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Strong Solutions of Doubly Nonlinear Parabolic Equations

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Abstract. The aim of this article is to discuss strong solutions of doubly nonlinear parabolic equations

$$\frac{\partial Bu}{\partial t} + Au = f,$$

where $A : X \to X^*$ and $B : Y \to Y^*$ are operators satisfying standard assumptions on boundedness, coercivity and monotonicity. Six different situations are identified which allow to prove the existence of a solution $u \in L^\infty(0, T; X \cap Y)$ to an initial value $u_0 \in X \cap Y$, but only in some of these situations the equation is valid in a stronger space than $(X \cap Y)^*$.

Keywords. Doubly nonlinear evolution equations, elliptic-parabolic problems, strong solutions, regularity

Mathematics Subject Classification (2010). Primary 35K90, secondary 35D35

1. Introduction

An abstract doubly nonlinear parabolic equation has the form

$$\frac{\partial Bu}{\partial t} + Au = f \tag{1}$$

with operators $A : X \to X^*$, $B : Y \to Y^*$ on Banach spaces X, Y and an inhomogeneity (or nonlinearity) f.

In applications, doubly nonlinear parabolic equations occur as models of physical phenomena like the filtration of non-Newtonian fluids through porous media or the evolution of reaction-diffusion systems, and in many other fields like e.g. population dynamics. Further, in many cases these equations are degenerate or singular. Let us explicitly mention two examples.

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Example 1.1. The density ρ of a fluid in a homogeneous isotropic porous medium is governed by the continuity equation $\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho v)=0$, where v is the seepage velocity of the fluid. Assume that v depends only on the gradient of the pressure π within the fluid, i.e., $v=-a(\operatorname{grad}\pi)$. For a Newtonian fluid a is supposed to be linear (Darcy's law), but for a non-Newtonian fluid a is allowed to be a nonlinear monotone mapping. For simplicity, assume that a is the derivative of the convex function $\phi_a(\operatorname{grad}\pi):=\frac{1}{p}|\operatorname{grad}\pi|^p$, $1<p<\infty$, given by $a(\operatorname{grad}\pi)=(\operatorname{grad}\pi)^{p-1}:=|\operatorname{grad}\pi|^{p-2}\operatorname{grad}\pi$, where $|\cdot|$ is the Euclidean norm and $(\cdot)^{p-1}$ denotes signed power (of vectors). Further, assume that the pressure π satisfies the power law $\pi(\rho)=A\rho^\gamma$ with constants $A>0$, $\gamma>1$, then

$$\rho\, a(\operatorname{grad}\pi)=\left(\frac{\gamma A}{p'+\gamma-1}\operatorname{grad}\rho^{p'+\gamma-1}\right)^{p-1}.$$

Thus, using $m:=\frac{p'+\gamma}{p'+\gamma-1}=\frac{p+\gamma(p-1)}{1+\gamma(p-1)}$ and $u:=\rho^{p'+\gamma-1}=\rho^{m'-1}$ the equation governing the fluid reads as

$$\frac{\partial u^{m-1}}{\partial t}=\operatorname{div}\left(\left(\frac{\gamma A}{m'-1}\operatorname{grad}u\right)^{p-1}\right).$$

This equation is doubly degenerate under the conditions $\gamma>1$ and $p>2$, while in the case $\gamma>1$ and $1<p<2$ it is degenerate at points x with $u(x)=0$ and singular at points x with $(\operatorname{grad}u)(x)=0$.

Example 1.2. Assume that the kinetic energy of a non-Newtonian incompressible fluid in a domain $\Omega\subset\mathbb{R}^3$ is modeled by $\int_\Omega\phi_b(u)\,dx$, where $\phi_b:\mathbb{R}^3\to\mathbb{R}$ is the convex potential of the nonlinear momentum mapping b, and that the viscous stress tensor is modeled by $a(\nabla^{\mathrm{sym}}u)$ with a nonlinear mapping a depending only on the symmetric part of the (3×3)-matrix ∇u. Then the velocity vector field u of the fluid is governed by doubly nonlinear Navier-Stokes equations

$$\frac{\partial b(u)}{\partial t}+\operatorname{div}(b(u)\otimes u)=-d\pi+\operatorname{div}(a(\nabla^{\mathrm{sym}}u)),$$

where the pressure density π is implicitly determined by the incompressibility condition $\operatorname{div}(u)=0$. Note that these equations are up to the viscosity term Lie-Poisson equations, and mathematically it is not prohibited to choose a non-quadratic Hamiltonian in such equations. Thus, mathematically doubly nonlinear Navier-Stokes equations make sense. Physically, at least the question may be allowed whether it is justified to assume in every situation that particles within a fluid have a quadratic kinetic energy, e.g. in the situation of a fluid which flows in a porous medium so that the particles of the fluid interact with the particles of the medium.

Abstracting from these particular examples, in this article the abstract equation (1) is studied under the following structural assumptions:

(A1) X and Y are reflexive Banach spaces with a dense and separable intersection $X \cap Y$ [1], which is compactly embedded into Y.

(A2) $B : Y \to Y^*$ is a continuous strictly monotone potential operator, which is coercive and satisfies the growth condition $\|Bu\|_{Y^*} \le C\|u\|_Y^{m-1}$ with a constant $C < \infty$ and a parameter $1 < m < \infty$.

(A3) $A : X \to X^*$ is a pseudomonotone operator, which satisfies the semicoercivity condition $\langle Au, u\rangle \ge c_1\|u\|_X^p - c_2\|u\|_X - c_3\|Bu\|_{Y^*}^{m'}$ and the growth condition $\|Au\|_{X^*} \le C(\|u\|_Y)(1 + \|u\|_X^{p-1})$ for a parameter $1 < p < \infty$ with constants $c_1 > 0$, c_2, c_3 and an increasing function $C : \mathbb{R}_0^+ \to \mathbb{R}_0^+$.

(A4) $f \in L^{p'}(0, T; X^*)$ is an inhomogeneity.

Under these assumptions there exists to every initial value $u_0 \in Y$ a weak solution of equation (1) in the following sense.

Definition 1.3. A function $u \in L^p(0, T; X) \cap L^\infty(0, T; Y)$ is called a weak solution of equation (1) to the initial value $u_0 \in Y$, if $Bu \in L^\infty(0, T; Y^*)$ has the initial value $Bu_0 \in Y^*$ and a weak derivative $\frac{\partial Bu}{\partial t} \in L^{p'}(0, T; X^*)$ satisfying equation (1) as an equation in X^* for a.e. $t \in (0, T)$.

Like in [10], where the prototypical doubly nonlinear parabolic equation

$$\frac{\partial u^{m-1}}{\partial t} - \Delta_p u = f \tag{2}$$

was considered, the existence of weak solutions can be proved under these structural assumptions by a Faedo-Galerkin method (for proofs using Rothe's method see [1–3, 5, 7, 9] and the references therein).

In fact, similar to [14, 3.2] in Section 2 it is shown that the approximate equation obtained by restricting (1) to a finite-dimensional subspace can be solved under these assumptions with the help of the Leray-Schauder fixed point theorem. All other steps – the derivation of a priori estimates by testing the approximate equation with u, the extraction of weakly convergent subsequences and the proof that the weak limits are identical with their expected limits – are in complete analogy with the proof given in [10] for the prototypical equation (2). Hence, under the structural assumptions (A1)–(A4) there exists a weak solution of (1).

In a more general sense weak solutions even exist if (A4) is replaced by weaker conditions. This observation is crucial in the following, where equation (1) is discussed for inhomogeneities and nonlinearities $f = f(u)$ with $f \in L^2(0, T; Y^*)$ or $f \in L^2(0, T; H^*)$ for an intermediate Hilbert space H.

[1] More precisely, there are continuous linear embeddings of X and Y into a fixed complete locally convex space Z such that their intersection $X \cap Y$ within Z is dense in X resp. Y w.r.t. the norms $\|\cdot\|_X$ resp. $\|\cdot\|_Y$, and that $X \cap Y$ is separable w.r.t. the norm $\|\cdot\|_X + \|\cdot\|_Y$.

Remark 1.4. If $f \in L^{p'}(0,T;X^*) + L^2(0,T;Y^*)$, then under the additional assumption that B satisfies $\|u\|_Y \le C(1 + \|Bu\|_{Y^*}^{m'})$ with a constant $C < \infty$ there exists a weak solution in the more general sense that $\frac{\partial Bu}{\partial t} \in L^{p'}(0,T;X^*) + L^2(0,T;Y^*)$ and equation (1) is satisfied as an equation in $(X \cap Y)^*$ for a.e. $t \in [0,T]$. Moreover, if $f = f(t,u)$ is a nonlinearity satisfying $\|f(t,u)\|_{Y^*} \le C(\gamma(t) + \|u\|_Y^{m-1})$ with a constant $C < \infty$ and a function $\gamma \in L^2(0,T)$, then under the stronger additional assumption that B satisfies $\|u\|_Y \le C(1+\|Bu\|_{Y^*}^{m'-1})$ with a constant $C < \infty$ there exists a weak solution in the more general sense.

Remark 1.5. Let H be an intermediate Hilbert space such that $X \cap Y \subset H \subset Y$ is an interpolation triple, i.e., there is a $\theta \in [0,1]$ and a constant $C < \infty$ such that $\|u\|_H \le C\|u\|_{X\cap Y}^{\theta}\|u\|_Y^{1-\theta}$ for all $u \in X \cap Y$, and assume $p \ge 2$ or $\frac{1}{2} \le \theta \le \frac{p}{2}$.

If $f \in L^{p'}(0,T;X^*) + L^2(0,T;H^*)$, then under the additional assumption that B satisfies $\|u\|_Y \le C(1 + \|Bu\|_{Y^*}^{m'})$ with a constant $C < \infty$ there exists a weak solution in the more general sense that $\frac{\partial Bu}{\partial t} \in L^{p'}(0,T;X^*) + L^2(0,T;H^*)$ and equation (1) is satisfied as an equation in $(X \cap H)^*$ for a.e. $t \in [0,T]$.

Moreover, if $f = f(t,u)$ is a nonlinearity satisfying $\|f(t,u)\|_{H^*} \le C(\gamma(t) + \|u\|_Y^{(m-1)(1-\theta)})$ with a constant $C < \infty$ and a function $\gamma \in L^2(0,T)$, then under the stronger additional assumption that B satisfies $\|u\|_Y \le C(1 + \|Bu\|_{Y^*}^{m'-1})$ with a constant $C < \infty$ there exists a weak solution in the more general sense.

The question arises whether weak solutions have better properties than those mentioned in Definition 1.3. Here we are interested in strong solutions, i.e., we ask ourselves whether it is possible to derive additional a priori estimates by testing the approximate equation with $\frac{\partial u}{\partial t}$ under the additional assumptions that $A : X \to X^*$ is a potential operator, f has values in Y^* or H^*, and $u_0 \in X \cap Y$. Contrary to nonlinear parabolic equations $\frac{\partial u}{\partial t} + Au = f$, in the case of doubly nonlinear parabolic equations it makes sense to distinguish six different types of strong solutions. These types of strong solutions are explored in Section 3.

The first three types of strong solutions are derived under the condition that the derivative dB^{-1} of the inverse B^{-1} of B exists. The solution types are named so that solutions of first type are stronger than solutions of second type, and these again are stronger than solutions of third type. Let us start with the weakest type of strong solutions. The following theorem about the existence of what we call strong solutions of third type is proved in Section 3.1. It is applicable to the operator $Bu = u^{m-1}$ on $Y := L^m(\Omega)$ in the case $1 < m \le 2$.

Theorem 1.6. *Additionally to the structural assumptions* (A1)–(A3) *require that*

- $B^{-1} : Y^* \to Y$ *is* C^1, *satisfies* $dB^{-1}(0) = 0$ *and is uniformly monotone in the sense that* $\langle v^*, dB^{-1}(u^*)v^* \rangle \ge c(\|u^*\|_{Y^*})\|dB^{-1}(u^*)v^*\|_Y^2$ *holds for every* $u^*, v^* \in Y^*$, $u^* \neq 0$, *with a decreasing function* $c : \mathbb{R}^+ \to \mathbb{R}^+$,

- *$A : X \to X^*$ is a potential operator,*
- *$f \in L^2(0, T; Y^*)$.*

Then there exists to every initial value $u_0 \in X \cap Y$ a strong solution u of equation (1) *in the sense that u is a weak solution which additionally satisfies $u \in L^\infty(0, T; X)$, has the initial value $u_0 \in X \cap Y$ and a weak derivative $\frac{\partial u}{\partial t} \in L^2(0, T; Y)$.*

As a consequence, $u \in C(0, T; Y)$ and also $u \in C(0, T; (X, \text{weak}))$ because of $u \in L^\infty(0, T; X)$. In fact, $u(s_n) \to u(t)$ in Y as $s_n \to t$ and boundedness of $u(s_n)$ in X implies weak convergence of $u(s_n)$ in X at least for a subsequence. But due to denseness of $X^* \cap Y^*$ in X^* the weak limit has to coincide with $u(t)$ for every subsequence, so that really $u(s) \rightharpoonup u(t)$ weakly in X as $s \to t$. Further, by continuity of B also $Bu \in C(0, T; Y^*)$. However, it can not be concluded that equation (1) holds in a better space than $(X \cap Y)^*$.

The assumptions of the following theorem guarantee the existence of solutions for which equation (1) is valid in a better function space. The corresponding solutions are called strong solutions of second type. This theorem is applicable to operators B on the space $Y := L^m(\Omega)$ with potential $\Phi_B(u) := \int_\Omega b(u)\,dx$ for a convex C^2-function b which behaves like $\frac{1}{2}|u|^2$ as $|u| \to 0$ and like a multiple of $|u|^m$ as $|u| \to \infty$ for $1 < m \le 2$, The proof of the theorem is given in Section 3.2.

Theorem 1.7. *Additionally to the structural assumptions* (A1)–(A3) *assume that $X \cap Y \subset H \subset Y$ is an interpolation triple and $p \ge 2$ or $\frac{1}{2} \le \theta \le \frac{p}{2}$ as in Remark* 1.5*, and that*

- *$B^{-1} : Y^* \to Y$ is C^1, satisfies $\|u\|_Y \le C(1 + \|Bu\|_{Y^*}^{m'-1})$ with a constant $C < \infty$, and is strongly monotone in the sense that $\langle v^*, dB^{-1}(u^*)v^* \rangle \ge c\|v^*\|_{H^*}^2$ for all $u^*, v^* \in Y^*$ with a constant $c > 0$* [2]*,*
- *$A : X \to X^*$ is a potential operator such that the intersection of Y and the domain $D(A) := \{u \in X \mid Au \in H^*\}$ of A w.r.t. H^* is dense in $X \cap Y$,*
- *$f = f(u)$ is a nonlinearity such that $g := dB^{-1}(Bu)^* f(u) \in L^2(0, T; H)$ is independent of u.*

Then there exists to every initial value $u_0 \in X \cap Y$ a strong solution u of equation (1) *in the sense that u is a weak solution which additionally satisfies $u \in L^\infty(0, T; X)$, and $Bu \in L^\infty(0, T; Y^*)$ has the initial value $Bu_0 \in Y^*$ and a weak derivative $\frac{\partial Bu}{\partial t} \in L^2(0, T; H^*)$.*

As a consequence of this theorem $Au = f - \frac{\partial Bu}{\partial t} \in L^2(0, T; H^*)$, and equation (1) is valid as an equation in H^*. An analogous result is valid if B^{-1} is C^1

[2]Note that this condition is equivalent to strong monotonicity of B^{-1} as an operator $B^{-1} : Y^* \subset H^* \cong H \to Y$, i.e., to the condition $\langle u^* - v^*, B^{-1}u^* - B^{-1}v^* \rangle \ge c\|u^* - v^*\|_{H^*}^2$ for arbitrary $u^*, v^* \in Y^*$.

and strongly monotone as an operator $B^{-1} : Y^* \to Y$, i.e.,

$$\langle v^*, dB^{-1}(u^*)v^*\rangle \geq c\|v^*\|_{Y^*}^2 \tag{3}$$

or equivalently $\langle u^*-v^*, B^{-1}u^*-B^{-1}v^*\rangle \geq c\|u^*-v^*\|_{Y^*}^2$ is valid for all $u^*, v^* \in Y^*$ with a constant $c > 0$. Then $\frac{\partial Bu}{\partial t} \in L^2(0,T;Y^*)$, and equation (1) is even valid as an equation in the space Y^*. The corresponding theorem about these so-called strong solutions of first type is formulated in Section 3.3.

Finally, also the case is handeled where not B^{-1} but B is a continously differentiable operator, and in this case there again are three different types of strong solutions. But before we begin with a discussion of strong solutions, let us give a short summary how the existence of weak solutions of equation (1) can be proved by a Faedo-Galerkin method.

2. Weak solutions

To prove the existence of weak solutions by a Faedo-Galerkin method, let us consider the restriction of equation (1) to a finite-dimensional subspace $W_k \subset X \cap Y$. More precisely, denote by $\iota_k : W_k \to X \cap Y$ the inclusion of W_k, by $A_k := \iota_k^* \circ A \circ \iota_k$, $B_k := \iota_k^* \circ B \circ \iota_k$ and $f_k := \iota_k^* \circ f$ the restriction of A, B and f to W_k, and consider the approximate equation

$$\frac{\partial B_k u_k}{\partial t} + A_k u_k = f_k\,. \tag{4}$$

We want to show that the integral form of this equation, i.e., the equation

$$B_k u_k(t) = B_k u_k(0) + \int_0^t (f_k(s) - A_k u_k(s))\,ds,$$

has locally in time a solution u_k to the initial value $u_k(0) \in W_k$.

Note that the operator $B_k : W_k \to W_k^*$ has a continuous inverse, as B_k is continuous, bounded, strictly monotone and coercive due to assumption (A2). Thus, the integral equation can equivalently be written as

$$u_k(t) = B_k^{-1}\left(B_k u_k(0) + \int_0^t (f_k(s) - A_k u_k(s))\,ds\right).$$

Consider the right hand side as an operator $B_k^{-1} \circ K_k$ on $C([0,T], W_k)$, and recall that on the finite-dimensional space W_k all norms are equivalent. Let $r > 0$ be given, then there is a T_k such that the distance of $(B^{-1} \circ K_k)(u_k)(t)$ to $u_k(0)$ is smaller than r for every $t \in [0, T_k]$ provided that u_k satisfies the inequality $\sup_{s\in[0,T_k]} \|u_k(s) - u_k(0)\| \leq r$.

In fact, by continuity of B_k^{-1} there is a $\tilde{r} > 0$ such that if $v_k \in W_k$ and $\|v_k - B_k u_k(0)\| < \tilde{r}$, then $\|B_k^{-1}(v_k) - u_k(0)\| < r$. Further,

$$\begin{aligned}\left\|\int_0^t (f_k(s) - A_k u_k(s))\, ds\right\| &\le \int_0^t \|A_k u_k(s)\|\, ds + \int_0^t \|f_k(s)\|\, ds \\ &\le \int_0^{T_k} C(\|u_k\|)(1 + \|u_k\|^{p-1})\, ds + \left(\int_0^{T_k} \|f_k(s)\|^{p'} ds\right)^{\frac{1}{p'}} T_k^{\frac{1}{p}} \\ &\le M_k T_k + \|f\|_{L^{p'}(0,T;X^*)} T_k^{\frac{1}{p}},\end{aligned}$$

where $M_k := \max\{C(\|u_k\|)(1 + \|u_k\|^{p-1}) \mid \|u_k - u_k(0)\| \le r\}$. Thus for small T_k and $t \in [0, T_k]$ the norm $\|(K_k u_k)(t) - B_k u_k(0)\|$ becomes smaller than $\tilde{r}$, and hence the distance of $(B_k^{-1} \circ K_k)(u_k)(t)$ and $u_k(0)$ is smaller than r for all $t \in [0, T_k]$.

Moreover, $K_k : C(0, T_k; W_k) \to C(0, T_k; W_k^*)$ is a compact operator, as bounded subsets of $C(0, T_k; W_k) \subset L^p(0, T_k; W_k)$ are mapped by K_k to bounded subsets of $W^{1,p'}(0, T_k; W_k^*)$, which is compactly embedded into $C(0, T_k; W_k^*)$. Because B_k^{-1} is continuous, also $B_k^{-1} \circ K_k$ is a compact operator on $C(0, T_k; W_k)$. Thus, $\mathrm{Id} - B_k^{-1} \circ K_k$ is a compact perturbation of the identity, so that there is a fixed point by the Leray-Schauder fixed point theorem (see e.g [6, 11.4]). This shows the local existence in time of a solution $u_k \in C(0, T_k; W_k)$ of the approximate equation.

Note further that if f_k is not merely the restriction of $f \in L^{p'}(0, T; X^*)$ to W_k, but $f_k \in C(0, T; W^*)$ is a continuous approximation of f in the sense that $f_k \rightharpoonup f$ in $L^{p'}(0, T; X^*)$ as $k \to \infty$, then $u_k \in C^1(0, T_k; W_k)$ is valid, because by demicontinuity of A the function $s \mapsto A_k u_k(s)$ is continuous for a continuous $u_k \in C(0, T_k; W_k)$. This observation will be important in the next section, where strong solutions are considered.

All other steps – the derivation of a priori estimates by testing the approximate equation with u_k, the extraction of a weakly convergent subsequence which among others satisfies $u_k \rightharpoonup u$ in $L^p(0, T; X)$, $u_k \overset{*}{\rightharpoonup} u$ in $L^\infty(0, T; Y)$ and $Bu_k \overset{*}{\rightharpoonup} (Bu)_{ex}$ in $L^\infty(0, T; Y^*)$, and the proof that weak limits and expected limits are identical (e.g. $(Bu)_{ex} = Bu$) – are in complete analogy with the proof for the prototypical case in [10]. Thus, under the assumptions (A1)–(A4) existence of weak solutions of the doubly nonlinear parabolic equation (1) in the sense of Definition 1.3 can be proved for initial values $u_0 \in Y$.

3. Strong solutions

To prove the existence of strong solutions, we would like to test the approximate equation (4) by $\frac{\partial u}{\partial t}$ (hereby, we suppress the index k of approximate solutions to make the following calculations better readable).

However, the existence of $\frac{\partial u}{\partial t}$ is not at all obvious, because only the existence of $\frac{\partial Bu}{\partial t}$ is known from the approximate equation. But if the approximation f_k of f is a continuous function $f_k \in C(0,T;W^*)$ then due to demicontinuity of A the function $t \mapsto f_k(t) - Au(t)$ is continuous for $u \in C(0,T;W)$, and from the approximate equation

$$Bu(t) = Bu(0) + \int_0^t (f(s) - Au(s))\, ds$$

we can conclude that $\frac{\partial Bu}{\partial t} = f - Au$ exists and is a continuous function.

Therefore, let us assume that B^{-1} is continuously differentiable, then the existence of $\frac{\partial Bu}{\partial t}$ implies by the chain rule the existence of the time derivative of $(B^{-1} \circ B)(u) = u$, and the formula

$$\frac{\partial u}{\partial t} = dB^{-1}(Bu)\frac{\partial Bu}{\partial t} \tag{5}$$

is valid.

3.1. Strong solutions of third type. For continuously differentiable B^{-1} the approximate equation (4) can be tested by $\frac{\partial u}{\partial t}$ to obtain

$$\left\langle \frac{\partial Bu}{\partial t}, \frac{\partial u}{\partial t} \right\rangle + \left\langle Au, \frac{\partial u}{\partial t} \right\rangle = \left\langle f, \frac{\partial u}{\partial t} \right\rangle .$$

Now $\frac{d}{dt}\Phi_A(u) = \langle Au, \frac{\partial u}{\partial t}\rangle$ due to $A = d\Phi_A$ and $\langle f, \frac{\partial u}{\partial t}\rangle \leq \|f\|_{Y^*}\|\frac{\partial u}{\partial t}\|_Y$, but to obtain an a priori estimate, we also have to estimate the first term from below.

Therefore, assume $dB^{-1}(0) = 0$ and

$$\langle v^*, dB^{-1}(u^*)v^* \rangle \geq c(\|u^*\|_{Y^*})\|dB^{-1}(u^*)v^*\|_Y^2 \tag{6}$$

for all $u^*, v^* \in Y^*$, $u^* \neq 0$, where $c : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing function of $\|u^*\|_{Y^*}$. Put $u^* := Bu$ and $v^* := \frac{\partial Bu}{\partial t}$ in this inequality, then

$$\left\langle \frac{\partial Bu}{\partial t}, \frac{\partial u}{\partial t} \right\rangle \geq c(\|Bu\|_{Y^*}) \left\| \frac{\partial u}{\partial t} \right\|_Y^2$$

by equation (5). Thus, because we already know from the discussion of weak solutions that $\|Bu\|_{Y^*}$ is uniformly bounded in time, there is a constant $c > 0$ such that

$$c \left\| \frac{\partial u}{\partial t} \right\|_Y^2 + \frac{d}{dt}\Phi_A(u) \leq \|f\|_{Y^*} \left\| \frac{\partial u}{\partial t} \right\|_Y \leq \frac{1}{2\epsilon^2}\|f\|_{Y^*}^2 + \frac{\epsilon^2}{2} \left\| \frac{\partial u}{\partial t} \right\|_Y^2 ,$$

and by choosing $\epsilon > 0$ so small that $c > \frac{\epsilon^2}{2}$, an a priori estimate of $\frac{\partial u}{\partial t}$ in $L^2(0,T;Y)$ and of u in $L^\infty(0,T;X)$ can be derived. Recall that we spoke the whole time about solutions of the approximate equations (we only suppressed the index k for better readability, but now we are going to mention it again).

Due to the a priori estimates of $\frac{\partial u_k}{\partial t}$ in $L^2(0,T;Y)$ and of u_k in $L^\infty(0,T;X)$ for the approximates solutions u_k, we are able to extract a subsequence of the approximate solution u_k such that additionally to the weak convergences mentioned in Section 2 also $u_k \overset{*}{\rightharpoonup} (u)_{ex}$ in $L^\infty(0,T;X)$ and $\frac{\partial u_k}{\partial t} \rightharpoonup (\frac{\partial u}{\partial t})_{ex}$ in $L^2(0,T;Y)$.

It is easy to verify that these weak limits are identical with their expected values. In fact, as $u_k \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;Y)$, on the one hand $(u)_{ex} = u$ by denseness of $X^* \cap Y^*$ in X^*. On the other hand, as $\int_0^T \langle \frac{\partial v}{\partial t}, u_k - u_k(0)\rangle\, dt = -\int_0^T \langle v - v(T), \frac{\partial u_k}{\partial t}\rangle\, dt$ holds for all $v \in L^2(0,T;W_k^*)$ with final value $v(T) \in W_k^*$ and weak derivative $\frac{\partial v}{\partial t} \in L^1(0,T;W_k^*)$, let $k \to \infty$ to obtain by density of $\bigcup_k W_k$ in Y the equation

$$\int_0^T \left\langle \frac{\partial v}{\partial t}, u - u(0) \right\rangle_Y dt = -\int_0^T \left\langle v - v(T), \left(\frac{\partial u}{\partial t}\right)_{ex} \right\rangle dt$$

for all $v \in L^2(0,T;Y^*)$ with final value $v(T) \in Y^*$ and weak derivative $\frac{\partial v}{\partial t} \in L^1(0,T;Y^*)$. Therefore, the weak derivative $\frac{\partial u}{\partial t}$ of u is identical with $(\frac{\partial u}{\partial t})_{ex}$.

Thus, we proved Theorem 1.6 and obtained what we call a strong solution of third type. Finally, let us show that in the case of the prototypical equation the required inequality (7) is valid.

Example 3.1. Consider the prototypical equation (2), where $Y = L^m(\Omega)$ and $Bu = u^{m-1}$. Thus, if $m \leq 2$, then B^{-1} has the Frechet-derivative $dB^{-1}(u^*) = \frac{1}{m-1}|u^*|^{\frac{2-m}{m-1}}$ at u^*. Therefore, by Hölder inequalities

$$\begin{aligned}
\|dB^{-1}(u^*)v^*\|_m^m &= \frac{1}{(m-1)^m}\int_\Omega (|u^*|^{\frac{2-m}{m-1}}|v^*|)^m dx \\
&= \frac{1}{(m-1)^m}\int_\Omega (|u^*|^{\frac{2-m}{m-1}}|v^*|^2)^{\frac{m}{2}}|u^*|^{\frac{m(2-m)}{2(m-1)}}\, dx \\
&\leq \frac{1}{(m-1)^m}\left(\int_\Omega |u^*|^{\frac{2-m}{m-1}}|v^*|^2 dx\right)^{\frac{m}{2}} \left(\int_\Omega |u^*|^{\frac{m}{m-1}}\, dx\right)^{\frac{2-m}{2}}
\end{aligned}$$

and due to $\langle v^*, dB^{-1}(u^*)v^*\rangle = \frac{1}{m-1}\int_\Omega |u^*|^{\frac{2-m}{m-1}}|v^*|^2 dx$ hence

$$\langle v^*, dB^{-1}(u^*)v^*\rangle \geq (m-1)\|dB^{-1}(u^*)v^*\|_m^2 \|u^*\|_{m'}^{\frac{m-2}{m-1}}.$$

Thus inequality (6) holds with the function $c(\|u^*\|_{m'}) = (m-1)\|u^*\|_{m'}^{\frac{m-2}{m-1}}$.

3.2. Strong solutions of second type. Assume that additionally to the spaces X, Y a Hilbert space H is given such that $X \cap Y \subset H \subset Y$ is an interpolation triple, i.e., there is a $\theta \in [0,1]$ and a constant $C < \infty$ such that $\|u\|_H \leq C\|u\|_{X\cap Y}^{\theta}\|u\|_Y^{1-\theta}$ for every $u \in X \cap Y$. Further, assume $p \geq 2$ or $\frac{1}{2} \leq \theta \leq \frac{p}{2}$. Strong solutions of second type correspond to the validity of an inequality

$$\langle v^*, dB^{-1}(u^*)v^*\rangle \geq c\|v^*\|_{H^*}^2 \tag{7}$$

for all $u^*, v^* \in Y^*$ with a constant $c > 0$.

For the approximate equation consider finite-dimensional subspaces $W \subset D(A) \cap Y \subset X \cap Y$, and assume further that $f = f(u)$ is a nonlinearity such that $g := dB^{-1}(Bu)^* f(u) \in L^2(0,T;H)$ is – for simplicity – an inhomogeneity, i.e., g does not depend on u. Note that like in Remark 1.5 in this case $f(u)$ satisfies a growth assumption such that weak solutions of equation (1) exists. In fact, apply inequality (7) to $u^* := Bu$, $v^* := f(u)$, to obtain

$$c\|f(u)\|_{H^*}^2 \leq \langle f(u), dB^{-1}(Bu)f(u)\rangle = \langle f(u), g\rangle \leq \|f(u)\|_{H^*}\|g\|_H\,.$$

Thus $\|f(u)\|_{H^*} \leq \frac{1}{c}\|g\|_H$, is dominated uniformly in u by a function in $L^2(0,T)$. Especially, $f(u) \in L^2(0,T;H^*)$ for every $u \in L^\infty(0,T;Y)$, and weak solutions satisfying $\frac{\partial Bu}{\partial t} \in L^{p'}(0,T;X^*) + L^2(0,T;H^*)$ exist due to Remark 1.5.

Due to $W \subset D(A)$ and $f \in L^2(0,T;H^*)$ a solution $u \in W$ of the approximate equation $\frac{\partial Bu}{\partial t} + Au = f$ satisfies $\frac{\partial Bu}{\partial t} \in H^*$ for a.e. t. Especially, inequality (7) can be applied to $v^* = \frac{\partial Bu}{\partial t}$ to obtain

$$\left\langle \frac{\partial Bu}{\partial t}, \frac{\partial u}{\partial t}\right\rangle \geq c\left\|\frac{\partial Bu}{\partial t}\right\|_{H^*}^2.$$

Further, as $g := dB^{-1}(Bu)^* f(u) \in L^2(0,T;H)$ is an inhomogeneity,

$$\begin{aligned}\left\langle f, \frac{\partial u}{\partial t}\right\rangle_Y &= \left\langle f, dB^{-1}(Bu)\frac{\partial Bu}{\partial t}\right\rangle_Y \\ &= \left\langle \frac{\partial Bu}{\partial t}, g\right\rangle_H \\ &\leq \left\|\frac{\partial Bu}{\partial t}\right\|_{H^*}\|g\|_H \\ &\leq \frac{\epsilon^2}{2}\left\|\frac{\partial Bu}{\partial t}\right\|_{H^*}^2 + \frac{1}{2\epsilon^2}\|g\|_H^2\end{aligned}$$

for all $\epsilon > 0$. Thus, by choosing $\epsilon > 0$ so small that $c > \frac{\epsilon^2}{2}$, we obtain from

$$c\left\|\frac{\partial Bu}{\partial t}\right\|_{H^*}^2 + \frac{d}{dt}\Phi_A(u) \leq \frac{\epsilon^2}{2}\left\|\frac{\partial Bu}{\partial t}\right\|_{H^*}^2 + \frac{1}{2\epsilon^2}\|g\|_H^2$$

a priori estimates of $\frac{\partial Bu}{\partial t}$ in $L^2(0,T;H^*)$ and of u in $L^\infty(0,T;X)$.

Therefore, we are able to guarantee weak* convergence of a subsequence of the approximate solutions u_k in $L^\infty(0,T;X)$ and weak convergence of $\frac{\partial Bu_k}{\partial t}$ in $L^2(0,T;H^*)$. Again, it is easy to verify that the weak limits of these sequences are identical with their expected values. Hence, Theorem 3.3 has been proved, i.e., there exist what we call strong solutions of second type.

Especially, because $\frac{\partial Bu}{\partial t}$ and f lie in $L^2(0,T;H^*)$, also $Au = f - \frac{\partial Bu}{\partial t}$ lies in $L^2(0,T;H^*)$, so that equation (1) holds as an equation in H^* for a.e. $t \in [0,T]$, and thus $u(t) \in D(A)$ for a.e. $t \in [0,T]$.

As another consequence, $Bu \in W^{1,2}(0,T;H^*) \subset C(0,T;H^*)$ due to $Bu \in L^\infty(0,T;Y^*) \subset L^2(0,T;H^*)$ and $\frac{\partial Bu}{\partial t} \in L^2(0,T;H^*)$. Therefore, strong solutions of second type are very similar to strong solutions of nonlinear parabolic equations $\frac{\partial u}{\partial t} + Au = f$. Finally, let us give an example of an operator B where the required inequality (3) is valid, and let us discuss which nonlinearities $f(u)$ are allowed in this case.

Example 3.2. Let $Y := L^m(\Omega)$ for a bounded domain Ω, $1 < m < 2$, so that $H := L^2(\Omega)$ is continuously embedded into Y. Consider the potential operator $B = d\Phi_B : Y \to Y^*$ induced by the functional $\Phi_B(u) := \int_\Omega b(u)\,dx$ on Y, where $u \mapsto b(u)$ is a convex C^2-function which behaves like $\frac{1}{2}|u|^2$ as $|u| \to 0$ and like a multiple of $|u|^m$ as $|u| \to \infty$. For example, we may choose $b(u) = \frac{|u|^{2+m}}{2|u|^m + m|u|^2}$.

Then $b'(u)$ behaves like u as $|u| \to 0$ and like u^{m-1} as $|u| \to \infty$. Thus, $(b')^{-1}(u)$ behaves like u as $|u| \to 0$ and like $u^{m'-1}$ as $|u| \to \infty$, so that $((b')^{-1})'(u)$ behaves like 1 as $|u| \to 0$ and like $\frac{1}{m-1}|u|^{\frac{2-m}{m-1}}$ as $|u| \to \infty$. Especially, pointwisely $((b')^{-1})'(u) \geq c$ for a constant $c > 0$.

Therefore,

$$c\|v^*\|_2^2 = \int_\Omega c|v^*|^2 dx \leq \int_\Omega ((b')^{-1})'(u^*)|v^*|^2 dx,$$

and as a consequence $\langle v^*, dB^{-1}(u^*)v^*\rangle \geq c\|v^*\|_2^2$ for all $u^*, v^* \in Y^*$, so that inequality (3) is valid.

Further, every nonlinearity f with $f(u) \in L^2(0,T;H^*)$ for $u \in L^\infty(0,T;Y)$ such that $g := dB^{-1}(Bu)^* f(u) \in L^2(0,T;H)$ is an inhomogeneity has the form

$$f(t,u) = \frac{g(t)}{((b')^{-1})'(u)}$$

with a pregiven $g \in L^2(0,T;H)$. In fact, note that $u \mapsto ((b')^{-1})'(u)$ is a function which is bounded away from zero and stays bounded for bounded u, and $dB^{-1}(Bu)^* f(u) = g$ holds because $dB^{-1}(Bu)$ is merely a multiplication operator.

3.3. Strong solutions of first type. Strong solutions of first type correspond to the validity of an inequality (3) for all $u^*, v^* \in Y^*$ with a constant $c > 0$. This assumption is stronger that (7), and it allows to prove the existence of solutions with even better properties by a minor modification of the proof of Theorem 1.7.

Theorem 3.3. *Additionally to the structural assumptions* (A1)–(A3) *require that*

- $B^{-1} : Y^* \to Y$ *is* C^1, *satisfies* $\|u\|_Y \leq C(1 + \|Bu\|_{Y^*}^{m'-1})$ *with a constant* $C < \infty$, *and is strongly monotone in the sense that* $\langle v^*, dB^{-1}(u^*)v^* \rangle \geq c\|v^*\|_{Y^*}^2$ *for all* $u^*, v^* \in Y^*$ *with a constant* $c > 0$ [3],
- $A : X \to X^*$ *is a potential operator such that the intersection of* Y *and the domain* $D(A) := \{u \in X \mid Au \in Y^*\}$ *of* A *w.r.t.* Y^* *is dense in* $X \cap Y$,
- $f = f(u)$ *is a nonlinearity such that* $g := dB^{-1}(Bu)^* f(u) \in L^2(0, T; Y)$ *is independent of* u.

Then there exists to every initial value $u_0 \in X \cap Y$ *a strong solution* u *of equation* (1) *in the sense that* u *is a weak solution which additionally satisfies* $u \in L^\infty(0, T; X)$, *and* $Bu \in L^\infty(0, T; Y^*)$ *has the initial value* $Bu_0 \in Y^*$ *and a weak derivative* $\frac{\partial Bu}{\partial t} \in L^2(0, T; Y^*)$.

However, while this theorem even guarantees the validity of equation (1) as an equation in Y^* for a.e. $t \in [0, T]$ – and especially $Au = f - \frac{\partial Bu}{\partial t} \in L^2(0, T; Y^*)$, so that $u(t)$ lies in the domain $D(A)$ of A w.r.t. Y^* for a.e. $t \in [0, T]$ – it is not directly applicable to nonlinear superposition operators B due to lack of strong monotonicity. In contrast, Theorem 3.3 seems to be applicable to the situation where Y is a Sobolev-Slobodetskii space and B^{-1} is a fractional differential operator.

3.4. Strong solutions of sixth type. While differentiability of B^{-1} was required to prove existence of strong solutions of first, second and third type, strong solutions of fourth, fifth and sixth type are related to differentiability of B. However, in this case the existence of $\frac{\partial u}{\partial t}$ can not be concluded from the approximate equation (4). Therefore, the approximate equation should be changed to

$$dB_k(u_k)\frac{\partial u_k}{\partial t} + A_k u_k = f_k \tag{8}$$

(again, in the following we suppress the index k). Provided that $dB(u) : W \to W^*$ is invertible for every $u \in W$, this equation can be written as $\frac{\partial u}{\partial t} = (dB(u))^{-1}(f - Au)$ or in integral form as

$$u(t) = u(0) + \int_0^t (dB(u(s)))^{-1}(f(s) - Au(s))\, ds.$$

[3]Note that this condition is equivalent to strong monotonicity of $B^{-1} : Y^* \to Y$, i.e., to the condition $\langle u^* - v^*, B^{-1}u^* - B^{-1}v^* \rangle \geq c\|u^* - v^*\|_{Y^*}^2$ for arbitrary $u^*, v^* \in Y^*$.

By continuity of dB, continuity of the approximations of f and demicontinuity of A this integral equation has locally in time a continuously differentiable solution $u(t) \in W$. Further, by the chain rule also Bu is continuously differentiable w.r.t. time and $\frac{\partial Bu}{\partial t} = dB(u)\frac{\partial u}{\partial t}$. Especially, the approximate equation can be written as $\frac{\partial Bu}{\partial t} + Au = f$, so that existence of a weak solution follows in the same way as before.

Again, to obtain existence of a certain kind of strong solutions, test the new approximate equation (8) by $\frac{\partial u}{\partial t}$. Assume that additionally to X, Y a Hilbert space H is given such that $Y \subset H \cong H^* \subset Y^*$ is a Gelfand triple, let $f \in L^2(0,T;H^*)$ and assume the validity of

$$\langle dB(u)v, v\rangle \geq c\|v\|_H^2 \tag{9}$$

for all $u, v \in Y$ with a constant $c > 0$. Note that this inequality expresses that $dB(u)$ is uniformly coercive in u, so that $dB(u) : W \to W^*$ is invertible for every $u \in W$. Therefore, the new approximate equation (8) is solvable, and testing it by $v = \frac{\partial u}{\partial t}$ gives

$$c\left\|\frac{\partial u}{\partial t}\right\|_H^2 + \frac{d}{dt}\Phi_A(u) \leq \frac{1}{2\epsilon^2}\|f\|_{H^*}^2 + \frac{\epsilon^2}{2}\left\|\frac{\partial u}{\partial t}\right\|_H^2 .$$

Thus, a priori estimates of $\frac{\partial u}{\partial t}$ in $L^2(0,T;H)$ and of u in $L^\infty(0,T;X)$ can be obtained. Hence, the following theorem about the existence of strong solutions of sixth type has been proved.

Theorem 3.4. *Additionally to the structural assumptions* (A1)–(A3) *require that there is a Hilbert space H such that $Y \subset H \cong H^* \subset Y^*$ is a Gelfand triple and that*

- *$B : Y^* \to Y$ is C^1 and satisfies $\langle dB(u)v, v\rangle \geq c\|v\|_H^2$ for all $u, v \in Y$ with a constant $c > 0$.*
- *$A : X \to X^*$ is a potential operator,*
- *$f \in L^2(0,T;H^*)$*

Then there exists to every initial value $u_0 \in X \cap Y$ a strong solution u of equation (1) *in the sense that u is a weak solution which additionally satisfies $u \in L^\infty(0,T;X)$, has the initial value $u_0 \in X \cap Y$ and a weak derivative $\frac{\partial u}{\partial t} \in L^2(0,T;H)$.*

As a consequence, $u \in C(0,T;H)$ and hence also $u \in C(0,T;(X,\text{weak}))$, but equation (1) is not valid in a better space than $(X \cap Y)^*$.

Example 3.5. Let $Y := L^m(\Omega)$ for a bounded domain Ω, $1 < m < 2$, so that Y is continuously embedded into $H := L^2(\Omega)$. Consider the potential operator $B = d\Phi_B : Y \to Y^*$ induced by the functional $\Phi_B(u) := \int_\Omega b(u)\,dx$, where b is a convex C^2-function which behaves like $\frac{1}{2}|u|^2$ as $|u| \to 0$ and like a multiple of $\frac{1}{m}|u|^m$ as $|u| \to \infty$.

Then $b'(u)$ behaves like u as $|u| \to 0$ and like u^{m-1} as $|u| \to \infty$, thus $b''(u)$ behaves like 1 as $|u| \to 0$ and like $(m-1)|u|^{m-2}$ as $|u| \to \infty$. Especially, pointwisely $b''(u) \geq c$ for a constant $c > 0$. Therefore,

$$c\|v\|_2^2 = \int_\Omega c|v|^2\,dx \leq \int_\Omega b''(u)|v|^2 dx,$$

and as a consequence $\langle dB(u)v, v\rangle \geq c\|v\|_2^2$ for all $u, v \in Y$, so that inequality (9) is valid.

3.5. Strong solutions of fifth type. Without an intermediate Hilbert space assume the validity of

$$\langle dB(u)v, v\rangle \geq c\|v\|_Y^2$$

for all $u, v \in Y$ with a constant $c > 0$, then the following theorem about the so-called strong solutions of fifth type can be proved in a similar way as Theorem 3.4.

Theorem 3.6. *Additionally to the structural assumptions* (A1)–(A3) *require that*

- *$B : Y^* \to Y$ is C^1 and satisfies $\langle dB(u)v, v\rangle \geq c\|v\|_Y^2$ for all $u, v \in Y$ with a constant $c > 0$.*
- *$A : X \to X^*$ is a potential operator,*
- *$f \in L^2(0, T; Y^*)$.*

Then there exists to every initial value $u_0 \in X \cap Y$ a strong solution u of equation (1) *in the sense that u is a weak solution which additionally satisfies $u \in L^\infty(0, T; X)$, has the initial value $u_0 \in X \cap Y$ and a weak derivative $\frac{\partial u}{\partial t} \in L^2(0, T; Y)$.*

3.6. Strong solutions of fourth type. Strong solutions of the fourth type are related to differentiable operators B satisfying $dB(0) = 0$ and

$$\langle dB(u)v, v\rangle \geq c(\|u\|_Y)\|dB(u)v\|_{Y^*}^2 \tag{10}$$

for all $u, v \in Y$, $u \neq 0$, with a decreasing function $c : \mathbb{R}^+ \to \mathbb{R}^+$ of $\|u\|_Y$.

However, this inequality does not guarantee that $dB(u)$ has an inverse, so that local solvability in time of the approximate equation (8) has to be shown by a different method, namely by discretization in time like in [13, 11.2]. To use this method we additionally require that A is a monotone p-coercive potential operator. But before we discretize, let us first discuss the right hand side f, which in this case should be a nonlinearity like in the case of strong solutions of first type.

Let us assume that f is a nonlinearity induced by an inhomogeneity $g \in L^2(0,T;Y)$ via $f(u) := dB(u)g$. Note that $dB(u) = dB(u)^*$, because dB is the second order derivative of the C^2-function Φ_B. Further, $f(u) \in L^2(0,T;Y^*)$ holds for every $u \in L^\infty(0,T;Y)$, because $u \in L^\infty(0,T;Y)$ implies $dB(u) \in L^\infty(0,T;L(Y,Y^*))$ due to

$$c(\|u\|_Y)\|dB(u)v\|_{Y^*}^2 \leq \langle dB(u)v, v\rangle \leq \|dB(u)v\|_{Y^*}\|v\|_Y$$

for all $v \in Y$ by inequality (10), i.e., the inequality $\|dB(u)v\|_{Y^*} \leq \frac{1}{c(\|u\|_Y)}\|v\|_Y$ and hence $\|dB(u)\|_{L(Y,Y^*)} \leq \frac{1}{c(\|u\|_Y)}$ is valid.

Moreover, if the function $c(\|u\|_Y)$ from inequality (10) does not decrease faster than $\frac{1}{1+\|u\|_Y^{m-2}}$ as $\|u\|_Y \to \infty$, then $f(u)$ automatically satisfies assumptions similar to those of Remark 1.4, so that weak solutions of equation (1) exist. Indeed, apply inequality (10) to $v = g$ to obtain

$$c(\|u\|_Y)\|f(u)\|_{Y^*}^2 = c(\|u\|_Y)\|dB(u)g\|_{Y^*}^2 \leq \langle dB(u)g, g\rangle = \langle f(u), g\rangle \leq \|f(u)\|_{Y^*}\|g\|_Y$$

and thus $\|f(u)\|_{Y^*} \leq C\|g\|_Y(1 + \|u\|_Y^{m-2})$.

Now let us consider the time-discretization

$$dB(u_{l-1})\frac{u_l - u_{l-1}}{h} + Au_l = f(u_{l-1}) = dB(u_{l-1})g_l \tag{11}$$

of the new approximate equation (8) in W. By p-coercivity of A a solution of the time-discretized equation exists for every step size $h > 0$. In fact, inequality (10) guarantees $\langle dB(u_{l-1})v, v\rangle \geq 0$, and hence the operator on the left hand side of the equation $(hA + dB(u_{l-1}))u_l = dB(u_{l-1})(hg_l + u_{l-1})$ is p-coercive due to

$$\langle (hA + dB(u_{l-1}))v, v\rangle \geq h\langle Av, v\rangle \geq h\|v\|_X^p .$$

Denote by u_h the piecewise affine interpolant to the points u_l, by $\bar{u}_h$ the piecewise constant interpolant and by $\bar{u}_h^R$ the retarded piecewise constant interpolant. Then u_h is bounded in $L^\infty(0,T;Y)$ due to the discrete analogon

$$\begin{aligned}\frac{\hat{\Phi}_B(u_l) - \hat{\Phi}_B(u_{l-1})}{h} &\leq \langle dB(u_{l-1})\frac{u_l - u_{l-1}}{h}, u_l\rangle + \langle Au_l, u_l\rangle \\ &= \langle f(u_{l-1}), u_l\rangle \\ &\leq C\|g_l\|_Y(1 + \|u_{l-1}\|_Y^{m-2})\|u_l\|_Y\end{aligned}$$

of testing the new approximate equation by u, where $\hat{\Phi}_B$ denotes the Legendre transform of a convex potential Φ_B of B in dependence of u. Here $\hat{\Phi}_B(u) \geq c\|u\|_Y^m$ is satisfied for a constant $c > 0$, and hence $\|u_l\|_Y \leq C$ holds for all l with a constant $C < \infty$.

Moreover, if A is additionally assumed to be monotone, then u_h is bounded in $L^\infty(0,T;X)$ and $dB(\bar{u}_h^R)\frac{\partial u_h}{\partial t}$ is bounded $L^2(0,T;Y^*)$, as is shown by the discrete analogon

$$\begin{aligned}
&c(\|u_{l-1}\|_Y)\left\|dB(u_{l-1})\frac{u_l-u_{l-1}}{h}\right\|_{Y^*}^2+\frac{\Phi_A(u_l)-\Phi_A(u_{l-1})}{h}\\
&\le\left\langle dB(u_{l-1})\frac{u_l-u_{l-1}}{h},\frac{u_l-u_{l-1}}{h}\right\rangle+\left\langle Au_l,\frac{u_l-u_{l-1}}{h}\right\rangle\\
&=\left\langle f(u_{l-1}),\frac{u_l-u_{l-1}}{h}\right\rangle\\
&=\left\langle dB(u_{l-1})g,\frac{u_l-u_{l-1}}{h}\right\rangle\\
&\le\frac{\epsilon^2}{2}\left\|dB(u_{l-1})\frac{u_l-u_{l-1}}{h}\right\|_{Y^*}^2+\frac{1}{2\epsilon^2}\|g_l\|_Y^2\,.
\end{aligned}$$

of testing the new approximate equation by $\frac{\partial u}{\partial t}$. Note that $u_h\in L^\infty(0,T;Y)$ implies $dB(\bar{u}_h^R)\in L^\infty(0,T;L(Y,Y^*))$, hence also $\frac{\partial u_h}{\partial t}$ is bounded in $L^2(0,T;Y)$.

Thus, there is a subsequence of u_{h_n} such that $u_{h_n}\overset{*}{\rightharpoonup}u$ in $L^\infty(0,T;X\cap Y)$ and $\frac{\partial u_{h_n}}{\partial t}\rightharpoonup(\frac{\partial u}{\partial t})_{ex}$ in $L^2(0,T;Y)$ as $h_n\to 0$ (in the following we suppress the index n), and it is easy to verify that $(\frac{\partial u}{\partial t})_{ex}=\frac{\partial u}{\partial t}$.

Let us write the discretized equation (11) as

$$dB(\bar{u}_h^R)\frac{\partial u_h}{\partial t}+A\bar{u}_h=f(\bar{u}_h^R)=dB(\bar{u}_h^R)\bar{g}_h\,.$$

It remains to verify, that we can form the limit $h\to 0$ in this equation.

By $u_h\in L^\infty(0,T;X\cap Y)$, compactness of $X\cap Y\subset Y$ and Aubins-Lions' lemma a subsequence of u_h convergences strongly to u in $L^q(0,T;Y)$ for an arbitrary index $q<\infty$. Because $\|\bar{u}_h^R-u_h\|_{L^2(0,T;Y)}\le Ch\|\frac{\partial u_h}{\partial t}\|_{L^2(0,T;Y)}$ and $\frac{\partial u_h}{\partial t}$ is uniformly bounded in $L^2(0,T;Y)$ w.r.t. h, also $\bar{u}_h^R\to u$ in $L^2(0,T;Y)$. Thus even $\bar{u}_h^R\to u$ in $L^q(0,T;Y)$ by interpolation, and in a similar way $\bar{u}_h\to u$ in $L^q(0,T;Y)$.

Now $u\mapsto dB(u)$ is a mapping from $L^q(0,T;Y)$ to $L^{\frac{q}{m-2}}(0,T;L(Y,Y^*))$ due to the validity of $\|dB(u)\|_{L(Y,Y^*)}\le\frac{1}{c(\|u\|_Y)}\le C(1+\|u\|_Y^{m-2})$, and $dB(\bar{u}_h^R)\to dB(u)$ in $L^{\frac{q}{m-2}}(0,T;L(Y,Y^*))$ by continuity of $dB:Y\to L(Y,Y^*)$ and compactness of $X\cap Y\subset Y$. Further, $\frac{\partial u_h}{\partial t}\rightharpoonup\frac{\partial u}{\partial t}$ weakly in $L^2(0,T;Y)$, so that $dB(\bar{u}_h^R)\frac{\partial u_h}{\partial t}\rightharpoonup dB(u)\frac{\partial u}{\partial t}$ weakly in $L^{\frac{2q}{q-2(m-2)}}(0,T;Y)$ provided that q is chosen so large that $2q\ge q-2(m-2)$. By construction $\bar{g}_h\rightharpoonup g$, and by (pseudo)monotonicity of A also $A\bar{u}_h\rightharpoonup Au$. Thus, every term in the discrete version of the new approximate equation converges to the expected limit, and hence $u\in W^{1,p}(0,T;W)$ solves the new approximate equation (8).

Further, although dB is not bounded, the function $dB(u)$ is bounded in time, and therefore the chain rule $\frac{\partial Bu}{\partial t} = dB(u)\frac{\partial u}{\partial t}$ holds, i.e., $Bu \in L^\infty(0,T;Y^*)$ has a weak derivative $\frac{\partial Bu}{\partial t} \in L^2(0,T;Y^*)$ and the former formula holds. Therefore, u satisfies also the original approximate equation (4), i.e., $\frac{\partial Bu}{\partial t} + Au = f(u)$ holds. However, the former calculations additionally proved the existence of $\frac{\partial u}{\partial t}$ and $\frac{\partial Bu}{\partial t} = dB(u)\frac{\partial u}{\partial t}$.

Thus, as in the other cases it is possible to test the approximate equation by $\frac{\partial u}{\partial t}$ to conclude

$$c\left\|\frac{\partial Bu}{\partial t}\right\|_{Y^*}^2 + \frac{d}{dt}\Phi_A(u) \le \frac{\epsilon^2}{2}\left\|\frac{\partial Bu}{\partial t}\right\|_{Y^*}^2 + \frac{1}{2\epsilon^2}\|g\|_Y^2,$$

and hence a priori estimates of $\frac{\partial Bu}{\partial t}$ in $L^2(0,T;Y^*)$ and of u in $L^\infty(0,T;X)$ are valid. Therefore, the following theorem about the existence of strong solutions of fourth type holds.

Theorem 3.7. *Additionally to the structural assumptions* (A1)–(A3) *require that*

- *$B : Y^* \to Y$ is C^1, satisfies $\|u\|_Y \le C\|Bu\|_{Y^*}^{m'-1}$ with a constant $C < \infty$, and is uniformly monotone in the sense that $dB(0) = 0$ and $\langle dB(u)v, v\rangle \ge c(\|u\|_Y)\|dB(u)v\|_{Y^*}^2$ holds for all $u, v \in Y$, $u \ne 0$, with a decreasing function $c : \mathbb{R}^+ \to \mathbb{R}^+$ of $\|u\|_Y$ which does not decrease faster than $\frac{1}{1+\|u\|_Y^{m-2}}$ as $\|u\|_Y \to \infty$,*
- *$A : X \to X^*$ is a monotone p-coercive potential operator,*
- *$f(u) := dB(u)g$ for a $g \in L^2(0,T;Y)$.*

Then there exists to every initial value $u_0 \in X \cap Y$ a strong solution u of equation (1) *in the sense that u is a weak solution which additionally satisfies $u \in L^\infty(0,T;X)$ and $\frac{\partial Bu}{\partial t} \in L^2(0,T;Y^*)$.*

Due to $\frac{\partial Bu}{\partial t}, f \in L^2(0,T;Y^*)$ also $Au = f - \frac{\partial Bu}{\partial t}$ lies in $L^2(0,T;Y^*)$, and equation (1) holds as an equation in Y^* for a.e. $t \in [0,T]$. Especially, $u(t) \in D(A)$ for a.e. $t \in [0,T]$, where the domain of A w.r.t. Y^* is defined by $D(A) := \{u \in X \,|\, Au \in Y^*\}$.

Another consequence is that $Bu \in W^{1,2}(0,T;Y^*) \subset C(0,T;Y^*)$. Thus, if B^{-1} is continuous, then also $u \in C(0,T;Y)$, and $u \in L^\infty(0,T;X)$ implies $u \in C(0,T;(X,\text{weak}))$.

Example 3.8. Consider the prototypical equation (2), where $Y = L^m(\Omega)$ and $Bu = u^{m-1}$. Thus, if $m \ge 2$, then B has the Frechet-derivative

$dB(u) = (m-1)|u|^{m-2}$ at u. Therefore, by Hölder inequalities

$$\begin{aligned}\|dB(u)v\|_{m'}^{m'} &= (m-1)^{m'} \int_\Omega (|u|^{m-2}|v|)^{m'} dx \\ &= (m-1)^{m'} \int_\Omega (|u|^{m-2}|v|^2)^{m'/2} |u|^{(m-2)\frac{m'}{2}} dx \\ &\le (m-1)^{m'} \left(\int_\Omega |u|^{m-2}|v|^2 dx \right)^{\frac{m'}{2}} \left(\int_\Omega |u|^m \, dx \right)^{\frac{m-2}{2(m-1)}}\end{aligned}$$

and due to $\langle dB(u)v, v\rangle = (m-1)\int_\Omega |u|^{m-2}|v|^2 \, dx$ hence

$$\langle dB(u)v, v\rangle \ge \frac{\|dB(u)v\|_{m'}^2}{(m-1)\|u\|_m^{m-2}} .$$

Thus $\langle dB(u)v, v\rangle \ge c(\|u\|_m)\|dB(u)v\|_{m'}^2$ holds with the function $c(\|u\|_m) = \frac{1}{m-1}\|u\|_m^{2-m}$. Since $X := W_0^{1,p}(\Omega)$ is compactly embedded into $Y = L^m(\Omega)$ for a bounded domain iff $m < p^*$, in the case $2 \le m < p^*$ the prototypical equation (2) admits a strong solution of fourth type.

4. Conclusion

The aim of this article was to discuss existence of strong solutions of doubly nonlinear parabolic equations. Six different situations have been identified, and in each of these situations it has been shown that there are solutions of equation (1) which satisfy $u \in L^\infty(0,T;X)$ and have other better properties than ordinary weak solutions. The most important types of strong solutions are what we call the first, second and fourth type, as these imply the validity of equation (1) in Y^* (resp. H^*).

As a consequence, for a given operator A higher regularity of solutions may be proved by embedding the domain $D(A) \subset X$ of $A : D(A) \to Y^*$ (resp. H^*) into an appropriate space. Further, uniqueness of strong solutions may be proved in an elementary way by the method of [8] (see also [13, 11.2.3] and [11]), while it is much harder to prove uniqueness of weak solutions, see [12].

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