An Inhomogeneous, L^2-Critical, Nonlinear Schrödinger Equation

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Abstract. An inhomogeneous nonlinear Schrödinger equation is considered, which is invariant under L^2-scaling. The sharp condition for global existence of H^1-solutions is established, involving the L^2-norm of the ground state of the stationary equation. Strong instability of standing waves is proved by constructing self-similar solutions blowing up in finite time.

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1. Introduction

The purpose of this note is to point out the case of an inhomogeneous nonlinear Schrödinger equation having L^2-scaling invariance. Namely, we consider the Cauchy problem

$$i\partial_t\phi + \Delta\phi + |x|^{-b}|\phi|^{2\sigma}\phi = 0, \quad \phi(0,\cdot) = \phi_0 \in H^1(\mathbb{R}^N) \qquad \text{(NLS)}$$

with $\sigma = \frac{2-b}{N}$, in any dimension $N \geq 1$. Here and henceforth, $H^1(\mathbb{R}^N)$ denotes the Sobolev space of complex-valued functions $H^1(\mathbb{R}^N, \mathbb{C})$, with its usual norm. We suppose that $0 < b < \min\{2, N\}$. The case $b = 0$ is the classical (focusing) nonlinear Schrödinger equation with L^2-critical nonlinearity. In the above setting, it turns out that (NLS) is also invariant under the L^2-scaling

$$\begin{aligned} \phi \to \phi_\lambda(t, x) &:= \lambda^{\frac{N}{2}}\phi(\lambda^2 t, \lambda x) \\ \phi_0 \to (\phi_0)_\lambda(x) &:= \lambda^{\frac{N}{2}}\phi_0(\lambda x) \end{aligned} \qquad \text{for } \lambda > 0. \qquad (1)$$

We came across this inhomogeneous critical nonlinearity for (NLS) while studying stability of standing waves for some classes of nonlinear Schrödinger

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equations, where (NLS) arises both as a model and a limiting case, see [5,6,8]. In particular, the Cauchy problem (NLS) is studied there, and it is found that, for $0 < b < \min\{2, N\}$, it is well-posed in $H^1(\mathbb{R}^N)$,

- locally if $0 < \sigma < \widetilde{2} := \begin{cases} \frac{2-b}{N-2} & \text{if } N \geq 3 \\ \infty & \text{if } N \in \{1, 2\}; \end{cases}$
- globally for small initial conditions if $0 < \sigma < \widetilde{2}$;
- globally for any initial condition in $H^1(\mathbb{R}^N)$ if $0 < \sigma < \frac{2-b}{N}$.

Theorem 2.5 below answers the natural question: in the limit case $\sigma = \frac{2-b}{N}$, how small should the initial condition be to have global existence? We consider here strong solutions $\phi = \phi(t, x) \in C_t^0 H_x^1([0, T) \times \mathbb{R}^N)$ for some $T > 0$, and the notion of well-posedness as defined in [3]. Our notation for the space-time function spaces comes from [15]. We may simply denote by $\phi(t) \in H^1(\mathbb{R}^N)$ the function $x \to \phi(t, x)$. The solution is called global (in time) if we can take $T = \infty$. If it is not the case, the blow-up alternative states that $\|\phi(t)\|_{H^1} \to \infty$ as $t \uparrow T$. Moreover, we have conservation of the L^2-norm along the flow,

$$\|\phi(t)\|_{L_x^2} = \|\phi_0\|_{L_x^2}, \quad t \in [0, T),$$

and of the energy,

$$E(\phi(t)) := \int_{\mathbb{R}^N} |\nabla \phi(t)|^2 \, \mathrm{d}x - \frac{1}{\sigma+1} \int_{\mathbb{R}^N} |x|^{-b} |\phi(t)|^{2\sigma+2} \, \mathrm{d}x = E(\phi_0), \quad t \in [0, T). \quad (2)$$

Also, the L^2-norm of $\phi(t)$ is invariant under the transformation (1), i.e.,

$$\|\phi(t)\|_{L_x^2} = \|\phi_\lambda(t)\|_{L_x^2}, \quad t \in [0, T).$$

This is why it is called the L^2-scaling.

A standing wave for (NLS) is a (global) solution of the form $\varphi_\omega(t, x) = e^{i\omega^2 t} u_\omega(x)$ for some $\omega \in \mathbb{R}$, with $u_\omega \in H^1(\mathbb{R}^N)$ satisfying the stationary equation

$$\Delta u - \omega^2 u + |x|^{-b} |u|^{2\sigma} u = 0. \quad (\mathrm{E}_\omega)$$

In [5,6,8], we were concerned with bifurcation and orbital stability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities of the form $V(x)|\phi|^{2\sigma}\phi$ with $V(x) \sim |x|^{-b}$ at infinity or around the origin. These equations have important applications in nonlinear optics (see [6]). The limiting problem (NLS) turned out to play a central role in our analysis. For this model case, a global branch of positive solutions of (E_ω) is simply given by the mapping $u \in C^1((0, \infty), H^1(\mathbb{R}^N))$,

$$\omega \mapsto u_\omega(x) = u(\omega)(x) := \omega^{\frac{2-b}{2\sigma}} u_1(\omega x), \quad (3)$$

where u_1 is the unique positive radial solution (ground state) of (E$_\omega$) with $\omega = 1$. The existence of the ground state is proved in [5, 8] by variational methods in dimension $N \geq 2$, and in [6] for $N = 1$. Uniqueness is a delicate problem, handled in dimension $N \geq 3$ by a theorem of Yanagida [19] (see [5]), in dimension $N = 2$ by a shooting argument [7], and in dimension $N = 1$ by the method of horizontal separation of graphs of Peletier and Serrin [12], as used in [16]. These existence and uniqueness results hold for $0 < b < \min\{2, N\}$ and $0 < \sigma < \widetilde{2}$.

Using the general theory of orbital stability of Grillakis, Shatah and Strauss [10], we obtained in [5, 6, 8] various stability/instability results for general nonlinearities $V(x)|\phi|^{2\sigma}\phi$ by studying the monotonicity of the L^2-norm of the standing waves, as a function of $\omega > 0$. It turned out that $\sigma = \frac{2-b}{N}$ is a threshold for stability in the regimes we considered. For this value of σ, we could not determine if the standing waves are stable or not, even in the model case $V(x) = |x|^{-b}$. In fact, if $\sigma = \frac{2-b}{N}$, we have $\|u_\omega\|_{L^2} = \|u_1\|_{L^2}$ along the curve of solutions (3), for u_ω is then an L^2-scaling of u_1. In Section 3, we prove a strong instability result for standing waves of (NLS), without requiring that u_ω be the ground state of (E$_\omega$).

Section 2 is devoted to a sharp global existence result in the spirit of Weinstein [17]. For $\sigma = \frac{2-b}{N}$ we prove that the solutions of (NLS) are global in time provided $\|\phi_0\|_{L^2} < \|\psi\|_{L^2}$, where ψ is the ground state of (E$_1$). This is done by computing the best constant for an interpolation inequality. The sharpness of the result is proved in Section 3 where we construct self-similar solutions blowing up in finite time, in particular with the critical mass $\|\psi\|_{L^2}$.

Related results for inhomogeneous nonlinear Schrödinger equations can be found in the literature, see for instance [1, 4, 11, 14]. However, no one seems to have noticed the possibility of L^2-scaling invariance. The results established here use basic ideas going back to [17, 18]. The classical L^2-critical case ($b = 0$) has been studied extensively, and in particular the properties of the blow-up solutions are well-known (see [13] for a survey). The case $b \neq 0$ certainly deserves further investigation.

Notation. In Section 2 we work in the Sobolev space of real-valued functions $H := H^1(\mathbb{R}^N, \mathbb{R})$. We use the shorthand notation $\| \cdot \|_p := \| \cdot \|_{L^p}$ for the usual Lebesgue norms throughout.

2. Critical mass and global existence

We start by solving the minimization problem

$$\inf_{u \in H \setminus \{0\}} J(u) \tag{4}$$

where $J : H \setminus \{0\} \to \mathbb{R}$ is the Weinstein functional defined by

$$J(u) = J_{N,b}(u) = \frac{\|\nabla u\|_2^2 \|u\|_2^{2\sigma}}{I(u)} \quad \text{for } \sigma = \frac{2-b}{N}, \tag{5}$$

with

$$I(u) = \int_{\mathbb{R}^N} |x|^{-b} |u|^{2\sigma+2} \, \mathrm{d}x. \tag{6}$$

Lemma 2.1. *For $N \geq 1$, $0 < b < \min\{2, N\}$ and $0 < \sigma < \widetilde{2}$, the functional $I : H \to \mathbb{R}$ defined in (6) is of class $C^1(H, \mathbb{R})$ and is weakly sequentially continuous. In particular, it follows that $J \in C^1(H \setminus \{0\}, \mathbb{R})$.*

Proof. See [8, Section 2.1] and [5, Section 1.1] for $N \geq 2$, [6, Section 2] for $N = 1$. $\qquad\qquad\square$

Proposition 2.2. *Let $N \geq 1$, $0 < b < \min\{2, N\}$ and $\sigma = \frac{2-b}{N}$. There exists a positive radial function $\psi \in H$ such that:*

(i) *ψ is a minimizer for (4), that is, $J_{N,b}(\psi) = \inf_{u \in H \setminus \{0\}} J_{N,b}(u)$;*

(ii) *ψ is the unique ground state of $(\mathrm{E}_{\sqrt{\sigma}})$. Furthermore, the minimum value is $J_{N,b}(\psi) = \frac{\|\psi\|_2^{2\sigma}}{\sigma+1} = \frac{\|\psi\|_2^{\frac{4-2b}{N}}}{\frac{2-b}{N}+1}$.*

Proof. We follow Weinstein [17]. Let $\{u_n\} \subset H \setminus \{0\}$ be a minimizing sequence for (4):

$$J(u_n) \to m := \inf J \geq 0 \quad \text{as } n \to \infty.$$

Clearly, we can choose $u_n \geq 0$. Moreover, by Schwarz symmetrization (see [8, p. 146]) we can suppose that u_n is radial and radially non-increasing for all n. It follows from the structure of $J = J_{N,b}$ that J is invariant under the scaling $u \to u_{\lambda,\mu}(x) := \lambda u(\mu x)$, $\lambda, \mu > 0$. (This is not the case for $\sigma \neq \frac{2-b}{N}$.) This allows us to choose u_n such that $\|\nabla u_n\|_2 = \|u_n\|_2 = 1$ for all n. Hence there exists $u^* \in H$ such that, up to a subsequence, $u_n \rightharpoonup u^*$ weakly in H. Furthermore, u^* is non-negative, spherically symmetric, radially non-increasing, with

$$\|\nabla u^*\|_2 \leq 1 \quad \text{and} \quad \|u^*\|_2 \leq 1. \tag{7}$$

Now by Lemma 2.1 and (7) we have

$$m = \lim J(u_n) = \lim \frac{1}{I(u_n)} = \frac{1}{I(u^*)} \geq J(u^*) \tag{8}$$

so that, in fact, $J(u^*) = m$ and $\|\nabla u^*\|_2 = \|u^*\|_2 = 1$. In particular, $u_n \to u^*$ strongly in H. (Note that (8) prevents $u^* = 0$.) This concludes the proof of (i).

To show that ψ can be chosen so as to satisfy $(\mathrm{E}_{\sqrt{\sigma}})$, we first remark that u^* is a solution of the Euler-Lagrange equation corresponding to (4), which reads

$$\Delta u^* - \sigma u^* + m(\sigma + 1)|x|^{-b}(u^*)^{2\sigma+1} = 0.$$

Setting $u^* = [m(\sigma + 1)]^{-\frac{\sigma}{2}}\psi$, it follows that ψ is a solution of $(E_{\sqrt{\sigma}})$. Furthermore, ψ is positive and radial, so it is the unique ground state of $(E_{\sqrt{\sigma}})$.   □

As an immediate consequence we have

Corollary 2.3. $C_{N,b} := \dfrac{\frac{2-b}{N}+1}{\|\psi\|_2^{\frac{4-2b}{N}}}$ *is the best constant for the inequality*

$$\int_{\mathbb{R}^N} |x|^{-b}|u|^{\frac{4-2b}{N}+2}\,\mathrm{d}x \leq C\|\nabla u\|_2^2\|u\|_2^{\frac{4-2b}{N}}, \quad u \in H. \tag{9}$$

Remark 2.4. Note that (9) is a special case of the interpolation inequalities obtained in [2].

We now turn to the global existence result.

Theorem 2.5. *Set $\sigma = \frac{2-b}{N}$ and let ψ be the ground state of (E_1). If*

$$\|\phi_0\|_2 < \|\psi\|_2,$$

the solution of (NLS) *is global and bounded in H^1.*

Proof. Local existence of solutions to (NLS) is ensured by results in [3] (see [8, Appendix K] for precise statements and references). So the maximal solution $\phi(t,x)$ of (NLS) with initial condition ϕ_0 is defined on a time interval $[0,T)$ with $T \in (0,\infty]$. Moreover, we have the conservation laws

$$E(\phi(t)) = E(\phi_0) \quad \text{and} \quad \|\phi(t)\|_2 = \|\phi_0\|_2 \quad \text{for all } t \in [0,T),$$

where E is defined in (2). It is well-known since [9] that the boundedness of $\|\nabla\phi(t)\|_2$ is then sufficient to conclude global existence. Using the constants of motion, we have

$$\|\nabla\phi(t)\|_2^2 = E(\phi(t)) + \frac{1}{\sigma+1}\int_{\mathbb{R}^N} |x|^{-b}|\phi(t)|^{2\sigma+2}\,\mathrm{d}x$$

$$\leq E(\phi_0) + \frac{C}{\sigma+1}\|\nabla\phi(t)\|_2^2\|\phi_0\|_2^{2\sigma},$$

where $C = C_{N,b} > 0$ is the constant given by Corollary 2.3. Hence,

$$\left(1 - \frac{C_{N,b}}{\frac{2-b}{N}+1}\|\phi_0\|_2^{\frac{4-2b}{N}}\right)\|\nabla\phi(t)\|_2^2 \leq E(\phi_0). \tag{10}$$

Using the formula for $C_{N,b}$, it follows from (10) that the solution is global if $\|\phi_0\|_2 < \|\psi\|_2$ where ψ is the ground state of $(E_{\sqrt{\sigma}})$. But for $\sigma = \frac{2-b}{N}$, $(E_{\sqrt{\sigma}})$ is transformed into (E_1) by the scaling

$$\psi \to \psi_{\lambda^{-1}}(x) = \lambda^{-\frac{N}{2}}\psi(\lambda^{-1}x) \quad \text{with } \lambda = \sqrt{\sigma}.$$

Since this transformation leaves the L^2-norm unchanged, we can indeed choose ψ to be the ground state of (E_1). The proof is complete.   □

Remark 2.6. We call $\|\psi\|_2$ the *critical mass* for (NLS). As we show below, the condition for global existence given by Theorem 2.5 is sharp in the sense that we can find solutions with critical mass which blow up in finite time.

3. Instability of standing waves

It is a lenghty but straightforward calculation to show that (NLS) is invariant under the pseudo-conformal transformation, as defined in [3, Section 6.7]. Namely, for any $a \in \mathbb{R}$, if $\phi(s,y) \in C_s^0 H_y^1([0,S) \times \mathbb{R}^N)$ is a solution to (NLS) (with the obvious modification of the variables), then the function $\phi_a(t,x) \in C_t^0 H_x^1([0,T) \times \mathbb{R}^N)$ defined by

$$\phi_a(t,x) = (1 - at)^{-\frac{N}{2}} \mathrm{e}^{-i\frac{a|x|^2}{4(1-at)}} \phi\left(\frac{t}{1-at}, \frac{x}{1-at}\right), \tag{11}$$

$$T = \begin{cases} \infty & \text{if } aS \leq -1 \\ \frac{S}{1+aS} & \text{if } aS > -1, \end{cases}$$

is also a solution. The fact that (NLS) with $\sigma = \frac{2-b}{N}$ behaves nicely under (11) when $b > 0$ is closely related to the L^2-scaling invariance of the equation. In fact, the pseudo-conformal transformation conserves the L^2-norm:

$$\|\phi_a(t)\|_2 \equiv \|\phi(s)\|_2.$$

We shall now use this transformation to show that all standing waves for (NLS) with $\sigma = \frac{2-b}{N}$ are strongly unstable in the following sense. We only consider the case $\omega = 1$ for simplicity of notation.

Theorem 3.1. *Let $u \in H$ be a nontrivial solution of (E_1). For any $\delta > 0$ there exists a solution $\varphi \in C_t^0 H_x^1([0,T) \times \mathbb{R}^N)$ of (NLS) such that $\|\varphi(0) - u\|_{H^1} < \delta$ and $\|\varphi(t)\|_{H^1} \to \infty$ as $t \uparrow T$.*

Proof. Let $a > 0$ to be tuned later. We apply the transformation (11) to the standing wave $\phi(t,x) = \mathrm{e}^{it} u(x)$, defining $\varphi \in C_t^0 H_x^1([0,a^{-1}) \times \mathbb{R}^N)$ by ($S = \infty$ for ϕ):

$$\varphi(t,x) = (1 - at)^{-\frac{N}{2}} \mathrm{e}^{-i\frac{a|x|^2}{4(1-at)}} \mathrm{e}^{i\frac{t}{1-at}} u\left(\frac{x}{1-at}\right). \tag{12}$$

It is easy to check that $(1 - at)\|\nabla\varphi(t)\|_2 \to \|\nabla u\|_2$ as $t \uparrow a^{-1}$ and so φ blows up at finite time $T := a^{-1}$. Furthermore, $\varphi(0,x) = \mathrm{e}^{-i\frac{a|x|^2}{4}} u(x)$ and we have:

$$\|\varphi(0) - u\|_2^2 = \int_{\mathbb{R}^N} |\mathrm{e}^{-i\frac{a|x|^2}{4}} - 1|^2 u(x)^2 \, \mathrm{d}x \tag{13}$$

$$\text{and} \quad \|\nabla\varphi(0) - \nabla u\|_2^2 = \int_{\mathbb{R}^N} |\mathrm{e}^{-i\frac{a|x|^2}{4}} - 1|^2 |\nabla u(x)|^2 + \frac{a^2}{4}|x|^2 u(x)^2 \, \mathrm{d}x. \tag{14}$$

It is standard to show that u decays exponentially and it follows by dominated convergence that both (13) and (14) go to zero as $a \to 0$. Hence, for any $\delta > 0$, there is $a_\delta > 0$ such that $\|\varphi(0) - u\|_{H^1} < \delta$ whenever $0 < a < a_\delta$. This concludes the proof. □

Remark 3.2.   (i) We know precisely the blow-up rate of φ,

$$\|\varphi(t)\|_{H^1} \sim (1 - at)^{-1} \quad \text{and} \quad \|\varphi(t)\|_\infty \sim (1 - at)^{-\frac{N}{2}} \quad \text{as } t \uparrow a^{-1}.$$

(ii) The type of solutions constructed in (12) are often called 'self-similar' in the literature. In fact, the modulus $|\varphi(t, x)| = (1 - at)^{-\frac{N}{2}} \left| u \left(\frac{x}{1-at} \right) \right|$ presents a self-similar profile in the usual sense: at any time t, there is a scaling parameter $\lambda(t) > 0$ such that $|u(x)| = \lambda(t)^{\frac{N}{2}} |\varphi(t, \lambda(t)x)|$. Thus $|\varphi(t)|$ retains the shape of $|u|$ while blowing up.

Corollary 3.3. *There exists a solution of* (NLS) *with critical mass that blows up in finite time.*

Proof. Take φ defined by (12) with $u = \psi$, the ground state of (E_1). □

Remark 3.4. Note that (12) yields blow-up solutions with self-similar profiles corresponding to any solution of (E_1). In particular, it follows from Theorem 2.5 that ψ is the solution of (E_1) with minimal L^2-norm; the corresponding result is well-known in the case $b = 0$.

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