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# Characterization of Besov Spaces on Nested Fractals by Piecewise Harmonic Functions

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**Abstract.** In the present paper we characterize the Besov spaces $B^s_{pq}(\Gamma,\mu)$ on nested fractals in terms of the coefficients of functions with respect to the piecewise harmonic basis.

**Keywords.** Besov spaces, traces, nested fractals

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## 1. Introduction

Besov spaces $B^s_{pq}(\Gamma,\mu)$ on $d$-sets in $\mathbb{R}^n$ can be defined by traces of Besov spaces $B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$. When the smoothness parameter $s$ is small, $B^s_{pq}(\Gamma,\mu)$ can be characterized by intrinsic building blocks, namely atoms. In the present paper first we give a characterization of Besov spaces by a new type of atoms, which we call $(s,p,\sigma)$-atoms.

On the other hand Besov spaces on the most trivial example of a $d$-set, the unit interval, can be described by means of Faber-Schauder basis [18]. We are looking for its counterpart for the $d$-set $\Gamma$. So we need to find the description of functions in Faber-Schauder basis in such a way that it can be transferred to other sets. Our approach is to start with a Dirichlet form $(\mathcal{E},\mathcal{D})$, see e.g. [8, 15]. Then the harmonic function on $\Gamma$ with given boundary values can be defined as the unique function that minimizes $\mathcal{E}(f)$. Similarly we can define piecewise harmonic functions. Piecewise harmonic functions on the unit interval are exactly the functions forming the Faber-Schauder basis. Thus the family of piecewise harmonic functions may be regarded as the counterpart of Faber-Schauder basis. Piecewise harmonic functions are Lipschitz with respect to the

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effective resistance metric. We additionally assume that $\Gamma$ is a nested fractal. Then the effective resistance metric is equivalent to the Euclidean metric taken to some power and this enables us to treat piecewise harmonic functions as $(s,p,\sigma)$-atoms. Thus functions from $B^s_{pq}(\Gamma,\mu)$ can be characterized in terms of the coefficients of its expansion in a piecewise harmonic basis.

The main result of the paper is contained in the Theorem 5.1. Our proof is based on the atomic characterization of Besov spaces. A similar result is also presented in the paper [13], where the harmonic representation of Lipschitz spaces $(\Lambda^{p,q}_\alpha)^{(1)}(\Gamma)$ introduced by Strichartz is stated. It was shown in [1] that $(\Lambda^{p,q}_\alpha)^{(1)}(\Gamma)$ coincide with $\mathrm{Lip}(\alpha/\alpha_0,p,q,\Gamma)$, when $\Gamma$ is a nested fractal. Thus the harmonic representation of Besov spaces might be also proved by using the discrete characterizations of Besov spaces.

## 2. Preliminaries

**2.1. Basic notation and classical Besov spaces.** Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. $\mathbb{Z}$ is the set of all integers. Let $\mathbb{R}^n$ be Euclidean $n$-space, where $n\in\mathbb{N}$. The scalar product of $x,y\in\mathbb{R}^n$ is given by $xy=\sum_{i=1}^n x_iy_i$. Put $\mathbb{R}=\mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. By $S'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. $L_p(\mathbb{R}^n)$ with $0<p\le\infty$, is the standard quasi-Banach space with respect to Lebesgue measure, quasi-normed by

$$\|f|L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p\,dx\right)^{\frac{1}{p}}, \quad 0<p<\infty,$$
$$\|f|L_\infty(\mathbb{R}^n)\| = \operatorname*{ess\text{-}sup}_{x\in\mathbb{R}^n}|f(x)|\,.$$

If $\varphi\in S(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi)=\mathcal{F}\varphi(\xi)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\varphi(x)e^{-ix\xi}\,dx,\quad \xi\in\mathbb{R}^n,$$

denotes the Fourier transform of $\varphi$. The inverse Fourier transform is given by

$$\varphi^\vee(x)=\mathcal{F}^{-1}\varphi(x)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\varphi(\xi)e^{ix\xi}\,d\xi,\quad x\in\mathbb{R}^n.$$

One extends $\mathcal{F}$ and $\mathcal{F}^{-1}$ in the usual way from $S$ to $S'$. For $f\in S'(\mathbb{R}^n)$, $\mathcal{F}f(\varphi)=f(\mathcal{F}\varphi)$, where $\varphi\in S(\mathbb{R}^n)$.

Let $\varphi_0\in S(\mathbb{R}^n)$ with

$$\varphi_0(x)=1,\ |x|\le 1 \quad\text{and}\quad \varphi_0(x)=0,\ |x|\ge\frac{3}{2}, \tag{1}$$

and let
$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n,\ k \in \mathbb{N}. \tag{2}$$
Then, since
$$1 = \sum_{j=0}^{\infty} \varphi_j(x) \quad \text{for all } x \in \mathbb{R}^n, \tag{3}$$
the $\varphi_j$ form a dyadic resolution of unity in $\mathbb{R}^n$. According to the Paley-Wiener-Schwartz theorem $\left(\varphi_k \widehat{f}\right)^\vee$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$. In particular, $\left(\varphi_k \widehat{f}\right)^\vee(x)$ makes sense pointwise.

**Definition 2.1.** Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the dyadic resolution of unity according to (1)-(3), $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ and
$$\left\|f|B^s_{pq}(\mathbb{R}^n)\right\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\|\left(\varphi_k \widehat{f}\right)^\vee \Big| L_p(\mathbb{R}^n)\right\|^q\right)^{\frac{1}{q}}$$
(with the usual modification if $q = \infty$). Then the Besov space $B^s_{pq}(\mathbb{R}^n)$ consists of all $f \in S'(\mathbb{R}^n)$ such that $\left\|f|B^s_{pq}(\mathbb{R}^n)\right\|_\varphi < \infty$.

**2.2. Trace spaces $B^s_{pq}(\Gamma, \mu)$.** We proceed with defining Besov spaces on $d$-sets.

**Definition 2.2.** A compact set $\Gamma$ in $\mathbb{R}^n$ is called a $d$-set with $0 < d < n$ if there is a Radon measure $\mu$ in $\mathbb{R}^n$ with support $\Gamma$ such that for some positive constants $c_1$ and $c_2$, holds
$$c_1 r^d \leq \mu(B(\gamma, r)) \leq c_2 r^d, \quad \gamma \in \Gamma,\ 0 < r < 1. \tag{4}$$
where $B(x, r)$ is a ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ and of radius $r > 0$. The measure $\mu$ satisfying (4) is called a $d$-measure.

If $\Gamma$ is a $d$-set, then the restriction to $\Gamma$ of the $d$-dimensional Hausdorff measure $\mathrm{H}^d$ satisfies (4) and any measure $\mu$ satisfying (4) is equivalent to $\mathrm{H}^d|_\Gamma$.

**Definition 2.3.** Let $\mu$ be a Radon measure in $\mathbb{R}^n$. Let
$$s > 0,\ 1 < p < \infty,\ 0 < q < \infty. \tag{5}$$
Let for some $c > 0$,
$$\int_\Gamma |\varphi(\gamma)|\ \mu(d\gamma) \leq c \left\|\varphi|B^s_{pq}(\mathbb{R}^n)\right\| \quad \text{for all } \varphi \in S(\mathbb{R}^n). \tag{6}$$
Then the trace operator $\mathrm{tr}_\mu$,
$$\mathrm{tr}_\mu : B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_1(\Gamma, \mu),$$

is the completion of the pointwise trace $(\mathrm{tr}_\mu \varphi)(\gamma) = \varphi(\gamma)$, $\varphi \in S(\mathbb{R}^n)$. Furthermore, the image of $\mathrm{tr}_\mu$ is denoted by $\mathrm{tr}_\mu B^s_{pq}(\mathbb{R}^n)$ and is quasi-normed by

$$\left\| g | \mathrm{tr}_\mu B^s_{pq}(\mathbb{R}^n) \right\| = \inf \left\{ \left\| f | B^s_{pq}(\mathbb{R}^n) \right\| : \mathrm{tr}_\mu f = g \right\}.$$

**Remark 2.4.** The above definition is justified since $S(\mathbb{R}^n)$ is dense in $B^s_{pq}(\mathbb{R}^n)$ with (5). We refer to [16, Theorem 2.3.3, p. 48]. Due to (6), the trace of $f$ is independent of the approximation of $f$ in $B^s_{pq}(\mathbb{R}^n)$ by $S(\mathbb{R}^n)$-functions.

**Definition 2.5.** Let $\Gamma$ be a $d$-set in $\mathbb{R}^n$. Let $s > 0$, $1 < p < \infty$, $0 < q < \infty$. Then

$$B^s_{pq}(\Gamma, \mu) = \mathrm{tr}_\mu B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$$

with

$$\left\| f | B^s_{pq}(\Gamma, \mu) \right\| = \inf \left\{ \left\| g | B^s_{pq}(\mathbb{R}^n) \right\| : \mathrm{tr}_\mu g = f \right\}.$$

There is the extension operator which is closely connected to the trace operator. The following assertion is covered by [6, Theorem 3, p. 155], we also refer to [17, Section 1.17.2].

**Theorem 2.6.** *Let $\Gamma$ be a compact $d$-set in $\mathbb{R}^n$ with $0 < d < n$ and let $\mu$ be a corresponding Radon measure. Let*

$$0 < s < 1,\ 1 < p < \infty,\ 1 \le q \le \infty,\ t = s + \frac{n-d}{p},$$

*and let $\mathrm{tr}_\mu$ be the trace operator. Then there is a linear and bounded extension operator $\mathrm{ext}_\mu$ with*

$$\mathrm{ext}_\mu : B^s_{pq}(\Gamma, \mu) \hookrightarrow B^t_{pq}(\mathbb{R}^n) \tag{7}$$

*and*

$$\mathrm{tr}_\mu \circ \mathrm{ext}_\mu = \mathrm{id} \quad (\textit{identity in } B^s_{pq}(\Gamma, \mu)). \tag{8}$$

All our reasoning strongly uses the fact that $B^s_{pp}(\Gamma, \mu)$ with $1 < p < \infty$ and $0 < s < 1$ can be equivalently normed by $\left\| f | B^s_{pp}(\Gamma, \mu) \right\|_*$ with

$$\left\| f | B^s_{pp}(\Gamma, \mu) \right\|^p_* = \int_\Gamma |f(\gamma)|^p \, \mu(d\gamma) + \int_\Gamma \int_\Gamma \frac{|f(\gamma) - f(\delta)|^p}{|\gamma - \delta|^{d+sp}} \mu(d\gamma)\mu(d\delta), \tag{9}$$

we refer to [6]. We simplify the notation and write $B^s_p(\Gamma)$ instead of $B^s_{pp}(\Gamma, \mu)$.

## 3. Atomic characterizations of $B^s_p(\Gamma)$

Besov spaces $B^s_p(\Gamma)$ with $0 < s < 1$ and $1 < p < \infty$ can be characterized in terms of intrinsic building blocks, namely atoms.

Let for $\delta > 0$, $\Gamma_\delta = \bigcup_{\gamma\in\Gamma} B(\gamma,\delta)$ where

$$B(\gamma,\delta) = \{x \in \mathbb{R}^n : |x-\gamma| < \delta\}, \tag{10}$$

be a $\delta$-neighbourhood of $\Gamma$. Let $0<r<1$ be fixed. Let for $j\in\mathbb{N}_0$, $\{\gamma_{j,m}\}_{m=1}^{M_j}\subset\Gamma$ be the lattice of points with the following properties:

- For some $c_1 > 0$

$$|\gamma_{j,m_1} - \gamma_{j,m_2}| \geq c_1 r^j, \quad j \in \mathbb{N}_0,\ m_1 \neq m_2. \tag{11}$$

- For some some $c_2 > 0$

$$\Gamma_{c_2 r^j} \subset \bigcup_{m=1}^{M_j} B(\gamma_{j,m}, r^j), \quad j \in \mathbb{N}_0, \tag{12}$$

  where $B(\gamma_{j,m}, r^j)$ are given by (10).

Let

$$B^\Gamma_{j,m} = \{\gamma \in \Gamma : |\gamma - \gamma_{j,m}| < r^j\}, \quad j \in \mathbb{N}_0,\ m = 1,\dots,M_j, \tag{13}$$

be the intersection of balls $B(\gamma_{j,m}, r^j)$ with $\Gamma$.

**Definition 3.1.** Let $\Gamma$ be a $d$-set in $\mathbb{R}^n$. Let $1 < p < \infty$ and $0 < s < 1$. Then a continuous function $a_{jm}$ on $\Gamma$ is called an $(s,p)^*$-atom, if for $j \in \mathbb{N}_0$ and $m = 1,\dots,M_j$,

$$\operatorname{supp} a_{jm} \subset B^\Gamma_{j,m}, \tag{14}$$

$$|a_{jm}(\gamma)| \leq \mathrm{H}^d\left(B^\Gamma_{j,m}\right)^{\frac{s}{d}-\frac{1}{p}}, \quad \gamma \in \Gamma, \tag{15}$$

and

$$|a_{jm}(\gamma) - a_{jm}(\delta)| \leq \mathrm{H}^d\left(B^\Gamma_{j,m}\right)^{\frac{s-1}{d}-\frac{1}{p}} |\gamma - \delta| \tag{16}$$

with $\gamma,\delta \in \Gamma$, [17, Section 8.1.3].

Since $\Gamma$ is a $d$-set, we can reformulate (15) and (16) as

$$|a_{jm}(\gamma)| \leq c r^{j\left(s-\frac{d}{p}\right)},$$
$$|a_{jm}(\gamma) - a_{jm}(\delta)| \leq c r^{j\left(s-1-\frac{d}{p}\right)} |\gamma - \delta|.$$

For our further purposes we need the following assertion which is covered by [17, Proposition 8.10].

**Lemma 3.2.** *Let $\Gamma$ be a $d$-set. Let $r > 0$ and*

$$B^\Gamma(r) = \{\gamma \in \Gamma : |\gamma - \gamma_0| < r\} \quad \textit{for some } \gamma_0 \in \Gamma,$$

*and* $B(2r) = \{x \in \mathbb{R}^n : |x - \gamma_0| < 2r\}$. *Let* $f \in B^s_p(\Gamma)$ *with* $\operatorname{supp} f \subset B^\Gamma(r)$. *Then*

$$\left\| f|B^s_p(\Gamma) \right\| = \inf \left\| g|B^t_p(\mathbb{R}^n) \right\|, \quad t = s + \frac{n-d}{p},$$

*where the infimum is taken over all* $g \in B^t_p(\mathbb{R}^n)$, $g|_\Gamma = f$, $\operatorname{supp} g \subset B(2r)$.

Now we can formulate an intrinsic atomic decomposition of the trace spaces $B^s_p(\Gamma)$.

**Theorem 3.3.** *Let* $1 < p < \infty$, *and* $0 < s < 1$. *Then* $B^s_p(\Gamma)$ *is the collection of all* $f \in L_1(\Gamma, \mu)$ *which can be represented as*

$$f(\gamma) = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda^j_m a_{jm}(\gamma), \quad \gamma \in \Gamma, \tag{17}$$

*where* $\|\lambda\| = \left(\sum_{j=0}^{\infty} \sum_{m=1}^{M_j} |\lambda^j_m|^p\right)^{\frac{1}{p}} < \infty$, $a_{jm}$ *are* $(s,p)^*$*-atoms according to Definition* 3.1 *and* (17) *converges absolutely in* $L_1(\Gamma, \mu)$. *Furthermore,*

$$\left\| f|B^s_p(\Gamma) \right\| \sim \inf \|\lambda\| \tag{18}$$

*where infimum is taken over all admissible representations* (17)*,* [17, Chapter 8.1.3].

We introduce new type of atoms, that we call $(s,p,\sigma)$-atoms.

**Definition 3.4.** Let $1 < p < \infty$, $0 < \sigma < 1$ and $0 < s < \sigma$. Then a continuous function $a_{jm}$ on $\Gamma$ is called an $(s,p,\sigma)$-atom, if for $j \in \mathbb{N}_0$ and $m = 1, \ldots, M_j$,

$$\operatorname{supp} a_{jm} \subset B^\Gamma_{j,m}, \tag{19}$$

$$|a_{jm}(\gamma)| \le c r^{j\left(s - \frac{d}{p}\right)}, \quad \gamma \in \Gamma, \tag{20}$$

and

$$|a_{jm}(\gamma) - a_{jm}(\delta)| \le c r^{j\left(s - \sigma - \frac{d}{p}\right)} |\gamma - \delta|^\sigma \tag{21}$$

with $\gamma, \delta \in \Gamma$.

Let $a_{jm}$ be an $(s,p)^*$-atom and $0 < s < \sigma$. Then

$$\begin{aligned}
|a_{jm}(\gamma) - a_{jm}(\delta)| &\le c r^{j\left(s-1-\frac{d}{p}\right)} |\gamma - \delta| \\
&= c r^{j\left(s-1-\frac{d}{p}\right)} |\gamma - \delta|^{1-\sigma} |\gamma - \delta|^\sigma \\
&\le c r^{j\left(s-1-\frac{d}{p}\right)} r^{j(1-\sigma)} |\gamma - \delta|^\sigma \\
&= c r^{j\left(s-\sigma-\frac{d}{p}\right)} |\gamma - \delta|^\sigma,
\end{aligned}$$

which shows that any $(s,p)^*$-atom is an $(s,p,\sigma)$-atom.

**Theorem 3.5.** *Let* $1 < p < \infty$, $0 < \sigma < 1$ *and* $0 < s < \sigma$. *Then* $B^s_p(\Gamma)$ *is the collection of all* $f \in L_1(\Gamma,\mu)$ *which can be represented as*

$$f(\gamma) = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda^j_m a_{jm}(\gamma), \ \gamma \in \Gamma, \tag{22}$$

*where* $\|\lambda\| = \left( \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} |\lambda^j_m|^p \right)^{\frac{1}{p}} < \infty$, $a_{jm}$ *are* $(s,p,\sigma)$*-atoms according to Definition* 3.4 *and* (22) *converges absolutely in* $L_1(\Gamma,\mu)$. *Furthermore,*

$$\left\| f | B^s_p(\Gamma) \right\| \sim \inf \|\lambda\|$$

*where infimum is taken over all admissible representations* (22).

*Proof.* The proof is the adaption of reasoning in [17, Section 8.1.3]. The representation (17) with $(s,p)^*$-atoms is a special case of the representation (22) and it holds (18). Hence it remains to show that from the representation (22) follows that

$$f \in B^s_p(\Gamma) \quad \text{and} \quad \left\| f | B^s_p(\Gamma) \right\| \le c \, \|\lambda\| \, .$$

First we estimate the norm of $(s,p,\sigma)$-atoms in $B^s_p(\Gamma)$. Let $L$ be a number such that $\operatorname{diam} \Gamma \le 2^L$. Then

$$\begin{aligned}
&\int_\Gamma \int_\Gamma \frac{|a_{jm}(\gamma) - a_{jm}(\delta)|^p}{|\gamma - \delta|^{d+sp}} \mu(d\delta)\mu(d\gamma) \\
&\le c \int_\Gamma \int_\Gamma \frac{1}{|\gamma - \delta|^{d+(s-\sigma)p}} \mu(d\delta)\mu(d\gamma) \\
&= c \int_\Gamma \sum_{i=-\infty}^{L} \int_{B(\gamma,2^i)\setminus B(\gamma,2^{i-1})} \frac{1}{|\gamma - \delta|^{d+(s-\sigma)p}} \mu(d\delta)\mu(d\gamma) \\
&\le c \int_\Gamma \sum_{i=-\infty}^{L} \int_{B(\gamma,2^i)\setminus B(\gamma,2^{i-1})} \frac{1}{2^{i(d+(s-\sigma)p)}} \mu(d\delta)\mu(d\gamma) \\
&\le c\mu(\Gamma) \sum_{i=-\infty}^{L} \frac{2^{id}}{2^{i(d+(s-\sigma)p)}} \\
&= c\mu(\Gamma) \frac{2^{L(s-\sigma)p}}{1 - 2^{(s-\sigma)p}} \\
&\le C.
\end{aligned}$$

Moreover,

$$\int_\Gamma |a_{jm}(\gamma)|^p \, \mu(d\gamma) \le \int_{B_{jm}} \mu(B_{jm})^{\frac{sp}{d}-1} \mu(d\gamma) \le \mu(\Gamma)^{\frac{sp}{d}} = C.$$

This means that there is a constant $C > 0$ such that $\left\| a_{jm} | B_p^s(\Gamma) \right\| \leq C$ for all $(s, p, \sigma)$-atoms. Furthermore, for $0 < s \leq \bar{s} < \sigma$ we can write $a_{jm}(\gamma) = r^{j(s-\bar{s})} b_{jm}(\gamma)$, where $b_{jm}(\gamma) = r^{j(\bar{s}-s)} a_{jm}(\gamma)$. For each $j \in \mathbb{N}_0$ and $m = 1, \ldots, M_j$ we have

$$\operatorname{supp} b_{jm} = \operatorname{supp} a_{jm} \subset B_{jm}^{\Gamma}, \quad |b_{jm}(\gamma)| \leq c r^{j\left(\bar{s} - \frac{d}{p}\right)}$$

and

$$|b_{jm}(\gamma) - b_{jm}(\delta)| \leq c r^{j\left(\bar{s} - \sigma - \frac{d}{p}\right)} |\gamma - \delta|^{\sigma}.$$

This shows that $b_{jm}$ are $(\bar{s}, p, \sigma)$-atoms and $\left\| b_{jm} | B_p^{\bar{s}}(\Gamma) \right\| \leq C$. Hence

$$\left\| a_{jm} | B_p^{\bar{s}}(\Gamma) \right\| \leq C r^{j(s-\bar{s})}.$$

We apply Lemma 3.2 to $a_{jm}$. Then it follows that there are functions $A_{jm} \in B_{pp}^{\bar{t}}(\mathbb{R}^n)$, where $\bar{t} = \bar{s} + \frac{n-d}{p}$, such that

$$\operatorname{tr}_\mu A_{jm} = a_{jm}, \quad \operatorname{supp} A_{jm} \subset \left\{ x \in \mathbb{R}^n : |x - \gamma_{jm}| \leq c_1 r^j \right\}$$

and

$$\left\| A_{jm} | B_{pp}^{\bar{t}}(\mathbb{R}^n) \right\| \leq c_2 r^{j(t-\bar{t})}, \quad t = s + \frac{n-d}{p}.$$

Then according to [17, Definition 2.7] $A_{jm}$ are non-smooth atoms for $B_{pp}^t(\mathbb{R}^n)$ and from [17, Theorem 2.3] it follows that

$$F = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_m^j A_{jm} \quad \text{with } \|\lambda\| < \infty$$

belongs to $B_{pp}^t(\mathbb{R}^n)$ and $\left\| F | B_{pp}^t(\mathbb{R}^n) \right\| \leq c \|\lambda\|$. Taking into account that $f = \operatorname{tr}_\mu F$, we may conclude $\left\| f | B_p^s(\Gamma) \right\| \leq c \|\lambda\|$. □

## 4. Self-similar sets

Typical examples of $d$-sets are self-similar sets with invariant measure $\mu$. Generally speaking, a self-similar set is a set that is made up of parts which are similar to the whole. The mathematical definition was given by Hutchinson in [4].

**Definition 4.1.** A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called a similarity (similitude), if there is a constant $0 < \rho < 1$ such that for all $x, y \in \mathbb{R}^n$ holds

$$|F(x) - F(y)| = \rho |x - y|.$$

The constant $\rho$ is called the contraction ratio of $F$.

**Theorem 4.2.** *Let* $\{F_i\}_{i=1}^N$ *be similarities in* $\mathbb{R}^n$*. Then there exists a unique non-empty compact set* $\Gamma \subset \mathbb{R}^n$ *that satisfies*

$$\Gamma = \bigcup_{i=1}^{N} F_i(\Gamma). \tag{23}$$

$\Gamma$ *is called a self-similar set with respect to* $\{F_i\}_{i=1}^N$.

There are many books and papers dealing with self-similar sets, we refer to [2, 4, 8].

A set $\Gamma_w$ with $w = (w_1, w_2, \ldots, w_j)$, $w_i \in \{1, \ldots, N\}$ defined by

$$\Gamma_w = F_w(\Gamma) = F_{w_1} \circ F_{w_2} \circ \ldots \circ F_{w_j}(\Gamma),$$

is called $j$-simplex. We call $w$ a word of length $j = |w|$. Then it holds $\Gamma = \bigcup_{|w|=j} F_w(\Gamma)$.

Let $\Sigma = \{(\omega_1, \omega_2, \ldots) : \omega_i \in \{1, 2, \ldots, N\}\}$ be a set of all infinite sequence. For any $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma$ define a continuous surjective map $\pi : \Sigma \to \Gamma$ by

$$\pi(\omega) = \bigcap_{m=1}^{\infty} \Gamma_{\omega_1 \omega_2 \ldots \omega_m}.$$

Let $C = \bigcup_{i \neq j} (\Gamma_i \cap \Gamma_j)$, $\mathcal{C} = \pi^{-1}(C)$ and $\mathcal{P} = \bigcup_{n \geq 1} \sigma^n(\mathcal{C})$, where $\sigma : \Sigma \to \Sigma$ is the shift map defined by

$$\sigma(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots).$$

Let $V_0 = \pi(\mathcal{P})$ and $V_j = \bigcup_{i=1}^N F_i(V_{j-1})$, or equivalently

$$V_j = \bigcup_{|w|=j} F_w(V_0).$$

Then $V_j$ describes the set of boundary points of simplexes of fixed level $j$. It is clear that $V_j \subset V_{j+1}$. Let $V_* = \bigcup_{j=0}^{\infty} V_j$, then $\Gamma = \overline{V_*}$ in the Euclidean topology. When $j$ is fixed, $V_j$ is the natural lattice of points in the self-similar set $\Gamma$ that satisfies (11) and (12). We followed [8, Sections 1.2–1.3]. We form a graph $G_j$ with vertices $V_j$ and edge relation $\xi \sim_j \eta$ holding if and only if there exists a $j$-simplex containing both $\xi$ and $\eta$ as boundary points.

In this paper we consider sets $\Gamma$ such that they are self-similar with respect to the similarities with the same contraction ratio $0 < \rho < 1$, that is

$$|F_i(x) - F_i(y)| = \rho \, |x - y| \, . \tag{24}$$

The unit interval $\mathrm{I} = [0, 1]$ is a self-similar set with respect to the similarities $F_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$,

$$F_1(x) = \frac{1}{2}x, \quad F_2(x) = \frac{1}{2}x + \frac{1}{2}.$$

The Koch curve $K$ is a self-similar set with respect to the similarities $F_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2$,

$$F_1(x,y) = \left(\frac{1}{2}x + \frac{1}{2\sqrt{3}}y, \frac{1}{2\sqrt{3}}x - \frac{1}{2}y\right),$$
$$F_2(x,y) = \left(\frac{1}{2}x - \frac{1}{2\sqrt{3}}y + \frac{1}{2}, -\frac{1}{2\sqrt{3}}x - \frac{1}{2}y + \frac{1}{2\sqrt{3}}\right),$$

see [8], where mappings $F_1$, $F_2$ are given in a complex form. The self-similar structure of the unit interval I and the Koch curve $K$ can be used to establish the transform

$$H : \mathrm{I} \to K, \tag{25}$$

such that

$$|H(x) - H(y)|^d \sim |x - y|, \tag{26}$$

where $d = \frac{\ln 4}{\ln 3}$ is the Hausdorff dimension of the Koch curve $K$. For the analytical expression of $H$ we refer to [8, Example 1.2.7], some information can be also find in [17, Section 8.2.2].

There is a special kind of sets that are self-similar with respect to similarities (24), satisfying some additional properties, known as nested fractals. They were first introduced by Lindstrøm [11], and afterwards were studied by many authors, e.g. [10, 12]. Nested fractals should satisfy following conditions:

C0. $\#V_0 \geq 2$.

C1. **Open set condition.** The family of similarities $\{F_i\}_{i=1}^N$ satisfies the open set condition if there exists an open, bounded, nonempty set $O \subset \mathbb{R}^n$ such that

$$F_i(O) \cap F_j(O) = \emptyset \text{ for } i \neq j \quad \text{and} \quad \bigcup_{i=1}^N F_i(O) \subset O.$$

When the open set condition is satisfied, the Hausdorff dimension $d$ of $\Gamma$ is

$$d = \frac{\log N}{\log \frac{1}{\rho}},$$

we refer to [2, 4].

C2. **Nesting.** If $j \geq 1$ and $w = (w_1, w_2, \ldots, w_j)$ and $w' = (w'_1, w'_2, \ldots, w'_j)$ are distinct elements of $\{1, 2, \ldots, N\}^n$, then

$$\Gamma_w \cap \Gamma_{w'} = F_w(V_0) \cap F_{w'}(V_0).$$

C3. **Connectivity.** The graph $(V_1, G_1)$ is connected.

C4. **Symmetry.** For any $x, y \in \mathbb{R}^n$ with $x \neq y$, let $H_{xy}$ denote the hyperplane given by

$$H_{xy} = \{z \in \mathbb{R}^n : |z - x| = |z - y|\}$$

and let $R_{xy}$ denote the reflection with respect to $H_{xy}$. Then for any $x, y \in V_0$ with $x \neq y$, $R_{xy}$ maps $j$-cells to $j$-cells, and maps any $j$-cell which contains elements in both sides of $H_{xy}$ to itself for each $j \geq 0$.

The simplest example of the nested fractal is the Sierpinski gasket SG, which is generated by three similarities in the plane $F_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2, 3$, defined by

$$F_i(x) = \frac{1}{2}(x - \xi_i) + \xi_i, \tag{27}$$

where $\xi_i$ are the vertices of an equilateral triangle, see [15, Section 1.1].

Further on we assume that the diameter of $\Gamma$ is 1. Then the diameter of each $j$-simplex $\Gamma_{w_1 \ldots w_j}$ is $\rho^j$, where $\rho$ is from (24). In case of $I$ and SG we get $\rho = \frac{1}{2}$, in case of $K$ we have $\rho = \frac{1}{\sqrt{3}}$.

Suppose a real-valued function $u$ is given on the vertices $V_j$. Then there is a natural Dirichlet form

$$E_j(u) = \sum_{\xi \sim_j \eta} \left(u(\xi) - u(\eta)\right)^2 .$$

We need to multiply $E_j$ by the renormalization factor $\alpha^j$ in order to have the following consistency property:

**Lemma 4.3.** *For every function $u$ on $V_j$ there exists a unique extension $\tilde{u}$ to $V_{j+1}$ minimizing $E_{j+1}$, i.e.,*

$$E_{j+1}(\tilde{u}) = \min \left\{ E_{j+1}(u') : u'|_{V_j} = u \right\},$$

*and*

$$\alpha^j E_j(u) = \alpha^{j+1} E_{j+1}(\tilde{u}). \tag{28}$$

For I and $K$ the renormalization factor $\alpha$ is equal to 2, for SG we have $\alpha = \frac{5}{3}$, [15, Section 1.3]. The number $d_w = \frac{\log N\alpha}{\log \frac{1}{\rho}}$ is called the walk dimension of $\Gamma$. The renormalized graph energies are defined by $\mathcal{E}_j(u) = \alpha^j E_j(u)$. Then (28) can be reformulated as

$$\mathcal{E}_j(u) = \mathcal{E}_{j+1}(\tilde{u}).$$

The function $\tilde{u}$ is called a harmonic extension of $u$.

**Definition 4.4.** A continuous function $h : V_* \to \mathbb{R}$ is called harmonic if it minimizes $\mathcal{E}_j$ at all levels for given boundary values on $V_0$:

$$\mathcal{E}_j(h) = \min \left\{ \mathcal{E}_j(u) : u|_{V_0} = \rho \right\}.$$

According to [8, Theorem 3.2.4] for any harmonic function $u$ there exists a unique extension $\tilde{u} \in C(\Gamma)$ such that $\tilde{u}|_{V_*} = u|_{V_*}$. Thus, we identify $u$ with its extension $\tilde{u}$ and think of a harmonic function as a continuous function on $\Gamma$. The maximum and the minimum of the harmonic function are attained at the boundary $V_0$. This assertion is known as the maximum principle [8].

**Definition 4.5.** A continuous function $\psi : V_* \to \mathbb{R}$ is called piecewise harmonic of level $j$ if $\psi \circ F_w$ is harmonic for all $|w| = j$.

We denote the set of piecewise harmonic functions of level $j$ by $H_j$. These functions minimize $\mathcal{E}_m$ at all levels $m \geq j$ for given boundary values on $V_j$.

For $f : V_* \to \mathbb{R}$ define

$$\mathcal{E}(f) = \lim_{j\to\infty} \mathcal{E}_j(f), \quad \tilde{\mathcal{D}} = \{f : V_* \to \mathbb{R}, \mathcal{E}(f) < \infty\}.$$

If $f \in \tilde{\mathcal{D}}$, then it is uniformly continuous on $V_*$, hence it has a unique continuous extension to $\Gamma$. Let

$$\mathcal{D} = \{f \in C(\Gamma) : \mathcal{E}(f) < \infty\}.$$

Then $(\mathcal{E}, \mathcal{D})$ is regular Dirichlet form on $L_2(\Gamma, \mu)$.

By effective resistance metric on the set $\Gamma$ we mean a function $R : \Gamma \times \Gamma \to [0, \infty]$ defined by $R(x, x) = 0$ for $x \in \Gamma$ and

$$R(x, y)^{-1} = \inf \{\mathcal{E}(f) : f(x) = 0, f(y) = 1\}.$$

Let $\psi^j_\xi$, $\xi \in V_j$, be a piecewise harmonic function of level $j$ which equals 1 at $\xi$ and 0 at any other vertex of $V_j$:

$$\psi^j_\xi(x) = \delta_{\xi x} = \begin{cases} 1, & x = \xi \\ 0, & x \in V_j \setminus \{\xi\}. \end{cases}$$

Note that $\operatorname{supp} \psi^j_\xi \subset B(\xi, \rho^j)$.

In the case of the unit interval I piecewise harmonic functions are just piecewise linear functions. In fact, for $x = \frac{m}{2^{j-1}} + \frac{1}{2^j} \in V_j \setminus V_{j-1}$

$$\psi^j_x(t) = \begin{cases} 2^j(t - \frac{m}{2^{j-1}}), & \frac{m}{2^{j-1}} \leq t < \frac{m}{2^{j-1}} + \frac{1}{2^j} \\ 2^j(\frac{m+1}{2^{j-1}} - t), & \frac{m}{2^{j-1}} + \frac{1}{2^j} \leq t < \frac{m+1}{2^{j-1}} \\ 0, & \text{otherwise}, \end{cases}$$

and it holds

$$\left|\psi^j_x(t) - \psi^j_x(s)\right| \leq c\,|t - s| \quad \text{for all } t, s \in \mathrm{I}. \tag{29}$$

For the Koch curve $\Gamma$ piecewise harmonic functions $\widetilde{\psi}^j_\xi$ with $\xi = H(x)$ are the composition $\psi^j_x$ with the transform $H^{-1}$ from (25), $\widetilde{\psi}^j_\xi = \psi^j_x \circ H^{-1}$. Taking into account (29) and (26) we get

$$\left|\widetilde{\psi}^j_\xi(\gamma) - \widetilde{\psi}^j_\xi(\delta)\right| \leq c\,|\gamma - \delta|^d, \tag{30}$$

where $d = \frac{\ln 4}{\ln 3}$ is the Hausdorff dimension of $\Gamma$.

In general it was shown in [9] that harmonic functions on $\Gamma$ are uniformly Lipschitz continuous with respect to the resistance metric $R(x,y)$. From [3] follows that for a certain class of nested fractals, that satisfy Assumption 2.2 in [10], there exist constants $c, c' > 0$ such that for all $x, y \in \Gamma$

$$c' \left|x-y\right|^{\frac{\log \frac{1}{\alpha}}{\log \rho}} \leq R(x,y) \leq c \left|x-y\right|^{\frac{\log \frac{1}{\alpha}}{\log \rho}},$$

note that $\frac{\log \frac{1}{\alpha}}{\log \rho} = d_w - d$. Thus piecewise harmonic functions on certain nested fractals satisfy

$$\left|\psi_\xi^j(x) - \psi_\xi^j(y)\right| \leq c \left|x-y\right|^\sigma, \tag{31}$$

with $\sigma = d_w - d$. In particular, piecewise harmonic functions on the Sierpinski gasket satisfy

$$\left|\psi_\xi^j(x) - \psi_\xi^j(y)\right| \leq c \left|x-y\right|^\beta, \quad \text{for all } x, y \in \mathrm{SG},$$

where $\beta = \frac{\ln(5/3)}{\ln 2}$.

## 5. Characterization of Besov spaces $B^s_{pq}(\Gamma, \mu)$ by piecewise harmonic functions

Let $f \in C(\Gamma)$ and let $P_n f$, $n \geq 0$, be the unique piecewise harmonic function in $H_n$ which interpolates $f$ at all points in $V_n$:

$$P_0 f = \sum_{\xi \in V_0} f(\xi) \psi_\xi^0,$$

$$P_n f = \sum_{\xi \in V_0} f(\xi) \psi_\xi^0 + \sum_{j=1}^{n} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi_\xi^j, \; n \geq 1,$$

with

$$c_\xi(f) = f(\xi) - P_{j-1} f(\xi), \quad \xi \in V_j \setminus V_{j-1}, \; 1 \leq j \leq n, \tag{32}$$

[8, Definition 3.2.18]. For $\xi \in V_j \setminus V_{j-1}$ there is an $\omega \in \Sigma$ such that

$$\xi = \pi(\omega). \tag{33}$$

We define $\Delta(\xi)$ by $\Delta(\xi) = \left\{\eta \in V_{j-1} : \eta \in F_{\omega_1 \omega_2 \ldots \omega_{j-1}}(\Gamma)\right\}$, where $\omega$ is chosen according to (33). $\Delta(\xi)$ consists of vertices of $(j-1)$-simplex that $\xi$ belongs to. It is the same as the one defined in [5, Section 4.1]. Then (32) can be equivalently calculated as

$$c_\xi(f) = f(\xi) - \sum_{\eta \in \Delta(\xi)} \alpha_{\xi\eta} f(\eta), \quad \xi \in V_j \setminus V_{j-1}, \; 1 \leq j \leq n, \tag{34}$$

with

$$\sum_{\eta\in\Delta(\xi)} \alpha_{\xi\eta} = 1. \tag{35}$$

Let $V_{-1} = \emptyset$ and $P_{-1}f \equiv 0$, then $P_n f = \sum_{j=0}^{n} \sum_{\xi\in V_j\setminus V_{j-1}} c_\xi(f)\psi_\xi^j$. From the maximum principle for harmonic functions and from [14, Proposition 1.3.2] it follows that $P_n f$ tends to $f$ uniformly on $\Gamma$ as $n \to \infty$ and $f \in C(\Gamma)$ has the unique representation

$$f = \sum_{j=0}^{\infty} \sum_{\xi\in V_j\setminus V_{j-1}} c_\xi(f)\psi_\xi^j.$$

The question arises whether Besov spaces with a certain range of parameters can be characterized by coefficients $c_\xi(f)$. We give an affirmative answer in the following theorem.

**Theorem 5.1.** *Let $\Gamma$ be the above $d$-set with $\rho$ as in* (24) *and $\sigma$ as in* (31). *Let*

$$1 < p < \infty \quad \text{and} \quad \frac{d}{p} < s < \min\{1, \sigma\}. \tag{36}$$

*Then $f \in C(\Gamma)$ belongs to $B_p^s(\Gamma)$ if and only if it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{\xi\in V_j\setminus V_{j-1}} c_\xi(f)\psi_\xi^j, \tag{37}$$

*where $C_p^s(f) = \left(\sum_{j=0}^{\infty} \rho^{j\left(\frac{d}{p}-s\right)p} \sum_{\xi\in V_j\setminus V_{j-1}} |c_\xi(f)|^p\right)^{\frac{1}{p}} < \infty$, unconditional convergence being in $C(\Gamma)$. Furthermore,*

$$\left\| f|B_p^s(\Gamma)\right\| \sim C_p^s(f).$$

*Proof.* The idea of the proof is the same as in [5, Theorem 5.1]. Let

$$a_{j\xi}(x) = \rho^{j\left(s-\frac{d}{p}\right)}\psi_\xi^j(x), \quad j \in \mathbb{N}_0,\ \xi \in V_j \setminus V_{j-1}.$$

Then $a_{j\xi}$ satisfy (19)-(21). Taking into account that $C(\Gamma) \subset L_1(\Gamma)$ we get that (37) is an atomic representation of $f$ and from the Theorem 3.5 it follows that

$$\left\| f|B_p^s(\Gamma)\right\| \le c\, C_p^s(f).$$

To prove the converse, let $f \in B_p^s(\Gamma)$ and let $f = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_m^j a_{jm}$ be an atomic decomposition of $f$ into $(s,p,\sigma)$-atoms with $r = \rho$ in (13), (20) and (21) such that

$$\|\lambda\| \le c \left\| f|B_p^s(\Gamma)\right\|. \tag{38}$$

Then taking into account (20) and that $s > \frac{d}{p}$ we get

$$\left|\sum_{m=1}^{M_j} \lambda_m^j a_{jm}\right| \leq \sup_m \left|\lambda_m^j\right| \sum_{m=1}^{M_j} \rho^{j\left(s-\frac{d}{p}\right)} \leq c\rho^{\left(s-\frac{d}{p}\right)} \sup_m \left|\lambda_m^j\right| \leq c\rho^{\left(s-\frac{d}{p}\right)} \left(\sum_{m=1}^{M_j} \left|\lambda_m^j\right|^p\right)^{\frac{1}{p}}.$$

The Weierstrass test together with the estimate (38) imply that the series $\sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_m^j a_{jm}$ converges uniformly and it follows

$$c_\xi(f) = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_m^j c_\xi(a_{jm}).$$

From the formula (34) together with (35) and the property (20) of $(s, p, \sigma)$-atoms follows

$$|c_\xi(a_{jm})| \leq 2\rho^{j\left(s-\frac{d}{p}\right)}. \tag{39}$$

Moreover, for $i > 0$ the property (21) implies

$$\begin{aligned} |c_\xi(a_{jm})| &= \left| a_{jm}(\xi) - \sum_{\eta \in \Delta(\xi)} \alpha_{\xi\eta} a_{jm}(\eta) \right| \\ &= \left| \sum_{\eta \in \Delta(\xi)} \alpha_{\xi\eta} (a_{jm}(\xi) - a_{jm}(\eta)) \right| \\ &\leq \rho^{i\sigma} \rho^{j\left(s-\sigma-\frac{d}{p}\right)}, \quad \xi \in V_i \setminus V_{i-1}. \end{aligned} \tag{40}$$

Let us split $c_\xi(f)$ into two parts

$$c_\xi(f) = \sum_{j=0}^{i} \sum_{m=1}^{M_j} \lambda_m^j c_\xi(a_{jm}) + \sum_{j=i+1}^{\infty} \sum_{m=1}^{M_j} \lambda_m^j c_\xi(a_{jm}) = x_\xi(f) + y_\xi(f).$$

Taking into account the support condition for atoms (19), we get that for all $\xi$ and $j$ the number of atoms such that $c_\xi(a_{jm}) \neq 0$ is finite.

First we deal with

$$X_{i,p} = \left(\sum_{\xi \in V_i \setminus V_{i-1}} |x_\xi(f)|^p\right)^{\frac{1}{p}}.$$

Note that $\{\xi \in V_i \setminus V_{i-1} : c_\xi(a_{jm}) \neq 0\} \subset \left\{\xi \in V_i \cap B^\Gamma(\gamma_{jm}, \rho^j)\right\}$. The balls $B^\Gamma(\xi, \frac{\rho^i}{2})$ corresponding to different $\xi \in V_i \cap B^\Gamma(\gamma_{jm}, \rho^j)$ are disjoint and for $j < i$ they are contained in $B^\Gamma(\gamma_{jm}, 2\rho^j)$. Thus

$$\sum_{\xi \in V_i \cap B^\Gamma(\gamma_{jm}, \rho^j)} \mu\left(B^\Gamma\left(\xi, \frac{\rho^i}{2}\right)\right) \leq \mu\left(B^\Gamma\left(\gamma_{j,m}, 2\rho^j\right)\right).$$

Since $\mu$ is a $d$-measure this implies that $\left\{\xi \in V_i \cap B^\Gamma(\gamma_{jm}, \rho^j)\right\}$ can have at most $c\left(\frac{\rho^j}{\rho^i}\right)^d$ elements. Hence

$$\#\left\{\xi \in V_i \setminus V_{i-1} : c_\xi(a_{jm}) \neq 0\right\} \leq c\rho^{(j-i)d}, \quad j < i.$$

By Minskowski's and Hölder's inequalities together with (40)

$$\begin{aligned} X_{i,p} &\leq \sum_{j=0}^{i}\left(\sum_{\xi\in V_i\setminus V_{i-1}}\left(\sum_{m=1}^{M_j}\left|\lambda_m^j\right|\left|c_\xi(a_{jm})\right|\right)^p\right)^{\frac{1}{p}} \\ &\leq \sum_{j=0}^{i}\sum_{m=1}^{M_j}\left(\sum_{\xi\in V_i\setminus V_{i-1}}\left|\lambda_m^j\right|^p\left|c_\xi(a_{jm})\right|^p\right)^{\frac{1}{p}} \\ &\leq \sum_{j=0}^{i}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|\rho^{i\sigma}\rho^{j\left(s-\sigma-\frac{d}{p}\right)}\rho^{(j-i)\frac{d}{p}} \\ &\leq c\rho^{i\left(\sigma-\frac{d}{p}\right)}\sum_{j=0}^{i}\rho^{j(s-\sigma)}\left(\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{\frac{1}{p}} \end{aligned}$$

and it follows $X_{i,p,s} = \rho^{i\left(\frac{d}{p}-s\right)}X_{i,p} \leq c\rho^{i(\sigma-s)}\sum_{j=0}^{i}\rho^{j(s-\sigma)}\left(\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{\frac{1}{p}}$. Jensen's inequality implies

$$X_{i,p,s}^p \leq c\rho^{i(\sigma-s)p}\rho^{i(s-\sigma)(p-1)}\sum_{j=0}^{i}\rho^{j(s-\sigma)}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p = c\rho^{i(\sigma-s)}\sum_{j=0}^{i}\rho^{j(s-\sigma)}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p.$$

Then

$$\begin{aligned} \left(\sum_{i=0}^{\infty}X_{i,p,s}^p\right)^{\frac{1}{p}} &\leq c\left(\sum_{i=0}^{\infty}\rho^{i(\sigma-s)}\sum_{j=0}^{i}\rho^{j(s-\sigma)}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{\frac{1}{p}} \\ &= c\left(\sum_{j=0}^{\infty}\left(\sum_{i=j}^{\infty}\rho^{i(\sigma-s)}\right)\rho^{j(s-\sigma)}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{\frac{1}{p}} \\ &\leq c\left(\sum_{j=0}^{\infty}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{\frac{1}{p}} = c\left\|\lambda\right\|. \end{aligned}$$

To estimate

$$Y_{i,p} = \left(\sum_{\xi\in V_i\setminus V_{i-1}}\left|y_\xi(f)\right|^p\right)^{\frac{1}{p}}$$

we use Minkowski's and Hölder's inequalities together with the property (39). Then we get

$$
\begin{aligned}
Y_{i,p} &\leq \sum_{j=i}^{\infty} \left( \sum_{\xi \in V_i \setminus V_{i-1}} \left( \sum_{m=1}^{M_j} \left| \lambda_m^j \right| |c_\xi(a_{jm})| \right)^p \right)^{\frac{1}{p}} \\
&\leq \sum_{j=i}^{\infty} \sum_{m=1}^{M_j} \left( \sum_{\xi \in V_i \setminus V_{i-1}} \left| \lambda_m^j \right|^p |c_\xi(a_{jm})|^p \right)^{\frac{1}{p}} \\
&\leq c \sum_{j=i}^{\infty} \sum_{m=1}^{M_j} \left| \lambda_m^j \right| \rho^{j\left(s-\frac{d}{p}\right)} \\
&\leq c \sum_{j=i}^{\infty} \rho^{j\left(s-\frac{d}{p}\right)} \left( \sum_{m=1}^{M_j} \left| \lambda_m^j \right|^p \right)^{\frac{1}{p}} .
\end{aligned}
$$

Hence we have $Y_{i,p,s} = \rho^{i\left(\frac{d}{p}-s\right)} Y_{i,p} \leq c \rho^{i\left(\frac{d}{p}-s\right)} \sum_{j=i}^{\infty} \rho^{j\left(s-\frac{d}{p}\right)} \left( \sum_{m=1}^{M_j} |\lambda_m^j|^p \right)^{\frac{1}{p}}$. Applying Jensen's inequality we get

$$
Y_{i,p,s}^p \leq c \rho^{i\left(\frac{d}{p}-s\right)p} \rho^{i\left(\frac{d}{p}-s\right)(p-1)} \sum_{j=i}^{\infty} \rho^{j\left(s-\frac{d}{p}\right)} \sum_{m=1}^{M_j} \left| \lambda_m^j \right|^p \leq c \rho^{i\left(\frac{d}{p}-s\right)} \sum_{j=i}^{\infty} \rho^{j\left(s-\frac{d}{p}\right)} \sum_{m=1}^{M_j} \left| \lambda_m^j \right|^p .
$$

Then

$$
\begin{aligned}
\left( \sum_{i=0}^{\infty} Y_{i,p,s}^p \right)^{\frac{1}{p}} &\leq c \left( \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \rho^{i\left(\frac{d}{p}-s\right)} \rho^{j\left(s-\frac{d}{p}\right)} \sum_{m=1}^{M_j} \left| \lambda_m^j \right|^p \right)^{\frac{1}{p}} \\
&\leq c \left( \sum_{j=0}^{\infty} \left( \sum_{i=0}^{j} \rho^{i\left(\frac{d}{p}-s\right)} \right) \rho^{j\left(s-\frac{d}{p}\right)} \sum_{m=1}^{M_j} \left| \lambda_m^j \right|^p \right)^{\frac{1}{p}} \\
&\leq c \left( \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \left| \lambda_m^j \right|^p \right)^{\frac{1}{p}} = c \left\| \lambda \right\| .
\end{aligned}
$$

Thus $C_p^s(f) = \left( \sum_{i=0}^{\infty} \rho^{i\left(\frac{d}{p}-s\right)p} \sum_{\xi \in V_i \setminus V_{i-1}} |c_\xi(f)|^p \right)^{\frac{1}{p}}$ can estimated by

$$
C_p^s(f) \leq \left( \sum_{i=0}^{\infty} X_{i,p,s}^p \right)^{\frac{1}{p}} + \left( \sum_{i=0}^{\infty} Y_{i,p,s}^p \right)^{\frac{1}{p}} \leq c \left\| \lambda \right\| \leq c \left\| f | B_p^s(\Gamma) \right\| . \qquad \square
$$

**Corollary 5.2.** *Let*

$$1 < p < \infty \quad \text{and} \quad \frac{d}{p} < s < \min\{1, \sigma\}.$$

*The system of functions* $\{\psi_\xi^j, j \in \mathbb{N}_0, \xi \in V_j \setminus V_{j-1}\}$ *is an unconditional basis in* $B_p^s(\Gamma)$.

*Proof.* Let $f \in B_p^s(\Gamma)$. Then $f$ has the unique representation

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi_\xi^j \tag{41}$$

with the convergence first being in $C(\Gamma)$. It is left to show that (41) converges in $B_p^s(\Gamma)$.

Let us show that the sequence of partial sums $S_n = \sum_{j=0}^n \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi_\xi^j$ is a Cauchy sequence in $B_p^s(\Gamma)$. For $n > m$

$$\begin{aligned} \left\| S_n - S_m | B_p^s(\Gamma) \right\| &= \left\| \sum_{j=m+1}^{n} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi_\xi^j | B_p^s(\Gamma) \right\| \\ &\sim \left( \sum_{j=m+1}^{n} \rho^{j\left(\frac{d}{p} - s\right)p} \sum_{\xi \in V_j \setminus V_{j-1}} |c_\xi(f)|^p \right)^{\frac{1}{p}} \to 0, \quad n, m \to \infty. \end{aligned}$$

Since $B_p^s(\Gamma)$ is complete, the series (41) converges to $f$ in $B_p^s(\Gamma)$. □

**Corollary 5.3.** *Let*

$$\frac{d}{p} < s < \min\{1, \sigma\}, \quad 1 < p < \infty, \quad 1 \le q < \infty.$$

*Then the Theorem* 5.1 *remains valid for* $B_{pq}^s(\Gamma, \mu)$.

The proof follows by the same arguments that were used in [7, Section 3.3].

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