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On the Unique Solvability of Certain Nonlinear Singular Partial Differential Equations

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Abstract. We study the singular nonlinear equation $tu_t = F(t,x,u,u_x)$, where the function F is assumed to be continuous in t and holomorphic in the other variables. Under some growth conditions on the coefficients of the partial Taylor expansion of F, we show that if $F(t,x,0,0)$ is of order $O(\mu(t)^\alpha)$ for some $\alpha \in [0,1]$ as $t \to 0$ uniformly in some neighborhood of $x = 0$, then the equation has a unique solution $u(t,x)$ with the same growth order.

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1. Statement of the problem

Given any $r > 0$, we denote by B_r be the closed disc $\{x \in \mathbb{C};\ |x| \le r\}$ and by D_r the closed polydisc $\{x \in \mathbb{C}^n;\ |x| \le r\}$, where $|x| = \max_{1\le i\le n} |x_i|$. Let $T > 0$, $0 < R < 1$, $\rho > 0$ and set $\Omega = [0,T] \times D_R \times B_\rho \times D_\rho$.

We say that $\mu(t)$ is a *weight function* on $[0,T]$ if it is a continuous, nonnegative, increasing function on $(0,T)$ such that $\frac{\mu(t)}{t}$ is integrable on $(0,T)$. Note that such a function must satisfy $\lim_{t\to 0}\mu(t) = 0$. Examples of weight functions are t^δ, $\frac{1}{(-\log t)^{1+\delta}}$ and $\frac{1}{(-\log t)(\log(-\log t))^\delta}$ for any positive δ. We use the name coined by Tahara in [12]. Similarly defined functions have been referred to as *Dini functions* in [5,11].

Let $(t,x) \in \mathbb{R} \times \mathbb{C}^n$. We are interested in the singular nonlinear partial differential equation
$$t\frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right), \tag{1}$$

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where the function $F(t,x,u,v)$ is a continuous function on Ω and is holomorphic in the variables (x,u,v) for any fixed t, and $\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$. Gérard and Tahara [3, 4] have conducted extensive investigations on this singular equation which is modeled after the ordinary differential equation first studied by Briot and Bouquet in 1856. Assuming that $F(t,x,u,v)$ is holomorphic with respect to all the variables, they have proved the unique existence of a holomorphic as well as a type of singular solution to (1). They also provided extensions of their results to higher order nonlinear singular equations.

Meanwhile, assuming holomorphy in x but only continuity in t, Nagumo type existence and uniqueness theorems for linear singular equations of general order have also been established by Baouendi and Goulaouic [1] in 1973. In 1999, Lope [8] extended their results using weight functions and showed that the two essentially proved the case corresponding to $\mu(t) = t^\delta$. It was also pointed out in [8] that the integrability of $\frac{\mu(t)}{t}$ is a crucial property as it turned out to be almost necessary for the unique solvability of a class of higher order linear Fuchsian equations.

In addition to the above-mentioned results, there are also results for nonlinear equations under different growth assumptions due again to Baouendi and Goulaouic [2] and for nonlinear systems due to Koike [7]. In the former paper, the weight function used was $\mu(t) = t$, while in the latter, a stronger integrability condition was imposed on the function $\mu(t)$.

We point out that in this work, we not only give an existence and uniqueness theorem for nonlinear equations, we also provide a growth order estimate in terms of $\mu(t)^\alpha$, where the power α can be 0, 1 or anywhere in between. Work is currently being undertaken to extend our results to higher order equations.

Now let $\mu(t)$ be any weight function and $\alpha \in [0,1]$. Set $a(t,x) = F(t,x,0,0)$ and $\lambda(t,x) = F_u(t,x,0,0)$. We shall study (1) under the following assumptions:

(A_1) $a(t,x)$ and $a_{x_i}(t,x)$ for $1 \le i \le n$ are both bounded by $A\mu(t)$ on $[0,T]\times D_R$

(A_2) $F_{v_i}(t,x,0,0) = O(\mu(t))$ (as $t \to 0$) for $1 \le i \le n$ uniformly on D_R

(A_3) $\operatorname{Re}\lambda(t,x) \le -c$ on $[0,T] \times D_R$ for some $c > 0$

(A_4) For all $1 \le i,j \le n$, $F_{uv_i}(t,x,u,v)$ and $F_{v_iv_j}(t,x,u,v)$ are of order $O(\mu(t)^{1-\alpha})$ (as $t \to 0$) uniformly on $D_R \times B_\rho \times D_\rho$.

Since $\mu(t)$ is a weight function, the function $\varphi(t) = \int_0^t \mu(\tau)\,\frac{d\tau}{\tau}$ is well-defined on $[0,T]$. For any $r > 0$, let the region W_r be given by

$$W_r = \left\{(t,x);\ 0 \le t \le T \text{ and } \frac{\varphi(t)}{r} + |x| < R\right\}. \tag{2}$$

Note that the size of the region W_r also depends on T, although we will not explicitly indicate this in our notation for the sake of simplicity. We define two spaces on W_r: the space $X_0(W_r)$ is composed of all functions in $C^0(W)$ that are holomorphic in x for any fixed t, while the space $X_1(W_r)$ is composed of all functions in $C^1(W_r \cap \{t > 0\}) \cap C^0(W_r)$ that are also holomorphic in x for any fixed t. Observe that if $r_1 < r_2$, then $W_{r_1} \subset W_{r_2}$ and $X_j(W_{r_2}) \subset X_j(W_{r_1})$ for $j = 0, 1$.

The following is our main result.

Theorem 1.1 (Main Theorem)**.** *Suppose* (A_1)–(A_4) *hold. If* $\alpha \in (0,1]$ *and* T *is sufficiently small, or if* $\alpha = 0$ *and both* T *and* A *are sufficiently small, then there exists an* $r > 0$ *such that* (1) *has a unique solution* $u(t,x) \in X_1(W_r)$ *that satisfies*

$$|u(t,x)| \leq \frac{4A}{c}\mu(t)^\alpha \quad \textit{and} \quad \max_{1\leq i\leq n}\left\{\left|\frac{\partial u}{\partial x_i}(t,x)\right|\right\} \leq \rho\left(\frac{\mu(t)}{\mu(T)}\right)^\alpha \quad \textit{on } W_r. \quad (3)$$

Remark 1.2. The modifier "sufficiently small" in the statement of the Main Theorem will be made precise in Section 3.

Example 1.3. Let $(t,x) \in \mathbb{R} \times \mathbb{C}$, $\mu(t) = t$ and consider the equation

$$\left(t\frac{\partial}{\partial t} + 1\right)u + tx^2\frac{\partial u}{\partial x} = (1 + tx^2)u^2 - x^2u^2\frac{\partial u}{\partial x}. \quad (4)$$

Note that $F(t,x,0,0) \equiv 0$, which is of order $O(t^\alpha)$ for any $\alpha \in (0,1]$ as $t \to 0$. Note further that $u \equiv 0$ and $u(t,x) = 1 + tx$ are both solutions to (4) but the unique solution being referred to in our theorem is the former, because the latter is not of order $O(t^\alpha)$ for any $\alpha \in (0,1]$ as $t \to 0$.

Example 1.4. Let $(t,x) \in \mathbb{R} \times \mathbb{C}$, $\mu(t)$ be any weight function and consider the equation

$$\left(t\frac{\partial}{\partial t} + \frac{5}{8}\right)u + \mu(t)\frac{\partial u}{\partial x} = \frac{1}{16} + g(t,x) + u^2,$$

where the function $g(t,x)$ is continuous in t, holomorphic in x and is of order $O(\mu(t)^\beta)$ for some $\beta \in (0,1]$ as $t \to 0$ uniformly in a neighborhood of $x = 0$. Note that for small values of t, the quantity $\frac{1}{16} + g(t,x)$ is bounded by $\frac{1}{8}$. Our theorem guarantees the unique existence of the solution $u(t,x)$ whose magnitude is no more than $\frac{4(\frac{1}{8})}{\frac{5}{8}} = \frac{4}{5}$. This solution is in fact given by $\frac{1}{8} + w(t,x)$, where $w(t,x)$ is the unique solution with order $O(\mu(t)^\beta)$ of the equation

$$\left(t\frac{\partial}{\partial t} + \frac{3}{8}\right)w + \mu(t)\frac{\partial w}{\partial x} = g(t,x) + w^2.$$

The unique existence of such $w(t,x)$ is guaranteed again by Theorem 1.1.

2. Preliminaries

We rewrite (1) as

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x)\right) u = a(t,x) + \sum_{i=1}^{n} b_i(t,x)\frac{\partial u}{\partial x_i} + f(t,x,u) + G\left(t,x,u,\frac{\partial u}{\partial x}\right), \quad (5)$$

where $f(t,x,u) = \sum_{m=2}^{\infty} \frac{\partial^m F}{\partial u^m}(t,x,0,0)\frac{u^m}{m!}$ is holomorphic in (x,u) and $G(t,x,u,v)$ is the sum of terms in the partial Taylor expansion of $F(t,x,u,v)$ whose degree with respect to (u,v) is at least 2.

In view of the holomorphy of $F(t,x,u,v)$ with respect to (x,u,v) and Assumptions (A_1)–(A_4), there exist positive constants such that estimates (B_1)–(B_3) hold:

(B_1) $\max_{1\le i\le n}\{|b_i(t,x)|\} \le B\mu(t)$ on $[0,T]\times D_R$

(B_2) $\left|\frac{\partial \lambda}{\partial x}(t,x)\right| \le \Lambda$ on $[0,T]\times D_R$

(B_3) $\left|\frac{\partial^2 F}{\partial u \partial v_i}\right| \le B_{1,1}\mu(t)^{1-\alpha}$ and $\left|\frac{\partial^2 F}{\partial v_i \partial v_j}\right| \le B_{0,2}\mu(t)^{1-\alpha}$ for $1\le i,j\le n$ on Ω.

Here and in the following, $\left|\frac{\partial g}{\partial x}\right|$ denotes the quantity $\max_{1\le i\le n}\left|\frac{\partial g}{\partial x_i}\right|$. The constants appearing above shall not be used in any other context throughout the manuscript so as to avoid confusion.

Given $w \in X_0(W_r)$, we define

$$\Phi[w] = \sum_{i=1}^{n} b_i(t,x)\frac{\partial w}{\partial x_i} + G\left(t,x,w,\frac{\partial w}{\partial x}\right).$$

In view of (B_1) and (B_3), we have the following estimate for the modulus of the difference $\Phi[w_1] - \Phi[w_2]$.

Lemma 2.1. *Let $w_j(t,x)$ $(j=1,2)$ be in $X_0(W_r)$. If for some $\alpha \in [0,1]$,* both $|w_j|$ and $\left|\frac{\partial w_j}{\partial x}\right|$ are bounded by $\rho\left(\frac{\mu(t)}{\mu(T)}\right)^{\alpha}$ *on W_r then*

$$\left|\Phi[w_1] - \Phi[w_2]\right| \le \sum_{i=1}^{n} B\mu(t)\left|\frac{\partial w_1}{\partial x_i} - \frac{\partial w_2}{\partial x_i}\right| + \frac{nB_{1,1}\rho}{\mu(T)^{\alpha}}\mu(t)\,|w_1 - w_2|$$
$$+ \sum_{i=1}^{n} \frac{(B_{1,1} + nB_{0,2})\rho}{\mu(T)^{\alpha}}\mu(t)\left|\frac{\partial w_1}{\partial x_i} - \frac{\partial w_2}{\partial x_i}\right| \quad \text{on } W_r.$$

The following lemma provides a bound for the derivative of a holomorphic function in terms of the function itself. Walter [13] attributes a lemma of this type to Nagumo. It may be proved in the same way as Lemma 5.1.3 in [6].

Lemma 2.2. *Let $w(t,x) \in X_0(W_r)$ for some $r > 0$ and $\Psi(t)$ be a nonnegative function. If for $j = 0$ or 1, we have*

$$|w(t,x)| \le \frac{\Psi(t)(R-|x|)^j}{R-|x|-\frac{\varphi(t)}{r}} \quad \text{on } W_r,$$

then

$$\left|\frac{\partial w}{\partial x}(t,x)\right| \le \frac{4\Psi(t)(R-|x|)^j}{\left(R-|x|-\frac{\varphi(t)}{r}\right)^2} \quad \text{on } W_r.$$

The next lemma states some elementary results on linear Fuchsian equations.

Lemma 2.3. *Suppose* (A$_3$) *holds. For any $g(t,x) \in X_0(W_r)$, the equation*

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x)\right) w = g(t,x) \tag{6}$$

has a unique solution $w(t,x) \in X_1(W_r)$, and it is given by

$$w(t,x) = \int_0^t \exp\left(\int_\tau^t \lambda(s,x)\,\frac{ds}{s}\right) g(\tau,x)\,\frac{d\tau}{\tau}\,. \tag{7}$$

Moreover, the following estimates hold on W_r given any nondecreasing, nonnegative function $\psi(t)$:

(a) *If $|g(t,x)| \le M\psi(t)$, then $|w(t,x)| \le \dfrac{M}{c}\psi(t)$.*

(b) *If $|g(t,x)| \le \dfrac{M\mu(t)\psi(t)}{\left(R-|x|-\frac{\varphi(t)}{r}\right)^2}$, then $|w(t,x)| \le \dfrac{Mr\psi(t)}{R-|x|-\frac{\varphi(t)}{r}}$.*

(c) *If $|g(t,x)| \le \dfrac{M\mu(t)\psi(t)(R-|x|)}{\left(R-|x|-\frac{\varphi(t)}{r}\right)^2}$, then $|w(t,x)| \le \dfrac{M\varphi(t)\psi(t)}{R-|x|-\frac{\varphi(t)}{r}}$.*

Proof. The integral representation (7) of the solution is easily verified, and the estimate in (a) follows from it. To prove (b), we use the fact that $\varphi'(t) = \frac{\mu(t)}{t}$ and estimate as follows:

$$\begin{aligned}
|w(t,x)| &\le \int_0^t \left(\frac{\tau}{t}\right)^c \frac{M\mu(\tau)\psi(\tau)}{\left(R-|x|-\frac{\varphi(\tau)}{r}\right)^2}\frac{d\tau}{\tau} \\
&\le M\psi(t)\int_0^t \frac{\varphi'(\tau)}{\left(R-|x|-\frac{\varphi(\tau)}{r}\right)^2}d\tau \\
&= Mr\psi(t)\left(\frac{1}{R-|x|-\frac{\varphi(t)}{r}} - \frac{1}{R-|x|}\right) \\
&\le \frac{Mr\psi(t)}{R-|x|-\frac{\varphi(t)}{r}},
\end{aligned}$$

after simply ignoring the nonnegative subtrahend. As for (c), we estimate as in (b) but instead of dropping the subtrahend, we make use of the presence of $(R - |x|)$ to cancel the unwanted term in the denominator. $\square$

Let us now consider the semilinear version of (6) given by

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x)\right) w = g(t,x) + h(t,x,w). \tag{8}$$

The function $h(t,x,w)$ is assumed to be continuous on $\omega = \{(t,x,w) \in \mathbb{R} \times \mathbb{C}^n \times \mathbb{C};\ 0 \le t \le T,\ |x| \le R,\ |w| \le \rho\}$ and holomorphic in (x,w) for each t. Moreover, we will assume that there are constants $L_1, L_2 > 0$ such that if on W_r, we have $|w(t,x)| \le J\mu(t)^\alpha \le \rho$ for some $J > 0$ and $\alpha \in [0,1]$, then these must also hold on W_r:

(H_1) $\big|h\big(t,x,w(t,x)\big)\big| \le L_1 \cdot \big[J\mu(t)^\alpha\big]^2$;

(H_2) $\left|\dfrac{\partial h}{\partial w}(t,x,w)\right| \le L_2 \cdot J\mu(t)^\alpha$.

The following proposition shows that, as in the linear case, this is uniquely solvable and its solution satisfies some estimates. It will be used in the construction of approximate solutions to (1).

Proposition 2.4. *Suppose* (A_3), (H_1) *and* (H_2) *hold. Suppose further that* $g(t,x) \in X_0(W_r)$*, and for some* $\alpha \in [0,1]$*, we have* $|g(t,x)| \le M\mu(t)^\alpha$ *on* W_r*. If* T *or* M *is small enough so that*

$$M\mu(T)^\alpha \le \min\left\{\frac{c^2}{4L_1}, \frac{c^2}{4L_2}, \frac{\rho c}{2}\right\}, \tag{9}$$

then Equation (8) *has a unique solution* $w(t,x) \in X_1(W_r)$ *that satisfies*

$$|w(t,x)| \le \frac{2M}{c}\mu(t)^\alpha \quad \text{on } W_r. \tag{10}$$

Proof. We construct approximate solutions to (8) as follows:

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x)\right) w_0 = g(t,x),$$
$$\left(t\frac{\partial}{\partial t} - \lambda(t,x)\right) w_k = g(t,x) + h(t,x,w_{k-1}) \quad (k \ge 1).$$

For $k \ge 1$, we define $v_k(t,x) = w_k(t,x) - w_{k-1}(t,x)$. We claim that for all $k \ge 0$, the following hold:

$$|w_k(t,x)| \le \frac{2M}{c}\mu(t)^\alpha \quad \text{and} \quad |v_{k+1}(t,x)| \le \frac{M}{c}\frac{\mu(t)^\alpha}{2^{k+1}}. \tag{11}$$

With these estimates, it is easy to see that the approximate solutions converge to a solution $w(t,x)$ of (8), and that, by construction, $|w(t,x)| \le \frac{2M}{c}\mu(t)^\alpha$ on W_r. Note that (9) ensures that all the approximate solutions fall within the domain of definition of $h(t,x,w)$.

We now prove the estimates in (11) by induction. Let us consider the case when $k=0$. Since $|g(t,x)| \le M\mu(t)^\alpha$ on W_r, we have $|w_0(t,x)| \le \frac{M}{c}\mu(t)^\alpha$ by Lemma 2.3(a). To proceed to $w_1(t,x)$, we consider the difference $v_1 = w_1 - w_0$. Note that $\big(t\frac{\partial}{\partial t} - \lambda(t,x)\big)\, v_1(t,x) = h(t,x,w_0)$, where, by ($H_1$) and (9), the right-hand side is bounded by $\frac{c}{2}\frac{M}{c}\mu(t)^\alpha$. Thus, by Lemma 2.3(a) again, we have

$$|v_1(t,x)| \le \frac{M}{2c}\mu(t)^\alpha \quad \text{on } W_r.$$

From this, we see that $|w_1(t,x)| \le |w_0(t,x)| + |v_1(t,x)| \le \frac{2M}{c}\mu(t)^\alpha$ on W_r.

Suppose the claim is true for all $k = 0,1,\ldots,j$. We now show that it is also true when $k=j+1$. The difference $v_{j+1} = w_{j+1} - w_j$ satisfies

$$\begin{aligned}\left(t\frac{\partial}{\partial t} - \lambda(t,x)\right) v_{j+1}(t,x) &= h(t,x,w_j) - h(t,x,w_{j-1}) \\ &= v_j(t,x)\cdot \int_0^1 \frac{\partial h}{\partial w}(t,x,w_{j-1}+sv_j)\,ds.\end{aligned} \tag{12}$$

Applying the induction hypothesis, we see that $|w_{j-1}+sv_j| \le (1-s)|w_{j-1}|+s|w_j| \le \frac{2M}{c}\mu(t)^\alpha \le \rho$. Thus, by ($H_2$) and (9), the modulus of the right-hand side is at most $M\frac{\mu(t)^\alpha}{2^j}$ on W_r. By Lemma 2.3(a), we see that

$$|v_{k+1}(t,x)| \le \frac{M}{c}\frac{\mu(t)^\alpha}{2^{j+1}} \quad \text{on } W_r.$$

This then leads to an estimate for $|w_{k+1}|$, namely,

$$|w_{k+1}(t,x)| \le |w_0(t,x)| + \sum_{i=1}^{k+1} |v_i(t,x)| \le \frac{2M}{c}\mu(t)^\alpha \quad \text{on } W_r.$$

This completes the induction, and also the construction of a solution $w(t,x)$ satisfying (10).

To prove the uniqueness of the solution, we suppose that w and u are two solutions of (8) in $X_1(W_r)$ satisfying (10) on W_r. Using the same technique as in (12), we have

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x) - \int_0^1 \frac{\partial h}{\partial w}\big(t,x,u+s(w-u)\big)\,ds\right)(w-u) = 0.$$

Note that $\operatorname{Re}\big(\lambda(t,x) + \int_0^1 h_w\big(t,x,u+s(w-u)\big)\,ds\big) \le -\frac{c}{2}$ on W_r, thanks again to (H_2). We finish off by applying Lemma 2.3 to obtain $w \equiv u$ on W_r. □

3. Proof of Main Theorem

3.1. Existence of a solution. We will prove the existence of a solution by the method of successive approximations. We define the approximate solutions as follows:

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x)\right) u_0 = a(t,x) + f(t,x,u_0) \tag{13}$$

and for $k \geq 1$,

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x)\right) u_k = a(t,x) + \Phi[u_{k-1}] + f(t,x,u_k). \tag{14}$$

In the following, we will establish their domains of existence and prove their convergence to the desired solution using the method of Nirenberg [9] and Nishida [10].

Recall that by definition, $f(t,x,u) = u^2\widetilde{f}(t,x,u)$ where $\widetilde{f}$ is also holomorphic with respect to (x,u), so (H_1) and (H_2) are obviously satisfied. We now pose some restrictions on A and T in order to apply Proposition 2.4. Suppose $\widetilde{f}$, $\widetilde{f}_{x_i}$ $(1 \leq i \leq n)$ and $\widetilde{f}_u$ are bounded by K_1, K_2 and K_3, respectively, on ω. We require that these hold:

$$\left(1 + \frac{\Lambda}{c}\right) \cdot \frac{8A}{c}\mu(T)^\alpha \leq \rho, \tag{15}$$

$$\max\left\{\frac{K_2}{2}, 2K_1 + K_3\rho\right\} \cdot \frac{4A}{c}\mu(T)^\alpha \leq \frac{c}{2}, \tag{16}$$

$$\left(1 + \frac{\Lambda}{c}\right) \cdot \frac{8C_1}{c}\mu(T) \leq 1, \tag{17}$$

where

$$C_1 = nB + (2nB_{1,1} + n^2B_{0,2})\frac{\rho}{\mu(T)^\alpha}.$$

Condition (9) is implied by (15) and (16). Moreover, (15) ensures that the constructed approximate solutions are in the domain of definition of f and G. Note that if $\alpha \in (0,1]$, choosing a small T will ensure that all three conditions hold. However, if $\alpha = 0$, we also have to choose a small A. With these conditions, we have made precise what we meant by "sufficiently small" in the statement of the main theorem.

Playing an essential role in the proof of convergence is a decreasing sequence $\{r_k\}_{k\geq 0}$ of numbers tending to a positive limit r_∞. Let K_4 be a bound for the holomorphic function $\frac{\partial^2 f}{\partial u^2}$ on ω. We define the sequence by

$$\begin{aligned} r_0 &< \frac{1}{2C}, \\ r_k &= r_{k-1}\big(1 - (2Cr_0)^k\big), \end{aligned} \tag{18}$$

where

$$C = C_1 \left(4 + \frac{\Lambda + K_4 \rho}{\frac{c}{2}} \right). \tag{19}$$

By our choice of r_0, the series $\sum_{k \geq 1} (2Cr_0)^k$ is convergent, and so r_∞ is well-defined and positive.

Applying Proposition 2.4 to (13) and using the fact that $|a(t,x)| \leq A\mu(t)^\alpha$ on $[0,T] \times D_R$, we obtain a unique $u_0(t,x) \in X_1(W_{r_0})$ satisfying

$$|u_0(t,x)| \leq \frac{2A}{c} \mu(t)^\alpha \quad \text{on } W_{r_0}. \tag{20}$$

Differentiating (13) with respect to x_i, we see that

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x) - \frac{\partial f}{\partial u}(t,x,u_0) \right) \frac{\partial u_0}{\partial x_i} = \frac{\partial a}{\partial x_i} + \frac{\partial \lambda}{\partial x_i} w + \frac{\partial f}{\partial x_i}(t,x,u_0). \tag{21}$$

In view of (16) and (20), we see that $\operatorname{Re}\big(\lambda(t,x) + f_u(t,x,u_0)\big) \leq -\frac{c}{2} < 0$, and that $\left|\frac{\partial f}{\partial x_i}(t,x,u_0)\right| \leq \frac{c}{2}\frac{2A}{c}\mu(t)^\alpha = A\mu(t)^\alpha$. Applying Lemma 2.3(a) to (21), we obtain

$$\left| \frac{\partial u_0}{\partial x}(t,x) \right| \leq \frac{A + 2A\frac{\Lambda}{c} + A}{\frac{c}{2}} \mu(t)^\alpha = \left(1 + \frac{\Lambda}{c} \right) \frac{4A}{c} \mu(t)^\alpha \quad \text{on } W_{r_0}. \tag{22}$$

Observe that by (15), both $|u_0|$ and $\left|\frac{\partial u_0}{\partial x}\right|$ are bounded by $\frac{\rho}{2}$ on W_{r_0}.

As for the estimates of the succeeding approximate solutions, we have the following proposition.

Proposition 3.1. *For $k \geq 1$, the following are true:*

(a) *There exists a unique $u_k \in X_1(W_{r_{k-1}})$ satisfying* (14) *and*

$$|u_k(t,x)| \leq \frac{4A}{c} \mu(t)^\alpha \quad \text{on } W_{r_{k-1}}.$$

(b) *On $W_{r_{k-1}}$, the difference $u_k - u_{k-1}$ is bounded by*

$$|u_k - u_{k-1}| \leq \frac{(Cr_0)^{k-1} C_1 \varphi}{R - |x| - \frac{\varphi(t)}{r_{k-1}}} \cdot \frac{4A}{c} \mu(t)^\alpha.$$

(c) *On $W_{r_{k-1}}$, the partial derivatives of the difference are bounded by*

$$\left| \frac{\partial}{\partial x}(u_k - u_{k-1}) \right| \leq \frac{(Cr_0)^{k-1} C_1 \varphi}{R - |x| - \frac{\varphi(t)}{r_{k-1}}} \left(4 + \frac{\Lambda + K_4 \rho}{\frac{c}{2}} \right) \cdot \frac{4A}{c} \mu(t)^\alpha.$$

(d) *On W_{r_k}, we have*

$$\max\left\{|u_k - u_{k-1}|, \left|\frac{\partial}{\partial x}(u_k - u_{k-1})\right|\right\} \leq \frac{1}{2^k} \cdot \frac{4A}{c}\mu(t)^\alpha$$

and thus

$$|u_k| \leq \frac{8A}{c}\mu(t)^\alpha \quad \textit{and} \quad \left|\frac{\partial u_k}{\partial x}\right| \leq \left(1 + \frac{\Lambda}{c}\right)\frac{8A}{c}\mu(t)^\alpha \quad \textit{on } W_{r_k}.$$

Before proceeding to the proof, we note that the existence of a solution follows from this proposition. Observe that

$$u(t,x) = \lim_{k\to\infty} u_k(t,x) = u_0(t,x) + \lim_{k\to\infty}\sum_{j=1}^{k}(u_j - u_{j-1})(t,x),$$

where the sum is convergent in W_{r_∞} because of the first set of estimates in (d). Moreover, the limit function must also satisfy $|u(t,x)| \leq \frac{4A}{c}\mu(t)^\alpha$, since according to (a) each term of the sequence does. This is sharper than the one in (d) and is used in the statement of the theorem.

Similarly, we have the convergence of the sequence of partial derivatives to $\frac{\partial u}{\partial x_i}$ $(1 \leq i \leq n)$ with $|\frac{\partial u}{\partial x}| \leq 8\big(1 + \frac{\Lambda}{c}\big)\frac{A}{c}\mu(t)^\alpha$, in view again of (d) and (22). Note that Condition (15) implies that both $|u|$ and $|\frac{\partial u}{\partial x}|$ do not exceed ρ and also yields the estimate for $|\frac{\partial u}{\partial x}|$ that is given in Theorem 1.1.

Finally, since each $u_k(t,x)$ may be expressed in the form

$$u_k(t,x) = \int_0^t \exp\left(\int_\tau^t \lambda(s,x)\,\frac{ds}{s}\right)\big[a(\tau,x) + \Phi[u_{k-1}](\tau,x) + f(\tau,x,u_k)\big]\,\frac{d\tau}{\tau},$$

we simply take the limit as $k \to \infty$ to we deduce that $u(t,x) \in X_1(W_{r_\infty})$ and that it does solve (5).

Proof of Proposition 3.1. The proof is by induction. Recall that u_1 satisfies

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x)\right)u_1 = a(t,x) + \Phi[u_0] + f(t,x,u_1). \tag{23}$$

Using Lemma 2.1, the estimates in (20) and (22), and the condition on T in (17),

$$|a(t,x) + \Phi[u_0]| \leq A\mu(t)^\alpha + C_1\mu(t)\cdot\left(1 + \frac{\Lambda}{c}\right)\frac{4A}{c}\mu(t)^\alpha \leq \frac{2A}{c}\mu(t)^\alpha,$$

so by Proposition 2.4, there exists a unique $u_1 \in W_{r_0}$ satisfying (23) with $|u_1(t,x)| \leq \frac{4A}{c}\mu(t)^\alpha$. Thus (a) holds when $k = 1$.

Subtracting (13) from (23) yields

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x) - \int_0^1 \frac{\partial f}{\partial u}\big(t,x,u_0+s(u_1-u_0)\big)\,ds\right)(u_1-u_0) = \Phi[u_0]. \tag{24}$$

Using the bounds for u_0 and u_1, we see that $|u_0+s(u_1-u_0)| \le (1-s)|u_0|+s|u_1| \le \frac{4A}{c}\mu(t)^\alpha$, which, combined with (16), tells us that the modulus of the integral is at most $\frac{c}{2}$. Therefore, if we let $\widetilde{\lambda} = \lambda(t,x) + \int_0^1 f_u\big(t,x,u_0+s(u_1-u_0)\big)\,ds$, we have $\operatorname{Re}\widetilde{\lambda}(t,x) \le -c+\frac{c}{2} = -\frac{c}{2}$. The right-hand side satisfies

$$|\Phi[u_0]| \le \frac{4A}{c}C_1\mu(t)^{1+\alpha}\cdot\frac{R-|x|}{R-|x|} \le \frac{4A}{c}\cdot\frac{C_1\mu(t)^{1+\alpha}(R-|x|)}{R-|x|-\frac{\varphi(t)}{r_0}} \quad \text{on } W_{r_0}, \tag{25}$$

so by Lemma 2.3(c),

$$|u_1-u_0| \le \frac{C_1\varphi}{R-|x|-\frac{\varphi(t)}{r_0}}\cdot\frac{4A}{c}\mu(t)^\alpha \quad \text{on } W_{r_0}. \tag{26}$$

This proves (b).

To estimate the partial derivatives of the difference, we differentiate (13) and (23) with respect to x_i, and subtract the former from the latter to obtain $\big(t\frac{\partial}{\partial t}-\lambda(t,x)\big)\frac{\partial(u_1-u_0)}{\partial x_i} = (u_1-u_0)\frac{\partial\lambda}{\partial x_i} + \frac{\partial}{\partial x_i}\Phi[u_0] + \frac{\partial f}{\partial u}(t,x,u_1)\frac{\partial u_1}{\partial x_i} - \frac{\partial f}{\partial u}(t,x,u_0)\frac{\partial u_0}{\partial x_i}$. Dropping the variables t and x for brevity, we rewrite this as

$$\begin{aligned}&\left(t\frac{\partial}{\partial t} - \lambda - \frac{\partial f}{\partial u}(u_1)\right)\frac{\partial(u_1-u_0)}{\partial x_i}\\ &= \frac{\partial}{\partial x_i}\Phi[u_0] + (u_1-u_0)\left(\frac{\partial\lambda}{\partial x_i} + \frac{\partial u_0}{\partial x_i}\int_0^1 \frac{\partial^2 f}{\partial u^2}\big(u_1+s(u_0-u_1)\big)\,ds\right).\end{aligned}$$

Using the bound for u_1 and (16), we see again that the real part of $\lambda + f_u(u_1)$ is no more than $-\frac{c}{2}$. We bound the first term on the right-hand side by using Lemma 2.2 on (25); we use (26) and the previously introduced constants for the second term. Following the computations in Lemma 2.3, we see that

$$\begin{aligned}\left|\frac{\partial}{\partial x}(u_1-u_0)\right| &\le \frac{4AC_1}{c}\mu(t)^\alpha\int_0^t\left(\frac{\tau}{t}\right)^{\frac{c}{2}}\left[\frac{4(R-|x|)\mu(\tau)}{\big(R-|x|-\frac{\varphi(\tau)}{r_0}\big)^2} + \frac{(\Lambda+K_4\rho)\varphi(\tau)}{R-|x|-\frac{\varphi(\tau)}{r_0}}\right]\frac{d\tau}{\tau}\\ &\le \frac{C_1\varphi(t)}{R-|x|-\frac{\varphi(\tau)}{r_0}}\left(4+\frac{\Lambda+K_4\rho}{\frac{c}{2}}\right)\cdot\frac{4A}{c}\mu(t)^\alpha,\end{aligned} \tag{27}$$

proving (c). Finally, to prove (d), we note that by our choice of C in (19), we may combine (26) and (27). Recalling (2), we see that on W_{r_1},

$$\begin{aligned}\max\left\{|u_1-u_0|,\left|\frac{\partial}{\partial x}(u_1-u_0)\right|\right\} &\le \frac{C\varphi(t)}{R-|x|-\frac{\varphi(t)}{r_0}}\cdot\frac{4A}{c}\mu(t)^\alpha\\ &\le \frac{Cr_1(R-|x|)}{R-|x|-r_1\frac{R-|x|}{r_0}}\cdot\frac{4A}{c}\mu(t)^\alpha\\ &\le \frac{Cr_1}{(2Cr_0)^1}\cdot\frac{4A}{c}\mu(t)^\alpha\\ &\le \frac{1}{2}\cdot\frac{4A}{c}\mu(t)^\alpha.\end{aligned}$$

We then apply the triangle inequality to obtain bounds for $|u_1|$ and $\left|\frac{\partial u_1}{\partial x}\right|$.

Suppose now that (a)–(d) hold when $k=1,2,\ldots,j$. Let us show that they remain valid when $k=j+1$. By definition, u_{j+1} satisfies

$$\left(t\frac{\partial}{\partial t}-\lambda(t,x)\right)u_{j+1}=a(t,x)+\Phi[u_j]+f(t,x,u_{j+1}). \tag{28}$$

Since $\max\left\{|u_j|,\left|\frac{\partial u_j}{\partial x}\right|\right\}\le\left(1+\frac{\Lambda}{c}\right)\frac{8A}{c}\mu(t)^\alpha\le\rho\left(\frac{\mu(t)}{\mu(T)}\right)^\alpha$ on W_{r_j}, Lemma 2.1 and (17) yield

$$|a(t,x)+\Phi[u_j]|\le A\mu(t)^\alpha+C_1\mu(t)\cdot\left(1+\frac{\Lambda}{c}\right)\frac{8A}{c}\mu(t)^\alpha\le 2A\mu(t)^\alpha.$$

By Proposition 2.4, there exists a unique $u_{j+1}\in W_{r_j}$ satisfying (28) with $|u_{j+1}(t,x)|\le\frac{4A}{c}\mu(t)^\alpha$ on W_{r_j}, proving (a).

To prove (b), we note that the difference $u_{j+1}-u_j$ satisfies on W_{r_j} the equation

$$\left(t\frac{\partial}{\partial t}-\lambda-\int_0^1\frac{\partial f}{\partial u}\big(u_j+s(u_{j+1}-u_j)\big)\,ds\right)(u_{j+1}-u_j)=\Phi[u_j]-\Phi[u_{j-1}],$$

where we have temporarily dropped the variables t and x for brevity. From (a) and the induction hypothesis, $|u_j+s(u_{j+1}-u_j)|\le(1-s)|u_j|+s|u_{j+1}|\le\frac{4A}{c}\mu(t)^\alpha$, so by (16), the modulus of the integral is no more than $\frac{c}{2}$. Therefore, the real part of $\lambda(t,x)+\int_0^1 f_u\big(t,x,u_j+s(u_{j+1}-u_j)\big)\,ds$ is no more than $-\frac{c}{2}$. As for the right-hand side, we use (b) and (c) of the induction hypothesis and Lemma 2.1 to obtain

$$\begin{aligned}|\Phi[u_j]-\Phi[u_{j-1}]| &\le C_1\mu(t)\cdot\frac{(Cr_0)^{j-1}C_1\varphi}{R-|x|-\frac{\varphi(t)}{r_j}}\left(4+\frac{\Lambda+K_4\rho}{\frac{c}{2}}\right)\frac{4A}{c}\mu(t)^\alpha\\ &\le C\mu(t)\cdot\frac{(Cr_0)^{j-1}C_1\varphi}{R-|x|-\frac{\varphi(t)}{r_j}}\frac{4A}{c}\mu(t)^\alpha\quad\text{on } W_{r_j},\end{aligned} \tag{29}$$

where we have made use of the constant C in the second line. This holds on W_{r_j} because while the estimates in (b) and (c) hold on $W_{r_{j-1}}$, the assumptions of Lemma 2.1 are satisfied only on W_{r_j}. Applying now Lemma 2.3(b), recalling in the process that $R<1$, we get the desired estimate:

$$|u_{j+1}-u_j| \leq \frac{Cr_j(Cr_0)^{j-1}C_1\varphi}{R-|x|-\frac{\varphi(t)}{r_j}} \cdot \frac{4A}{c}\mu(t)^\alpha \leq \frac{(Cr_0)^j C_1\varphi}{R-|x|-\frac{\varphi(t)}{r_j}} \cdot \frac{4A}{c}\mu(t)^\alpha \quad \text{on } W_{r_j}. \quad (30)$$

To establish (c), we do as in the case when $k=1$, differentiating (14) with respect to x_i and taking the difference from the previous such equation. We obtain

$$\left(t\frac{\partial}{\partial t}-\lambda-\frac{\partial f}{\partial u}(u_{j+1})\right)\frac{\partial(u_{j+1}-u_j)}{\partial x_i}$$
$$=\frac{\partial}{\partial x_i}(\Phi[u_j]-\Phi[u_{j-1}])+(u_{j+1}-u_j)\left(\frac{\partial\lambda}{\partial x_i}+\frac{\partial u_j}{\partial x_i}\int_0^1\frac{\partial^2 f}{\partial u^2}\big(u_{j+1}+s(u_j-u_{j+1})\big)\,ds\right).$$

Using (16) and the bound for u_{j+1} in (a), we see again that the real part of $\lambda+f_u(u_{j+1})$ is no more than $-\frac{c}{2}$. We bound the first term on the right-hand side by using Lemma 2.2 on (29); we use (30) and the previously introduced constants for the second term. Following the computations in Lemma 2.3, we see that

$$\begin{aligned}
&\left|\frac{\partial}{\partial x}(u_{j+1}-u_j)\right| \\
&\leq \frac{4A}{c}\mu(t)^\alpha(Cr_0)^{j-1}CC_1\varphi(t)\int_0^t\left(\frac{\tau}{t}\right)^{\frac{c}{2}}\left[\frac{4\mu(\tau)}{\left(R-|x|-\frac{\varphi(\tau)}{r_j}\right)^2}+\frac{r_0(\Lambda+K_4\rho)}{R-|x|-\frac{\varphi(\tau)}{r_j}}\right]\frac{d\tau}{\tau} \\
&\leq \frac{(Cr_0)^{j-1}CC_1\varphi(t)}{R-|x|-\frac{\varphi(\tau)}{r_j}}\left(4r_j+r_0\frac{\Lambda+K_4\rho}{\frac{c}{2}}\right)\cdot\frac{4A}{c}\mu(t)^\alpha \\
&\leq \frac{(Cr_0)^j C_1\varphi(t)}{R-|x|-\frac{\varphi(\tau)}{r_j}}\left(4+\frac{\Lambda+K_4\rho}{\frac{c}{2}}\right)\cdot\frac{4A}{c}\mu(t)^\alpha.
\end{aligned} \quad (31)$$

Finally, to prove (d), we use (19) to combine (30) and (31). On $W_{r_{j+1}}$, we have

$$\begin{aligned}
\max\left\{|u_{j+1}-u_j|,\left|\frac{\partial}{\partial x}(u_{j+1}-u_j)\right|\right\} &\leq \frac{(Cr_0)^j C\varphi(t)}{R-|x|-\frac{\varphi(t)}{r_j}}\cdot\frac{4A}{c}\mu(t)^\alpha \\
&\leq \frac{(Cr_0)^j Cr_{j+1}(R-|x|)}{R-|x|-r_{j+1}\frac{R-|x|}{r_j}}\cdot\frac{4A}{c}\mu(t)^\alpha \\
&\leq \frac{(Cr_0)^{j+1}}{(2Cr_0)^{j+1}}\cdot\frac{4A}{c}\mu(t)^\alpha \\
&\leq \frac{1}{2^{j+1}}\cdot\frac{4A}{c}\mu(t)^\alpha.
\end{aligned}$$

The other estimates are obtained by applying the triangle inequality. This concludes the induction, the proof of Proposition 3.1, and the proof of the existence. □

3.2. Uniqueness of the solution. Suppose $u(t,x)$ and $v(t,x)$ are two solutions of (1) in $X_1(W_{r_\infty})$ satisfying (3). We claim that for $k = 0, 1, 2, \dots$ and for any $1 \le i \le n$, we have

$$\max\left\{|u-v|\,, \left|\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right|\right\} \le \frac{2\rho}{\mu(T)^\alpha} \cdot \frac{(Cr_\infty)^k \mu(t)^\alpha}{R - |x| - \frac{\varphi(t)}{r_\infty}} \quad \text{on } W_{r_\infty}.$$

Letting k approach infinity implies that $u \equiv v$ on W_{r_∞} since $Cr_\infty < Cr_0 < \frac{1}{2}$ by our choice of r_0 in (18).

The case when $k = 0$ is clear from (3) and the fact that $R < 1$. Now suppose the claim is true when $k = j$. Employing the same technique as in (24), we see that

$$\left(t\frac{\partial}{\partial t} - \lambda(t,x) - \int_0^1 \frac{\partial f}{\partial u}\big(t,x,u+s(v-u)\big)\,ds\right)(u-v) = \Phi[u] - \Phi[v],$$

where the real part of $\lambda(t,x) + \int_0^1 f_u\big(t,x,u+s(v-u)\big)\,ds$ is at most $-\frac{c}{2}$. This is guaranteed because both u and v satisfy (3) and T has been chosen to satisfy (16). Using Lemma 2.1, the induction hypothesis and $R < 1$ again, the right-hand side may be bounded by

$$|\Phi[u] - \Phi[v]| \le \frac{2\rho}{\mu(T)^\alpha} \cdot \frac{(Cr_\infty)^j C_1 \mu(t)^{1+\alpha}}{R - |x| - \frac{\varphi(t)}{r_\infty}} \le \frac{2\rho}{\mu(T)^\alpha} \cdot \frac{(Cr_\infty)^j C_1 \mu(t)^{1+\alpha}}{\big(R - |x| - \frac{\varphi(t)}{r_\infty}\big)^2}.$$

Thus, by Lemma 2.3(b), we obtain

$$\max\left\{|u-v|, \left|\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\right|\right\} \le \frac{2\rho}{\mu(T)^\alpha} \cdot \frac{(Cr_\infty)^j C_1 r_\infty \mu(t)^\alpha}{R - |x| - \frac{\varphi(t)}{r_\infty}} \le \frac{2\rho}{\mu(T)^\alpha} \cdot \frac{(Cr_\infty)^{j+1} \mu(t)^\alpha}{R - |x| - \frac{\varphi(t)}{r_\infty}},$$

as desired.

Remark 3.2. The bound for the unique solution may be slightly improved if, instead of (17), we first fix a $\delta > 0$ and require T to satisfy $\big(1 + \frac{\Lambda}{c}\big) \cdot \frac{8C_1}{c} \mu(T) \le \delta$. Under this condition, the solution now satisfies

$$|u(t,x)| \le (1+\delta)\frac{2A}{c}\mu(t)^\alpha.$$

It should be noted that δ comes from the nonlinear terms involving the partial derivatives with respect to the "space" variable x. Thus in the special case of Example 1.4 where there is no nonlinearity involving $\frac{\partial u}{\partial x}$, we may take $\delta = 0$ and assert that the solution should be bounded by $\frac{2}{5}$.

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