Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 31 (2012), 203–216
DOI: 10.4171/ZAA/1455


# On Monotonicity of Nonoscillation Properties of Dynamic Equations in Time Scales

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**Abstract.** For equations on time scales, we consider the following problem: when will nonoscillation on time scale $\mathbb{T}$ imply nonoscillation of the same equation on any time scale $\widetilde{\mathbb{T}}$ including $\mathbb{T}$ as a subset? The main result of the paper is the following. If nonnegative coefficients $A_k(t)$ are nonincreasing and $\alpha_k(t) \leq t$ are nondecreasing in $t \in \mathbb{R}$, then nonoscillation of the equation

$$x^{\Delta}(t) + \sum_{k=1}^{m} A_k(t)x(\alpha_k(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}$$

yields nonoscillation of the same equation on any time scale $\widetilde{\mathbb{T}} \supset \mathbb{T}$.

**Keywords.** Nonoscillation, time scales, dependence of properties on time scales, finite difference approximations on various grids

**Mathematics Subject Classification (2010).** Primary 34N05, secondary 39A21, 34K11, 65L12

## 1. Introduction

Most publications on time scales are either focused on the generalization of results well known for either differential or difference equations (or both) to the relevant models on time scales or develop the theory of equations on time scales independently, thus contributing to the theory of both discrete and continuous

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On leave at the University of Calgary. Partially supported by Doctoral Research Scholarship of the Council of Higher Education of Turkey.

equations. There are very few publications concerned with the relation between different time scales for the same equation, or between $\Delta$-integrals and usual Lebesgue integrals, see, for example, [1–3].

On the other hand, there are many known facts on the connection of nonoscillation properties of differential and difference equations. Let us compare the sharp nonoscillation conditions

$$A\tau \leq \frac{1}{\mathrm{e}} \tag{1.1}$$

for the delay differential equation

$$x'(t) + A\,x(t-\tau) = 0 \quad \text{for } t \in \mathbb{R}_0^+ := [0,\infty) \tag{1.2}$$

and

$$A\tau \leq \left(\frac{k}{k+1}\right)^{k+1} \tag{1.3}$$

for the delay difference equation

$$\Delta x(n) + \frac{A\tau}{k} x(n-k) = 0 \quad \text{for } n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{1.4}$$

where $\Delta x(n) := x(n+1) - x(n)$ for $n \in \mathbb{N}_0$. Equation (1.4) can be considered as a finite difference approximation of (1.2) with the time step $\tau$. In fact, denoting by $y(t)$ the Euler approximation of $x(t)$ in (1.2) at points $nh$, where $h = \frac{\tau}{k}$ ($\tau = kh$), we have

$$\frac{y((n+1)h) - y(nh)}{h} = -A\,y(nh-\tau) = -A\,y(h(n-k))\,.$$

Further, assuming $x(n) = y(nh)$, we obtain difference equation (1.4). We observe that

$$\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^{k+1} = \frac{1}{\mathrm{e}} \quad \text{and} \quad \left(\frac{k}{k+1}\right)^{k+1} \leq \frac{1}{\mathrm{e}}.$$

In fact, the sequence $\big\{\big(\frac{k}{k+1}\big)^{k+1}\big\}_{n\in\mathbb{N}_0}$ is increasing, so it does not exceed its limit. To justify that the function $f(x) := \big(\frac{x}{x+1}\big)^{x+1}$ for $x \in \mathbb{R}_0^+$ is increasing, it is enough to demonstrate that

$$g(x) := -\ln\big(f(x)\big) = (x+1)\ln\left(1 + \frac{1}{x}\right) \quad \text{for } x \in \mathbb{R}^+$$

is decreasing, which follows from the derivative estimation

$$g'(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x} < 0 \quad \text{for all } x \in \mathbb{R}^+,$$

as $\ln(1+\lambda) < \lambda$ for any $\lambda \in \mathbb{R}^+$.

Thus, the nonoscillation condition for difference equation (1.4) is more restrictive than for corresponding delay differential equation (1.2), and the constant in the right hand side of (1.3) tends to $\frac{1}{\mathrm{e}}$ in (1.1) as the number $(k+1)$ of grid points covering the segment $[t-\tau, t]_{\mathbb{R}}$ tends to infinity.

The main purpose of the present paper is to extend this result to a more general class of delay equations than (1.2) (in particular, to equations with variable delays and coefficients) and to more general grids than uniform grids proportional to a single delay (any time scales).

Everywhere we will use the following definition of nonoscillation.

**Definition 1.1.** An equation is *nonoscillatory* if it has an eventually positive or an eventually negative solution (with some initial condition for an ordinary differential equation and some initial function for a delay differential equation).

Our paper is also concerned with ordinary differential equations and relevant difference equations. Let us start with a simple observation. Consider the linear differential equation

$$x'(t) + A(t)x(t) = 0 \quad \text{for } t \in \mathbb{R}_0^+, \quad \text{where } A : \mathbb{R} \to \mathbb{R}_0^+. \tag{1.5}$$

Then all solutions of (1.5) (not identically equal to zero) are nonoscillatory. Everywhere we assume that all coefficients are locally bounded.

The equation on time scales

$$x^{\Delta}(t) + A(t)x(t) = 0 \quad \text{for } t \in \mathbb{T}, \quad \text{where } A : \mathbb{T} \to \mathbb{R}_0^+, \tag{1.6}$$

can be treated as a numerical approximation of (1.5); this equation can be oscillatory or not. The objective of the present paper is to study the dependency of nonoscillation properties of ordinary differential and delay differential dynamic equations on time scales. In particular, for ordinary differential equations we explore the existence of a nonoscillatory analogue of the continuous equation (1.5). For both delay and ordinary differential equations we investigate conditions which imply that nonoscillation of an equation on time scale $\mathbb{T}$ implies nonoscillation of the same dynamic equation on any finer time scale $\widetilde{\mathbb{T}} \supset \mathbb{T}$.

The paper is organized as follows. Section 2 explores the dependence of nonoscillation properties on time scales for ordinary differential equations, Section 3 is concerned with delay equations. Finally, Section 4 involves discussion and outlines some open problems.

## 2. Linear ordinary differential equations

Let us start with equation (1.6). Since equation (1.5) is nonoscillatory, dynamic equation (1.6) is considered as an approximation of (1.5) and we are trying to mimic the properties of the original equation, it is natural to consider nonoscillation of (1.6) as a function of time scale $\mathbb{T}$. In particular, we would like to answer the following questions:

(Q1) Is nonoscillation property scale monotone, i.e., does nonoscillation of (1.6) for scale $\mathbb{T}$ imply nonoscillation for any scale $\widetilde{\mathbb{T}}$ such that $\mathbb{T} \subset \widetilde{\mathbb{T}}$?

(Q2) If the answer to the previous question is negative, can we find a scale $\mathbb{T}$ such that under certain limitations on $\mathbb{T}$ nonoscillation of (1.6) over time scale $\mathbb{T}$ implies nonoscillation over any time scale $\widetilde{\mathbb{T}} \supset \mathbb{T}$?

(Q3) If the answer to the first question is negative, can we modify (1.6) so that the nonoscillation property is monotonic over time scales?

(Q4) Is it possible to modify (1.6) so that it preserves nonoscillation of (1.5)?

First, let us demonstrate that, generally, a scale refinement for a nonoscillatory equation does not imply that the equation over the refined time scale is still nonoscillatory.

**Example 2.1.** Consider two time scales $\mathbb{T} = \mathbb{N}_0$ and $\widetilde{\mathbb{T}} = \mathbb{N}_0/2 \supset \mathbb{T}$, where $\mathbb{N}_0/2 = \{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \cdots\}$. Let the continuous coefficient be

$$A(t) := \begin{cases} 4(2t - 2\lfloor t \rfloor), & t \in \big[n, n + \frac{1}{2}\big)_{\mathbb{R}}, \quad n \in \mathbb{N}_0 \\ 4(2\lfloor t \rfloor + 2 - 2t), & t \in \big[n + \frac{1}{2}, n + 1\big)_{\mathbb{R}}, \quad n \in \mathbb{N}_0, \end{cases}$$

where $\lfloor \cdot \rfloor$ is the least integer function. Then $A\big(\frac{n}{2}\big) = 4\big(1 - (-1)^n\big)$ for any $n \in \mathbb{N}_0$, equation (1.6) over time scale $\mathbb{T}$ is nonoscillatory since $A(n) = 0$ for any $n \in \mathbb{N}_0$, so any solution is constant. However, on $\widetilde{\mathbb{T}}$ we have $A\big(n + \frac{1}{2}\big) = 4$ for $n \in \mathbb{N}_0$, thus

$$x^{\Delta}\left(n+\frac{1}{2}\right) = 2\left[x(n+1) - x\left(n+\frac{1}{2}\right)\right] = -A\left(n+\frac{1}{2}\right)x\left(n+\frac{1}{2}\right) = -4x\left(n+\frac{1}{2}\right)$$

and $x$ changes sign on each segment $\big[n + \frac{1}{2}, n + 1\big)_{\mathbb{R}}$ for all $n \in \mathbb{N}_0$, since $x(n + 1) = -x\big(n + \frac{1}{2}\big)$ for all $n \in \mathbb{N}_0$.

Example 2.1 gives the negative answer to Question 1. Further, let us present answers to Questions 2 and 3.

As we assumed, the coefficients $A$ are locally bounded, thus the answer to Question 2 is positive. Let us note that the following condition

$$1 - A(t)\mu(t) > 0 \quad \text{for all } t \in \mathbb{T} \tag{2.1}$$

is necessary and sufficient for nonoscillation of (1.6) (see [4, Theorems 2.48 and 2.62]).

**Theorem 2.2.** *If $\mathbb{T}$ is a time scale satisfying*

$$1 - A(s)\mu(t) > 0 \quad \textit{for all } s \in [t, \sigma(t))_{\mathbb{R}} \textit{ and all } t \in \mathbb{T}, \tag{2.2}$$

*then equation* (1.6) *is nonoscillatory for any* $\widetilde{\mathbb{T}} \supset \mathbb{T}$.

*Proof.* Let us prove that (2.1) is satisfied for any $t \in \widetilde{\mathbb{T}}$. We define the function $\rho^* : \mathbb{R} \to \mathbb{T}$ by

$$\rho^*(t) := \max(-\infty, t]_{\mathbb{T}} \quad \text{for } t \in \mathbb{R}, \tag{2.3}$$

with the convention that $\max \emptyset := -\infty$. Note that $\rho^*(t) = t$ for all $t \in \mathbb{T}$ and $\rho^*(t) < t$ for all $t \in \widetilde{\mathbb{T}} \backslash \mathbb{T}$, in particular, $\rho^*(t) \leq t$ for all $t \in \widetilde{\mathbb{T}}$. Also recall that $\widetilde{\sigma}(t) \leq \sigma(t)$ for all $t \in \mathbb{T}$ since $\mathbb{T} \subset \widetilde{\mathbb{T}}$. Now, let $t \in \widetilde{\mathbb{T}}$. If $t \in \mathbb{T}$, then (2.1) obviously holds due to (2.2). Let $t \in \widetilde{\mathbb{T}} \backslash \mathbb{T}$. It is clear that $[t, \widetilde{\sigma}(t))_{\widetilde{\mathbb{T}}} \subset [\rho^*(t), \sigma(\rho^*(t)))_{\widetilde{\mathbb{T}}}$. Then, we have $\sigma(\rho^*(t)) \geq \widetilde{\sigma}(t)$, hence

$$\widetilde{\mu}(t) = \widetilde{\sigma}(t) - t < \sigma(\rho^*(t)) - \rho^*(t) = \mu(\rho^*(t)).$$

By (2.2) and $t \in [\rho^*(t), \sigma(\rho^*(t)))_{\widetilde{\mathbb{T}}}$, we conclude $1 - A(t)\mu(\rho^*(t)) > 0$, thus $1 - A(t)\widetilde{\mu}(t) > 1 - A(t)\mu(\rho^*(t)) > 0$, which completes the proof. $\square$

**Remark 2.3.** Let us note that if the coefficient $A$ is bounded, then time scales satisfying (2.1) and (2.2) exist. While $\mathbb{T} = \mathbb{R}$ presents an obvious (and thus not interesting example) since $\mu(t) \equiv 0$ for all $t \in \mathbb{R}$, the discrete scale with a step $h_j$ on $[t_j, t_{j+1}]_{\mathbb{R}}$, where $h_j < \frac{1}{M_j}$ and $M_j := \sup_{t \in [t_j, t_{j+1})_{\mathbb{R}}} A(t)$, satisfies condition (2.2) of the theorem.

**Remark 2.4.** If the coefficient $A$ is unbounded, then an adaptive time scale can be constructed to have (2.2) satisfied, as the following examples demonstrate.

**Example 2.5.** The function $A(t) = 0.4t$ is increasing and unbounded on $[1, \infty)_{\mathbb{R}}$. Consider the initial value problem

$$\begin{cases} x^{\Delta}(t) + 0.4t\, x(t) = 0 \quad \text{for } t \in [1, \infty)_{\mathbb{T}} \\ \qquad\qquad x(1) = x_0. \end{cases} \tag{2.4}$$

If we introduce the time scale

$$\mathbb{T} = \left\{ n + \frac{k}{n} : \ k \in [0, n)_{\mathbb{N}_0} \text{ and } n \in \mathbb{N} \right\},$$

then $A(s)\mu(1) \leq 0.8 < 1$ for $s \in [1, \sigma(1))_{\mathbb{R}} = [1, 2)_{\mathbb{R}}$, $\mu(t) = \frac{1}{\lfloor t \rfloor}$ for $t \in [1, \infty)_{\mathbb{T}}$ and for $s \in [t, \sigma(t))$ we have

$$A(s)\mu(t) \leq A(\lfloor t \rfloor + 1)\frac{1}{\lfloor t \rfloor} = \frac{0.4(\lfloor t \rfloor + 1)}{\lfloor t \rfloor} < 1 \quad \text{for } t \in [2, \infty)_{\mathbb{T}}.$$

Thus the solution of equation (2.4) on the time scale $\mathbb{T}$ (and any $\widetilde{\mathbb{T}} \supset \mathbb{T}$) is nonoscillatory and tends to zero at infinity. To demonstrate that any solution tends to zero, let us denote by I the identity function and note that $-0.4\,\mathrm{I}$ is positively regressive on $\mathbb{T}$ and thus on any $\widetilde{\mathbb{T}} \supset \mathbb{T}$. For all $t \in [1,\infty)_{\widetilde{\mathbb{T}}}$, we have

$$|x(t)| = |x_0|\widetilde{\mathrm{e}}_{-0.4\,\mathrm{I}}(t,t_0) \leq |x_0| \exp\Big\{-0.4\int_1^t \eta\widetilde{\Delta}\eta\Big\} \leq |x_0| \exp\big\{-0.4(t-1)\big\}. \quad (2.5)$$

It should be noted that we have used [5, Lemma 2] while obtaining the first inequality in (2.5) and that the generalized exponential function of a positively regressive function is positive [4, Theorem 2.44]. This proves the equality $\lim_{t\to\infty} x(t) = 0$.

**Example 2.6.** The initial value problem

$$\begin{cases} x'(t) + \dfrac{1}{(t-5)^2}x(t) = 0 & \text{for } t \in [0,5)_{\mathbb{R}} \\ x(0) = x_0 \end{cases} \quad (2.6)$$

has a nonoscillatory unbounded solution $x(t) = x_0 \exp\left\{\frac{1}{t-5} + \frac{1}{5}\right\}$ for $t \in [0,5)_{\mathbb{R}}$, which tends to zero as $t \to 5^-$, $A(t) = \frac{1}{(t-5)^2}$ is unbounded and increasing on $[0,5)_{\mathbb{R}}$. Let us choose the time scale

$$\mathbb{T} := \mathbb{N} \cup \big\{5 - 2^{1-n} + 2^{-2n-1}k : \ k \in [0,2^{n+1})_{\mathbb{N}_0} \text{ and } n \in \mathbb{N}_0\big\}.$$

Consider (2.6) on $[0,5)_{\mathbb{T}}$. Then, we have

$$A(s)\mu(j) = \frac{1}{(j-4)^2} < 1 \quad \text{for all } s \in [j,\sigma(j))_{\mathbb{R}} = [j,j+1)_{\mathbb{R}},$$

where $j = 0,1,2$. For any $n \in \mathbb{N}_0$ and each $s \in [5-2^{1-n}, 5-2^{-n})_{\mathbb{R}}$, we get

$$\mu(5 - 2^{1-n} + 2^{-2n-1}k) = 2^{-2n-1} \quad \text{and} \quad A(s) < A(5-2^{-n}) = 2^{2n},$$

which implies $A(s)\mu(5-2^{1-n}+2^{-2n-1}k) \leq 2^{2n}2^{-2n-1} = 0.5 < 1$ for any $n \in \mathbb{N}_0$, any $k \in [0,2^{n+1})_{\mathbb{N}_0}$ and all $s \in [5-2^{1-n}, 5-2^{-n})_{\mathbb{R}}$. Thus the solution of approximated equation (2.6) on time scale $\mathbb{T}$ (and any $\widetilde{\mathbb{T}} \supset \mathbb{T}$) is nonoscillatory and tends to 0 as $t \to 5^-$.

To answer Question 3, let us introduce the following version of equation (1.6) with an "averaged-coefficient"

$$x^{\Delta}(t) + B(t)x(t) = 0 \quad \text{for } t \in [1,\infty)_{\mathbb{T}}, \quad (2.7)$$

where

$$B(t) := \begin{cases} A(t), & \text{if } \mu(t) = 0 \\ \dfrac{1}{\mu(t)}\displaystyle\int_t^{\sigma(t)} A(\eta)\,\mathrm{d}\eta, & \text{otherwise.} \end{cases} \quad (2.8)$$

This model better matches a finite difference approximation for equations with steeply changing coefficients. It should be noted here that for a continuous coefficient $A$,

$$\lim_{s\to t}\frac{1}{t-s}\int_s^t A(\eta)\,\mathrm{d}\eta = A(t) \quad \text{for } t\in\mathbb{R},$$

which shows that the coefficient $B$ is closer to $A$ if the step size ($\mu(t)=\sigma(t)-t$) between consecutive points ($t$ and $\sigma(t)$) becomes smaller.

**Theorem 2.7.** *If equation* (2.7) *is nonoscillatory on time scale* $\mathbb{T}$*, where* $B(t)$ *is defined by* (2.8)*, then it is also nonoscillatory on any time scale* $\widetilde{\mathbb{T}}\supset\mathbb{T}$*.*

*Proof.* By (2.1), the necessary and sufficient nonoscillation condition for (2.7) is $B(t)\mu(t)<1$ for all $t\in\mathbb{T}$ with $\sigma(t)>t$, which can be rewritten as

$$\frac{\mu(t)}{\mu(t)}\int_t^{\sigma(t)} A(\eta)\,\mathrm{d}\eta = \int_t^{\sigma(t)} A(\eta)\,\mathrm{d}\eta < 1 \quad \text{for any } t\in\mathbb{T} \text{ with } \mu(t)>0. \tag{2.9}$$

However, if $\mathbb{T}\subset\widetilde{\mathbb{T}}$, then $\widetilde{\sigma}(t)\le\sigma(t)$ for any $t\in\mathbb{T}$. If $t\in\widetilde{\mathbb{T}}\backslash\mathbb{T}$, then $[t,\widetilde{\sigma}(t))_{\mathbb{R}}\subset[\rho^*(t),\sigma(\rho^*(t)))_{\mathbb{R}}$, where $\rho^*$ was defined in (2.3). Thus by (2.9) we have

$$\int_t^{\widetilde{\sigma}(t)} A(\eta)\,\mathrm{d}\eta \le \int_{\rho^*(t)}^{\sigma(\rho^*(t))} A(\eta)\,\mathrm{d}\eta < 1 \quad \text{for any } t\in\widetilde{\mathbb{T}} \text{ with } \widetilde{\mu}(t)>0.$$

This means that (2.7) is nonoscillatory on time scale $\widetilde{\mathbb{T}}$. □

Finally, let us comment on Question 4.

**Theorem 2.8.** *All solutions of the equation*

$$x^{\Delta}(t)+A(t)x(\sigma(t))=0 \quad \textit{for } t\in\mathbb{T} \tag{2.10}$$

*are nonoscillatory.*

*Proof.* Using the so-called "simple formula" (see [4, Theorem 1.6(iv)]), we can rewrite (2.10) in the form $x^{\Delta}(t)+\frac{A(t)}{1+A(t)\mu(t)}x(t)=0$ for $t\in\mathbb{T}$. Condition (2.1) for this equation can be rewritten as

$$1-\frac{A(t)}{1+A(t)\mu(t)}\mu(t)=\frac{1}{1+A(t)\,\mu(t)}>0 \quad \text{for all } t\in\mathbb{T},$$

which implies that (2.10) is nonoscillatory since $A:\mathbb{T}\to\mathbb{R}_0^+$. □

Theorem 2.8 can be compared to the well-known result on stability of implicit finite difference schemes (in contrast to unstable explicit schemes).

**Remark 2.9.** Let us note that equation (2.10) is always stable (we assume $A : \mathbb{T} \to \mathbb{R}_0^+$ and $\int^\infty A(\eta)\Delta\eta = \infty$). By Theorem 2.8 all solutions of (2.10) are nonoscillatory, without loss of generality we consider positive solutions, which are nonincreasing by (2.10) and the assumption $A(t) \geq 0$. Any monotone bounded function has a limit $\lim_{t\to\infty} x(t) = \ell$, where $x(t) \geq \ell$ for all $t \in [t_0, \infty)_{\mathbb{T}}$; assuming $\ell > 0$, we get

$$x(t) \leq x(t_0) - \ell \int_{t_0}^{t} A(\eta)\Delta\eta \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}},$$

which yields a contradiction since the right-hand side becomes negative for all sufficiently large $t$. However, (1.6) can be stable or not: for example, the equation

$$x^\Delta(t) + 3\,x(t) = 0 \quad \text{for } t \in \mathbb{N},$$

is unstable, since $x(t+1) = -2\,x(t)$ for $t \in \mathbb{N}$, and a small deviation of the initial condition can become infinitely large as $n \to \infty$.

## 3. Delay equations

Consider the delay equation on a time scale $\mathbb{T}$ unbounded above

$$x^\Delta(t) + \sum_{k=1}^{m} A_k(t)x(\alpha_k(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \text{ where } A_k : \mathbb{T} \to \mathbb{R}_0^+, \tag{3.1}$$

which will also be treated as an approximation of the delay differential equation

$$x'(t) + \sum_{k=1}^{m} A_k(t)x(\alpha_k(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{R}}, \text{ where } A_k : \mathbb{R} \to \mathbb{R}_0^+. \tag{3.2}$$

Equations (3.1) and (3.2) will be considered with the initial conditions

$$x(t) = \varphi(t) \quad \text{for } t \in (-\infty, t_0]_{\mathbb{T}} \tag{3.3}$$

and

$$x(t) = \varphi(t) \quad \text{for } t \in (-\infty, t_0]_{\mathbb{R}}, \tag{3.4}$$

respectively.

The main result of the present paper is that as far as $A_k$ are positive and nonincreasing, any scale refinement will keep nonoscillation property of the original equation. The proof is based on the reduction of an equation on time scales to a delay differential equation with delays and coefficients of a special form. This idea was, for instance, widely applied to reduce solutions of difference equations to the values of solutions of delay differential equations with piecewise constant delays at integer points. Below we present some auxiliary results as lemmas.

**Lemma 3.1** (See [6, 7]). *If $A_k(t) \geq B_k(t) \geq 0$ and $\alpha_k(t) \leq \beta_k(t) \leq t$ for all $t \in [t_0, \infty)_{\mathbb{R}}$, then nonoscillation of* (3.2) *implies nonoscillation of the delay equation*

$$x'(t) + \sum_{k=1}^{m} B_k(t)x(\beta_k(t)) = 0 \quad \textit{for } t \in [t_0, \infty)_{\mathbb{R}}.$$

For construction of a delay differential equation which has the same solution at the points of $\mathbb{T}$ as the equation on the time scale, we introduce the following notation:

$$B_k(t) = A_k(\rho^*(t)) \quad \text{and} \quad \beta_k(t) = \alpha_k(\rho^*(t)) \quad \text{for } t \in [t_0, \infty)_{\mathbb{R}}, \tag{3.5}$$

where $\rho^*$ is defined in (2.3).

**Lemma 3.2.** *A solution of the delay differential equation*

$$x'(t) + \sum_{k=1}^{m} B_k(t)x(\beta_k(t)) = 0 \quad \textit{for } t \in [t_0, \infty)_{\mathbb{R}}, \tag{3.6}$$

*at points $t \in \mathbb{T}$ coincides with the solution of* (3.1)*, assuming that the initial functions $\varphi$ in* (3.3) *and* (3.4) *coincide at all points $t \in \mathbb{T}$.*

*Proof.* Let $x$ be a solution of (3.6), then it is obvious that $x$ satisfies (3.1) at right-dense points in $\mathbb{T}$. To complete the proof, we have to show that this is also true for right-scattered points in $\mathbb{T}$. Let $t \in \mathbb{T}$ with $\sigma(t) > t$, then it follows from definitions (3.5), (3.6) and $\rho^*(t) = t$ for $t \in \mathbb{T}$ that

$$\begin{aligned} x(\sigma(t)) &= x(t) + \int_t^{\sigma(t)} x'(\eta)\,\mathrm{d}\eta \\ &= x(t) - \sum_{k=1}^{m} \int_t^{\sigma(t)} B_k(\eta)x(\beta_k(\eta))\,\mathrm{d}\eta \\ &= x(t) - \sum_{k=1}^{m} \int_t^{\sigma(t)} A_k(t)x(\alpha_k(t))\,\mathrm{d}\eta \\ &= x(t) - \sum_{k=1}^{m} A_k(t)x(\alpha_k(t))\big(\sigma(t) - t\big) \\ &= x(t) - \mu(t)\sum_{k=1}^{m} A_k(t)x(\alpha_k(t)), \end{aligned}$$

which yields $x^{\Delta}(t) = \frac{x(\sigma(t))-x(t)}{\mu(t)} = -\sum_{k=1}^{m} A_k(t)x(\alpha_k(t))$. Therefore, $x$ satisfies (3.1) at right-scattered points too. Hence, the reference to the uniqueness of the solution of initial problems (3.1), (3.3) and (3.6), (3.4) completes the proof. □

Further we consider only time scales and delays such that $\alpha_k(t) \in \mathbb{T}$ for any $t \in \mathbb{T}$.

**Theorem 3.3.** *Suppose that* $A_k(t) \geq 0$ *are nonincreasing functions for all* $t \in \mathbb{R}$, $\alpha_k(t) \leq t$ *are nondecreasing and equation* (3.1) *is nonoscillatory. Then for any* $\widetilde{\mathbb{T}} \supset \mathbb{T}$ *the equation*

$$x^{\Delta}(t) + \sum_{k=1}^{m} A_k(t)x(\alpha_k(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\widetilde{\mathbb{T}}} \tag{3.7}$$

*is also nonoscillatory.*

*Proof.* We define $\widetilde{\rho}^* : \mathbb{R} \to \widetilde{\mathbb{T}}$ by $\widetilde{\rho}^*(t) := \max(-\infty, t]_{\widetilde{\mathbb{T}}}$ for $t \in \mathbb{R}$ with the convention that $\max \emptyset := -\infty$. We also define $\widetilde{B}_k(t) := B_k(\widetilde{\rho}^*(t))$ and $\widetilde{\beta}_k(t) := \beta_k(\widetilde{\rho}^*(t))$ for $t \in [t_0, \infty)_{\mathbb{R}}$. Then as $\widetilde{\mathbb{T}}$ is finer than $\mathbb{T}$, i.e., $\mathbb{T} \subset \widetilde{\mathbb{T}}$, we have $\widetilde{\rho}^*(t) \geq \rho^*(t)$ for all $t \in [t_0, \infty)_{\mathbb{R}}$. By Lemma 3.2, a nonoscillatory solution of (3.1) at $t \in \mathbb{T}$ also satisfies (3.6). The definitions of $\widetilde{B}_k$, $\widetilde{\beta}_k$ and the assumptions of the theorem imply

$$B_k(t) \geq \widetilde{B}_k(t) \geq A_k(t) \quad \text{and} \quad \beta_k(t) \leq \widetilde{\beta}_k(t) \leq \alpha_k(t) \leq t \quad \text{for all } t \in \mathbb{R}.$$

Thus, by comparison Lemma 3.1, the equation

$$x'(t) + \sum_{k=1}^{m} \widetilde{B}_k(t)x(\widetilde{\beta}_k(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{R}}$$

has a nonoscillatory solution which (again, by Lemma 3.2) coincides at $t \in \widetilde{\mathbb{T}}$ with a solution of (3.7), which has the same initial function. Consequently, (3.7) is also nonoscillatory. $\square$

Obviously the results of Theorem 3.3 are valid for equations with constant delays (if relevant time scales allow) and coefficients. For differential and difference equations Theorem 3.3 immediately implies the following result.

**Corollary 3.4.** *Assume that* $A_k : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *are nonincreasing functions,* $\alpha_k(t) \leq t$ *for* $t \in \mathbb{R}_0^+$ *are nondecreasing,* $\alpha_k(n) \in \mathbb{Z}$ *for all* $n \in \mathbb{N}_0$ *and the difference equation*

$$\Delta x(n) + \sum_{k=1}^{m} A_k(n)x(\alpha_k(n)) = 0 \quad \text{for } n \in \mathbb{N}_0 \tag{3.8}$$

*is nonoscillatory, then differential equation* (3.2) *is also nonoscillatory. If all solutions of* (3.2) *are oscillatory then* (3.8) *is also oscillatory.*

Using some special construction which is similar to that applied above to ordinary differential equations we can consider the case when $A_k$ are not necessarily nonincreasing.

**Corollary 3.5.** *Suppose that $A_k : \mathbb{R} \to \mathbb{R}_0^+$ are nonincreasing functions, $\alpha_k$ are nonincreasing with $\alpha_k(t) \le t$ for $t \in \mathbb{R}_0^+$. Suppose also that the equation*

$$x^{\Delta}(t) + \sum_{k=1}^{m} B_k(t)x(\alpha_k(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

*where $\alpha_k(t) \in \mathbb{T}$ and $B_k(t) := \sup_{s\in[t,\sigma(t))_{\mathbb{R}}} A_k(s)$ for $t \in [t_0, \infty)_{\mathbb{T}}$, is nonoscillatory. Then for any $\widetilde{\mathbb{T}} \supset \mathbb{T}$ equation* (3.7) *on the refined scale is also nonoscillatory.*

Let us demonstrate sharpness of the conditions of Theorem 3.3. For simplicity, we will everywhere below compare solutions of differential and difference equations, i.e., $\mathbb{T} = \mathbb{Z}$ and $\widetilde{\mathbb{T}} = \mathbb{R}$. The requirement that the coefficient $A$ is nonincreasing (or that the supremum is taken as a coefficient of the approximating equation, similar to Corollary 3.5) is necessary, as Example 3.6 demonstrates.

Let us note that in Example 3.6, we apply the following sufficient oscillation condition [7, Theorem 3.4.3]: If $A(t)$ is positive, $\alpha(t) \le t$ is nondecreasing with $\lim_{t\to\infty} \alpha(t) = \infty$, and $\limsup_{t\to\infty} \int_{\alpha(t)}^{t} A(\eta)\, \mathrm{d}\eta > 1$, then all solutions of the equation

$$x'(t) + A(t)x(\alpha(t)) = 0 \quad \text{for } t \in \mathbb{R}_0^+$$

are oscillatory.

**Example 3.6.** Consider the delay differential equation

$$x'(t) + \big(0.1 + 10\sin^2(\pi t)\big)x(t-2) = 0 \quad \text{for } t \in \mathbb{R}_0^+. \tag{3.9}$$

with a non-monotone coefficient. Since

$$\int_{t-2}^{t} \big(0.1 + 10\sin^2(\pi\eta)\big)\, \mathrm{d}\eta = \int_{t-2}^{t} \big(5.1 - 5\cos(2\pi\eta)\big)\, \mathrm{d}\eta = 10.2 > 1 \quad \text{for all } t \in \mathbb{R}_0^+,$$

equation (3.9) is oscillatory. However, for the time scale $\mathbb{N}_0 \subset \mathbb{R}$ we have the difference equation

$$\Delta x(n) + \big(0.1 + 10\sin^2(\pi n)\big)x(n-2) = \Delta x(n) + 0.1\, x(n-2) = 0, \quad \text{for } n \in \mathbb{N}_0. \tag{3.10}$$

Equation (3.10) is nonoscillatory since $0.1 < \frac{2^2}{3^3} = \frac{4}{27}$. We recall that $\Delta x(n) + A\, x(n-k) = 0$, where $A \in \mathbb{R}_0^+$ and $k \in \mathbb{N}$, is nonoscillatory if and only if $A \le \frac{k^k}{(k+1)^{k+1}}$ [7, Theorem 7.2.1], compare to nonoscillation condition (1.3) of equation (1.4).

The following example demonstrates that the requirement of delay monotonicity also cannot be omitted.

**Example 3.7.** The delay differential equation

$$x'(t) + 0.2\,x\big(t - 1 - 100\{t\}(1 - \{t\})\big) = 0 \quad \text{for } t \in \mathbb{R}_0^+, \tag{3.11}$$

where $\{t\}$ is the fractional part of number $t$, is oscillatory. To prove it, let us assume the contrary that $x$ is a nonoscillatory solution. Without loss of generality, we assume that $x(t) > 0$ for all $t \in \mathbb{R}_0^+$. Then $x$ is decreasing and satisfies $x'(t) + 0.2\,x(t) \le 0$ for all $t \in \mathbb{R}_0^+$. Then, by Grönwall's inequality, we have

$$x(t) \le x(s)\,\mathrm{e}^{-0.2(t-s)} \quad \text{for all } t \in [s, \infty)_{\mathbb{R}}, \text{ where } s \in \mathbb{R}_0^+. \tag{3.12}$$

We now consider the solution on the intervals of the form $[n + 0.25, n + 0.75]_{\mathbb{R}}$, where $n \in \mathbb{N}_0$. If $t \in [n+0.25, n+0.75]_{\mathbb{R}}$ for some $n \in \mathbb{N}_0$, then $t-1-100\{t\}(1-\{t\}) \le n - 19.75$ and

$$x'(t) = -0.2\,x\big(t - 1 - 100\{t\}(1 - \{t\})\big) \le -0.2\,x(n - 19.75),$$

which yields by integrating from $(n + 0.25)$ to $(n + 0.75)$ and using (3.12) that

$$x(n+0.75) \le x(n+0.25) - 0.2 \cdot 0.5\,x(n-19.75) \le \big(\mathrm{e}^{-4} - 0.1\big)x(n-19.75) < 0$$

because of $\mathrm{e}^{-4} - 0.1 \approx -0.081 < 0$. The contradiction proves that all the solutions of (3.11) are oscillatory. However, the relevant difference equation

$$\Delta x(n) + 0.2\,x\big(n - 1 - 10\sin^2(\pi n)\big) = \Delta x(n) + 0.2\,x(n-1) = 0 \quad \text{for } n \in \mathbb{N}_0$$

is nonoscillatory since $0.2 < \frac{1^1}{2^2} = 0.25$.

## 4. Discussion and open problems

In the present paper, we have considered the dependency of oscillation properties on the time scale. Under certain conditions, we have demonstrated that if the equation is nonoscillatory on time scale $\mathbb{T}$, then this property is preserved on any finer time scale. The main result of the paper is Theorem 3.3 and, as Examples 3.6 and 3.7 illustrate, its conditions are sharp.

So far we have considered positive coefficients $A$ only. Obviously, both delay and nondelay equations with negative coefficients are nonoscillatory. Consider the nondelay equation

$$\begin{cases} x^{\Delta}(t) + A(t)x(t) = 0 & \text{for } t \in \mathbb{T} \\ x(t_0) = x_0 \end{cases} \tag{4.1}$$

with a continuous oscillatory coefficient $A : \mathbb{R} \to \mathbb{R}$. Then, picking up the points of $\mathbb{T}$ such that $A(t) < 0$, $-A(t)\mu(t) > \lambda$ for some constant $\lambda > 0$ and $t \in [t_0, \infty)_{\mathbb{T}}$ (which is possible, since for any $A(t) < 0$, there are points exceeding $t - \frac{\lambda}{A(t)}$ where the coefficient $A$ is negative), we obtain for the solution of (4.1):

$$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}}} x(t) = \pm\infty,$$

where the limit of $\infty$ corresponds to $x_0 > 0$ and $-\infty$ is for any $x_0 < 0$. Choosing $\mathbb{T} \subset \{t \in \mathbb{R} : \ A(t) = 0\}$, we obtain a solution which is constant on $[t_0, \infty)_{\mathbb{T}}$.

Finally, let us present some relevant problems, exercises and topics for research and discussion.

(P1) Under which condition on an oscillatory coefficient $A$, for any $\ell \in \mathbb{R}$ will there exist a time scale including $t_0$ and $x_0$ such that the solution of (4.1) satisfies

$$\lim_{\substack{t\to\infty \\ t\in\mathbb{T}}} x(t) = \ell\ ?$$

(P2) By Theorem 2.8, the modified equation

$$x^{\Delta}(t) + A(t)x(\sigma(t)) = 0 \quad \text{for } t \in \mathbb{T}, \text{ where } A : \mathbb{T} \to \mathbb{R}_0^+,$$

is nonoscillatory on any time scale $\mathbb{T}$. Consider the following equation

$$x^{\Delta}(t) + A(t)\big[\theta\, x(t) + (1-\theta)x(\sigma(t))\big] = 0 \quad \text{for } t \in \mathbb{T}, \qquad (4.2)$$

where $A : \mathbb{T} \to \mathbb{R}_0^+$ and $\theta \in [0,1]_{\mathbb{R}}$ which can be obtained by the $\theta$-approximation of an ordinary differential equation. For $\theta = 0$, equation (4.2) is nonoscillatory. For each $A$, compute the maximal $\theta$ such that $x$ is nonoscillatory.

(P3) Under which conditions will all nonoscillatory solutions on the time scales considered in this paper tend to zero?

(P4) In Lemma 3.2 to each equation on time scales we matched a differential equation such that the solution of the differential equation coincides with the solution of the equation on time scale $\mathbb{T}$ for any $t \in \mathbb{T}$. Solve the inverse problem: for each differential equation, find (if possible) an unbounded above time scale $\mathbb{T}$ such that the solutions of two equations coincide at any $t \in \mathbb{T}$. If, generally, the answer is negative, find sufficient conditions and/or modify the coefficient such that this coincidence is possible.

(P5) Extend the results of Theorem 3.3 to the integrodifferential equation on time scales

$$x^{\Delta}(t) + \int_{h(t)}^{t} K(t,\eta)x(\eta)\Delta\eta = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}$$

and/or

$$x^{\Delta}(t) + \int_{h(t)}^{t} K(t, \sigma(\eta))x(\eta)\Delta\eta = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

(P6) Formulate conditions under which stability and global boundedness of solutions is monotone in time scales.

(P7) Extend the results of the present paper to nonlinear equations with a unique positive equilibrium. In addition to nonoscillation, consider stability, solution persistence and solution estimates.

**Acknowledgement.** We are grateful to the reviewers of the paper for valuable remarks and comments.

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Received November 18, 2010; revised May 12, 2011