrow \infty} \mu^*(\{s \in S : y_{\lambda,n}(s) > \varepsilon\}) = 0 \quad \text{uniformly w.r.t. } \lambda \in \Lambda_0,$$

or

$$\lim_{(\lambda,n) \rightarrow (\lambda_0, \infty)} \mu^*(\{s \in S : y_{\lambda,n}(s) > \varepsilon\}) = 0,$$

respectively, holds for each $\varepsilon > 0$, where μ^* denotes the outer measure for μ . We point out that this definition does not require that the functions $y_{\lambda,n}$ are measurable or that $\mu(S) < \infty$. However, if this is the case, the condition (19) can be described much easier.

Proposition 4.10. *If $\mu(S) < \infty$, $y_{\lambda,n}$ are measurable (and λ_0 has a countable base of neighborhoods), then (19) holds if and only if for every sequence $\lambda_n \in \Lambda_0$ (resp. $\lambda_n \rightarrow \lambda_0$) every subsequence of $y_{\lambda_n, n}$ contains a subsequence n_k with $y_{\lambda_{n_k}, n_k}(s) \rightarrow 0$ for almost all $s \in S$.*

Proof. By Egorov's and Riesz' theorems, the condition in the claim is equivalent to the fact that for every sequence $\lambda_n \in \Lambda_0$ (resp. $\lambda_n \rightarrow \lambda_0$) and each $\varepsilon > 0$ we have

$$\lim_{n \rightarrow \infty} \mu(\{s \in S : y_{\lambda_n, n}(s) > \varepsilon\}) = 0,$$

see e.g. [23, Proposition 2.2]. □

Roughly speaking, the following result states that under the condition that $f(\lambda, \cdot, u) = o(|u|^\alpha)$ in measure, one does not need to require in Theorem 4.8 that $\|b_{\lambda, n}\|_{L^q} \rightarrow 0$, but mere boundedness suffices. However, we are able to prove this only if the functions $b_{\lambda, n}$ are constant and independent of λ :

Proposition 4.11. *Let $X = L_{p_1}(S, U_1) \times \dots \times L_{p_m}(S, U_m)$ with $p_j \geq 1$, and $Y = L_q(S, V)$ with $q \geq 1$. Suppose $\alpha \in (0, \infty)$, and $\alpha q < p_j < \infty$ for all j . Let $b, C_j \in [0, \infty)$ and a monotone null sequence $\delta_n > 0$ with bounded $\frac{\delta_n}{\delta_{n+1}}$ be given such that for any $\lambda \in \Lambda_0$ (resp. any $\lambda \in \Lambda$) and any index n the following holds for almost all $s \in S$:*

$$\sup_{v \in f(\lambda, s, u_1, \dots, u_m)} |v|_V \leq \delta_n^\alpha b + \sum_{j=1}^m \delta_n^{\alpha - \frac{p_j}{q}} C_j |u_j|_{U_j}^{\frac{p_j}{q}} \tag{20}$$

for all $u = (u_1, \dots, u_m) \in U$. If

$$\mu\text{-}\limsup_{\substack{\lambda \in \Lambda_0 \\ n \rightarrow \infty}} \frac{\sup\{|v|_V : v \in f(\lambda, \cdot, u), |u|_U < \delta_n\}}{\delta_n^\alpha} = 0 \tag{21}$$

(or

$$\mu\text{-}\limsup_{(\lambda, n) \rightarrow (\lambda_0, \infty)} \frac{\sup\{|v|_V : v \in f(\lambda, \cdot, u), |u|_U < \delta_n\}}{\delta_n^\alpha} = 0, \tag{22}$$

respectively) holds, then the hypotheses of Theorem 4.8 are satisfied for some $b_{\lambda, n}$ and c_j .

Proof. Without loss of generality, we may assume that the product space U is equipped with the max-norm. By (21) resp. (22), there are a null sequence $d_n > 0$ and sets $E_{\lambda, n} \in \Sigma$ with $\mu(E_{\lambda, n}) \rightarrow 0$ uniformly w.r.t. $\lambda \in \Lambda_0$ as $n \rightarrow \infty$ (resp. as $(\lambda, n) \rightarrow (\lambda_0, \infty)$) and

$$S \setminus E_{\lambda, n} \subseteq \{s \in S : |v|_V \leq d_n \delta_n^\alpha \text{ whenever } v \in f(\lambda, s, u), |u|_U < \delta_n\}.$$

We show that the hypotheses of Theorem 4.8 hold with $c_j := C_j + b$ and

$$b_{\lambda, n}(s) := \begin{cases} b & \text{if } s \in E_{\lambda, n} \\ d_n & \text{if } s \notin E_{\lambda, n}. \end{cases}$$

Indeed, $\|b_{\lambda,n}\|_{L^q} \rightarrow 0$ follows from $\mu(E_{\lambda,n}) \rightarrow 0$ and $d_n \rightarrow 0$. We have to prove that (17) holds. In case $s \in E_{\lambda,n}$, (17) follows in view of $b = b_{\lambda,n}(s)$ and $C_j \leq c_j$ immediately from (20). In case $s \in S \setminus E_{\lambda,n}$, we assume first $|u|_U < \delta_n$. Then our choice of $E_{\lambda,n}$ implies

$$\sup_{v \in f(\lambda,s,u)} |v|_V \leq d_n \delta_n^\alpha = \delta_n^\alpha b_{\lambda,n}(s)$$

which in turn implies (17) also in this case. In the remaining case $|u|_U \geq \delta_n$ there is some index $j_0 = j$ with $|u_j|_{U_j} \geq \delta_n$. With this j_0 , the first term on the right-hand side of (20) is

$$\delta_n^\alpha b = \delta_n^{\alpha - \frac{p_{j_0}}{q}} \delta_n^{\frac{p_{j_0}}{q}} b \leq \delta_n^{\alpha - \frac{p_{j_0}}{q}} b |u_{j_0}|_{U_{j_0}}^{\frac{p_{j_0}}{q}},$$

hence combining this estimate with the j_0 -th summand in (20), we obtain (17) also in this case. \square

As [1, Example at end of Section 1] shows, the criteria of Proposition 4.11 and Theorem 4.9 are far from being necessary. However, hypotheses (20), (21), and (22) of Proposition 4.11 are rather simple to verify as we show in the following two lemmas.

The first of these lemmas means that Proposition 4.11 allows to treat the limit cases $\beta_{\lambda,j,k} \in \{\alpha, \frac{p_j}{q}\}$ which had to be omitted in Theorem 4.9. Since we are restricted to the case of constant coefficients and $\mu(S) < \infty$ anyway, it is no loss of generality to consider estimates which involve *only* these limit cases, i.e., it is reasonable to formulate the hypothesis without further interpolating numbers $\beta_{\lambda,j,k}$ as we did in Proposition 4.11.

Lemma 4.12. *Let $X = L_{p_1}(S, U_1) \times \cdots \times L_{p_m}(S, U_m)$ with $p_j \geq 1$, and $Y = L_q(S, V)$ with $q \geq 1$, and $\alpha \in (0, \infty)$, $\alpha q < p_j < \infty$. Assume that there are $c_{j,1}, c_{j,2} \in [0, \infty)$ such that for every $\lambda \in \Lambda$ we have that for almost all $s \in S$ the estimates*

$$\sup_{v \in f(\lambda,s,u_1,\dots,u_m)} |v|_V \leq \sum_{j=1}^m (c_{j,1} |u_j|_{U_j}^{\frac{p_j}{q}} + c_{j,2} |u_j|_{U_j}^\alpha) \quad (23)$$

for every $(u_1, \dots, u_m) \in U$. Then there are $b, C_j \in [0, \infty)$ such that (20) holds for every $\delta_n \in (0, 1]$.

Proof. For any $\delta \in (0, 1]$ we have

$$c_{j,1} |u_j|_{U_j}^{\frac{p_j}{q}} \leq c_{j,1} \delta^{\alpha - \frac{p_j}{q}} |u_j|_{U_j}^{\frac{p_j}{q}},$$

and, moreover, the same calculation as in the proof of Theorem 4.9 with $\beta_{\lambda,j,k} := \alpha$ and $a_{j,n}(s) := c_{j,2}$ (using Young's inequality) shows that for any $\delta > 0$

$$c_{j,2} |u_j|_{U_j}^\alpha \leq \delta^\alpha c_{j,2}^{\frac{p_j}{p_j - \alpha q}} + \delta^{\alpha - \frac{p_j}{q}} |u_j|_{U_j}^{\frac{p_j}{q}}.$$

Adding both estimates, we obtain (20) from (23) for every $\delta_n = \delta \in (0, 1]$. \square

Lemma 4.13. *Let $\alpha \in (0, \infty)$. Assume $\mu(S) < \infty$ (and that λ_0 has a countable base of neighborhoods). Suppose that for any sequence $\lambda_n \in \Lambda_0$ (resp. $\lambda_n \rightarrow \lambda_0$), there are a monotone null sequence $r_n > 0$, with $\frac{r_n}{r_{n+1}}$ being bounded, and measurable functions $f_n: S \rightarrow [0, \infty]$ satisfying*

$$\sup_{|u|_U < r_n} \sup_{v \in f(\lambda_n, s, u)} |v|_V \leq f_n(s) \tag{24}$$

for almost all $s \in S$ and such that every subsequence contains a subsequence n_k such that

$$\lim_{k \rightarrow \infty} \frac{f_{n_k}(s)}{r_{n_k}^\alpha} = 0 \tag{25}$$

for almost all $s \in S$. Then (21) (resp. (22)) holds for all null sequences $\delta_n > 0$.

Proof. If (21) resp. (22) fails for a null sequence $\delta_n > 0$ then there are $\varepsilon, \delta > 0$, $\lambda_n \in \Lambda_0$ (resp. $\lambda_n \rightarrow \lambda_0$) and $u_n \in U$ with $|u_n|_U < \delta_n$, and

$$\mu^* \left(\left\{ s \in S : \sup_{v \in f(\lambda_n, s, u_n)} |v|_V > \varepsilon \delta_n^\alpha \right\} \right) > \delta \tag{26}$$

for infinitely many n . Let $r_n > 0$ be the corresponding sequence of the hypothesis. Since $\delta_n \rightarrow 0$ and r_n is a monotone null sequence with $\frac{r_n}{r_{n+1}} \leq c_0$, there are numbers $n_k \rightarrow \infty$ with $c_0^{-1} r_{n_k} \leq \delta_k < r_{n_k}$ for all large k . By (24), we obtain from (26) that $\mu^* \left(\left\{ s \in S : f_{n_k}(s) > \varepsilon c_0^{-\alpha} r_{n_k}^\alpha \right\} \right) > \delta$ for infinitely many k which is a contradiction in case $\mu(S) < \infty$, since Egorov's theorem implies by (24) that $f_{n_k} r_{n_k}^{-\alpha} \rightarrow 0$ in measure (cf. proof of Proposition 4.10). \square

Summarizing the previous observations, we obtain for instance the following convenient condition.

Theorem 4.14. *Let $X = L_{p_1}(S, U_1) \times \dots \times L_{p_m}(S, U_m)$ with $p_j \geq 1$, and $Y = L_q(S, V)$ with $q \geq 1$. Let $\alpha \in (0, \infty)$, $\Lambda_0 \subseteq \Lambda$, and λ_0 have a countable base of neighborhoods in Λ . Suppose that $\alpha q < p_j < \infty$ for all j and that $\mu(S) < \infty$. Assume that for any sequence $\lambda_n \in \Lambda_0$ (resp. $\lambda_n \rightarrow \lambda_0$) there are a monotone null sequence $r_n > 0$, $\frac{r_n}{r_{n+1}}$ being bounded, and measurable functions $f_n: S \rightarrow [0, \infty]$ satisfying (24) for almost all $s \in S$ and such that every subsequence contains a subsequence with (25) for almost all $s \in S$. Finally, assume that for almost all $s \in S$ the estimate (23) holds for all $(u_1, \dots, u_m) \in U$.*

Then (9) (resp. (10)) holds for all small $r > 0$ with a function $c: (0, \infty) \rightarrow [0, \infty]$ satisfying $\frac{c(r)}{r^\alpha} \rightarrow 0$ as $r \rightarrow 0$.

The proof follows from Proposition 4.11 whose assumptions are verified by the previous lemmas.

For easier reference, we give some particular application of Remark 4.7.

Theorem 4.15. *Let the hypotheses of Theorem 4.14 be satisfied only on $S \setminus E$ where E is a finite union of atoms S_{i_1}, \dots, S_{i_n} , and let $X_E \subseteq X$ denote the subset of all functions which are constant on each S_{i_j} . If*

$$\lim_{r \rightarrow 0} \sup_{|u|_U < r} \sup_{\lambda \in \Lambda_0} \sup_{s \in E} \frac{\sup_{v \in f(\lambda, s, u)} |v|}{r^\alpha} = 0$$

(or $\lim_{\substack{(\lambda, r) \rightarrow (\lambda_0, 0) \\ \lambda \in \Lambda, r > 0}} \sup_{|u|_U < r} \sup_{s \in E} \frac{\sup_{v \in f(\lambda, s, u)} |v|}{r^\alpha} = 0$ respectively),

then (13) (resp. (14)) holds with $D = \Lambda \times X_E$.

As described in the beginning of this section, all results of this section could be formulated in terms of classical Fréchet derivatives (in the single-valued and parameter-independent case) by applying them in case $\alpha = 1$ to the auxiliary function (8). As an example, here is such a reformulation of Theorem 4.14. To simplify the measurability hypotheses, we assume separability of the spaces U_j .

Theorem 4.16. *Let $U := U_1 \times \dots \times U_m$ be separable, $X = L_{p_1}(S, U_1) \times \dots \times L_{p_m}(S, U_m)$, and $Y = L_q(S, V)$ with $1 \leq q < p_j < \infty$ for all j . Let $\mu(S) < \infty$. Suppose that $f: S \times U \rightarrow V$ is such that $f(\cdot, u)$ is measurable for all $u \in U$ and that there is $a \in L_q(S)$ such that for almost all $s \in S$ the growth condition*

$$|f(s, u)| \leq a(s) + \sum_{j=1}^m b_j |u_j|_{U_j}^{\frac{p_j}{q}}$$

holds for all $u = (u_1, \dots, u_m) \in X$. Let $x_0 \in X$ be such that $f(s, \cdot)$ is for almost all $s \in S$ continuous on U and Fréchet differentiable at $x_0(s)$ with derivative $f_u(s, x_0(s)): U \rightarrow V$, and that there are $c_{j,i} \in [0, \infty)$ such that for almost all $s \in S$ the estimate

$$|f(s, u + x_0(s)) - f(s, x_0(s)) - f_u(s, x_0(s))u| \leq \sum_{j=1}^m (c_{j,1} |u_j|_{U_j}^{\frac{p_j}{q}} + c_{j,2} |u_j|_{U_j})$$

holds for every $u = (u_1, \dots, u_m) \in U$. Then $F(x)(s) := f(s, x_0(s))$ maps X into Y and is Fréchet differentiable at x_0 with derivative $(F'(x)h)(s) = f_u(s, x_0(s))(h(s))$.

Proof. It is well-known that $F: X \rightarrow Y$. The claim about the derivative follows from Theorem 4.14 with $\alpha = 1$ and the function (8). Here we use that the function $f_r(s) := \sup_{|u| < r} |\tilde{f}(\lambda, s, u)|_V$ occurring in (24) is measurable for every $r > 0$, since U is separable. Indeed, if $\{\tilde{u}_1, \tilde{u}_2, \dots\}$ is a countable dense subset of $\{u \in U : |u| < r\}$, the functions $f_{r,n}(s) := |\tilde{f}(\lambda, s, u_n)|_V$ are measurable and satisfy $f_r(s) = \sup_n f_{r,n}(s)$ for almost all $s \in S$. \square

For $m = 1$ and $U_1 = V = \mathbb{R}$, Theorem 4.16 becomes a variant of [27, Theorem 7] (for the latter also some s -dependency of $c_{1,2}$ is admissible and thus actually also a corresponding variant of Theorem 4.9 is covered). Various similar differentiability criteria for the classical superposition operator can be found in literature, see e.g. [18, 19] or [7, Proposition 1.1.4] where, however, usually the continuity (in [7, Proposition 1.1.4] even Hölder continuity) of $f_u(s, \cdot)$ on the whole space is required (in contrast, we only require that this function is defined on $u = x_0(s)$).

However, the main new contribution of this section concerns the dependency on the parameter λ which was apparently only studied in some very particular cases in the setting of Sobolev spaces (see the comments in the next part of the paper). The multivalued case even seems to be completely new.

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