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Continuity and Differentiability of Multivalued Superposition Operators with Atoms and Parameters II

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Abstract. For a given single- or multivalued function f and "atoms" S_i, let $S_f(\lambda, x)$ be the set of all measurable selections of the function $s \mapsto f(\lambda, s, x(s))$ which are constant on each S_i. It is discussed how this definition must be extended so that S_f can serve as a right-hand side for PDEs when one is looking for weak solutions in Sobolev spaces. Continuity and differentiability of the corresponding operators are studied.
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1. Introduction

This paper is a continuation of [19], where multivalued superposition operators with atoms in spaces of measurable functions had been studied. The aim of this paper is to apply that study in spaces of Sobolev functions. Let us briefly recall the motivation of atoms which originates from obstacle problems for PDEs. Consider on a domain S an equation like

$$-\Delta u(s) \in f(\lambda, s, u(s), \nabla u(s)) \quad \text{on } S, \quad u|_{\partial S} = 0, \tag{1}$$

where f is either single-valued or also contains some "jumps", e.g. for some real-valued functions g, h, u_0

$$f(\lambda, s, u, v) = \begin{cases} \{g(\lambda, s, u, v)\} & \text{if } u < u_0(\lambda, s, v) \\ [g(\lambda, s, u, v) - h(\lambda, s, u, v), g(\lambda, s, u, v)] & \text{if } u = u_0(\lambda, s, v) \\ \{g(\lambda, s, u, v) - h(\lambda, s, u, v)\} & \text{if } u > u_0(\lambda, s, v); \end{cases}$$

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such jumps are an important tool for modeling unilateral obstacles, e.g. a source or sink working under some conditions (see [8, 9, 18] for more realistic such problems described by *systems* of equations). Now it can happen that on some disjoint subsets $S_i \subseteq S$ $(i \in I)$ the obstacle does not act "pointwise" but only in an averaged sense, mathematically e.g. described by integrals like

$$\begin{aligned} -\Delta u(s) \equiv const \in f\left(\lambda, s, \int_{S_i} u(t)\,dt, \int_{S_i} \nabla u(t)\,dt\right) \quad \text{on } S_i, \\ -\Delta u(s) \in f(\lambda, s, u(s), \nabla u(s)) \quad \text{on } S \setminus \bigcup_{i \in I} S_i, \quad u|_{\partial\Omega} = 0, \end{aligned} \tag{2}$$

see [18]. Heuristically, on the obstacle S_i the "obstacle's cause" $(u, \nabla u)$ is averaged on the right-hand side. Hence, it makes sense, heuristically, to require as in (2) that the "obstacle's effect" $-\Delta u$ should (in the simplest case) be constant on S_i. In fact, the latter follows even automatically in similar problems from a natural weak formulation [7, 10].

Now the operator on the right-hand side of (2) can be described as the composition of differential and integral operators and of the (multivalued) operator

$$\begin{aligned} S_f(\lambda, u, v) :=\{y : y \text{ measurable}, y(s) \in f(\lambda, s, u(s), v(s)) \text{ a.e.}, \\ \text{and } y|_{S_i} \text{ a.e. constant for every } i\}. \end{aligned}$$

We call S_f the superposition operator with parameter λ and *atoms* S_i. Mathematically, the meaning of the atoms is that we do not consider *arbitrary measurable* selections of $f(\lambda, \cdot, u(\cdot), v(\cdot))$, but only those selections which are measurable on the "reduced" measure space where we identify S_i as atoms of the measure space (recall that measurable functions are by definition a.e. constant on atoms of a measure space).

As mentioned above, the aim of the paper is to study continuity and differentiability properties of S_f in Sobolev spaces. Usually, the latter would follow by standard procedures (embedding theorems) from the corresponding results in L_p spaces which was studied in [19]. Unfortunately, due to the multivaluedness of f, there can occur a surprising difficulty: A natural way to define weak solutions of (1) in the Sobolev space $W_0^{1,2}(S)$ is to define them as those u for which the variational equation

$$\int_S \nabla u(s) \cdot \nabla \varphi(s)\,ds = \int_S y(s)\varphi(s)\,ds \quad \text{for all } \varphi \in W_0^{1,2}(S) \tag{3}$$

holds with some $y \in S_f(\lambda, u, \nabla u)$. If $S_f(\lambda, u, \nabla u)$ is single-valued, it is equivalent to require

$$\int_S \nabla u(s) \cdot \nabla \varphi(s)\,ds \in \left\{\int_S y(s)\varphi(s)\,ds : y \in S_f(\lambda, u, \nabla u)\right\} \quad \text{for all } \varphi \in W_0^{1,2}(S), \tag{4}$$

but in the multivalued case, it is not clear whether these equations are actually equivalent (i.e., whether y in (4) can be chosen independent of φ). On the one hand, we thus have a problem of heuristics: Is (3) or (4) the "natural" weak formulation of (1)? On the other hand, no matter how we answer this question, we have a serious mathematical problem if we want to apply e.g. degree theory for multivalued maps: To apply this theory, one needs that the multivalued map is upper semicontinuous *and* has closed values. However, roughly speaking, under natural hypotheses, the former can be proved for the right-hand side of (3), and the latter for the right-hand side of (4), but not vice versa. To solve the mathematical and the heuristic problem simultaneously, we will show that actually the corresponding operators are the same, and so (cf. Examples 3.2 and 3.4) the problems (3) and (4) are actually equivalent (under natural hypotheses), and the corresponding maps are upper semicontinuous *and* have closed values. To the author's knowledge this equivalence is a new result (for multivalued superposition operators) even without parameters and atoms, but the atoms S_i bring an additional aspect into its proof: Note that it is crucial that the test function φ in (3) and (4) is *not* assumed to be constant on each S_i, but only the selection y of S_f should have this property. This explains that it is not sufficient to just pass to a measure space with atoms, but that the set S_i must be considered as a feature of S_f only.

2. General Notations

By a multivalued function $F\colon X \multimap Y$, we mean a function from X into the powerset of Y. In contrast to usual practice, we will allow that $F(x)$ is empty. As usual for multivalued functions, we will use the notation

$$F(X_0) := \bigcup_{x\in X_0} F(x) \quad (X_0 \subseteq X).$$

In [19], we had given the definition of upper/lower semicontinuity (in the uniform sense) in a general setting. We repeat this definition here for the simpler setting that Y is metric space. In this case, an ε-uniform neighborhood of a set $M \subseteq Y$ is the set of all $y \in Y$ which have distance less than ε to M. In contrast, a topological neighborhood of M is just any set $N \subseteq Y$ containing an open subset containing M.

Definition 2.1. Let $F\colon X \multimap Y$ be a multivalued function between a topological space X and a topological (metric) space Y.

1. F is *upper semicontinuous at* $x_0 \in X$ (*in the uniform sense*) if for each topological (ε-uniform) neighborhood $N \subseteq Y$ of $F(x_0)$ there is a neighborhood $M \subseteq X$ of x_0 with $F(M) \subseteq N$.

2. F is *lower semicontinuous at* $x_0 \in X$ if for each $y \in F(x_0)$ and each topological neighborhood $N \subseteq Y$ of y there is a neighborhood $M \subseteq X$ of x_0 with $F(x) \cap M \neq \varnothing$ for each $x \in M$.
3. F is *lower semicontinuous at* $x_0 \in X$ *in the uniform sense* if for each $\varepsilon > 0$ there is a neighborhood $M \subseteq X$ of x_0 such that for each $x \in M$ and each $y \in F(x_0)$ there is some $z \in F(x)$ with $d(y, z) < \varepsilon$.

What we call upper/lower semicontinuous at x_0 in the uniform sense is in literature sometimes called upper/lower semicontinuous at x_0 in the ε-sense [1] or (δ, ε)-upper/lower semicontinuous [3]. The following result was obtained in [19] in a more general setting.

Proposition 2.2. *If F is upper semicontinuous at x_0 and Y is a metric space, then F is upper semicontinuous at x_0 in the uniform sense; the converse holds if $F(x_0)$ is compact. If F is lower semicontinuous at x_0 in the uniform sense then it is lower semicontinuous at x_0; the converse holds if Y is a metric space and $F(x_0)$ is precompact.*

Throughout this paper, (S, Σ, μ) will denote a complete σ-finite measure space, and $S_i \in \Sigma$ $(i \in I)$ will denote a fixed family of pairwise disjoint sets with $\mu(S_i) > 0$ $(i \in I)$ which will play the role of atoms mentioned in the introduction. It is explicitly admissible that $I = \varnothing$, in which case the results of this paper deal with "ordinary" multivalued superposition operators. As shown in the first part of the paper, I is at most countable.

Let $(U, |\cdot|)$ and $(V, |\cdot|)$ be normed spaces, and Λ is a topological space. Then every map $f\colon \Lambda \times S \times U \multimap V$ induces a (multivalued and parameter-dependent) "superposition operator S_f with the atoms $(S_i)_{i \in I}$" in the following sense.

Definition 2.3. For $f\colon \Lambda \times U \multimap V$, let $S_f\colon \Lambda \times \mathscr{M}(S, U) \multimap \mathscr{M}(S, V)$ be defined as follows. For $\lambda \in \Lambda$ and $x \in \mathscr{M}(S, U)$, let $S_f(\lambda, x)$ denote the set of all $y \in \mathscr{M}(S, V)$ with the following two properties.

1. $y(s) \in f(\lambda, s, x(s))$ for almost all $s \in S$.
2. $y|_{S_i}$ is constant (almost everywhere) for every $i \in I$.

In particular, S_f depends on the atoms S_i $(i \in I)$, although we do not mark this dependency explicitly in the notation.

3. Continuity and Differentiability in Sobolev Spaces

In the weak formulation of PDEs, one is usually not directly interested in the superposition operator but with a related operator which occurs e.g. in (3) or (4). In this section, we consider, roughly speaking, "superposition operators" F and F_K defined by the right-hand side of (3) and (4), respectively. For F, we can transfer the continuity results of the first part of the paper in a straightforward manner (Theorem 3.3), and concerning F_K, we can prove that the values are closed or even compact (Theorem 3.10). The main task of this section will be to show that both operators actually coincide (Theorem 3.7) so that we can conclude in Theorems 3.11 and 3.13 that F simultaneously is continuous and has closed or compact values (and thus is e.g. upper semicontinuous *not* only in the uniform sense). As a side result, the equality of the corresponding operators implies that (3) and (4) are actually equivalent. At the end of the section, we also show how the differentiability results of [19] carry over to F.

In order to not overburden this paper with even more technicalities, we restrict ourselves to the setting of classical (nonweighted) Sobolev spaces and measure spaces of finite measure, although generalizations would be possible and are actually straightforward.

Let $\Omega \subseteq \mathbb{R}^N$ be a (not necessarily bounded) domain with a Lipschitz boundary. Let $\Omega_0 \subseteq \Omega$ be a subset of finite measure and $\Gamma \subseteq \partial\Omega$ be a subset of finite $(N-1)$-dimensional Hausdorff measure. Here, the hypotheses $\Omega_0 \subseteq \Omega$ and $\Gamma \subseteq \partial\Omega$ are only for the sake of simplicity of notation; all results in this section remain true (with appropriate change in the notation) if $\Omega_0 \subseteq \overline{\Omega}$ has finite measure and $\Gamma \subseteq \overline{\Omega}$ is a subset of a Lipschitz path with finite $(N-1)$-dimensional Hausdorff measure.

We will consider the measure space $S := \Omega_0 \cup \Gamma$. The atoms $S_i \subseteq S$ $(i \in I)$ can even be measurable subsets of S, i.e., S_i can even be the union of a subset of Ω_0 and of Γ.

Let $U_1, \dots, U_m$ be real normed spaces, $p_1, \dots, p_m \in [1, \infty]$, and W_0 be a linear subspace of $W^{1,p_1}(\Omega, U_1) \times \cdots \times W^{1,p_m}(\Omega, U_m)$. We use the notations

$$p_{0,i}^* := \begin{cases} \frac{p_i N}{N-p_i} & \text{if } N > p_i \\ \in [1,\infty) & \text{if } N = p_i > 1 \\ \infty & \text{if } N < p_i \text{ or } N = p_i = 1, \end{cases}$$

$$p_{1,i}^* := \begin{cases} \frac{p_i (N-1)}{N-p_i} & \text{if } N > p_i \\ \in [1,\infty) & \text{if } N = p_i > 1 \\ \infty & \text{if } N < p_i \text{ or } N = p_i = 1, \end{cases}$$

where in case $N = p_i > 1$ we mean that one can choose any number from the interval $[1, \infty)$ in the definition. We put

$$\begin{aligned} X := &\big(L_{p_{0,1}^*}(\Omega_0, U_1) \times \cdots \times L_{p_{0,m}^*}(\Omega_0, U_m)\big) \times \big(L_{p_1}(\Omega_0, (U_1^*)^N) \times \cdots \times L_{p_m}(\Omega_0, (U_m^*)^N)\big) \\ &\oplus \big(L_{p_{1,1}^*}(\Gamma, U_1) \times \cdots \times L_{p_{1,m}^*}(\Gamma, U_m)\big); \end{aligned}$$

here and in the following, we denote for a normed space U its dual space by U^*.

Similarly, let $V_1, \dots, V_n$ be real normed spaces, $q_1, \dots, q_n \in [1, \infty]$, and let W be a linear subspace of $W^{1,q_1}(\Omega, V_1) \times \cdots \times W^{1,q_n}(\Omega, V_n)$. We put

$V := V_1 \times \cdots \times V_n$ and use the notations

$$q_{0,i}^* := \begin{cases} \frac{q_i N}{q_i(N+1)-N} & \text{if } N > q_i \\ \in (1,\infty] & \text{if } N = q_i > 1 \\ 1 & \text{if } N < q_i \text{ or } N = q_i = 1, \end{cases}$$

$$q_{1,i}^* := \begin{cases} \frac{q_i(N-1)}{q_i N - N} & \text{if } N > q_i \\ \in (1,\infty] & \text{if } N = q_i > 1 \\ 1 & \text{if } N < q_i \text{ or } N = q_i = 1, \end{cases}$$

choose $q_{j,i} \geq q_{j,i}^*$, and put

$$Y := \big(L_{q_{0,1}}(\Omega_0, V_1^*) \times \cdots \times L_{q_{0,n}}(\Omega_0, V_n^*)\big) \oplus \big(L_{q_{1,1}}(\Gamma, V_1^*) \times \cdots \times L_{q_{1,n}}(\Gamma, V_n^*)\big).$$

Occasionally, we will also consider the space

$$Y' := \big(L_{q'_{0,1}}(\Omega_0, V_1) \times \cdots \times L_{q'_{0,n}}(\Omega_0, V_n)\big) \oplus \big(L_{q'_{1,1}}(\Gamma, V_1) \times \cdots \times L_{q'_{1,n}}(\Gamma, V_n)\big)$$

where $\frac{1}{q'_{j,i}} + \frac{1}{q_{j,i}} = 1$.

We will assume throughout that $q_{0,i} < \infty$ (if $\Omega_0 \neq \varnothing$) and that $q_{1,i} < \infty$ (if $\Gamma \neq \varnothing$). In particular, in the limit case $q_i = 1 < N$, our considerations only apply when $\Gamma = \varnothing$ (although one could treat this case by considering Orlicz spaces, but we will not do this here). The reason for the choice of the constants are of course the Sobolev and trace embedding theorems which in our case can be summarized in the following result.

Proposition 3.1. *The linear maps $T_0\colon W_0 \to X$, $T_1\colon W \to Y'$, and $T\colon Y \to W^*$, defined (in the usual sense of traces) by $T_0 u := (u|_{\Omega_0}, \nabla u|_{\Omega_0}, u|_\Gamma)$, $T_1 v := (v|_{\Omega_0}, \nabla v|_\Gamma)$, and*

$$(Tv)w := \int_{\Omega_0} v(s)w(s)\,ds + \int_\Gamma v(s)w(s)\,ds$$

are bounded. Moreover, T_1 and T are compact maps if $V_1, \ldots, V_n$ are finite-dimensional and

$$N \geq q_i \implies \big(q_{0,i} > q_{0,i}^* \quad \textit{and} \quad q_{1,i} > q_{1,i}^*\big) \quad (i = 1, \ldots, n). \tag{5}$$

Proof. The claims about T and T_0 follow from classical embedding theorems (see e.g. [20]). Note that $Y \subseteq (Y')^*$, and T is the restriction of the adjoint operator T_1^* to Y and thus bounded resp. compact by Schauder's theorem. □

Let now Λ be a topological space, and for $j = 1, \ldots, n$, let

$$f_{0,j} \colon \Lambda \times \Omega_0 \times \prod_{i=1}^m U_i \times \prod_{i=1}^m (U_i^*)^N \multimap V_j^*$$
$$f_{1,j} \colon \Lambda \times \Gamma \times \prod_{i=1}^m U_i \multimap V_j^*.$$

Recall that we assumed $S = \Omega \cup \Gamma$. We will later consider $D \subseteq \Lambda \times X$. Let $E \subseteq S$ be a (possibly empty) union of finitely many atoms such that for any $(\lambda, x) \in D$ the function x is constant on the atoms in E. For almost all $s \in S \backslash E$, we require the growth estimates

$$\begin{aligned} \sup_{y \in f_{0,j}(\lambda, s, u_1, \ldots, u_m, v_1, \ldots, v_m)} |y| &\leq a_{0,\lambda,j}(s) + \sum_{i=1}^m b_{0,i} \, |u_i|^{\frac{p_{0,i}^*}{q_{0,i}}} + \sum_{i=1}^m c_i \, |v_i|^{\frac{p_i}{q_{0,i}}} \\ \sup_{y \in f_{1,j}(\lambda, s, u_1, \ldots, u_m)} |y| &\leq a_{1,\lambda,j}(s) + \sum_{i=1}^m b_{1,i} \, |u_i|^{\frac{p_{1,i}^*}{q_{1,i}}} \end{aligned} \tag{6}$$

where $a_{0,\lambda,j} \in L_{q_{0,j}}(\Omega_0)$, $a_{1,\lambda,j} \in L_{q_{1,j}}(\Gamma)$ and $b_{k,i}, c_i \in [0, \infty)$. For those i with $N < p_i$ (i.e., with $p_{k,i}^* = \infty$), we replace the terms $b_{k,i} \, |u_i|^{p_{k,i}^* q_{k,i}^{-1}}$ in (6) by arbitrary functions $b_{k,i}(s, u)$ with the property that for each $r > 0$ there are $b_{0,i,r} \in L_{q_{0,i}}(\Omega_0)$ and $b_{1,i,r} \in L_{q_{1,i}}(\Gamma)$ with $|b_{k,i}(s, u)| \leq b_{k,i,r}(s)$ for $|u| \leq r$.

On E, we require instead of (6) only that $f_{0,j}(\cdot, s, \cdot)$ and $f_{1,j}(\cdot, s, \cdot)$ are locally bounded (actually we need the latter only in a neighborhood of $(\lambda_0, x_0(s))$ with $(\lambda_0, x_0) \in D$ in Theorem 3.3). We define now piecewise

$$f \colon \Lambda \times S \times \prod_{i=1}^m U_i \times \prod_{i=1}^m (U_i^*)^N \times \prod_{i=1}^m U_i \multimap \prod_{j=1}^n V_j^*$$

by

$$f(\lambda, s, u, v, w) := \begin{cases} \prod_{j=1}^n f_{0,j}(\lambda, s, u, v) & \text{if } s \in \Omega_0 \\ \prod_{j=1}^n f_{1,j}(\lambda, s, w) & \text{if } s \in \Gamma. \end{cases}$$

Finally, with T, T_0 as in Proposition 3.1, we are able to define the map we are actually interested in: We define $F \colon \Lambda \times W_0 \multimap W^*$ by $F(\lambda, u) := T(S_f(\lambda, T_0(u)))$.

Example 3.2. Consider for simplicity $m = n = 1$, $U_1 = V_1 = \mathbb{R}$. In the space $W = W_0 = W_0^{1,2}(\Omega)$, the operator $J \colon W_0 \to W^*$, defined by $J(u)(\varphi) := \int_S \nabla u(s) \cdot \nabla \varphi(s) \, ds$, is an isomorphism ($J = id$ after an identification $W^* = W_0$ with an an equivalent scalar product), and the problem (3) is equivalent to $J(u) \in F(\lambda, u)$. In this sense, the weak formulation of (1) is the inclusion $u \in F(\lambda, u)$ (with $\Omega_0 = \Omega$, $\Gamma = \varnothing$, $f_{0,1} = f$).

Similar examples could be given for problems with boundary conditions of the type $\frac{\partial u}{\partial n}(s) \in f_{1,1}(\lambda, s, u(s))$ on $\Gamma \subseteq \partial\Omega$ (in the space $W = W_0 = \{u \in W^{1,2}(\Omega) : u|_{\partial\Omega\setminus\Gamma} = 0\}$) or for problems with the p-Laplacian instead of the Laplacian (working in $W = W_0 = W_0^{1,p}(\Omega)$ with a different auxiliary map J).

It will be convenient to introduce the functions

$$\begin{aligned} F_0(\lambda, x)(s) &:= f(\lambda, s, x(s)) \\ f_s(\lambda, u) &:= f(\lambda, s, u). \end{aligned} \tag{7}$$

Theorem 3.3 (Continuity and Boundedness/Compactness of F)**.** *Under the growth condition* (6) (*on $S \setminus E$ and the locally boundedness on E mentioned after* (6))*, the map F is well-defined. Moreover, we have for any $(\lambda_0, u_0) \in D \subseteq \Lambda \times W_0$:*

1. *If $M_0 \subseteq W_0$ is bounded and $\Lambda_0 \subseteq \Lambda$ is such that $\{\|a_{k,\lambda,j}\|_{L_{q_{k,j}}} : \lambda \in \Lambda_0\}$ are bounded, and that $f_{k,j}(\cdot, s, \cdot)$ are for $s \in E$ uniformly bounded on $D_1 := D \cap (\Lambda_0 \times M_0)$ $(k = 0, 1$ and $j = 1, \ldots, n)$, then $S_f(D_1) \subseteq Y$ and $F(D_1) \subseteq W^*$ are bounded. If additionally* (5) *holds and $V_1, \ldots, V_n$ are finite-dimensional, the set $F(D_0) \subseteq W^*$ is even precompact.*
2. *Let $F_0(\lambda_0, T_0u_0)$ be measurable in the Bochner sense with compact values. If $F_0(\lambda_0, T_0u_0)$ is constant on each S_i $(i \in I)$ and assumes only nonempty values, then $F(\lambda_0, u_0) \neq \varnothing$. If $\{a_{k,\lambda,j} : \lambda \in \Lambda_0\}$ has equicontinuous norm in $L_{q_{k,j}}$ for a neighborhood Λ_0 of a point λ_0 with a countable base of neighborhoods, and for almost all $s \in S$ the function* (7) *is upper semicontinuous at $(\lambda_0, T_0u_0(s))$ in the uniform sense, then $F\colon D \mapsto W^*$ is upper semicontinuous at (λ_0, u_0) in the uniform sense.*
3. *Let for each $(\lambda, u) \in D$ the function $F_0(\lambda, T_0u)$ be measurable in the Bochner sense with nonempty compact values, and be constant on each of the atoms S_i. Then F assumes only nonempty values on D. Suppose in addition that $\{a_{k,\lambda,j} : \lambda \in \Lambda_0\}$ has equicontinuous norm in $L_{q_{k,j}}$ for a neighborhood Λ_0 of a point λ_0 with a countable base of neighborhoods, and for almost all $s \in S$ the function* (7) *is lower semicontinuous at $(\lambda_0, T_0u_0(s))$. Then $F\colon D \mapsto W^*$ is lower semicontinuous at (λ_0, u_0). Moreover, $F\colon D \mapsto W^*$ is lower semicontinuous at (λ_0, u_0) in the uniform sense if either* (7) *is lower semicontinuous at $(\lambda_0, T_0u_0(s))$ in the uniform sense for almost all $s \in S$ or if* (5) *holds and $V_1, \ldots, V_n$ are finite-dimensional.*
4. *If all values of the functions* (7) *are convex for almost all $s \in S$, then also all values of S_f and F are convex.*

Analogous assertions hold even for the map $T \circ S_f\colon D_0 \multimap W^*$ at $(\lambda_0, x_0) \in D_0 \subseteq \Lambda \times X$.

Proof. It suffices to prove the last statement, i.e., to prove the assertions for the map $T \circ S_f$. Indeed, since $id \otimes T_0$ is bounded and continuous, the corresponding assertions for $F = T \circ S_f \circ (id \otimes T_0)$ then follow with the choice $D_0 := (id \otimes T_0)(D)$.

For $j = 1, \dots, n$ and $\lambda \in \Lambda_0$ let A_{Λ_0} denote the set of all functions

$$a_{\lambda,j}(s) := \begin{cases} a_{0,\lambda,j}(s) & \text{if } s \in \Omega_0 \\ a_{1,\lambda,j}(s) & \text{if } s \in \Gamma. \end{cases}$$

For $s \in \Omega_0$ resp. $s \in \Gamma$, let $B_{0,j}(s, u_1, \dots, u_m, v_1, \dots, v_m)$ resp. $B_{1,j}(s, u_1, \dots, u_m)$ denote the set of all $v \in V_j^*$ whose norm is bounded by the sum in (6), and put

$$B(\lambda, s, u, v, w) := \begin{cases} \prod_{j=1}^n B_{0,j}(\lambda, s, u, v) & \text{if } s \in \Omega_0 \\ \prod_{j=1}^n B_{1,j}(\lambda, s, w) & \text{if } s \in \Gamma. \end{cases}$$

Then $f|_{\Lambda_0 \times S \times V} \preceq A_{\Lambda_0} + B$ outside of E, and a straightforward estimate with Minkowski's inequality shows that $S_B^\circ \colon X \multimap Y$ and, moreover, if $X_0 \subseteq X$ is bounded then $S_B^\circ(X_0)$ is bounded in Y. Hence, also $S_f(D \cap (\Lambda_0 \times X_0)) \subseteq A_{\Lambda_0} + S_B^\circ(B_0)$ is bounded in Y if A_{Λ_0} is bounded in Y. This implies the first claim by Proposition 3.1. For the continuity claims, we obtain from [19] that $S_f(\lambda, x) \neq \varnothing$ resp. that $S_f \colon D_0 \multimap Y$ is upper/lower semicontinuous (in the uniform sense) at $(\lambda_0, x_0) \in D_0$. Hence, the cointinuitity carries over to the composition $T \circ S_f$. The second claim concerning lower semicontinuity follows from Proposition 2.2.

If the functions (7) assume only convex values for almost all s, then clearly also $S_f(\lambda, x)$ is convex for every $(\lambda, x) \in \Lambda \times X$. Since T is linear, it follows that $T(S_f(\lambda, x))$ is convex; in particular, F assumes only convex values. $\square$

Theorem 3.3 is essentially a straightforward application of the continuity results from [19], combined with standard embedding results. However, it seems not easy to show that the values $F(\lambda, u)$ are closed (hence compact under the additional hypotheses of the first part of Theorem 3.3). For this reason, we will show that F coincides with a map for which it is easy to show that the values are closed. For $K \subseteq W$, we define $F_K \colon \Lambda \times X \multimap W^*$ by

$$\begin{aligned} F_K(\lambda, x) &= \bigcap_{\varphi \in K} \bigcup_{y \in S_f(\lambda,x)} \Big\{ z \in W^* : z(\varphi) = \int_S y(s) T_1 \varphi(s)\, ds \Big\} \\ &= \bigcap_{\varphi \in K} \Big\{ z \in W^* : z(\varphi) = \int_{\Omega_0} y(s)\varphi(s)\, ds + \int_\Gamma y(s) \varphi|_\Gamma(s)\, ds \\ &\qquad \text{for some } y \in S_f(\lambda, x) \Big\}. \end{aligned} \tag{8}$$

A special case of that operator has been introduced in [15]. Of course, we understand $\varphi|_\Gamma$ in contexts as the above in the sense of traces. Note that under our growth hypothesis (6), the integrals in (8) are well-defined, because $S_f(\lambda, x)$ is contained in Y, and $T_1(\varphi) \in Y'$ (by Theorem 3.3 and Proposition 3.1, respectively).

Example 3.4. In the setting of Example 3.2, the problem (4) is equivalent to $J(u) \in F_K(\lambda, T_0 u)$ if $K = W$. Hence (according to Example 3.2), the problems (3) and (4) are equivalent if $F_K(\lambda, T_0 u) = F(\lambda, u)$ (for $K = W$).

We will show that, if K is "large enough" in a certain sense, the equality $F_K(\lambda, x) = T(S_f(\lambda, x))$ holds, which for $x = T_0 u$ means $F_K(\lambda, T_0 u) = F(\lambda, u)$. The difficulty of this equality (in case $K = W$) is that it requires an exchange of the order of unions and intersections. The case $K \neq W$ is interesting e.g. in the setting of Example 3.4, where it might be more natural to work with the cone K of nonnegative functions instead of the full space $K = W$ (in fact, in earlier results like e.g. [8] only this cone was considered). In order to formulate precisely what we require for K, we introduce the following notion.

Definition 3.5. Let Z be a projectable space of (classes of) measurable functions $x\colon S \to V$. We say that a subset $M \subseteq Z$ is *monotonically sub-dense in the characteristic functions in* Z if for any $e \in V$ and any $E_0 \in \Sigma$ with $\mu(E_0) > 0$ and $e\chi_{E_0} \in Z$ there is a subset $K_0 \subseteq E_0$ with $\mu(K_0) > 0$ such that $e\chi_{K_0}$ is contained in the closure of $M^+ \cup (-M^+)$ where M^+ denotes the set of all linear combinations of elements of M with nonnegative coefficients.

Our main hypothesis for the equality $F_K(\lambda, x) = T(S_f(\lambda, x))$ will be that $T_1(K)$ be monotonically sub-dense in the characteristic functions in Y'. The following observation by standard density arguments shows that this holds if K is not extremely degenerate on $S = \Omega_0 \cup \Gamma$.

Proposition 3.6. *Suppose that there are $V_0 \subseteq V$ and $u_0\colon \Omega \to [0,\infty]$, such that for any $e_0 \in V_0$ and any smooth function $\varphi\colon \mathbb{R}^N \to [0,1]$ with compact support the pointwise product $e_0\varphi u_0$ belongs to $K \subseteq W$. Suppose:*

1. *$q_{k,j} > 1$ unless possibly $k = N = 1$.*
2. *The set V_0^+ of all linear combinations of elements of V_0 with nonnegative coefficients has the property that $V_0^+ \cup (-V_0^+)$ is dense in V.*
3. *$u_0(x) \neq 0$ for almost all $x \in S$ (on $\Gamma \subseteq S$ in the sense of traces, of course).*

Then $T_1(K)$ is monotonically sub-dense in the characteristic functions in Y'.

Proof. If $e \in V$ and $E_0 \in \Sigma$ with $\mu(E_0) > 0$ are given, we have $\mathrm{mes}_N(E_0 \cap \Omega_0) > 0$ or $\mathrm{mes}_{N-1}(E_0 \cap \Gamma) > 0$. Shrinking E_0 if necessary, we can assume additionally that $u_0(x) \geq 1/m$ a.e. on E_0 for some $m \in \mathbb{N}$. We find by the inner-regularity of the measure a compact subset $K_0 \subseteq \Omega_0$ or $K_0 \subseteq \Gamma$ with $\operatorname{mes}_N(K_0) > 0$ or $\operatorname{mes}_{N-1}(K_0) > 0$. Then K_0 has the property required in Definition 3.5.

More precisely, we show that if $e \in \pm\overline{V}_0^+$ then $e\chi_{K_0} \in Y'$ can be approximated in norm by functions from $\pm T_1(K^+)$ where K^+ denotes the set of all linear combinations of K^+ with nonnegative coefficients. To see this, note that we have $q'_{k,j} < \infty$ unless Γ consists of at most two atoms. By Lebesgue's dominated convergence theorem and since $\mu(S) < \infty$, it thus suffices to show that we can approximate $e\chi_{K_0}$ almost everywhere by a uniformly bounded sequence of functions from $\pm T_1(K^+)$. To this end, we choose a sequence $e_i \in \pm V_0^+$, a decreasing sequence of open sets $U_i \supseteq K_0$ with $\bigcap_i U_i = K_0$ (here we use the compactness of K_0) and by standard arguments sequences of smooth functions $\varphi_i, \psi_i \colon \mathbb{R}^N \to \mathbb{R}$ with $\chi_{K_0} \le \varphi_i \le \chi_{U_i}$ and $\psi_i(x) \to u_0(x)$ for almost all $x \in K_0$. For the latter note that u_0 belongs to some Sobolev space and thus can be approximated by smooth functions in the norm of that space. Since $u_0(x) \ge 1/m$ for $x \in K_0$, we can also assume that $\psi_i(x) \ge 1/m$ for all $x \in \mathbb{R}^N$. Hence, the pointwise product $v_i = e_i\varphi_i u_0/\psi_i$ belongs to K^+, and $T_1(v_i)$ are uniformly bounded and converge a.e. to $e\chi_{K_0}$, as required. $\square$

We also have to introduce the measure space S_0 which we obtain from S by identifying the sets S_i $(i \in I)$ as actual atoms in S_0 with the same measure (this causes no measure theoretic difficulties since I is at most countable). We write $S_{0,0}$ and $S_{1,0}$ for that part of S_0 which comes from identifying the atoms in Ω_0 or Γ, respectively, and define

$$Y_0' := \big(L_{q'_{0,1}}(S_{0,0}, V_1) \times \cdots \times L_{q'_{0,n}}(S_{0,0}, V_n)\big) \oplus \big(L_{q'_{1,1}}(S_{0,1}, V_1) \times \cdots \times L_{q'_{1,n}}(S_{0,1}, V_n)\big).$$

We assume here that the number of atoms S_i which contain parts of Ω_0 and Γ simultaneously is finite. Under this hypothesis, we may even assume without loss of generality that $S_{0,0} \cap S_{1,0} = \varnothing$, since we can join the corresponding atoms to either $S_{0,0}$ or $S_{1,0}$, and a different choice only changes the norm in Y_0' to an equivalent norm.

There is a natural linear surjection $P \colon Y' \to Y_0'$, defined by

$$Px(s) := \begin{cases} x(s) & \text{if } s \notin \bigcup\limits_{i \in I} S_i \\ \dfrac{1}{\mu(S_i)} \displaystyle\int_{S_i} x(t)\,dt & \text{if } s \in S_i. \end{cases}$$

By Young's inequality, we have

$$\int_{S_i} |Px(s)|^p \, ds \le \int_{S_i} \frac{1}{\mu(S_i)} \int_{S_i} |x(t)|^p \, dt\, ds = \int_{S_i} |x(s)|^p \, ds \quad (1 \le p < \infty),$$

i.e., the map $P \colon Y' \to Y_0'$ is bounded (by 1 if there are no atoms which intersect Ω_0 and Γ simultaneously). P is really a surjection and actually has norm at least 1, since for any function $y_0 \in Y_0'$, we can define a corresponding preimage $y \in Y'$ with $\|y\|_{Y'} = \|y_0\|_{Y_0'}$ by the formula

$$y(s) := \begin{cases} y_0(s) & \text{if } s \notin \bigcup\limits_{i \in I} S_i \\ y_0(S_i) & \text{if } s \in S_i. \end{cases} \tag{9}$$

In particular, it follows that for any dense set $M \subseteq Y'$ the image $P(M)$ is dense in $P(Y') = Y_0'$.

Theorem 3.7. *Let* $(\lambda_0, x_0) \in \Lambda \times X$. *Assume the growth condition* (6) (*on* $S \setminus E$ *and the locally boundedness on* E *mentioned after* (6)), *at least for* $\lambda = \lambda_0$; *then* $T(S_f(\lambda_0, x_0)) \subseteq F_K(\lambda_0, x_0)$. *Moreover, assume one of the following:*

1. *The linear hull of* K *is dense in* W, *and* $S_f(\lambda_0, x_0)$ *is at most single-valued.* (*The latter holds in particular, if the functions* $f_{k,j}$ *are at most single-valued.*)
2. *There are only finitely many atoms* S_i *which intersect* Ω_0 *and* Γ *simultaneously. The function* $F_0(\lambda_0, x_0)$ *is Bochner measurable, and for almost all* $s \in S$ *the set* $F_0(\lambda_0, x_0)(s)$ *is convex and closed w.r.t. the weak* topology. The spaces* $V_1, \ldots, V_n$ *are separable. Moreover, if* $S_{k,0}$ ($k \in \{0, 1\}$) *does not consist of only finitely many atoms, then suppose that* $q_{k,j} > 1$ *and* V_j *has the Radon-Nikodym property w.r.t.* $S_{k,0}$ *for all* $j \in \{1, \ldots, n\}$. $T_1(K)$ *is monotonically sub-dense in the characteristic functions in* Y'. *Finally, assume that there is a finite constant* C *such that the family of all linear combinations of the form* $\varphi = \sum_{k=1}^{\ell} \mu_k \varphi_k$ *with* $\ell \in \mathbb{N}$, $v_k \in K$, *such that*
$$\sum_{k=1}^{\ell} |\mu_k| \, \|P(T_1(\varphi_k))\|_{Y_0'} \leq C \, \|P(T_1(\varphi))\|_{Y_0'} \tag{10}$$
is dense in W.
3. *If* $F_0(\lambda_0, x_0)$ *is constant on each* S_i ($i \in I$), *the hypothesis that* $T_1(K)$ *be monotonically sub-dense in the characteristic functions in* Y' *can be relaxed to the hypothesis that* $P(T_1(K_+))$ *be monotonically sub-dense in the characteristic functions in* Y_0'.

Then $T(S_f(\lambda_0, x_0)) = F_K(\lambda_0, x_0)$.

For the case $I = \varnothing$, i.e., if we consider superposition operators without atoms, the condition for K concerning (10) becomes exactly the condition imposed on K in [15].

In most applications, one will work with reflexive spaces $V_1, \ldots, V_n$ in which case there is no need to care about the Radon-Nikodym property or the weak* topology:

Remark 3.8. Let $V_1, \dots, V_n$ be separable and reflexive. Then $V_1, \dots, V_n$ have the Radon-Nikodym property w.r.t. any σ-finite measure space, and moreover, if $F_0(\lambda_0, x_0)(s)$ is convex and closed w.r.t. the norm topology then it is also closed w.r.t. the weak topology and thus also closed w.r.t. the weak* topology.

Remark 3.9. We are not able to show directly the continuity of F_K, but by the equality $F_K = T \circ S_f$ (Theorem 3.7), it suffices to study the continuity of $T \circ S_f$ (Theorem 3.3). In view of Proposition 2.2 and the subsequent Theorem 3.10, we thus find a result about the upper semicontinuity of F_K which essentially contains [15, Theorem 6.1].

This is important, since there is a gap in the beginning of the corresponding proof of [15, Lemma 8.10] (namely, it is not clear whether the functions $w_{n,i}$ used can be chosen independently of $v \in K$). Thus, our results not only overcome this gap in the proof of [15], but we also do not need the structure conditions about the dependency of parameters which were required in [15]. However, the reader should be aware that we need in Theorem 3.7 in the multivalued case two assumptions which were not supposed in [15, Theorem 6.1]:

1. $V_1, \dots, V_n$ are reflexive (or, at least, $V_1, \dots, V_n$ have the Radon-Nikodym property and the values of F_0 are even weakly*-closed).
2. $T_1(K)$ is monotonically sub-dense in the characteristic functions in Y', e.g. K satisfies the hypotheses of Proposition 3.6.

In all publications known to the author where the continuity claim from [15, Theorem 6.1] was used in the multivalued case (these are [8,9,16,17]), the spaces $V_1, \dots, V_n$ were even finite-dimensional and the hypotheses of Proposition 3.6 follow immediately from other hypotheses so that these results are not impacted by the mentioned gap in the proof of [15].

Proof of Theorem 3.7. If $z \in T(S_f(\lambda_0, x_0))$, then we have for some $y \in S_f(\lambda_0, x_0)$ that $z = T_1(y)$, and so

$$z(\varphi) = \int_S y(s)\varphi(s)\,ds \quad (\varphi \in W).$$

In particular, $z \in F_K(\lambda_0, x_0)$ for every $K \subseteq W$.

If $S_f(\lambda_0, x_0)$ is single-valued, we have at most one $y \in S_f(\lambda_0, x_0)$. If no such y exists then $F(\lambda_0, x_0) \subseteq F_K(\lambda_0, x_0) = \varnothing$. However, if a unique $y \in S_f(\lambda_0, x_0)$ exists, then we have for every $z \in F_K(\lambda_0, x_0)$ that

$$z(\varphi) = \int_S y(s)\varphi(s)\,ds \quad (\varphi \in K). \tag{11}$$

Since both sides of (11) are linear, we conclude that (11) holds even for all z in the linear hull of K. Moreover, since $y \in Y$ and $T_1 \colon K \to Y'$ is bounded, we can pass to the closure in (11), i.e., (11) even holds for all $\varphi \in W$, i.e., $z = T(y) \in T(S_f(\lambda_0, x_0))$.

Now we consider the last two two cases in the claim. We recall that Theorem 3.3 implies that $S_f(\lambda_0, x_0)$ is bounded in Y by some constant C_0. Hence, if $z \in F_K(\lambda_0, x_0)$, we have for any linear combination $\varphi = \sum_{k=1}^{\ell} \mu_k \varphi_k$ with $\varphi_k \in K$ and (10) that, for some $y_1, \dots, y_\ell \in S_f(\lambda_0, x_0)$,

$$\begin{aligned}
|z(\varphi)| &= \Big|\sum_{k=1}^{\ell} \mu_k \int_S y_k(s)\varphi_k(s)\, ds\Big| \\
&\leq \sum_{k=1}^{\ell} |\mu_k| \left(\int_{S\setminus\bigcup_{i\in I} S_i} |y_k(s)\varphi(s)|\, ds + \sum_{i\in I} |y_k(S_i)P\varphi(S_i)|\right) \\
&\leq \sum_{k=1}^{\ell} |\mu_k|\, \|y_k\|_{Y_0} \|P(T_1\varphi_k(s))\|_{Y_0'} \\
&\leq C_0 C \|P(T_1(\varphi))\|_{Y_0'},
\end{aligned}$$

where we used Hölder's inequality and the constant C from (10). Since the functions φ of the above form are dense in W and $P \circ T_1 \colon W \to Y_0'$ is bounded, we conclude that

$$|z(\varphi)| \leq C_0 C \|P(T_1(\varphi))\|_{Y_0'} \quad (\varphi \in W). \tag{12}$$

Let now $Z := P(T_1(W))$, endowed with the norm of Y_0'. By (12), the kernel N of the map $P \circ T_1 \colon W \to Z$ is contained in the null space of z, and so z induces some $z_0 \in (W/N)^*$ by means of the formula $z_0([\varphi]) := z(\varphi)$. Hence, we can define a linear functional h on Z by the formula $h(P(T_1(\varphi))) := z_0([\varphi])$. The formula (12) implies $h \in Z^*$. By Hahn-Banach theorem, we can extend h to some $h \in (Y_0')^*$.

Now we use the hypothesis that either $S_{k,0}$ consists of only finitely many atoms or that V_j has the Radon-Nikodym property w.r.t. $S_{k,0}$ and $q_{k,j}' < \infty$. In both cases, it follows that the dual space of $L_{q_{k,j}'}(S_{k,0}, V_j)$ is given by $L_{q_{k,j}}(S_{k,0}, V_j^*)$ with the usual interpretation of integrals as functionals, see e.g. [6] or [13, Section 2.22.5].

Hence, we can represent h as an integral functional, i.e., when we define Y_0 analogously to Y but with Ω_0 and Γ replaced by $S_{0,0}$ and $S_{1,0}$, respectively, there is some $y_0 \in Y_0$ with $h(v) = \int_{S_0} y_0(s)v(s)\, ds$ for $v \in Y_0'$. In particular, this holds for all $v \in Z = P(T_1(W))$. Defining $y \in Y$ by means of the formula (9), we obtain for all $\varphi \in W$

$$z(\varphi) = z_0([\varphi]) = h(P(T_1(\varphi))) = \int_{S_0} y_0(s)P(T_1\varphi)(s)\, ds = \int_S y(s)T_1\varphi(s)\, ds,$$

i.e., $z = T(y)$. Note that, by construction, y is measurable and constant on the atoms S_i. Hence, if we can show that $y(s) \in F_0(\lambda_0, x_0)(s)$ for almost all

$s \in S$, we have $y \in S_f(\lambda_0, x_0)$, and so $z = T(y) \subseteq T(S_f(\lambda_0, x_0))$ which implies the claim.

Assume by contradiction that we do not have $y(s) \in F_0(\lambda_0, x_0)(s)$ for almost all $s \in S$. Note first that (6) implies that the values $F_0(\lambda_0, x_0)(s)$ are for almost all $s \in S$ bounded in the norm of V^* and thus weak* compact and contain no line. Since the normed space V is automatically a barrelled locally convex space, it follows from [12, last Corollary] that the elements of $F_0(\lambda_0, x_0)(s)$ are for every dense subset $A_0 \subseteq V$ characterized by

$$w \in F_0(\lambda_0, x_0)(s) \iff w(e) \leq \alpha_e(s) := \sup_{v \in F_0(\lambda_0, x_0)(s)} v(e) \quad \text{for every } e \in A_0.$$

We apply this for a countable dense subset $A_0 \subseteq V$: If for any $e \in A_0$, we would have $y(s)e \leq \alpha_e(s)$ for almost all $s \in S$, then, since A_0 is countable, we would also have $y(s) \leq \alpha_e(s)$ for every $e \in A_0$ for almost all $s \in S$, and so $y(s) \in F_0(\lambda_0, x_0)(s)$ for almost all $s \in S$, contradicting our assumption. Hence, there is some $e \in A_0 \subseteq V$ such that the inequality

$$y(s)e > \alpha_e(s) = \sup\{ve : v \in F_0(\lambda_0, x_0)(s)\} \quad (s \in E_0) \tag{13}$$

holds on a non-null set $E_0 \subseteq S$. (We note that (13) and our above argument is a variant of [4, Proposition III.35] for the weak* topology.) Since F_0 has essentially separable range, and so there is a separable complete subspace V_0 of V^* which contains $F_0(\lambda_0, x_0)(s)$ for almost all $s \in S$, we can apply e.g. [11, Theorem 5.6] in V_0 to obtain that there is a countable family y_k of measurable functions with values in V_0 (hence, y_k are Bochner measurable) such that $F_0(\lambda_0, x_0)(s)$ is the closure of $\{y_k(s) : k\} \subseteq V_0$ (in the norm topology) for almost all $s \in S$. In particular, $\alpha_e(s) = \sup_k(y_k(s)e)$ is measurable. Our argument from above now implies that (13) actually holds for a measurable set $E_0 \subseteq S$ of positive measure. The existence of the above functions y_k also implies that for any $x \in Y'$, the functions

$$\begin{aligned} G^+(x)(s) &:= \sup\,(F_0(\lambda_0, x_0)(s)x(s)) = \sup_k\,(y_k(s)x(s)) \\ G^-(x)(s) &:= \inf\,(F_0(\lambda_0, x_0)(s)x(s)) \ = \inf_k\,(y_k(s)x(s)) \end{aligned}$$

are measurable. We recall that $\{y_k(s) : k\}$ is bounded in the norm topology for almost all s, and so $g^+(s, v) := \sup_k\,(y_k(s)v)$ and $g^-(s, v) := \inf_k\,(y_k(s)v)$ are Carathéodory functions. Since $G^\pm \colon Y' \to L_1(S, \mathbb{R})$ is the superposition operator generated by $g^\pm$ it follows from [14, Theorem 6.4] that $G^\pm$ is continuous. Note now that (13) implies that one of the strict inequalities

$$\int_S y(s)x(s)\,ds > \int_S G^+(x)(s)\,ds \quad \text{or} \quad \int_S y(s)x(s)\,ds < \int_S G^-(x)(s)\,ds \tag{14}$$

holds for $x = e\chi_{K_0}$ or $x = -e\chi_{K_0}$, respectively, whenever $K_0 \subseteq E$ has positive measure. Assuming that $T_1(K)$ is monotonically sub-dense in the characteristic functions in Y', using the notation of Definition 3.5, we can approximate at least one such function $\pm e\chi_{K_0}$ arbitrarily good in the space Y' by elements from $T_1(K)^+$. Since both sides of the inequalities (14) depend continuously on $x \in Y'$, we obtain that at least one of these strict inequalities also holds for a function $x \in T_1(K)^+$. Since T_1 is linear, we find some $\varphi \in K^+$ with $x = T_1\varphi$, hence $z(\varphi) = \int_S y(s)T_1\varphi(s)\,ds$ satisfies

$$z(\varphi) > \int_S G^+(T_1\varphi)(s)\,ds \quad \text{or} \quad z(\varphi) < \int_S G^-(T_1\varphi)(s)\,ds. \tag{15}$$

By $\varphi \in K^+$ we mean that there are $\mu_j \geq 0$ and $\varphi_j \in K$ with $\varphi = \sum_{j=1}^{\ell} \mu_j\varphi_j$. Since $z \in F_K(\lambda_0, x_0)$ there are $y_{\varphi_j} \in S_f(\lambda_0, x_0)$ with

$$z(\varphi) = \sum_{j=1}^{\ell} \mu_j z(\varphi_j) = \sum_{j=1}^{\ell} \mu_j \int_S y_{\varphi_j}(s)T_1\varphi_j(s)\,ds. \tag{16}$$

Since $y_{\varphi_j}(s) \in F_0(\lambda_0, x_0)(s)$ for almost all $s \in S$, we have

$$G^-(T_1\varphi_j)(s) \leq y_{\varphi_j}(s)T_1\varphi_j(s) \leq G^+(T_1\varphi_j)(s)$$

for almost almost all $s \in S$. Multiplying this inequality with $\mu_j \geq 0$ and integrating and summing up, we find by (16) that

$$z(\varphi) \leq \sum_{j=1}^{\ell} \int_S \mu_j G^+(T_1\varphi_j)(s)\,ds = \int_S G^+(T_1\varphi)(s)\,ds,$$

and analogously

$$z(\varphi) \geq \int_S G^-(T_1\varphi)(s)\,ds.$$

Both inequalities together contradict (15).

If we assume that $F_0(\lambda_0, x_0)$ is constant on each S_i ($i \in I$), then also $G^\pm(x)$ are constant on each S_i ($i \in I$), and we can consider $G^\pm$ as a superposition operator from Y_0' into $L_1(S_0, \mathbb{R})$ which is continuous by the same reasoning as above. Hence, essentially the same argument as above, only replacing throughout the measure space S by S_0 (and the function y by y_0, T_1 by PT_1, and Y' by Y_0'), we obtain a contradiction analogously as above if $P(T_1(K))$ is monotonically sub-dense in the characteristic functions in Y_0'. □

Theorem 3.10. *Let $(\lambda_0, x_0) \in D$ be such that the growth hypothesis* (6) (*on $S \setminus E$ and the locally boundedness on E mentioned after* (6)) *holds. Suppose that the function $F_0(\lambda_0, x_0)(s) := f(\lambda_0, s, x_0(s))$ is measurable in the Bochner sense and assumes closed convex values for almost all $s \in S$. Then $F_K(\lambda_0, x_0)$ is closed and convex.*

Proof. Since the intersection of closed convex sets is closed and convex, it suffices to show that for each fixed $\varphi \in K$ the set

$$\left\{ z \in W^* \;\middle|\; z(\varphi) = \int_S y(s) T_1 \varphi(s)\, ds \text{ for some } y \in S_f(\lambda_0, x_0) \right\}$$

is closed and convex. Since $S_f(\lambda_0, x_0)$ is convex, the proof of the convexity of this set is trivial. To show that the set is closed, it suffices to show that

$$M := \left\{ \int_S y(s) T_1 \varphi(s)\, ds : y \in S_f(\lambda, x_0) \right\}$$

is closed, since then also the considered set $\{z \in W^* : z(\varphi) \in M\}$ is closed. We define a multivalued function $G \colon S_0 \multimap V^*$ by

$$G(s) := \begin{cases} F_0(\lambda_0, x_0)(s) & \text{if } s \notin S_i \\ \{z : \text{there is } y \in S_f(\lambda, x_0) \text{ with } y|_{S_i} = z \text{ a.e.}\} & \text{if } s \in S_i. \end{cases}$$

Then the definition of S_f implies

$$M = \left\{ \int_{S_0} y(s) P T_1 \varphi(s)\, ds : y \text{ measurable and } y(s) \in G(s) \text{ a.e.} \right\}.$$

Since F_0 has essentially separable range, there is a separable complete subspace V_0 of V^* which contains $F_0(\lambda_0, x)(s)$ almost all $s \in S$, without loss of generality for all $s \in S$. Then $G \colon S_0 \multimap V_0$, and since I is countable, it follows that G is measurable in the Bochner sense. From [11, Theorem 3.5], we obtain that G has a measurable graph. Define $h \colon S \times V_0 \to \mathbb{R}$ by $h(s, \ell) := \ell(P T_1 \varphi(s))$. For any measurable function $x \colon S \to \mathbb{R}$ with $x(s) \in h(s, G(s)) = G(s) P T_1 \varphi(s)$ a.e., we find by the Fillipov's implicit function theorem e.g. in the form of [11, Theorem 7.2] that there is a measurable y with $y(s) \in G(s)$ and $x(s) = h(s, x(s)) = y(s) P T_1 \varphi(s)$. This shows

$$M = \left\{ \int_{S_0} x(s)\, ds \;\middle|\; x \colon S \to \mathbb{R} \text{ measurable with } x(s) \in G(s) P T_1 \varphi(s) \right\}.$$

Hence, putting $G_0(s) := G(s) P T_1 \varphi(s)$, we have $M = \int_{S_0} G_0(s)\, ds$, where the integral is understood in the Aumann sense [2], i.e., as the set of all integrals over measurable selections of G_0. Note that the values $G_0(s) \subseteq \mathbb{R}$ are closed and convex, even compact by (6), and the latter also implies that G_0 has a uniform integrable majorant. It is well-known that this implies that M is convex and compact, in particular closed. In our situation, one need not even invoke such a deep classical result to see this: The convexity of the values $G_0(s)$ implies that M is an interval, and to see that it is closed, it suffices to observe that the functions $s \mapsto \sup G_0(s)$ and $s \mapsto \inf G_0(s)$ are measurable selections of G_0, see [15, Proposition 3.2]. $\square$

Combining Theorem 3.7 with Theorem 3.10, we obtain that the values of F are closed:

Theorem 3.11. *Let $(\lambda_0, x_0) \in \Lambda \times X$. Assume the growth condition* (6) (*on $S \setminus E$ and the locally boundedness on E mentioned after* (6)) *at least for $\lambda = \lambda_0$. Suppose that only finitely many of the atoms S_i intersect Ω_0 and Γ simultaneously. Let $F_0(\lambda_0, x_0)$ be Bochner measurable, and for almost all $s \in S$ the set $F_0(\lambda_0, x_0)(s)$ be convex and closed w.r.t. the weak* topology. Suppose also that the spaces $V_1, \ldots, V_n$ are separable. For the case that $S_{k,0}$ ($k \in \{0, 1\}$) does not consist of only finitely many atoms, assume $q_{k,j} > 1$ and that V_j has the Radon-Nikodym property w.r.t. $S_{k,0}$ for all $j \in \{1, \ldots, n\}$. Finally, assume one of the following:*

1. *$T_1(W)$ is monotonically sub-dense in the characteristic functions in Y'.*
2. *$P(T_1(W))$ is monotonically sub-dense in the characteristic functions in Y_0', and the function $F_0(\lambda_0, x_0)$ is constant on each S_i.*

Then $T(S_f(\lambda_0, x_0))$ is convex and closed. In particular, if (5) *holds and if $V_1, \ldots, V_n$ are finite-dimensional, then $T(S_f(\lambda_0, x_0))$ is compact.*

For applications of Theorem 3.11 to problems involving also classical Neumann and Dirichlet conditions (e.g. the references in Remark 3.9) it is worth to note that it is sufficient that the support of the function u_0 in Proposition 3.6 only covers those parts of S for which multivaluedness occurs:

Corollary 3.12. *Let $S_1 \subseteq S$ be a measurable subset on which $F_0(\lambda_0, x_0)$ is a.e. single-valued. Then Theorem 3.11 holds even if one replaces in the hypothesis the space Y'* (*resp. Y_0'*) *by the subspace of functions vanishing on S_1 and on all atoms intersecting S_1 in a set of positive measure.*

Proof. There is a single-valued function f_1 and a multivalued function f_2 vanishing on S_1 such that

$$S_f(\lambda_0, x_0) = S_{f_1}(\lambda_0, x_0) + S_{f_2}(\lambda_0, x_0),$$

hence, it suffices to prove that the last term of

$$T(S_f(\lambda_0, x_0) = T(S_{f_1}(\lambda_0, x_0)) + T(S_{f_2}(\lambda_0, x_0))$$

is closed. Let S_2 denote the union of S_1 with the atoms intersecting S_1 in a set of positive measure. Then all functions from $S_{f_2}(\lambda_0, x_0)$ vanish outside S_2, and so $T(S_{f_2}(\lambda_0, x_0)) = \widetilde{T}(S_{f_2}(\lambda_0, x_0))$ where $\widetilde{T}$ is defined as T but with Ω_0 and Γ replaced by $\widetilde{\Omega}_0 := \Omega_0 \setminus S_2$ and $\widetilde{\Gamma} := \Gamma \setminus S_2$, respectively. Now the claim follows by applying Theorem 3.11 in the setting of these modified sets. □

Using Proposition 2.2 and Theorem 3.3, we obtain the upper semicontinuity of F:

Theorem 3.13. *Let* $(\lambda_0, u_0) \in D \subseteq \Lambda \times W_0$ *be such that the hypotheses of Theorem* 3.10 *or Corollary* 3.12 *hold with* $x_0 = T_0 u_0$. *Suppose also that* $V_1, \ldots, V_n$ *are finite-dimensional, and* (5) *holds. Let* λ_0 *have a countable base of neighborhoods, and suppose there is one neighborhood* $\Lambda_0 \subseteq \Lambda$ *of* λ_0 *such that* $\{a_{k,\lambda,j} : \lambda \in \Lambda_0\}$ *has equicontinuous norm in* $L_{q_{k,j}}$. *Finally, suppose that for almost all* $s \in S$ *the function* (7) *is upper semicontinuous at* $(\lambda_0, T_0 u_0(s))$ *in the uniform sense. Then* $F\colon D \multimap W^*$ *assumes at* (λ_0, u_0) *a compact convex value and is upper semicontinuous at* (λ_0, u_0).

Concerning the differentiability of $F\colon \Lambda \times W_0 \multimap W^*$ and $T \circ S_f\colon \Lambda \times X \multimap W^*$, we obtain a stronger result if we observe that the growth estimate is actually even independent of the choice of the auxiliary spaces Y (and similarly for X).

Theorem 3.14. *Let* $D \subseteq \Lambda \times X$, $\alpha \in (0, \infty)$, $1 \le p_{k,i} \le p^*_{k,i}$, $1 \le \widetilde{p}_i \le p_i$, *and* $\widetilde{q}_{k,i} \ge q^*_{k,i}$, *and*

$$\begin{aligned}\widetilde{X} := &\big(L_{p_{0,1}}(\Omega_0, U_1) \times \cdots \times L_{p_{0,m}}(\Omega_0, U_m)\big) \times \big(L_{\widetilde{p}_1}(\Omega_0, (U_1^*)^N) \times \cdots \times L_{\widetilde{p}_m}(\Omega_0, (U_m^*)^N)\big)\\ &\oplus \big(L_{p_{1,1}}(\Gamma, U_1) \times \cdots \times L_{p_{1,m}}(\Gamma, U_m)\big)\\ \widetilde{Y} := &\big(L_{\widetilde{q}_{0,1}}(\Omega_0, V_1^*) \times \cdots \times L_{\widetilde{q}_{0,n}}(\Omega_0, V_n^*)\big) \oplus \big(L_{\widetilde{q}_{1,1}}(\Gamma, V_1^*) \times \cdots \times L_{\widetilde{q}_{1,n}}(\Gamma, V_n^*)\big).\end{aligned}$$

Then

$$\lim_{r \to 0} \sup_{\substack{(\lambda,x) \in D\\ \lambda \in \Lambda_0,\ \|x\|_{\widetilde{X}} \le r}} \frac{\sup_{y \in S_f(\lambda,x)} \|y\|_{\widetilde{Y}}}{r^\alpha} = 0 \tag{17}$$

implies

$$\begin{aligned}\lim_{r \to 0} \sup_{\substack{(\lambda,T_0 u) \in D\\ \lambda \in \Lambda_0,\ \|u\|_{W_0} \le r}} \frac{\sup_{z \in F(\lambda,u)} \|z\|_{W^*}}{r^\alpha} &= \lim_{r \to 0} \sup_{\substack{(\lambda,x) \in D\\ \lambda \in \Lambda_0,\ \|x\|_{X} \le r}} \frac{\sup_{z \in T(S_f(\lambda,x))} \|z\|_{W^*}}{r^\alpha}\\ &= \lim_{r \to 0} \sup_{\substack{(\lambda,x) \in D\\ \lambda \in \Lambda_0,\ \|x\|_{\widetilde{X}} \le r}} \frac{\sup_{z \in T(S_f(\lambda,x))} \|z\|_{W^*}}{r^\alpha}\\ &= 0,\end{aligned} \tag{18}$$

and

$$\lim_{\substack{(\lambda,r) \to (\lambda_0,0)\\ \lambda \in \Lambda,\ r > 0}} \sup_{\substack{\|x\|_{\widetilde{X}} \le r\\ (\lambda,x) \in D}} \frac{\sup_{y \in S_f(\lambda,x)} \|y\|_{\widetilde{Y}}}{r^\alpha} = 0 \tag{19}$$

implies

$$\lim_{\substack{(\lambda,r)\to(\lambda_0,0)\\ \lambda\in\Lambda,\ r>0}} \sup_{\substack{\|u\|_{W_0}\le r\\ (\lambda,T_0u)\in D}} \frac{\sup_{z\in F(\lambda,u)}\|z\|_{W^*}}{r^\alpha} = \lim_{\substack{(\lambda,r)\to(\lambda_0,0)\\ \lambda\in\Lambda,\ r>0}} \sup_{\substack{\|x\|_X\le r\\ (\lambda,x)\in D}} \frac{\sup_{z\in T(S_f(\lambda,x))}\|z\|_{W^*}}{r^\alpha}$$
$$= \lim_{\substack{(\lambda,r)\to(\lambda_0,0)\\ \lambda\in\Lambda,\ r>0}} \sup_{\substack{\|x\|_{\widetilde{X}}\le r\\ (\lambda,x)\in D}} \frac{\sup_{z\in T(S_f(\lambda,x))}\|z\|_{W^*}}{r^\alpha} \tag{20}$$
$$=0.$$

Proof. Defining $\widetilde{T}\colon \widetilde{Y}\to W^*$ by the same formula as T, Proposition 3.1 implies that $\widetilde{T}$ is well-defined and bounded, and the definition implies $T\circ F=\widetilde{T}\circ F$. Hence, (17) resp. (19) implies by the boundedness of $\widetilde{T}$ the last equality in (18) resp. (20). The boundedness of the embedding $X\to\widetilde{X}$ then implies the second equality, and the first follows by the boundedness of $T_0\colon W_0\to X$. □

The motivation for changing the constants $\widetilde{q}_{j,i}$, in Theorem 3.14 is that the hypothesis (17) or (19) can be less restrictive for smaller $\widetilde{q}_{j,i}$. In particular, even if one is interested in the compact case (5) and thus assumes correspondingly the more restrictive growth condition (6), one need not work with the same restrictive constants $q_{j,i}$ if one is interested in proving (18) or (20), but one can instead switch to the less restrictive constants $\widetilde{q}_{j,i} := q^*_{j,i}$.

The motivation for choosing $\widetilde{p}_{k,i}\le p^*_{k,i}$ is that one can choose that former to be finite even if the latter is infinite.

How to verify (17) or (19) was already discussed in [19]. (One can of course consider the components of f in the product space separately.) For easier reference, we just formulate what one obtains from two special cases of [19].

Theorem 3.15. *Let $\alpha\in(0,\infty)$ satisfy $\alpha q^*_{i,j}<p_{i,k}\le p^*_{i,k}$ and $p_{i,k},p_k\in[1,\infty)$ for all $i=0,1$, all $j=1,\dots,n$, and all $k=1,\dots,m$. Let $\Lambda_0\subseteq\Lambda$ be fixed (resp. assume that λ_0 has a countable base of neighborhoods). Assume that for any sequence $\lambda_\nu\in\Lambda_0$ (resp. $\lambda_\nu\to\lambda_0$) there are a monotone null sequence $r_\nu>0$ with $\frac{r_\nu}{r_{\nu+1}}$ being bounded and measurable functions $g_{i,k,\nu}$ satisfying*

$$g_{0,k,\nu}(s)\ge\sup\{|w| : w\in f_{0,k}(\lambda_\nu,s,u,v),\ |u|,|v|\le r_\nu\}$$
$$g_{1,k,\nu}(s)\ge\sup\{|w| : w\in f_{1,k}(\lambda_\nu,s,u),\ |u|\le r_\nu\}$$

for almost all $s\in\Omega_0$ or $s\in\Gamma$, respectively, and such that for every subsequence there is a subsequence ν_i such that

$$\frac{g_{0,k,\nu_i}(s)}{r^\alpha_{\nu_i}}\to 0 \quad \text{and} \quad \frac{g_{1,k,\nu_i}(s)}{r^\alpha_{\nu_i}}\to 0 \tag{21}$$

for almost all $s\in\Omega_0$ or $s\in\Gamma$, respectively.

Finally, suppose that there are constants $c_{i,j,k} \in [0,\infty)$ *such that for any* $\lambda \in \Lambda_0$ *(resp.* $\lambda \in \Lambda$*) we have for almost all* $s \in \Omega_0$ *resp.* $s \in \Gamma$ *the uniform growth estimates*

$$\begin{aligned} &\sup_{w \in f_{0,k}(\lambda,s,u_1,\dots,u_m,v_1,\dots,v_m)} |w| \\ &\le \sum_{j=1}^{m} \left(c_{0,j,1} |u_j|^{\frac{p_{0,j}}{q_{0,k}^*}} + c_{0,j,2} |v_j|^{\frac{p_j}{q_{0,k}^*}} + c_{0,j,3} |u_j|^{\alpha} + c_{0,j,4} |v_j|^{\alpha} \right), \end{aligned} \tag{22}$$

$$\sup_{w \in f_{1,k}(\lambda,s,u_1,\dots,u_m)} |w| \le \sum_{j=1}^{m} \left(c_{1,j,1} |u_j|^{\frac{p_{1,j}}{q_{1,k}^*}} + c_{1,j,2} |u_j|^{\alpha} \right).$$

Then (18) (*resp.* (20)) *holds with* $D = \Lambda \times X$*.*

Moreover, if E *is the union of finitely many of the atoms and* $X_E \subseteq X$ *denotes the subset of all functions which are constant on these atoms, then* (18) (*resp.* (20)) *holds with* $D = \Lambda \times X_E$ *even if we assume the growth estimates only for* $s \in S \setminus E$*, and if* $g_{i,k,j}$ *are constant on the atoms in* E*.*

In the single-valued scalar case, $m = n = 1$, and $f = f_{0,1}$ independent of v, some special cases of Theorem 3.15 can be found in literature for particular applications, e.g. [5, Lemma 3.11] (for $\alpha+1 = p_1 = q_1$) or [8] (for $\alpha = 1$, $p_1 = q_1$). However, even in these special cases, the conditions of these results are more restrictive. For instance, in [5, Lemma 3.11] the limit $\lim_{u\to 0} \frac{f(\lambda,s,u)}{|u|^{\alpha}} = 0$ is required to be uniform also with respect to s (a.e.), while our corresponding requirement (21) is only pointwise (a.e.), and for our growth assumption (22) (for small u) it suffices that this limit is uniformly bounded w.r.t. s; also for large u our growth assumption (22) is weaker than that from [5, Lemma 3.11] where, roughly speaking, $|f(\lambda, s, u)| \le c_0 + c_1 |u|^{\gamma}$ with $\gamma < p_{0,1} - 1 = \frac{p_{0,1}}{q_{0,1}^*}$ is required while for (22) $\gamma = \frac{p_{0,1}}{q_{0,1}^*}$ is still acceptable (for large u).

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