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Coincidence and Calculation of some Strict s-Numbers

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Abstract. The paper considers the so-called strict s-numbers, which form an important subclass of the family of all s-numbers. For operators acting between Hilbert spaces the various s-numbers are known to coincide: here we give examples of linear maps T and non-Hilbert spaces X, Y such that all strict s-numbers of $T : X \to Y$ coincide. The maps considered are either simple integral operators acting in Lebesgue spaces or Sobolev embeddings; in these cases the exact value of the strict s-numbers is determined.

Keywords. s-Numbers, generalized trigonometric functions, Sobolev embedding, Hardy operator, widths, compact maps, asymptotic estimates

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1. Introduction

In 1974, Pietsch [12] introduced his axiomatic theory of s-numbers of bounded linear operators acting between Banach spaces. This theory plays an important rôle in approximation theory and also in operator theory, and offers a unified base for studying the approximation numbers and other important numbers such as those associated with Bernstein, Mityagin and Kolmogorov. To be more precise we now define s-numbers and mention some basic facts concerning them.

Given Banach spaces X, Y, the closed unit ball in X will be denoted by B_X, while $B(X, Y)$ will stand for the space of all bounded linear maps of X to Y; we shall write $B(X)$ instead of $B(X, X)$.

Let $s : T \longmapsto (s_n(T))$ be a rule that attaches to every bounded linear operator acting between any pair of Banach spaces a sequence of non-negative numbers that has the following properties:

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(S1) $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$.
(S2) $s_n(S+T) \leq s_n(S) + \|T\|$ for $S, T \in B(X,Y)$ and $n \in \mathbb{N}$.
(S3) $s_n(BTA) \leq \|B\|\, s_n(T)\, \|A\|$ whenever $A \in B(X_0, X)$, $T \in B(X,Y)$, $B \in B(Y, Y_0)$ and $n \in \mathbb{N}$.
(S4) $s_n(Id : l_2^n \to l_2^n) = 1$ for $n \in \mathbb{N}$.
(S5) $s_n(T) = 0$ when $\text{rank}(T) < n$.

We shall call $s_n(T)$ (or $s_n(T : X \to Y)$) the n^{th} s-number of T.

When the property (S4) is replaced by

(S6) $s_n(Id : E \to E) = 1$ for every Banach space E with dim $(E) \geq n$,

we say that $s_n(T)$ is the n^{th} s-number of T in the "strict" sense. It is obvious that (S6) implies (S4), and so for a given operator T the class of s-numbers is larger than that of strict s-numbers. Note that the original definition of s-numbers given in [12] coincides with that of strict s-numbers provided here.

Given $T \in B(X,Y)$ and $n \in \mathbb{N}$, the n^{th} approximation number of T is defined to be

$$a_n(T) = \inf\{\|T - F\| : F \in B(X,Y),\ \text{rank}(F) < n\};$$

the n^{th} isomorphism number of T is

$$i_n(T) = \sup\left\{\|A\|^{-1}\,\|B\|^{-1}\right\},$$

where the supremum is taken over all Banach spaces G with $\dim(G) \geq n$ and all maps $A \in B(Y,G)$, $B \in B(G,X)$ such that ATB is the identity on G. The approximation numbers are strict and are the largest s-numbers; the isomorphism numbers are the smallest strict s-numbers; for maps between Hilbert spaces, all s-numbers coincide. Further examples of s-numbers are given by the numbers associated with the names of Bernstein, Chang, Gelfand, Hilbert, Kolmogorov, Mityagin and Weyl; the Bernstein, Gelfand and Mityagin numbers are strict. When the spaces involved are not Hilbert spaces it is certainly not true that all s-numbers coincide: for example, if $I_1 : l_1 \to l_\infty$ is the identity, then the n^{th} Bernstein and Mityagin numbers of I_1 coincide and equal $\frac{1}{n}$, while the n^{th} Gelfand and Kolmogorov numbers of I_1 coincide and are $\geq \frac{1}{2}$; for the identity map $I_2 : l_1 \to l_1$ we have $a_n(I_2) = 1$ and the n^{th} Hilbert number of I_2 behaves like $\frac{1}{\sqrt{n}}$. For these results, together with more information about s-numbers, and those that are strict, we refer to [12] and the remarkable book [13].

In this paper integral operators of Hardy type, acting in L_p, and certain Sobolev embeddings are considered: for each of these it is shown that all strict numbers coincide and are given by an explicit formula: see Theorems 3.3 and 3.5, together with Theorems 4.1–4.4.

2. Preliminaries

Let $1 < p < \infty$ and define a (differentiable) function $F_p : [0,1] \to \mathbb{R}$ by

$$F_p(x) = \int_0^x \frac{1}{\sqrt[p]{1-t^p}}\,dt, \quad 0 \le x \le 1. \tag{2.1}$$

Since F_p is strictly increasing it is a one-to-one function on $[0,1]$ with range $\left[0, \frac{\pi_p}{2}\right]$, where

$$\pi_p = 2\int_0^1 \frac{1}{\sqrt[p]{1-t^p}}\,dt. \tag{2.2}$$

The inverse of F_p on $\left[0, \frac{\pi_p}{2}\right]$ we denote by $\sin_p$ and extend as in the case of sin (when $p = 2$) to $[0, \pi_p]$ by defining

$$\sin_p(x) = \sin_p(\pi_p - x) \quad \text{for } x \in \left[\frac{\pi_p}{2}, \pi_p\right];$$

further extension is achieved by oddness and $2\pi_p$-periodicity on the whole of $\mathbb{R}$. By this means we obtain a differentiable function on $\mathbb{R}$ which coincides with sin when $p = 2$.

Corresponding to this we define a function $\cos_p$ by the prescription

$$\cos_p(x) = \frac{d}{dx}\sin_p(x), \quad x \in \mathbb{R}. \tag{2.3}$$

Clearly $\cos_p$ is even, $2\pi_p$-periodic and odd about π_p; and $\cos_2 = \cos$. If $x \in \left[0, \frac{\pi_p}{2}\right]$, then from the definition it follows that $\cos_p(x) = \left(1 - (\sin_p(x))^p\right)^{\frac{1}{p}}$. Moreover, the antisymmetry and periodicity show that

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1, \quad x \in \mathbb{R}. \tag{2.4}$$

From (2.2) it follows that

$$\frac{\pi_p}{2} = p^{-1}\int_0^1 (1-s)^{-\frac{1}{p}} s^{\frac{1}{p}-1} ds = p^{-1} B\left(1 - \frac{1}{p}, \frac{1}{p}\right) = p^{-1}\Gamma\left(1 - \frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right),$$

where B is the Beta function, Γ is the Gamma function and

$$\pi_p = \frac{2\pi}{p \sin\left(\frac{\pi}{p}\right)}. \tag{2.5}$$

Clearly $\pi_2 = \pi$ and, with $p' = \frac{p}{p-1}$,

$$p\pi_p = 2\Gamma\left(\frac{1}{p'}\right)\Gamma\left(\frac{1}{p}\right) = p'\pi_{p'}. \tag{2.6}$$

More about the $\sin_p$ and $\cos_p$ functions can be found in [1,4] and [6–11]; in particular, an excellent historical review of generalized trigonometric functions is given in [9].

Consider the most simple integral operator. On the interval $I = (a, b)$ let

$$T_c f(x) := \int_c^x f(t)dt, \quad \text{where } c \in [a, b]. \tag{2.7}$$

At first we consider T_0 as a map from $L_2(0,1)$ into $L_2(0,1)$. It is obvious that T_0 is compact and that there exists a function in $L_2(0,1)$ at which the norm of T_0 is attained. In this case it is quite simple to show that $\|T_0|L_2(0,1) \to L_2(0,1)\| = \frac{2}{\pi}$ and that the norm is attained when $f(t) = \cos\left(\frac{\pi t}{2}\right)\frac{\pi}{2}$ so that $T_0 f(t) = \sin\left(\frac{\pi t}{2}\right)$.

When $p \neq 2$ then again T_0 is a compact map from $L_p(0,1)$ into $L_p(0,1)$ and there exists a function at which the norm is attained. In a classical paper [5], the following theorem was proved.

Theorem 2.1. *Let $p \in (1, \infty)$ and let I be the interval $(0,1)$. Then*

$$\|T_0 : L_p(I) \to L_p(I)\| = \frac{(p')^{\frac{1}{p}} p^{\frac{1}{p'}}}{\pi} \sin\left(\frac{\pi}{p}\right). \tag{2.8}$$

The extremals are the non-zero multiples of $f(x) = \frac{\pi_p}{2}\cos_p\left(\frac{\pi_p x}{2}\right)$ and $T_0 f(x) = \sin_p\left(\frac{\pi_p x}{2}\right)$.

A more general version of this theorem was independently proved in [14]. We define a quantity A_0 that plays a key rôle in the approximation of T_c.

Definition 2.2. Let $J := (c, d) \subset I = (a, b)$. We define

$$A_0(J) = \sup_{\|u\|_{p,J} > 0} \inf_{\alpha \in \mathbb{R}} \frac{\left\|\int_c^{\cdot} u(t)dt - \alpha\right\|_{p,J}}{\|u\|_{p,J}}.$$

The next two lemmas can be obtained, after some modifications, from results contained in the paper [3] but for the reader's convenience we prove them.

Lemma 2.3. *Let $(x, y) \subset I$. Then $A_0((x, y))$ is a continuous function of x and y.*

Proof. For simplicity we shall write $A_0(x, y)$ instead of $A_0((x, y))$. Suppose that there are $x, y \in I$ and $\varepsilon > 0$ such that $A_0(x, y + h_n) - A_0(x, y) > \varepsilon$ for some sequence $\{h_n\}$ with $0 < h_n \downarrow 0$ as $n \uparrow \infty$. Then there exists $\varepsilon_1 > 0$ such that $A_0^p(x, y + h_n) - A_0^p(x, y) > \varepsilon_1$ for all $n \in \mathbb{N}$. For economy of expression write

$$I_{w,z} = \inf_{\alpha \in \mathbb{R}} \frac{\left\|\int_x^{\cdot} u(s)ds - \alpha\right\|_{p,(x,w)}^p}{\|u\|_{p,(x,z)}^p}.$$

Then for all $h > 0$ we have

$$\begin{aligned} A_0^p(x,y+h) - A_0^p(x,y) &= \sup_{\|u\|_{p,(x,y+h)}>0} I_{y+h,y+h} - \sup_{\|u\|_{p,(x,y)}>0} I_{y,y} \\ &\le \sup_{\|u\|_{p,(x,y+h)}>0} \{I_{y+h,y+h} - I_{y,y+h}\} \\ &\le \sup_{\|u\|_{p,(x,y+h)}>0} \frac{\left\|\int_x^{\cdot} u(s)ds\right\|_{p,(y,y+h)}^p}{\|u\|_{p,(x,y+h)}^p} \\ &\le |(y,y+h)|^{\frac{p}{p'}} = h^{\frac{p}{p'}}, \end{aligned}$$

and we have a contradiction. Hence $A_0(x,y+h) \to A_0(x,y)$ as $h \to 0$. In the same way it can be shown that $A_0(x+h,y) \to A_0(x,y)$ as $h \to 0$. □

Lemma 2.4. *Let $J = (c,d) \subset I$. Then there is a function $f \in L_p(J)$ and a point $s \in [c,d]$ such that*

$$A_0(J) = \frac{\left\|\int_s^{\cdot} f(t)dt\right\|_{p,J}}{\|f\|_{p,J}} = \inf_{\alpha\in\mathbb{R}} \frac{\left\|\int_c^{\cdot} f(t)dt - \alpha\right\|_{p,J}}{\|f\|_{p,J}}.$$

Proof. There is a sequence $\{f_n\}$ of functions in $L_p(J)$, with $\|f_n\|_{p,J} = 1$ for each $n \in \mathbb{N}$, and a sequence of numbers $\{s_n\}$ from $[c,d]$ such that

$$\left\|\int_{s_n}^{\cdot} f_n(t)dt\right\|_{p,J} + \frac{1}{n} = \inf_{\alpha\in\mathbb{R}} \left\|\int_c^{\cdot} f_n(t)dt - \alpha\right\|_{p,J} + \frac{1}{n} > A_0(J).$$

Since $T_c : L_p(J) \to L_p(J)$ is compact, there is a subsequence of $\{f_n\}$, again denoted by $\{f_n\}$ for convenience, which converges weakly in $L_p(J)$, to f, say, and $T_c f_n \to T_c f$ in $L_p(I)$. As $T_c : L_p(J) \to L_p(J)$ is compact, T_c also acts compactly from $L_p(J)/\operatorname{sp}\{1\}$, the quotient space modulo constants, to itself, where $\|h\|_{L_p(J)/\operatorname{sp}\{1\}} := \inf_{\alpha\in\mathbb{R}} \|h-\alpha\|_{p,J}$; moreover, $T_c f_n \to T_c f$ in $L_p(J)/\operatorname{sp}\{1\}$. Using the facts that $\|f\|_{p,J} \le \lim\inf \|f_n\|_{p,J}$ and $\|T_c f\|_{L_p(J)\setminus\{1\}} = A_0(J)$, we conclude that $\|f\|_{p,J} = 1$. Because

$$F(u) := \frac{\left\|\int_u^{\cdot} f(t)dt\right\|_{p,J}}{\|f\|_{p,J}}$$

depends continuously on u, there exists $s \in [c,d]$ such that

$$\frac{\left\|\int_s^{\cdot} f(t)dt\right\|_{p,J}}{\|f\|_{p,J}} = \inf_{c\le u\le d} \frac{\left\|\int_u^{\cdot} f(t)dt\right\|_{p,J}}{\|f\|_{p,J}} = A_0(J).$$

Thus f has all the properties required in the theorem. □

The next lemma was also proved in [3].

Lemma 2.5. *Let $J = (c,d) \subset I$ and suppose that f and s are as in the last lemma. Then f may be chosen so that $s = \frac{c+d}{2}$, $f(c+) = f(d-) = 0$ and f is odd about $\frac{c+d}{2}$.*

Theorem 2.6. *Let $J = (c,d) \subset I$. Then*

$$A_0(J) = \frac{\left\| \int_{\frac{c+d}{2}}^{\cdot} u(t)dt \right\|_{p,J}}{\|u\|_{p,J}} = \inf_{\alpha\in\mathbb{R}} \frac{\left\| \int_{\frac{c+d}{2}}^{\cdot} u(t)dt - \alpha \right\|_{p,J}}{\|u\|_{p,J}} = \gamma_p\,|J|,$$

where

$$u(x) = \cos_p\left(\frac{\pi_p\left(x - \frac{c+d}{2}\right)}{d-c}\right) \quad \text{and} \quad \gamma_p = \frac{(p')^{\frac{1}{p}} p^{\frac{1}{p'}}}{2\pi} \sin\left(\frac{\pi}{p}\right).$$

Proof. From Lemma 2.5 it follows that the function f of that lemma is odd with respect to $\frac{c+d}{2}$ and has a derivative vanishing at c and d; moreover, it is an extremal for

$$\sup_g \frac{\left\|\int_s^{\cdot} g(t)dt\right\|_{p,(s,d)}}{\|g\|_{p,(s,d)}} \quad \text{and} \quad \sup_g \frac{\left\|\int_s^{\cdot} g(t)dt\right\|_{p,(c,s)}}{\|g\|_{p,(c,s)}}.$$

The result is now a consequence of Theorem 2.1. □

From Theorem 2.1 also follows the next remark.

Remark 2.7. The function ϕ defined by $\phi(x) = \sin_p\left(\pi_p \frac{x-a}{b-a}\right)$ satisfies

$$\frac{\|\phi\|_{p,I}}{\|\phi'\|_{p,I}} = \gamma_p\,|I|,$$

where γ_p is as in Theorem 2.6.

Three different partitions of $[a,b]$ will be useful in what follows. These are $J(n) := \{J_0, J_1, \ldots, J_n\}$, where

$$J_0 = \left[a, a + \frac{b-a}{2n+1}\right], \ J_i = \left[a + \frac{(2i-1)(b-a)}{2n+1}, a + \frac{(2i+1)(b-a)}{2n+1}\right] \tag{2.9}$$

for $i = 1, \ldots, n$; $S(n) := \{S_1, \ldots, S_n\}$, where

$$S_i = \left[a + \frac{(i-1)(b-a)}{n}, a + \frac{i(b-a)}{n}\right] \quad \text{for } i = 1, \ldots, n, \tag{2.10}$$

and $I(n) := \{I_0, \ldots, I_n\}$, where

$$\begin{aligned} I_0 &= \left[a, a + \frac{b-a}{2n}\right], \ I_n = \left[b - \frac{b-a}{2n}, b\right], \\ I_i &= \left[a + \frac{(2i-1)(b-a)}{2n}, a + \frac{(2i+1)(b-a)}{2n}\right] \quad \text{for } i = 1, \ldots, n-1. \end{aligned} \tag{2.11}$$

3. The Hardy operator T_c

We first determine the approximation numbers of the operator T_c on the interval $I = (a, b)$ where $c = \frac{a+b}{2}$.

Lemma 3.1. *Let n be an odd natural number and let $c = \frac{a+b}{2}$. Then*

$$a_{n+1}(T_c) = a_n(T_c) = \gamma_p \frac{|I|}{n},$$

where γ_p is as in Theorem 2.6. Moreover, the bounded linear operator P_{T_c} defined by

$$P_{T_c} f(x) = \sum \left(\int_c^{d_i} f(t)dt \right) \chi_{S_i}(x) + 0\chi_{S_{\frac{n+1}{2}}}(x), \tag{3.1}$$

(where the sum is over all $i \in \{1, 2, \ldots, n\}$ with $i \neq \frac{n+1}{2}$, $S(n) = \{S_i\}_{i=1}^n$ is the partition of $[a, b]$ given by (2.10) and d_i is the mid-point of S_i), is the optimal linear approximant to T_c among all n- and $(n-1)$-dimensional linear operators.

Proof. Let $S_i = [a_i, b_i]$, so that $d_i = \frac{a_i+b_i}{2}$, and note that $|S_i| = \frac{|I|}{n}$. The map P_{T_c} given by (3.1) has rank $n-1$. Let $f \in L_p(I)$. By Theorem 2.1, $\left(\frac{b-a}{n}\right) \gamma_p \|f\|_{p,(a_i,b_i)}$ is greater than or equal to

$$\left(\left\| \int_{d_i}^{\cdot} f(t)dt \right\|_{p,(d_i,b_i)}^p + \left\| \int_{d_i}^{\cdot} f(t)dt \right\|_{p,(a_i,d_i)}^p \right)^{\frac{1}{p}} \quad \text{if } i \neq \frac{n+1}{2},$$

and

$$\left(\|T_c f\|_{p,(d_i,b_i)}^p + \|T_c f\|_{p,(a_i,d_i)}^p \right)^{\frac{1}{p}} \quad \text{if } i = \frac{n+1}{2}.$$

Using $c = d_{\frac{n+1}{2}}$ we obtain

$$\begin{aligned} \|T_c f - P_{T_c} f\|_{p,I}^p &\leq \sum_{i=1}^n \|(T_c - P_{T_c})(f)\|_{p,I_i}^p \\ &\leq \sum_{i=1}^n \left(\left\| \int_{d_i}^{\cdot} f(t)dt \right\|_{p,(a_i,d_i)}^p + \left\| \int_{d_i}^{\cdot} f(t)dt \right\|_{p,(d_i,b_i)}^p \right) \\ &\leq \sum_{i=1}^n \left\{ \gamma_p \frac{b-a}{n} \right\}^p \|f\|_{p,(a_i,b_i)}^p \\ &\leq \left\{ \gamma_p \frac{b-a}{n} \right\}^p \|f\|_{p,I}^p, \end{aligned}$$

so that for odd n we have $a_n(T_c) \leq \gamma_p \frac{|I|}{n}$.

To estimate the approximation numbers from below we again use the partition $S(n) = \{S_i\}_{i=1}^n$ of I, and $\{d_i\}_{i=1}^n$ as above. Then using Theorems 2.1

and 2.6 for each $i \in \{1, 2, \dots, n\}$, $i \neq \frac{n+1}{2}$, we see that there are functions $\phi \in L_p(I)$, non-zero only on S_i, and functions $\phi_-, \phi_+ \in L_p(I)$, non-zero only on $\left(a_{\frac{n+1}{2}}, c\right)$ and $\left(c, b_{\frac{n+1}{2}}\right)$ respectively, such that

$$\inf_{\alpha \in \mathbb{R}} \frac{\|T_{d_i}\phi_i - \alpha\|_{p,S_i}}{\|\phi_i\|_{p,S_i}}, \quad \frac{\|T_c\phi_-\|_{p,\left(a_{\frac{n+1}{2}},c\right)}}{\|\phi_-\|_{p,\left(a_{\frac{n+1}{2}},c\right)}} \quad \text{and} \quad \frac{\|T_c\phi_+\|_{p,\left(c,b_{\frac{n+1}{2}}\right)}}{\|\phi_+\|_{p,\left(c,b_{\frac{n+1}{2}}\right)}}$$

are all equal to $\gamma_p \left|S_i\right|$. Let $P_n : L_p(I) \to L_p(I)$ be bounded and linear, with rank n. Then there are constants λ_i $\left(i \in \{1, 2, \dots, n\}, i \neq \frac{n+1}{2}\right)$, λ_-, λ_+ such that for $g = \sum \lambda_i\phi_i + \lambda_-\phi_- + \lambda_+\phi_+$ we have $P_n g = 0$. And we obtain

$$\begin{aligned}
\|T_c g - P_n g\|_{p,I}^p &= \|T_c g\|_{p,I}^p \\
&= \sum_{i=1}^{n} \|T_c g\|_{p,S_i}^p \\
&= \sum_{i=1}^{n} \|T_{d_i}(g) + (T_c g)(d_i)\|_{p,S_i}^p \\
&\geq \sum_{i \neq \frac{n+1}{2}} \inf_{\alpha \in \mathbb{R}} \|\lambda_i T_{d_i}\phi_i - \alpha\|_{p,S_i}^p + \|Tg\|_{p,\left(a_{\frac{n+1}{2}},c\right)}^p + \|Tg\|_{p,\left(c,b_{\frac{n+1}{2}}\right)}^p \\
&= \sum_{i \neq \frac{n+1}{2}} \|\lambda_i\phi_i\|_{p,S_i}^p \left(\gamma_p \frac{|I|}{n}\right)^p \\
&\quad + \left(\|\lambda_-\phi_-\|_{p,\left(a_{\frac{n+1}{2}},c\right)}^p + \|\lambda_+\phi_+\|_{p,\left(c,b_{\frac{n+1}{2}}\right)}^p\right) \left(\gamma_p \frac{|I|}{n}\right)^p \\
&\geq \left(\gamma_p \frac{|I|}{n}\right)^p \|g\|_{p,I}^p,
\end{aligned}$$

from which it follows that for odd n, $a_{n+1}(T_c) \geq \gamma_p \frac{|I|}{n}$. Hence for odd n,

$$\gamma_p \frac{|I|}{n} \leq a_{n+1}(T_c) \leq a_n(T_c) \leq \gamma_p \frac{|I|}{n},$$

and the proof is complete. □

Lemma 3.2. *Let n be an odd natural number and let $T_c : L_p(I) \to L_p(I)$ be the Hardy operator with $c = \frac{a+b}{2}$. Then*

$$\gamma_p \frac{|I|}{n} = a_{n+1}(T_c) \leq i_{n+1}(T_c),$$

where γ_p is as in Theorem 2.6.

Proof. It is enough to deal with the case when $I = (-1,1)$. Let $I(n) = \{I_i\}_{i=0}^n$ be the partition of $I = (-1,1)$ given by (2.11). Note that $2|I_0| = 2|I_n| = |I_i|$ when $0 < i < n$.

We introduce a sequence space $l_{p,w}^n$ with norm

$$\|\{c_i\}\|_{l_{p,w}^n} := \left\{ 2\sum_{i=1}^{n-1} |c_i|^p + |c_0|^p + |c_n|^p \right\}^{\frac{1}{p}}.$$

Maps $A : l_{p,w}^n \to L_p(0,1)$ and $B : L_p(0,1) \to l_{p,w}^n$ are defined by

$$A\left(\{c_i\}_{i=0}^n\right) = \sum_{i=0}^{n} (-1)^{i+1} c_i \chi_{I_i}(x) \cos_p\left(\frac{\pi_p n x}{2}\right)$$

and

$$B(g(x)) = \left\{ \frac{\left(n-\frac{1}{2}\right)\pi_p \int_{I_i} (-1)^{i+1} g(x) \left(\sin_p\left(\frac{\pi_p n x}{2}\right)\right)_{(p)} dx}{\left\|\sin_p\left(\frac{\pi_p n\,\cdot}{2}\right)\right\|_{p,I_i}} \right\}_{i=0}^{n},$$

where $(s)_p = |s|^{p-2}s$ $(s \in \mathbb{R}\setminus\{0\})$, $(0)_p = 0$, whenever $p \in (1,\infty)$. Then

$$T_c\left(c_i \chi_{I_i}(x) \cos_p\left(\frac{\pi_p n x}{2}\right)\right) = \frac{c_i \chi_{I_i}(x) \sin_p\left(\frac{\pi_p n x}{2}\right)}{\frac{\pi_p n}{2}},$$

from which it follows that

$$T_c\left(A\left(\{c_i\}_{i=0}^n\right)\right) = \sum_{i=0}^{n} \frac{(-1)^{i+1} c_i \chi_{I_i}(x) \sin_p\left(\frac{\pi_p n x}{2}\right)}{\frac{\pi_p n}{2}}.$$

Using the definition of B we obtain

$$B\left(T_c\left(A\left(\{c_i\}_{i=0}^n\right)\right)\right) = \left\{ c_i \int_{I_i} \frac{\left|\sin_p\left(\frac{\pi_p n t}{2}\right)\right|^p}{\left\|\sin_p\left(\frac{\pi_p n\,\cdot}{2}\right)\right\|_{p,I_i}} dt \right\}_{i=0}^{n} = \{c_i\}_{i=0}^n.$$

Thus BT_cA is the identity on $l_{p,w}^n$.

Moreover, $\left\{\frac{\|B:L_p(0,1)\to l_{p,w}^n\|}{\frac{\pi_p n}{2}}\right\}^p$ equals the supremum, over all $g \in L_p(0,1)$ with $\|g\|_{p,(0,1)} \le 1$, of

$$2\sum_{i=1}^{n-1} \left| \frac{\int_{I_i} g(t) \left(\sin_p\left(\frac{\pi_p n t}{2}\right)\right)_{(p)} dt}{\left\|\sin_p\left(\frac{\pi_p n\,\cdot}{2}\right)\right\|_{p,I_i}^p} \right|^p + \left| \frac{\int_{I_0} g(x) \left(\sin_p\left(\frac{\pi_p n t}{2}\right)\right)_{(p)} dt}{\left\|\sin_p\left(\frac{\pi_p n\,\cdot}{2}\right)\right\|_{p,I_0}^p} \right|^p + \left| \frac{\int_{I_n} g(x) \left(\sin_p\left(\frac{\pi_p n t}{2}\right)\right)_{(p)} dt}{\left\|\sin_p\left(\frac{\pi_p n\,\cdot}{2}\right)\right\|_{p,I_n}^p} \right|^p.$$

Note that the supremum is attained only when $g(x) = \sum_{i=0}^{n} c_i \chi_{I_i}(x) \sin_p\left(\frac{\pi_p n x}{2}\right)$. Hence $\frac{\left\|B : L_p(0,1) \to l^n_{p,w}\right\|}{\frac{\pi_p n}{2}}$ equals

$$
\begin{aligned}
&\sup_{\{c_i\} \in l^n_{p,w}} \frac{\left(2\sum_{i=1}^{n-1} |c_i|^p + |c_0|^p + |c_n|^p\right)^{\frac{1}{p}}}{\left\|\sum_{i=0}^{n} c_i \chi_{I_i}(\cdot) \sin_p\left(\frac{\pi_p n \cdot}{2}\right)\right\|_{p,(0,1)}} \\
&= \sup_{\{c_i\} \in l^n_{p,w}} \frac{\left(2\sum_{i=1}^{n-1} |c_i|^p + |c_0|^p + |c_n|^p\right)^{\frac{1}{p}}}{\left\{\sum_{i=0}^{n} \int_{I_i} \left|c_i \chi_{I_i}(x) \sin_p\left(\frac{\pi_p n x}{2}\right)\right|^p dx\right\}^{\frac{1}{p}}} \\
&= \sup_{\{c_i\} \in l^n_{p,w}} \frac{\left(2\sum_{i=1}^{n-1} |c_i|^p + |c_0|^p + |c_n|^p\right)^{\frac{1}{p}}}{\left(2\sum_{i=1}^{n-1} |c_i|^p + |c_0|^p + |c_n|^p\right)^{\frac{1}{p}} \left\{\int_{I_n} \left|\sin_p\left(\frac{\pi_p n x}{2}\right)\right|^p dx\right\}^{\frac{1}{p}}} \\
&= \frac{1}{\left\{\int_{I_n} \left|\sin_p\left(\frac{\pi_p n x}{2}\right)\right|^p dx\right\}^{\frac{1}{p}}},
\end{aligned}
$$

and $\left\|A : l^n_{p,w} \to L_p(0,1)\right\|$ equals

$$
\begin{aligned}
&\sup_{\|\{c_i\}\|_{l^n_{p,w}} \le 1} \left\{\int_I \left|\sum_{i=0}^{n} c_i \chi_{I_i}(x) \cos_p\left(\frac{\pi_p n x}{2}\right)\right|^p dx\right\}^{\frac{1}{p}} \\
&= \sup_{\|\{c_i\}\|_{l^n_{p,w}} \le 1} \left\{\sum_{i=1}^{n} |c_i|^p \int_{I_i} \left|\cos_p\left(\frac{\pi_p n x}{2}\right)\right|^p dx\right\}^{\frac{1}{p}} \\
&= \sup_{\|\{c_i\}\|_{l^n_{p,w}} \le 1} \left(2\sum_{i=1}^{n-1} |c_i|^p + |c_0|^p + |c_n|^p\right)^{\frac{1}{p}} \left(\int_{I_n} \left|\cos_p\left(\frac{\pi_p n x}{2}\right)\right|^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_{I_n} \left|\cos_p\left(\frac{\pi_p n x}{2}\right)\right|^p dx\right)^{\frac{1}{p}}.
\end{aligned}
$$

Thus

$$
i_n(T) \ge \|A\|^{-1} \|B\|^{-1} = \frac{\left(\int_{I_n} \left|\sin_p\left(\frac{\pi_p n x}{2}\right)\right|^p dx\right)^{\frac{1}{p}}}{\frac{\pi_p n x}{2} \left(\int_{I_n} \left|\cos_p\left(\frac{\pi_p n x}{2}\right)\right|^p dx\right)^{\frac{1}{p}}},
$$

which completes the proof. □

From Lemmas 3.1 and 3.2 we have, using the ordering of the strict s-numbers mentioned in the Introduction,

Theorem 3.3. *Let $T_c : L_p(I) \to L_p(I)$ be the Hardy operator with $c = \frac{a+b}{2}$ and let $\widetilde{s}_n$ stand for any strict s-number. If n is odd, then*

$$\widetilde{s}_n(T_c) = \widetilde{s}_{n+1}(T_c) = \gamma_p \frac{|I|}{n}, \tag{3.2}$$

where γ_p is as in Theorem 2.6. The bounded linear operator P_{T_c} defined in (3.1) is an optimal n-dimensional approximation of T_c.

By technical modification of Lemma 3.1 and Theorem 3.3 for the integral operator T_a on $I = (a, b)$ we obtain the next lemma and theorem.

Lemma 3.4. *For all $n \in \mathbb{N}$, the approximation numbers of the map $T_a : L_p(I) \to L_p(I)$ are given by*

$$a_{n+1}(T_a) = \gamma_p \frac{|I|}{n + \frac{1}{2}},$$

where γ_p is as in Theorem 2.6. Moreover, the bounded linear operators P_T, where

$$P_T f(x) = \sum_{i=1}^{n} \left(\int_a^{s_i} f(t)dt \right) \chi_{J_i}(x) + 0\chi_{J_0}(x), \tag{3.3}$$

the J_i are given by (2.9) and s_i is the mid-point of J_i, are optimal n-dimensional linear approximations of T_a.

Theorem 3.5. *Let $T_a : L_p(I) \to L_p(I)$ be the Hardy operator given by (2.7) and let $\widetilde{s}_n$ stand for any strict s-number. Then for all $n \in \mathbb{N}$,*

$$\widetilde{s}_n(T_a) = \gamma_p \frac{|I|}{n - \frac{1}{2}}, \tag{3.4}$$

where γ_p is as in Theorem 2.6.

4. Sobolev embeddings on intervals

Here we obtain the exact values of strict s-numbers of various Sobolev embeddings with arguments that again crucially depend on generalised trigonometric functions. Throughout this section $I = [a, b]$ will be a bounded interval and $\mathbb{T}$ will stand for the unit circle realised as the interval $[-\pi, \pi]$ with identified end-points; we always suppose that $1 < p < \infty$.

By $W_p^1(I)/\operatorname{sp}\{1\}$ we denote the factorisation of the usual Sobolev space $W_p^1(I)$ with respect to constants, equipped with the norm

$$\left\| [f] \mid W_p^1(I)/\operatorname{sp}\{1\} \right\| := \|f'\|_{p,I} \,;$$

note that elements of $W_p^1(I)/\operatorname{sp}\{1\}$ are equivalence classes $[\cdot]$ of functions which differ by a constant. In the same way, $L_p(I)/\operatorname{sp}\{1\}$ is given the norm

$$\|[f] \mid L_p(I)/\operatorname{sp}\{1\}\| := \inf \|f - c\|_{p,I},$$

where the infimum is taken over all scalars c. The spaces $W_p^1(\mathbb{T})/\operatorname{sp}\{1\}$ and $L_p(\mathbb{T})/\operatorname{sp}\{1\}$ are defined analogously. As usual, $\overset{0}{W}{}_p^1(I)$ is the space of all absolutely continuous functions f on I with norm $\|f'\|_{p,I}$ and zero values at a and b. By $\overset{a}{W}{}_p^1(I)$ (resp. $\overset{mid}{W}{}_p^1(I)$) we mean the space of all absolutely continuous functions f on I with norm $\|f'\|_{p,I}$ and zero value at a $\big($resp. at $\frac{a+b}{2}\big)$.

We consider the following Sobolev embeddings:

$$E_0 : \overset{0}{W}{}_p^1(I) \to L_p(I), \quad E_a : \overset{a}{W}{}_p^1(I) \to L_p(I), \quad E_{mid} : \overset{mid}{W}{}_p^1(I) \to L_p(I),$$

and

$$E_I : W_p^1(I)/\operatorname{sp}\{1\} \to L_p(I)/\operatorname{sp}\{1\}, \quad E_{\mathbb{T}} : W_p^1(\mathbb{T})/\operatorname{sp}\{1\} \to L_p(\mathbb{T})/\operatorname{sp}\{1\}.$$

The norm of E_0 is defined by

$$\|E_0\| = \sup_{\|f'\|_{p,I}>0, f(a)=f(b)=0} \frac{\|f'\|_{p,I}}{\|f\|_{p,I}};$$

the norms of E_a and E_{mid} we define in a similar way, while that of E_I is given by

$$\|E_I\| = \sup_{[f]\in W_p^1(I)/sp\{1\}} \frac{\|f'\|_{p,I}}{\|[f]\|_{p,I}},$$

with a corresponding definition for the norm of $E_{\mathbb{T}}$. Since the length $|I|$ of I is finite all these embeddings are compact (see, for example, [2, Theorem V.4.18]).

The closed unit ball in $\overset{a}{W}{}_p^1(I)$ is denoted by $B\overset{a}{W}{}_p^1(I)$; unit balls in the other spaces mentioned above are represented by similar expressions. Plainly

$$T_a\left(BL_p(I)\right) = B\overset{a}{W}{}_p^1(I), \quad T_c\left(BL_p(I)\right) = B\overset{mid}{W}{}_p^1(I),$$

where $c = \frac{a+b}{2}$. From this observation and Theorems 3.3 and 3.5 the next theorem follows.

Theorem 4.1. *Let $n \in \mathbb{N}$, let $\widetilde{s}_n$ stand for any strict s-number and let γ_p be as in Theorem* 2.6. *Then:*

(i) *if n is odd, $\widetilde{s}_n(E_{mid}) = \widetilde{s}_{n+1}(E_{mid}) = \gamma_p \frac{|I|}{n}$;*

(ii) *for all $n \in \mathbb{N}$, $\widetilde{s}_n(E_a) = \gamma_p \frac{|I|}{n+\frac{1}{2}}$.*

Next we focus on the strict s-numbers for the Sobolev embeddings E_I and $E_{\mathbb{T}}$.

Theorem 4.2. *Let $n \in \mathbb{N}$ and let $\widetilde{s}_n$ stand for any strict s-number. If n is even, then*

$$\widetilde{s}_n(E_{\mathbb{T}}) \geq \gamma_p \frac{2\pi}{n+1},$$

and when n is odd,

$$\widetilde{s}_n(E_{\mathbb{T}}) = \gamma_p \frac{2\pi}{n+1},$$

where γ_p is as in Theorem 2.6. Moreover, for given odd n, the bounded linear operator $P_{\mathbb{T}}$ given by

$$P_{\mathbb{T}}[f] = \left[\sum_{i=1}^{n+1} \frac{f(a_i) + f(b_i)}{2} \chi_{S_i}(.) \right], \tag{4.1}$$

where $\{S_i\}_1^{n+1} = S(n+1)$ is a partition of $I = [a,b] = \mathbb{T} = [-\pi, \pi]$ (see (2.10) with $S_i = [a_i, b_i]$, $a_0 = b_n$, and $a_{i+1} = b_i$), is an optimal linear operator for the Sobolev embedding $E_{\mathbb{T}}$ among all linear operators with rank $\leq n-1$.

Proof. Let n be odd and $\{S_i\}_{i=1}^{n+1} = S(n+1)$ be a partition of $[-\pi, \pi] = \mathbb{T} = I = [a,b]$. We can rewrite the operator $P_{\mathbb{T}}$ in the following way:

$$\begin{aligned} P_{\mathbb{T}}[f] = \Bigg[&\frac{f(a_1) + f(b_1)}{2} \chi_{\mathbb{T}}(.) + \sum_{i=2}^{n} \left(\frac{f(a_i) + f(b_i)}{2} - \frac{f(a_1) + f(b_1)}{2} \right) \chi_{S_i}(.) \\ &+ \left(\left(\sum_{i=1}^{n} (f(a_i) + f(b_i))(-1)^i \frac{1}{2} \right) - \frac{f(a_1) + f(b_1)}{2} \right) \chi_{S_{n+1}}(.) \Bigg]. \end{aligned}$$

From this we can see that the rank of $P_{\mathbb{T}}$ as a linear operator from $W_p^1(\mathbb{T})/\operatorname{sp}\{1\}$ into $L_p(\mathbb{T})/\operatorname{sp}\{1\}$ is equal to $n-1$. Let $f \in W_p^1(\mathbb{T})/\operatorname{sp}\{1\}$; then

$$\begin{aligned} \|[f] - P_{\mathbb{T}}[f]\|_{L_p(\mathbb{T})/\operatorname{sp}\{1\}} &= \inf_{c \in \mathbb{R}} \|f - P_{\mathbb{T}} f - c\|_{p,\mathbb{T}}^p \\ &\leq \|f - P_{\mathbb{T}} f\|_{p,\mathbb{T}}^p \\ &= \sum_{i=1}^{n+1} \left\| f - \frac{f(a_i) + f(b_i)}{2} \right\|_{p,S_i}^p. \end{aligned}$$

From Lemma 2.4 we have for any i with $1 \leq i \leq n+1$:

$$\begin{aligned} \sup_{\|f\|_{W_p^1(S_i)} \leq 1} \left\| f - \frac{f(a_i) + f(b_i)}{2} \right\|_{p,S_i}^p &= \sup_{\|f\|_{W_p^1(S_i)} \leq 1} \inf_{c \in \mathbb{R}} \|f - c\|_{p,S_i}^p \\ &= \sup_{\|f\|_{W_p^1(S_i)} \leq 1} \inf_{c \in \mathbb{R}} \left\| f - \frac{f(a_i) + f(b_i)}{2} - c \right\|_{p,S_i}^p \\ &= \sup_{\|f\|_{W_p^1(S_i)} \leq 1} (\gamma_p |S_i|)^p \, \|f'\|_{p,S_i}^p, \end{aligned}$$

and then $\|[f]-P_{\mathbb{T}}[f]\|_{L_p(\mathbb{T})/\operatorname{sp}\{1\}}=\|f-P_{\mathbb{T}}f\|_{L^p(\mathbb{T})/\operatorname{sp}\{1\}}\le\gamma_p\frac{2\pi}{n+1}\|f'\|_{p,\mathbb{T}}$. Thus $a_n(E_{\mathbb{T}})\le\gamma_p\frac{2\pi}{n+1}$.

To prove the lower estimate for $i_n(E_{\mathbb{T}})$, we introduce a sequence space $l_p^{n+1}/\operatorname{sp}\{1\}$ with norm

$$\|\ [\{c_i\}]\ \|_{l_p^{n+1}/\operatorname{sp}\{1\}}:=\inf_{c\in\mathbb{R}}\left\{\sum_{i=1}^{n+1}|c_i-c|^p\right\}^{\frac{1}{p}}.$$

Note that $\dim l_p^{n+1}/\operatorname{sp}\{1\}=n$.

Define a map $A: l_p^{n+1}/\operatorname{sp}\{1\}\to W_p^1(\mathbb{T})/\operatorname{sp}\{1\}$ by:

$$A[\{c_i\}_{i=1}^{n+1}]=\left[\sum_{i=1}^{n+1}(c_i-c)\chi_{I_i}(x)\sin_p\left((x-a_i)\frac{(n+1)\pi_p}{2\pi}\right)\right],$$

where c is a number for which

$$\|\ [\{c_i\}]\ \|_{l_p^{n+1}/\operatorname{sp}\{1\}}=\left\{\sum_{i=1}^{n+1}|c_i-c|^p\right\}^{\frac{1}{p}}.$$

Similarly, a map $B: L_p(\mathbb{T})/\operatorname{sp}\{1\}\to l_p^{n+1}/\operatorname{sp}\{1\}$ is defined by

$$B[g]=\left[\left\{\frac{\int_{I_i}(g(x)-c)\left(\sin_p\left((x-a_i)\frac{(n+1)\pi_p}{2\pi}\right)\right)_{(p)}dx}{\left\|\sin_p\left((\cdot-a_i)\frac{(n+1)\pi_p}{2\pi}\right)\right\|_{p,I_i}^p}\right\}_{i=1}^{n+1}\right],$$

where c is a constant such that $\|[g]\|_{L_p(\mathbb{T})/\operatorname{sp}\{1\}}=\|g-c\|_{p,\mathbb{T}}$.

Since $E_{\mathbb{T}}[g]=[g]$ we have $E_{\mathbb{T}}(A\,[\{c_i\}_{i=1}^{n+1}])=A\,[\{c_i\}_{i=1}^{n+1}]$. Thus using the definition of B we obtain

$$B\big(E_{\mathbb{T}}\big(A\,[\{c_i\}_{i=1}^{n+1}]\big)\big)=\left[\left\{c_i\int_{I_i}\frac{\left|\sin_p\left((x-a_i)\frac{(n+1)\pi_p}{2\pi}\right)\right|^p}{\left\|\sin_p\left((\cdot-a_i)\frac{(n+1)\pi_p}{2\pi}\right)\right\|_{p,I_i}^p}\,dx\right\}_{i=1}^{n+1}\right]=\left[\{c_i\}_{i=1}^{n+1}\right],$$

which means that $BE_{\mathbb{T}}A$ is the identity on $l_p^{n+1}/\operatorname{sp}\{1\}$.

Moreover, $\left\|B: L_p(\mathbb{T})/\operatorname{sp}\{1\}\to l_p^{n+1}/\operatorname{sp}\{1\}\right\|^p$ equals the supremum, over all $[g]\in L_p(\mathbb{T})/\operatorname{sp}\{1\}$ with $\|[g]\|_{L_p(\mathbb{T})/\operatorname{sp}\{1\}}\le 1$, of

$$\sum_{i=1}^{n+1}\left|\frac{\int_{I_i}(g(x)-c)\left(\sin_p\left(\frac{(n+1)\pi_p x}{2\pi}\right)\right)_{(p)}dx}{\left\|\sin_p\left(\frac{(n+1)\pi_p\,\cdot}{2\pi}\right)\right\|_{p,I_i}^p}\right|^p,$$

where c depends on g in such a way that $\|[g]\|_{L_p(\mathbb{T})/\operatorname{sp}\{1\}} = \|g-c\|_{p,\mathbb{T}}$. Note that then the supremum is attained only when $g(x)-c = \sum_{i=1}^{n+1} c_i \chi_{I_i}(x)\sin_p\left(\frac{(n+1)\pi_p x}{2\pi}\right)$ where c depends on g as above and $\| \; [\{c_i\}_{i=1}^{n+1}] \; \|_{l_p^{n+1}/\operatorname{sp}\{1\}} = \|\{c_i\}_{i=1}^{n+1}\|_{l_p^{n+1}}$. Then

$$\begin{aligned}
&\left\|B : L_p(\mathbb{T})/\operatorname{sp}\{1\} \to l_p^{n+1}/\operatorname{sp}\{1\}\right\| \\
&\le \sup_{\{c_i\}\in l_p^{n+1}} \frac{\left(\sum_{i=1}^{n+1}|c_i|^p\right)^{\frac{1}{p}}}{\left\|\sum_{i=1}^{n+1} c_i\chi_{I_i}(\cdot)\sin_p\left(\frac{(n+1)\pi_p\,\cdot}{2\pi}\right)\right\|_{p,\mathbb{T}}} \\
&= \sup_{\{c_i\}\in l_p^{n+1}} \frac{\left(\sum_{i=1}^{n+1}|c_i|^p\right)^{\frac{1}{p}}}{\left\{\sum_{i=1}^{n+1}\int_{I_i}\left|c_i\chi_{I_i}(x)\sin_p\left(\frac{(n+1)\pi_p x}{2\pi}\right)\right|^p dx\right\}^{\frac{1}{p}}} \\
&= \sup_{\{c_i\}\in l_p^{n+1}} \frac{\left(\sum_{i=1}^{n+1}|c_i|^p\right)^{\frac{1}{p}}}{\left(\sum_{i=1}^{n+1}|c_i|^p\right)^{\frac{1}{p}}\left\{\int_{I_1}\left|\sin_p\left(\frac{(n+1)\pi_p x}{2\pi}\right)\right|^p dx\right\}^{\frac{1}{p}}} \\
&= \left\{\int_{I_1}\left|\sin_p\left(\frac{(n+1)\pi_p x}{2\pi}\right)\right|^p dx\right\}^{-\frac{1}{p}},
\end{aligned}$$

and $\left\|A : l_p^{n+1}/\operatorname{sp}\{1\} \to W_p^1(\mathbb{T})/\operatorname{sp}\{1\}\right\|$ equals

$$\begin{aligned}
&\sup_{\|[\{c_i\}]\|_{l_p^{n+1}/\operatorname{sp}\{1\}}\le 1} \left\{\int_I \sum_{i=1}^{n+1}\left|(c_i-c)\chi_{I_i}(x)\frac{d}{dx}\left[\sin_p\left((x-a_i)\frac{(n+1)\pi_p}{2\pi}\right)\right]\right|^p dx\right\}^{\frac{1}{p}} \\
&= \sup_{\|[\{c_i\}]\|_{l_p^{n+1}/\operatorname{sp}\{1\}}\le 1} \left\{\sum_{i=1}^{n+1}|c_i-c|^p\int_{I_i}\left|\cos_p\left((x-a_i)\frac{(n+1)\pi_p}{2\pi}\right)\frac{(n+1)\pi_p}{2\pi}\right|^p dx\right\}^{\frac{1}{p}} \\
&= \frac{(n+1)\pi_p}{2\pi}\left\{\int_{I_1}\left|\cos_p\left((x-a_1)\frac{(n+1)\pi_p}{2\pi}\right)\right|^p dx\right\}^{\frac{1}{p}}.
\end{aligned}$$

Thus

$$i_n(T) \ge \|A\|^{-1}\|B\|^{-1} = \frac{2\pi\left(\int_{I_1}\left|\sin_p\left((x-a_1)\frac{(n+1)\pi_p}{2\pi}\right)\right|^p dx\right)^{\frac{1}{p}}}{(n+1)\pi_p\left(\int_{I_1}\left|\cos_p\left((x-a_1)\frac{(n+1)\pi_p}{2\pi}\right)\right|^p dx\right)^{\frac{1}{p}}},$$

which completes the proof.

When n is even then by using the above techniques we obtain

$$i_n(E_{\mathbb{T}}) \ge \gamma_p \frac{2\pi}{n+1}. \qquad \square$$

Now we focus on the Sobolev embedding E_I on an interval $I = [a, b]$.

Theorem 4.3. *Let $n \in \mathbb{N}$ and let $\widetilde{s}_n$ stand for any strict s-number. Then*

$$\widetilde{s}_n(E_I) = \gamma_p \frac{|I|}{n},$$

where γ_p is as in Theorem 2.6.

Proof. Let $S(n) = \{S_i\}_{i=1}^n$ be a partition of $I = [a, b]$ (see 2.10) with $S_i = [a_i, b_i]$, $a_1 = a$, $b_n = b$ and $a_{i+1} = b_i$. Clearly $|S_i| = \frac{|I|}{n}$ for $i = 1, \dots, n$. We define an operator $P_n : W_p^1(I)/\operatorname{sp}\{1\} \to L_p(I)/\operatorname{sp}\{1\}$ by:

$$P_n[f] := \left[\sum_{i=1}^n f\left(\frac{a_i + b_i}{2}\right) \chi_{S_i}(.) \right],$$

and we can see that rank $P_n = n - 1$. Thus using Theorem 2.6 we have

$$\begin{aligned}
(a_n(E_I))^p &\le \sup_{[f] \in W_p^1(I)/\operatorname{sp}\{1\}} \frac{\| (E_I - P_n)[f] \|_{L_p(I)/\operatorname{sp}\{1\}}^p}{\|f'\|_{L_p(I)}} \\
&= \sup_{f \in W_p^1(I)/\operatorname{sp}\{1\}} \inf_{c \in \mathbb{R}} \frac{\|(E_I - P_n)(f) - c\|_{L_p(I)}^p}{\|f'\|_{L_p(I)}} \\
&\le \sup_{f \in W_p^1(I)/\operatorname{sp}\{1\}} \frac{\|(E_I - P_n)(f)\|_{L_p(I)}^p}{\|f'\|_{L_p(I)}} \\
&\le \sup_{\|f'\|_{L_p(I)} \le 1} \left(\sum_{i=1}^n \left\| f(.) - f\left(\frac{a_i + b_i}{2}\right) \right\|_{p, S_i}^p \right) \\
&\le \sup_{\|u\|_{p,I} \le 1} \left(\sum_{i=1}^n \left\| \int_{\frac{a_i + b_i}{2}}^{.} u(t)dt \right\|_{p, S_i}^p \right) \\
&\le \sup_{\|u\|_{p,I} \le 1} \left(\sum_{i=1}^n (\gamma_p |I_i|)^p \|u\|_{p, S_i}^p \right) \\
&\le \left(\gamma_p \frac{|I|}{n} \right)^p,
\end{aligned}$$

so that $a_n(E_I) \le \gamma_p \frac{|I|}{n}$.

Now we shall prove the lower estimate for $i_n(E_1)$. Let $n \in \mathbb{N}$. We denote by $I(n) = \{I_i\}_{i=0}^n$ a partition of I (see (2.11)) where $I_i = (a_i, b_i)$, $a_0 = a$, $a_{i+1} = b_i$ and $b_n = b$. Note that $2|I_0| = 2|I_n| = |I_i| = \frac{|I|}{n}$ when $i = 1, \dots, n - 1$. By $l_{p,w}^n/\operatorname{sp}\{1\}$ we denote the n-dimensional sequence space with the norm

$$\|\{c_i\}_{i=0}^n\|_{l_{p,w}^n/\operatorname{sp}\{1\}} := \inf_{c \in \mathbb{R}} \left\{ \sum_{i=1}^{n-1} 2\,|c_i - c|^p + |c_0 - c|^p + |c_n - c|^p \right\}^{\frac{1}{p}}.$$

Maps $A : l^n_{p,w}/\operatorname{sp}\{1\} \to W^1_p(I)/\operatorname{sp}\{1\}$ and $B : L_p(I)/\operatorname{sp}\{1\} \to l^n_{p,w}/\operatorname{sp}\{1\}$ are defined by

$$A[\{c_i\}_{i=0}^n] = \left[\sum_{i=1}^n (c_i - c)\chi_{I_i}(x)\sin_p\left((x-a_i)\frac{n\pi_p}{|I|}\right) + (c_0 - c)\chi_{I_0}(x)\sin_p\left((b_0 - x)\frac{n\pi_p}{|I|}\right)\right],$$

where c is a number for which

$$\| \, [\{c_i\}_{i=0}^n] \, \|_{l^n_{p,w}/\operatorname{sp}\{1\}} = \left\{ \sum_{i=1}^{n-1} 2\,|c_i - c|^p + |c_0 - c|^p + |c_n - c|^p \right\}^{\frac{1}{p}}$$

and

$$B[g] = \left[\left\{ \frac{\int_{I_i} (g(x) - c) \left(\sin_p \left((x - a_i)\frac{n\pi_p}{|I|} \right) \right)_{(p)} dx}{\left\| \sin_p \left((\cdot - a_i)\frac{n\pi_p}{|I|} \right) \right\|^p_{p,I_i}} \right\}_{i=1}^n \cup \left\{ \frac{\int_{I_i} (g(x) - c) \left(\sin_p \left((b_0 - x)\frac{n\pi_p}{|I|} \right) \right)_{(p)} dx}{\left\| \sin_p \left((b_0 - \cdot)\frac{n\pi_p}{|I|} \right) \right\|^p_{p,I_i}} \right\}_{i=0} \right],$$

where c is a number for which $\|g\|_{L_p(I)/\operatorname{sp}\{1\}} = \|g - c\|_{L_p(I)}$. Obviously as in the previous proof we have

$$\begin{aligned} B\left(E_I\left(A\left[\{c_i\}_{i=0}^n\right]\right)\right) &= \left[\left\{ c_i \int_{I_i} \frac{\left| \sin_p \left((x - a_i)\frac{n\pi_p}{|I|} \right) \right|^p}{\left\| \sin_p \left((\cdot - a_i)\frac{n\pi_p}{|I|} \right) \right\|^p_{p,I_i}} dx \right\}_{i=1}^n \right. \\ &\qquad \left. \cup \left\{ c_i \int_{I_i} \frac{\left| \sin_p \left((b_0 - x)\frac{n\pi_p}{|I|} \right) \right|^p}{\left\| \sin_p \left((b_0 - \cdot)\frac{n\pi_p}{|I|} \right) \right\|^p_{p,I_i}} dx \right\}_{i=0} \right] \\ &= \left[\{c_i\}_{i=0}^n\right], \end{aligned}$$

which means that BE_IA is the identity on $l^n_{p,w}/\operatorname{sp}\{1\}$.

Note that $\|B[g]\|_{l^n_{p,w}/\operatorname{sp}\{1\}} = \big\|B\colon L_p(I)/\operatorname{sp}\{1\} \to l^n_{p,w}/\operatorname{sp}\{1\}\big\| \, \|[g]\|_{L^p(I)/\operatorname{sp}\{1\}}$ is true only when

$$g(x) - c = \sum_{i=1}^n c_i\chi_{I_i}(x)\sin_p\left((x - a_i)\frac{n\pi_p}{|I|}\right) + c_0\chi_{I_0}(x)\sin_p\left(\frac{n\pi_p(b_0 - x)}{|I|}\right)$$

where c is a constant such that $\|\{c_i - c\}\|_{l^n_{p,w}} = \| \, [\{c_i\}] \, \|_{l^n_{p,w}/\operatorname{sp}\{1\}}$.

Hence $\left\|B : L_p(I)/\operatorname{sp}\{1\} \to l^n_{p,w}/\operatorname{sp}\{1\}\right\|$ equals

$$\sup_{\{c_i\}\in l^n_{p,w}} \frac{\left(2\sum_{i=1}^{n-1}|c_i|^p+|c_0|^p+|c_n|^p\right)^{\frac{1}{p}}}{\left\|\sum_{i=1}^n c_i\chi_{I_i}(\cdot)\sin_p\left((\cdot-a_i)\frac{n\pi_p}{|I|}\right)+c_0\chi_{I_0}(\cdot)\sin_p\left((b_0-\cdot)\frac{n\pi_p}{|I|}\right)\right\|_{p,I}}$$

$$= \sup_{\{c_i\}\in l^n_{p,w}} \frac{\left(2\sum_{i=1}^{n-1}|c_i|^p+|c_0|^p+|c_n|^p\right)^{\frac{1}{p}}}{\left(2\sum_{i=1}^{n-1}|c_i|^p+|c_0|^p+|c_n|^p\right)^{\frac{1}{p}}\left\{\int_{I_n}\left|\sin_p\left((x-a_n)\frac{n\pi_p}{|I|}\right)\right|^p dx\right\}^{\frac{1}{p}}}$$

$$= \frac{1}{\left\{\int_{I_n}\left|\sin_p\left((x-a_i)\frac{n\pi_p}{|I|}\right)\right|^p dx\right\}^{\frac{1}{p}}},$$

and $\left\|A : l^n_{p,w}/\operatorname{sp}\{1\} \to W^1_p(I)/\operatorname{sp}\{1\}\right\|$ equals

$$\sup_{\|\ [\{c_i\}]\ \|_{l^n_{p,w}/\operatorname{sp}\{1\}}\le 1} \left\{\int_I\left[\sum_{i=1}^n\left|(c_i-c)\chi_{I_i}(x)\frac{d}{dx}\left[\sin_p\left((x-a_i)\frac{n\pi_p}{|I|}\right)\right]\right|^p\right.\right.$$
$$\left.\left.+\left|(c_o-c)\chi_{I_0}(x)\frac{d}{dx}\left[\sin_p\left((b_0-x)\frac{n\pi_p}{|I|}\right)\right]\right|^p\right]dx\right\}^{\frac{1}{p}}$$

$$= \sup_{\|\ [\{c_i\}]\ \|_{l^n_{p,w}/\operatorname{sp}\{1\}}\le 1} \left\{\sum_{i=1}^n|c_i-c|^p\int_{I_i}\left|\cos_p\left((x-a_i)\frac{n\pi_p}{|I|}\right)\left(\frac{n\pi_p}{|I|}\right)\right|^p dx\right.$$
$$\left.+|c_0-c|^p\int_{I_0}\left|\cos_p\left((b_0-x)\frac{n\pi_p}{|I|}\right)\left(\frac{n\pi_p}{|I|}\right)\right|^p dx\right\}^{\frac{1}{p}}$$

$$= \sup_{\|\ [\{c_i\}]\ \|_{l^n_{p,w}/\operatorname{sp}\{1\}}\le 1} \left(2\sum_{i=1}^{n-1}|c_i-c|^p+|c_0-c|^p+|c_n-c|^p\right)^{\frac{1}{p}}$$
$$\times\left(\int_{I_n}\left|\cos_p\left((x-a_n)\frac{n\pi_p}{|I|}\right)\left(\frac{n\pi_p}{|I|}\right)\right|^p dx\right)^{\frac{1}{p}}$$

$$= \frac{n\pi_p}{|I|}\left(\int_{I_n}\left|\cos_p\left((x-a_n)\frac{n\pi_p}{|I|}\right)\right|^p dx\right)^{\frac{1}{p}}.$$

Thus

$$i_n(T) \ge \|A\|^{-1}\|B\|^{-1} = \frac{|I|\left(\int_{I_n}\left|\sin_p\left((x-a_n)\frac{n\pi_p}{|I|}\right)\right|^p dx\right)^{\frac{1}{p}}}{n\pi_p\left(\int_{I_n}\left|\cos_p\left((x-a_n)\frac{n\pi_p}{|I|}\right)\right|^p dx\right)^{\frac{1}{p}}},$$

which completes the proof. □

Theorem 4.4. *Let $n \in \mathbb{N}$ and $\widetilde{s}_n$ stand for any strict s-number. Then*

$$\widetilde{s}_n(E_0) = \gamma_p \frac{|I|}{n},$$

where γ_p is as in Theorem 2.6.

Proof. Let $I(n) = \{I_i\}_{i=0}^n$ be a partition of $I = [a, b]$ (see 2.11) with $I_i = [a_i, b_i]$, $a_0 = a$, $b_n = b$ and $a_{i+1} = b_i$. Clearly $2|I_0| = 2|I_n| = |I_i| = \frac{|I|}{n}$ for $i = 1, \dots, n-1$. We define an operator $P_{n-1} : \overset{0}{W}{}_p^1(I) \to L_p(I)$ with $\mathrm{rank}(P_{n-1}) = n - 1$ by:

$$P_{n-1} f(x) := 0\chi_{I_0}(x) + 0\chi_{I_n} + \sum_{i=1}^{n-1} f\left(\frac{a_i + b_i}{2}\right) \chi_{I_i}(x).$$

Thus using Theorem 2.6 we have

$$\begin{aligned}
(a_n(E_0))^p &\le \sup_{f \in \overset{0}{W}{}_p^1(I)} \|(E_0 - P_{n-1})(f)\|_{L^p(I)}^p \\
&\le \sup_{f \in \overset{0}{W}{}_p^1(I)} \left(\left[\sum_{i=1}^{n-1} \left\| f(.) - f\left(\frac{a_i + b_i}{2}\right) \right\|_{p,I_i}^p \right] + \|f\|_{p,I_0}^p + \|f\|_{p,I_n}^p \right) \\
&\le \sup_{\|u\|_{p,I} \le 1} \left(\left[\sum_{i=1}^{n-1} \left\| \int_{\frac{a_i+b_i}{2}}^{.} u(t)dt \right\|_{p,I_i}^p \right] + \left\| \int_a^{.} u(t)dt \right\|_{p,I_0}^p + \left\| \int_{.}^{b} u(t)dt \right\|_{p,I_n}^p \right) \\
&\le \sup_{\|u\|_{p,I} \le 1} \left(\left[\sum_{i=1}^{n-1} (\gamma_p |I_i|)^p \|u\|_{p,I_i}^p \right] + (2\gamma_p |I_0|)^p \|u\|_{p,I_0}^p + (2\gamma_p |I_n|)^p \|u\|_{p,I_n}^p \right) \\
&\le \left[\gamma_p \frac{|I|}{n} \right]^p,
\end{aligned}$$

and then $a_n(E_0) \le \gamma_p \frac{|I|}{n}$.

Now we shall prove the lower estimate for $i_n(E_0)$. The map $A : l_p^n \to \overset{0}{W}{}_p^1(I)$ is defined by:

$$A\left(\{c_i\}_{i=1}^n\right) = \sum_{i=1}^n c_i \chi_{S_i}(x) \sin_p \left((x - a_i) \frac{n\pi_p}{|I|} \right),$$

where $\{S_i\}_{i=1}^n$ is a partition of I (see (2.10)) with $S_i = [a_i, b_i]$ and $|S_i| = \frac{|I|}{n}$. The map $B : L_p(I) \to l_p^n$ is defined by

$$Bg(x) = \left\{ \frac{\int_{S_i} g(x) \left(\sin_p \left((x - a_i) \frac{n\pi_p}{|I|} \right) \right)_{(p)} dx}{\left\| \sin_p \left((\cdot - a_i) \frac{n\pi_p}{|I|} \right) \right\|_{p,S_i}^p} \right\}_{i=1}^n .$$

Obviously we have $E_0(A(\{c_i\}_{i=1}^n)) = A(\{c_i\}_{i=1}^n)$ and then

$$B\left(E_0\left(A\left(\{c_i\}_{i=1}^n\right)\right)\right) = \left\{ c_i \int_{S_i} \frac{\left|\sin_p\left((x-a_i)\frac{n\pi_p}{|I|}\right)\right|^p}{\left\|\sin_p\left((\cdot - a_i)\frac{n\pi_p}{|I|}\right)\right\|_{p,S_i}^p} dx \right\}_{i=1}^n = \{c_i\}_{i=1}^n,$$

which means that BE_0A is the identity on l_p^n.

Note that $\left\|B : L_p(I) \to l_p^n\right\|$ equals the supremum of $\|Bg|l_p^n\|$ over all $g \in L_p(I)$ with $\|g\|_{L_p(I)} \le 1$, and the supremum is attained only when $g(x) = \sum_{i=1}^n c_i \chi_{S_i}(x) \sin_p\left(\frac{n\pi_p x}{|I|}\right)$. Then we have

$$\begin{aligned}\left\|B : L_p(I) \to l_p^n\right\| &\le \sup_{\{c_i\}\in l_p^n} \frac{\left(\sum_{i=1}^n |c_i|^p\right)^{\frac{1}{p}}}{\left\|\sum_{i=1}^n c_i \chi_{S_i}(\cdot)\sin_p\left(\frac{n\pi_p\,\cdot}{|I|}\right)\right\|_{p,I}} \\ &= \left\{\int_{S_1}\left|\sin_p\left(\frac{n\pi_p x}{|I|}\right)\right|^p dx\right\}^{-\frac{1}{p}},\end{aligned}$$

and $\left\|A : l_p^n \to: \overset{0}{W}{}_p^1(I)\right\|$ equals

$$\begin{aligned}&\sup_{\|\{c_i\}\|_{l_p^n}\le 1}\left\{\int_I \sum_{i=1}^n \left| c_i \chi_{S_i}(x) \frac{d}{dx}\left[\sin_p\left((x-a_i)\frac{n\pi_p}{|I|}\right)\right]\right|^p dx\right\}^{\frac{1}{p}} \\ &= \sup_{\|\{c_i\}\|_{l_p^n}\le 1}\left\{\sum_{i=1}^n |c_i|^p \int_{S_i}\left|\cos_p\left((x-a_i)\frac{n\pi_p}{|I|}\right)\left(\frac{n\pi_p}{|I|}\right)\right|^p dx\right\}^{\frac{1}{p}} \\ &= \frac{n\pi_p}{|I|}\left\{\int_{S_1}\left|\cos_p\left((x-a_1)\frac{n\pi_p}{|I|}\right)\right|^p dx\right\}^{\frac{1}{p}}.\end{aligned}$$

Thus

$$i_n(E_0) \ge \|A\|^{-1}\|B\|^{-1} = \frac{|I|\left(\int_{S_1}\left|\sin_p\left((x-a_1)\frac{n\pi_p}{|I|}\right)\right|^p dx\right)^{\frac{1}{p}}}{n\pi_p\left(\int_{S_1}\left|\cos_p\left((x-a_1)\frac{(n+1)\pi_p}{2\pi}\right)\right|^p dx\right)^{\frac{1}{p}}},$$

which completes the proof. □

Remark 4.5. The above results show that for the integral operators $T_{\frac{a+b}{2}}$ and T_a, viewed as maps from $L_p(I)$ to itself, all strict s-numbers coincide; their exact value is given. The same holds for certain Sobolev embeddings. Moreover, for $T_{\frac{a+b}{2}}$ and E_{mid} the strict s-numbers are not strictly decreasing. It is natural to ask whether such behaviour is exhibited by other integral operators, such as the weighted Hardy operator.

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