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Newton Polygons and Formal Gevrey Classes

By

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Introduction

Following to the fundamental study of Malgrange [7], Ramis elucidated the analytic meaning of slope of Newton polygon for ordinary differential operators [10]: In generic cases the index of operator in formal Gevrey class of order s equals to the ordinate at the origin of supporting line of Newton polygon with slope $k = 1/(s-1)$. He also demonstrated various comparison theorems.

The purpose of this note is to generalize one aspect of Ramis theory to partial differential operators. There seems to be three ways of generalization:

1. To consider holonomic systems.
2. To consider operators of Kashiwara-Kawai-Sjöstrand type [1, 3].
3. To consider Cauchy problems.

For 1, 2, we refer to Laurent theory [4, 5, 6]. We shall discuss from the standpoint 3.

On the other hand, our study is closely related to the Cauchy-Kowalewski theorem. Mizohata's inverse Cauchy-Kowalewski theorem asserts that if the operator is not Kowalewskian, there exists a divergent formal solution [8]. It is well known that the formal solution of heat equation belongs to Gevrey class of order 2. The problem is what determines the Gevrey order of formal solutions.

From a different point of view, Ōuchi developed the theory concerning the analytic meaning of formal solutions [9]. It is certain that his theory implies one part of our theorem. There exists, however, more elementary and straightforward method to our problem.

§ 1. Notations

For $x = (x_1, x_2, \dots, x_n) \in \mathbf{C}^n$, we set $|x| = \max_{1 \le j \le n} |x_j|$. Let $\mathcal{O}(|x| < r)$ be the set of all holomorphic functions in $\{x \in \mathbf{C}^n; |x| < r\}$. We also set

$$\mathcal{O}(|x| \le r) = \mathscr{C}^0(|x| \le r) \cap \mathcal{O}(|x| < r)$$

where $\mathscr{C}^0(|x| \le r)$ is the set of all continuous functions on $\{x \in \mathbf{C}^n; |x| \le r\}$.

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It is obvious that $\mathcal{O}(|x| \leq r)$ is a Banach space with maximum norm $\|\cdot\|_r$.

Let $\mathbb{C}[[t, x]]$ be the set of formal power series with complex coefficients in $n+1$ indeterminates t, x. Let $\mathbb{C}\{t, x\}$ be the set of convergent power series in $n+1$ variables $(t, x) = (t, x_1, \dots, x_n)$. When we set $A = \mathcal{O}(|x| \leq r)$ or $\mathbb{C}\{x\}$, we denote by $A[[t]]$ the set of formal power series in t with coefficients in A. These are subspaces of $\mathbb{C}[[t, x]]$.

We shall use standard multi-indices notations:

$$D_t = \frac{\partial}{\partial t}, \qquad D_j = \frac{\partial}{\partial x_j} \qquad (j = 1, 2, \dots, n),$$

$$D_x^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} \quad \text{for} \quad \alpha \in \mathbb{N}^n .$$

§2. Definitions

Let P be a differential operator with coefficients $\in \mathbb{C}[[t, x]]$:

$$P = P(t, x; D_t, D_x) = \sum_{j,\alpha} a_{j,\alpha}(t, x) D_t^j D_x^\alpha = \sum_{j,\alpha} t^{\sigma(j,\alpha)} \tilde{a}_{j,\alpha}(t, x) D_t^j D_x^\alpha$$

where $\tilde{a}_{j,\alpha}(0, x) \neq 0$ in $\mathbb{C}[[x]]$. Let Q be the second quadrant of $\mathbb{R}^2$ and for $(u, v) \in \mathbb{R}^2$, we set

$$Q(u, v) = (u, v) + Q .$$

Definition. The Newton polygon of P, denoted by $N(P)$, is defined by the convex hull of the union of $Q(j + |\alpha|, \sigma(j, \alpha) - j)$ for j, α such that $a_{j,\alpha} \neq 0$ in $\mathbb{C}[[t, x]]$:

$$N(P) = ch\left(\bigcup_{a_{j,\alpha} \neq 0} Q(j + |\alpha|, \sigma(j, \alpha) - j) \right).$$

Let $0 = k_0 < k_1 < \cdots < k_l$ be the slopes of sides of $N(P)$.

Remark. If P is a differential operator with holomorphic coefficients, this definition is a special case of more general one [4, 5, 6]: If we choose

$$X = \mathbf{C}^{n+1} = \mathbf{C}_t \times \mathbf{C}_x^n,\ Y = \{t = 0\} \subset X,\ \Lambda = T_Y^* X \text{ and } O = (o; o) \in X ,$$

then according to Laurent's notation [5] we have

$$N(P) = N_{\Lambda, O}(P) .$$

Let us notice that this definition is different from that of Mizohata [8]. For example, it suffices to consider the operator $P = D_t^2 + D_t D_x^2 + t^2 D_x^5$.

To examine the analytic meaning of k_j, we define the functions of formal Gevrey class.

Definition. Let $s \geq 1$, $\rho > 0$ and $r > 0$. Then we denoted by $G_{\rho,r}^s$ the set of all $u = \sum_{j=0}^\infty u_j t^j \in \mathcal{O}(|x| \leq r)[[t]]$ such that

$$|u|_{\rho,r}^s \overset{\text{def}}{=} \sum_{j=0}^{\infty} \frac{\|u_j\|_r}{(j!)^{s-1}} \rho^j < +\infty .$$

Lemma 1. *$G^s_{\rho,r}$ is a Banach space with norm $|\cdot|^s_{\rho,r}$.*

The proof is obvious.

We set

$$G^s_\rho = \bigcup_{r>0} G^s_{\rho,r} \text{ and } G^s = \bigcup_{\rho>0} G^s_\rho .$$

Note that $G^1 = \mathbf{C}\{t, x\}$. If we also set $G^\infty = \mathbf{C}\{x\}[[t]]$, then we have interpolation spaces G^s between the space of convergent power series and that of formal power series: for $1 < s < \infty$,

$$\mathbf{C}\{t, x\} = G^1 \subset G^s \subset G^\infty = \mathbf{C}\{x\}[[t]] \subset \mathbf{C}[[t, x]] .$$

§ 3. Statement of Theorem

Let P be a differential operator of the following form:

$$P = D_t^m + \sum_{0 \le j < m} a_{j,\alpha}(t, x) D_t^j D_x^\alpha ,$$

where $a_{j,\alpha} \in G^s$. We assume that P is not Kowalewskian:

$$\text{ord } P > m .$$

We consider the Cauchy problem

$$(CP)\begin{cases} Pu = f(t, x) & \\ D_t^j u|_{t=0} = g_j & \text{for} \quad 0 \le j \le m-1 \end{cases}$$

where

$$f \in G^s,\ g_j \in \mathbf{C}\{x\} .$$

There exists a unique formal solution $u \in G^\infty$. The Cauchy-Kowalewski theorem asserts that, if P is Kowalewskian, u is convergent. We investigate precisely the relation between the divergence order of u and the Newton polygon of P.

Theorem 1. *Let $s = 1 + 1/k_1$. Then there exists a unique solution $u \in G^s$, satisfying (CP).*

Remark 1. Particularly for f, $a_{j,\alpha} \in \mathbf{C}\{t, x\}$, *a fortiori* the assertion of theorem holds. We rediscover one corollary of Ōuchi's results [9].

Remark 2. This result is best possible: In general one cannot lower the Gevrey order s. For example, let

$$n = 1, \quad P = D_t - t^\sigma D_x^m, \quad f = 0 \quad \text{and} \quad g = \sum_{j=0}^{\infty} x^j \in \mathcal{O}(|x| < 1) ,$$

where $\sigma \in N$, $m \ge 2$. Then we have

$$u = \sum_{i,j \ge 0} \frac{(mi + j)!}{(\sigma + 1)^i i! j!} t^{(\sigma+1)i} x^j, \quad k_1 = \frac{\sigma + 1}{m - 1} \quad \text{and} \quad s_1 = \frac{\sigma + m}{\sigma + 1} .$$

It follows that

$$u \in G^s \quad \text{for} \quad s \geq s_1, \quad \text{but} \quad u \notin G^s \quad \text{for} \quad s < s_1 .$$

§ 4. Formal Norm and Lemmas

For $u \in G^s_{\rho,r}$, we shall use the formal norm:

$$N_r^s[u](t) \overset{\text{def}}{=} \sum_{j=0}^{\infty} \frac{\|u_j\|_r}{(j!)^{s-1}} t^j .$$

If $|t| \leq \rho$, then we have

$$|N_r^s[u](t)| \leq |u|^s_{\rho,r}, \quad N_r^s[u](\rho) = |u|^s_{\rho,r} .$$

We set

$$(D_t^{-1}u)(t) = \sum_{j=0}^{\infty} u_j \frac{t^{j+1}}{j+1} \quad \text{for} \quad u \in \mathcal{O}(|x| \leq r)[[t]] .$$

Lemma 2. *Let $a, u \in G^s_{\rho,r}$. The following properties hold for $0 \leq t \leq \rho$:*

(1) $$N_r^s[au](t) \leq N_r^s[a](t) \cdot N_r^s[u](t)$$

(2) $$N_{r'}^s[D_i u](t) \leq \frac{1}{r - r'} N_r^s[u](t)$$

for $0 < r' < r$, $i = 1, 2, \ldots, n$.

The proof is straightforward. Inequality (1) asserts that $G^s_{\rho,r}$ is a Banach algebra. Notice that in general D_t nor D_i do not operate on $G^s_{\rho,r}$.

We define the operators A_s, B_s acting on $\mathbb{R}\{t\}$:

(3) $$N_r^s[D_t^{-1}u](t) = A_s(N_r^s[u])(t)$$

(4) where $$A_s\colon \sum c_j t^j \mapsto \sum c_j \frac{t^{j+1}}{(j+1)^s}$$

(5) $$N_r^s[tu](t) = B_s(N_r^s[u])(t)$$

(6) where $$B_s\colon \sum c_j t^j \mapsto \sum c_j \frac{t^{j+1}}{(j+1)^{s-1}}$$

Proposition 1. *Let T and s be non-negative real numbers. Let $f(t) = \sum_{j=0}^{\infty} c_j t^j \in \mathbb{R}\{t\}$ with radius of convergence $> T$. If $f(t) \geq 0$ for $0 \leq t \leq T$, then*

$$(L_s f)(t) \overset{\text{def}}{=} \sum_{j=0}^{\infty} c_j \frac{t^j}{(j+1)^s} \geq 0$$

for $0 \leq t \leq T$.

Since the assertion is trivial for $s = 0$, we assume $s > 0$. It suffices to prove that L_s has the following integral representation: for f stated above,

$$(L_s f)(t) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\tau}\tau^{s-1} f(te^{-\tau})\, d\tau\,. \tag{7}$$

The convergence of integral is proved in the same way as that of Euler's expression of Gamma-function. For $f(t) = t^n$, we have

$$\begin{aligned}\frac{1}{\Gamma(s)} \int_0^\infty e^{-\tau}\tau^{s-1}(te^{-\tau})^n\, d\tau &= \frac{t^n}{\Gamma(s)} \int_0^\infty e^{-(n+1)\tau}\tau^{s-1}\, d\tau \\ &= \frac{t^n}{(n+1)^s} \frac{1}{\Gamma(s)} \int_0^\infty e^{-\tau}\tau^{s-1}\, d\tau \\ &= \frac{t^n}{(n+1)^s}\,.\end{aligned}$$

This implies that (7) holds for f polynomial. The right side of (7) is a continuous operator in $\mathscr{C}^0[0, T]$ and $f_n = \sum_{j=0}^{j=n} c_j t^j$ converges to f in $\mathscr{C}^0[0, T]$. In addition $L_s(f - f_n) \to 0 (n \to \infty)$ in $\mathscr{C}^0[0, T]$ by the fact that Taylor series are absolutely and uniformly convergent on any compact subset in the circle of convergence. Thus (7) holds for any f stated above.

Since we have $A_{s-1} = B_s$, $A_s f = t(L_s f)(t)$, the proposition means that *operators* A_s, B_s *preserve inequalities.*

§5. Proof of Theorem 1

First we show that the assumption $s = 1 + 1/k_1$ implies that

$$|\alpha| \le (s-1)\sigma(j, \alpha) + s(m-j)\,. \tag{8}$$

Indeed, Newton polygon of P has both vertex $(m, -m)$ and side of slope k_1 through $(m, -m)$. Since the points $(j + |\alpha|, \sigma(j, \alpha) - j)$ are included in the upper half plane defined by $y \ge k_1(x - m) - m$, we obtain

$$\sigma(j, \alpha) - j \ge k_1(j + |\alpha|) - (k_1 + 1)m \Longleftrightarrow |\alpha| \le \frac{1}{k_1}\sigma(j, \alpha) + \left(1 + \frac{1}{k_1}\right)(m - j)\,,$$

which proves (8).

Let $P = D_t^m - Q$ where

$$Q = -\sum_{j=0}^{m-1} \tilde{a}_{j,\alpha} D_x^\alpha t^{\sigma(j,\alpha)} D_t^j\,.$$

We define a sequence $\{u_k\}$ as follows:

$$\begin{cases} D_t^m u_0 = f \\ D_t^j u_0|_{t=0} = g_j \qquad (0 \le j \le m-1)\,. \end{cases}$$

For $k \ge 0$,

$$\begin{cases} D_t^m u_{k+1} = Q u_k + f \\ D_t^j u_{k+1}|_{t=0} = g_j \qquad (0 \le j \le m-1)\,. \end{cases}$$

Next we set

$$v_0 = u_0\,,$$

$$v_{k+1} = u_{k+1} - u_k \quad \text{if} \quad k \geq 0\,.$$

Then we have for $k \geq 1$,

$$\begin{cases} D_t^m v_k = Qv_{k-1}\,, \\ D_t^j v_k|_{t=0} = 0 \qquad (0 \leq j \leq m-1)\,. \end{cases}$$

We also set $w_k = D_t^m v_k$, then we have for $k \geq 1$, $v_k = D_t^{-m} w_k$. Then the sequence $\{w_k\}$ satifies the following equation:

$$w_0 = D_t^m u_0 = f\,,$$

$$w_{k+1} = QD_t^{-m} w_k \qquad (k \geq 0)$$

where

$$QD_t^{-m} = \sum_{0 \leq j < m, \alpha} \tilde{a}_{j,\alpha} D_x^\alpha t^{\sigma(j,\alpha)} D_t^{-(m-j)} w_k\,. \tag{9}$$

Let T and r_0 be positive real numbers such that f, $\tilde{a}_{j,\alpha} \in G^s_{T,r_0}$. We fix $r_1 \in]0, r_0[$. It follows immediately that for $0 < \rho < T$ and $0 < r < r_0$,

$$u_k,\ v_k,\ w_k \in G^s_{\rho,r}\,.$$

Let K and M denote positive constants such that

$$N^s_{r_0}[f](T) = K \text{ and } N^s_{r_0}[\tilde{a}_{j,\alpha}](T) \leq M$$

for any $\tilde{a}_{j,\alpha}$ which appears in P. We prove the following inequality by induction on k: There exist a positive constant C such that for $k \in \mathbf{N}$ and $r \in]r_1, r_0[$,

$$N^s_r[w_k] \leq KC^k \frac{e^{dk} t^k}{(r_0 - r)^{dk}} \tag{10}$$

where $d = \max\{|\alpha|;\, a_{j,\alpha} \neq 0\}$.

Let us take $r \in]r_1, r_0[$ and $r' > r$. From (1), (2), (9), we have

$$\begin{aligned} N^s_r[w_{k+1}] &\leq \sum \frac{M}{(r'-r)^{|\alpha|}} N^s_{r'}[t^{\sigma(j,\alpha)} D_t^{-(m-j)} w_k] \\ &= \sum \frac{M}{(r'-r)^{|\alpha|}} (B_s^{\sigma(j,\alpha)} A_s^{m-j}) N^s_{r'}[w_k] \\ &= \sum \frac{M}{(r'-r)^{|\alpha|}} (B_s^{\nu(j,\alpha)} A_s^j) N^s_{r'}[w_k] \end{aligned} \tag{11}$$

where we set $\nu(j, \alpha) = \sigma(m-j, \alpha)$ for $1 \leq j \leq m$. Then from (8), we have

$$|\alpha| \leq (s-1)\nu(j,\alpha) + sj\,. \tag{12}$$

If we assume that (10) holds for k, we get from Proposition 1 and (11),

$$N_r^s[w_{k+1}] \le KMC^k e^{dk} \sum \frac{1}{(r'-r)^{|\alpha|}(r_0-r')^{dk}} (B^{\nu(j,\alpha)}A^j)[t^k] \,.$$

We now choose $r' = r + (r_0 - r)/(k+1)$, so that $r_0 - r' = (r_0 - r)/(1 + 1/k)$. Then for the coefficients of $t^{k+j+\nu(j,\alpha)}$ under sigma sign, we have

$$\frac{1}{(r'-r)^{|\alpha|}(r_0-r')^{dk}} \frac{1}{((k+1)\dots(k+j))^s((k+j+1)\dots(k+j+\nu(j,\alpha))^{s-1}}$$
$$= \frac{(1+1/k)^{kd}}{(r_0-r)^{|\alpha|+dk}} \frac{(k+1)^{|\alpha|}}{((k+1)\dots(k+j))^s((k+j+1)\dots(k+j+\nu(j,\alpha))^{s-1}}$$

By (12), the second fraction is less than or equal to

$$\left(\frac{(k+1)^j}{(k+1)\dots(k+j)}\right)^s \left(\frac{(k+1)^{\nu(j,\alpha)}}{(k+j+1)\dots(k+j+\nu(j,\alpha))}\right)^{s-1},$$

which is less than or equal to 1. Thus we obtain

$$N_r^s[w_{k+1}] \le KC^k \frac{e^{d(k+1)}t^{k+1}}{(r_0-r)^{d(k+1)}} M \sum_{j \ge 1,\alpha} (r_0 - r)^{d-|\alpha|} t^{j-1+\nu(j,\alpha)} \,.$$

It suffices to take the constant C by

$$C = M \sum_{j \ge 1,\alpha} (r_0 - r_1)^{d-|\alpha|} T^{j-1+\nu(j,\alpha)} \,.$$

If we choose $\varepsilon \in]0, T]$ such that

$$\frac{Ce^d\varepsilon}{(r_0-r)^d} < 1 \,,$$

it follows from (10) that $\sum_{k=0}^{\infty} w_k$ is convergent in $G^s_{\varepsilon,r}$. Since D_t^{-m} is a continuous operator in $G^s_{\varepsilon,r}$ and that D_t^m, Q: $G^s_{\varepsilon,r} \to G^s_{\varepsilon_1,r_1}$ are continuous operators for $\varepsilon_1 \in]0, \varepsilon[$, it follows that

$$u = \lim_{k\to\infty} u_k = \sum_{k=0}^{\infty} v_k \in G^s_{\varepsilon,r} \subset G^s_{\varepsilon_1,r_1}$$

and u satisfies (CP) in $G^s_{\varepsilon_1,r_1}$. The proof is complete.

§6. Further Generalizations

To make the assertions clear, we stated Theorem 1 under more restrictive assumptions, which we shall make less strict as follows.

1. Theorem 1 also holds for operators of the following type:

$$P = \sum_{j,\alpha} a_{j,\alpha}(t,x) D_t^j D_x^\alpha$$

where $a_{m,0}(t,x)$ is a unit in $\mathbb{C}[[t,x]]$ and the point $(m, -m)$ is a vertex of $N(P)$. Notice that in this case order of P with respect to D_t may be larger than m.

2. For P, we denote its principal part by

$$\sigma(P)=\sum{}' a_{j,\alpha}D_t^jD_x^\alpha$$

where $\sum'$ means that sum is taken for all (j, α) such that $\sigma(j, \alpha)-j=\min[\sigma(j, \alpha)-j]$, namely sum of the terms of P which correspond to the points lying on the side of $N(P)$ parallel to abscissa. The operators discussed so far have the term D_t^m as principal part.

Theorem 2. *The assertion of Theorem* 1 *also holds for operators P such that* $\sigma(P)$ *is* Fuchsian *in the sense of Baouendi-Goulaouic under the usual conditions on characteristic exponents* [2].

Needless to say we have to modify the number of Cauchy data in this case.

These assertions are proved in the same way as Theorem 1.

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