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A Kinetic Approach in Nonlinear Parabolic Problems with L^1-Data

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Abstract. We consider the Cauchy-Dirichlet problem for a nonlinear parabolic equation with L^1 data. We show how the concept of kinetic formulation for conservation laws introduced by P.-L. Lions, B. Perthame and E. Tadmor [A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Amer. Math. Soc.* 7 (1994), 169 – 191] can be be used to give a new proof of the existence of renormalized solutions. To illustrate this approach, we also extend the method to the case where the equation involves an additional gradient term.

Keywords. Parabolic equations, renormalized solution, kinetic formulation

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1. Introduction

We consider the question of existence of solution to the nonlinear parabolic problem

$$u_t - \operatorname{div}(a(\nabla u)) = f \quad \text{in } \Omega \times (0,T) \tag{1a}$$

$$u = u_0 \quad \text{on } \Omega \times \{0\} \tag{1b}$$

$$u = 0 \quad \text{on } \Sigma, \tag{1c}$$

where Ω is a bounded subset of $\mathbb{R}^N$, $N \geq 1$, T is positive and $\Sigma = \partial\Omega \times (0,T)$. Let $p > 1$ be given. In (1), the operator $-\operatorname{div}(a(\nabla u))$ is assumed to be a Leray-Lions operator of exponent p (for example the p-Laplacian):

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Assumption 1. The function $a \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$ satisfies: There exists $\alpha > 0$, $\beta > 0$ such that

$$a(X) \cdot X \geq \alpha |X|^p \tag{2a}$$

$$|a(X)| \leq \beta |X|^{p-1} \tag{2b}$$

$$(a(X) - a(Y)) \cdot (X - Y) > 0 \tag{2c}$$

for all distinct $X, Y \in \mathbb{R}^N$, where $X \cdot Y$ is the canonical scalar product of two vectors of $\mathbb{R}^N$ and $|X|$ the associated euclidean norm of X.

The framework is L^1:

Assumption 2. The data u_0, f are L^1 functions on Ω and $\Omega \times (0, T)$ respectively.

Remark 1.1. The flux a may depend on x and u. More general problems also may be considered, with additional first-order terms $\operatorname{div}(\Phi(u))$, $\operatorname{div}(g)$ in Equation (1a), as in [5] for example.

The existence of solution (precisely, of renormalized solution, see Definition 2.1 below) to Problem (1) or quite more general problems has already been proved. We refer in particular to the paper by Blanchard, Murat, Redwane [5]. Our purpose here is to give a new proof of this fact. The cornerstone in the proof of existence of solution (by means of a process of approximation) of such a nonlinear parabolic problem as (1) is the proof of the strong convergence of the gradient. We give a new method (inspired from the kinetic formulation of conservation laws developed by Perthame and coauthors [18, 27, 29]) to prove this result.

Let us briefly summarize how and in which context the question of strong convergence of the gradient occurs: First, as soon as the problem under consideration involves a nonlinear function of the gradient. This is for example the case in the following problems:

$$-\Delta u + \gamma(u)|\nabla u|^2 = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{3}$$

or

$$-\operatorname{div}(a(\nabla u)) = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{4}$$

with given right-hand side $g \in L^2(\Omega)$, where a satisfies (2) with $p = 2$, and γ is a bounded continuous function satisfying the sign condition $u\gamma(u) \geq 0$ for all $u \in \mathbb{R}$. Indeed, in order to prove existence of a solution (in $H_0^1(\Omega)$) to (3) or (4), it is usual to prove existence by approximation, for example by Galerkin approximation, thus for a set of data g_n converging to g. Then weak convergence in H_0^1 of possibly a subsequence of u_n, the solution with datum g_n, although easily obtained by uniform estimate on $\|u_n\|_{H^1(\Omega)}$, is not enough to pass to the limit. One has to prove[1] the *strong convergence* of the gradient ∇u_n. This is done by use of monotonicity methods. We refer to [16, 24, 28], and [23] for a brief explanation of the technique.

Nonlinear expressions of the gradient also occur after renormalization of an elliptic or parabolic equation. Note actually that they occur even if the original equation is linear. Nevertheless, renormalization for elliptic or parabolic equation has been introduced to deal with nonlinear equations with data of low regularity, and as a consequence, once renormalized, the equation involves at least two nonlinear expressions of the gradient (see, e.g. Equation (6) below). In any case, it will be necessary to prove the strong convergence of the (truncates of) the gradient in order to get existence of a solution by approximation.

We give a new proof of the strong convergence of the gradient by use of an equation on the characteristic function on the level sets of the unknown, similar to the kinetic formulation for conservation laws introduced in [27] (see also [18, 29] concerning the kinetic formulation of second-order conservation laws). We intend to use it to study certain systems of reaction-diffusion equations (a forthcoming paper).

Let us conclude this introduction by a few words about the concept of renormalized solutions. Introduced by DiPerna and Lions for the study of ordinary differential equations and Boltzmann equation [21, 22], it has been extended to nonlinear elliptic equations in [11] in parallel with the equivalent notion of entropy solution [1] and has been extended to nonlinear parabolic equations in [4, 5, 26], in parallel with the equivalent notion of entropy solution [32]. It has also been extended to first-order conservation laws [2, 31].

The problem of strong convergence of the gradient, hence the question of existence of solution, has initially be solved by the method of Minty-Browder and Leray-Lions [16,24,28], then extended to the case of nonlinear elliptic, then parabolic equations with less and less regular data by several methods, see, e.g. [5–8, 10–12, 19, 20]. Note that this list of references to some works in the field of renormalized solutions for elliptic and parabolic equations is far from being complete.

The paper is organized as follows: In Section 2.1, we introduce the notion of renormalized solution and state the equivalent formulation by the so-called level-set P.D.E. In Section 2.2, we analyze this formulation and explain how it can be relaxed, although still characterizing renormalized solutions, see Theorem 2.5 and Lemma 2.6. In Section 2.3, we apply our tools to prove the convergence of an approximation to Problem (1) and thus existence of a renormalized solution to (1). In Section 3, we give the proofs of various results, which are reported at the end of the paper to let the main arguments of Section 2 stand out. Eventually, in Section 4, we extend the method to prove the existence of a

[1] Actually, in Problem (4) it is sufficient to obtain the weak convergence of $(a(\nabla u_n))$ to $a(\nabla u)$, see Remark 3.3

renormalized solution to the Cauchy-Dirichlet problem for a nonlinear parabolic equation with a term with natural growth.

Notations. We set $Q_T := \Omega \times (-1,T)$ and $U_T := Q_T \times \mathbb{R}$. Any measurable function $v\colon \Omega \times (0,T) \to \mathbb{R}^m$ is implicitly extended to a measurable function $Q_T \to \mathbb{R}^m$ still denoted by v, defined by $v \equiv 0$ on $\Omega \times (-1,0)$.

If ν is a Radon measure on U_T, we denote by ν_* be the push-forward of ν by the projection on $\mathbb{R}_\xi$:

$$\nu_*(E) = \nu(Q_T \times E), \quad \forall\, E \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$. More generally, if E is a topological space, $\mathcal{B}(E)$ denotes the σ-algebra of the Borel subsets of E.

If $q \geq 1$ and V is an open subset of $\mathbb{R}^q$ we denote by $\mathcal{D}(V)$ the set of smooth (C^∞) functions on V compactly supported in V and we denote by $\mathcal{D}'(V)$ the set of distributions on V.

2. Existence of a renormalized solution - strong convergence of the gradient

2.1. Renormalized solutions.

2.1.1. Definition. For $k > 0$, we let $T_k(u)$ be the truncate of a function u at level k: $T_k(u) := \min(u,k)$ if $u \geq 0$, T_k odd.

Definition 2.1. A function $u \in L^\infty(0,T;L^1(\Omega))$ is said to be a renormalized solution of the problem (1) if

1. (Regularity of the truncates)

$$T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega)), \quad \forall\, k > 0. \tag{5}$$

2. (Renormalized equation) For every function $S \in W^{2,\infty}(\mathbb{R})$ with $S(0) = 0$ such that S' has compact support, the equation

$$S(u)_t - \mathrm{div}(S'(u)a(\nabla u)) = S(u_0) \otimes \delta_{t=0} + S'(u)f - S''(u)a(\nabla u)\cdot\nabla u \tag{6}$$

is satisfied in the sense of distributions in Q_T.

3. (Recovering at infinity)

$$\lim_{k\to+\infty} \int_{Q_T\cap\{k<|u|<k+1\}} a(\nabla u)\cdot\nabla u dxdt = 0. \tag{7}$$

2.1.2. Level-set P.D.E. For $\alpha \in \mathbb{R}$, $\xi \in \mathbb{R}$, we set $\chi_\alpha(\xi) = \mathbf{1}_{0<\xi<\alpha} - \mathbf{1}_{\alpha<\xi<0}$. This is the "equilibrium function" in the kinetic formulation of conservation laws [27]. Let $u \in L^\infty(0,T;L^1(\Omega))$ satisfy (5). Then we define the (vector-valued) distribution $a(\nabla u)\delta_{u=\xi}$ on U_T by its restriction to each space $\mathcal{D}_K(U_T)^N$ (the set of smooth vector-valued functions with support in the compact subset K of U_T) as

$$\langle a(\nabla u)\delta_{u=\xi}, \alpha\rangle = \int_{Q_T} a(\nabla T_k(u)) \cdot \alpha(x,t,T_k(u))dxdt, \tag{8}$$

where $\alpha \in \mathcal{D}_K(U_T)^N$, $K \subset Q_T \times [-k,k]$. Similarly, we define the distribution $a(\nabla u)\cdot\nabla u\,\delta_{u=\xi}$ on U_T by

$$\langle a(\nabla u)\cdot\nabla u\,\delta_{u=\xi}, \alpha\rangle = \int_{Q_T} a(\nabla T_k(u))\cdot\nabla T_k(u)\alpha(x,t,T_k(u))dxdt, \tag{9}$$

for all $\alpha \in \mathcal{D}_K(U_T)$. By (5) and assumption (2b), we have

$$\begin{aligned}|\langle a(\nabla u)\delta_{u=\xi}, \alpha\rangle| &\le \|a(\nabla T_k(u))\|_{L^{p'}(Q_T)}\|\alpha\|_{L^p(K)}\\ &\le \beta C(K)\|T_k(u)\|_{L^p(0,T;W_0^{1,p}(\Omega))}\|\alpha\|_{L^\infty(K)},\end{aligned}$$

and

$$|\langle a(\nabla u)\cdot\nabla u\,\delta_{u=\xi}, \alpha\rangle| \le \beta\|T_k(u)\|_{L^p(0,T;W_0^{1,p}(\Omega))}\|\alpha\|_{L^\infty(K)}.$$

This shows that the right-hand sides of (8) and (9) are distributions on U_T of order 0. To prove that (8) and (9) makes sense, we must also show that their respective right-hand sides do not depend on the choice of k: Suppose $k < k'$ for example, with $K \subset Q_T \times [-k,k]$, then $\alpha(x,t,T_{k'}(u)) \neq 0$ for $|u| \le k$ only, in which case $T_k(u) = T_{k'}(u)$.

With these definitions at hand, we can give the "level-set" formulation of Definition 2.1.

Theorem 2.2. *A function $u \in L^\infty(0,T;L^1(\Omega))$ is a renormalized solution of the problem* (1) *if and only if it has the regularity of the truncates* (5) *and satisfies*

1. (*Level-set P.D.E.*) *The function $(x,t,\xi) \mapsto \chi_{u(x,t)}(\xi)$, denoted by χ_u, is solution in $\mathcal{D}'(U_T)$ of the equation*

$$\partial_t\chi_u - \operatorname{div}(a(\nabla u)\delta_{u=\xi}) = \chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi} + \partial_\xi\mu, \tag{10}$$

where μ is defined by

$$\mu := a(\nabla u)\cdot\nabla u\,\delta_{u=\xi}, \tag{11}$$

2. (*Recovering at infinity*)

$$\lim_{k\to+\infty}\int_{Q_T\cap\{k<|u|<k+1\}} a(\nabla u)\cdot\nabla u dxdt = 0. \tag{12}$$

The proof of Theorem 2.2 is given in Section 3.1.

2.2. Relaxation of the definition of renormalized solution - analysis of μ.

2.2.1. Analysis of μ. Let $u \in L^\infty(0,T;L^1(\Omega))$ be a renormalized solution to Problem (1) and let μ be defined by (11). Since $\mu \geq 0$, μ is represented by a nonnegative Radon measure on U_T. We study the properties of the push-forward μ_* of μ: $\mu_*(E) = \mu(Q_T \times E)$, $E \in \mathcal{B}(\mathbb{R})$.

Fact 1. For every $h \in C_c(\mathbb{R})$,

$$\int_{\mathbb{R}} h(\xi) d\mu_*(\xi) = \int_{U_T} h(\xi) d\mu(x,t,\xi). \tag{13}$$

Proof. By definition of μ_*, (13) is satisfied if $h = \mathbf{1}_E$ is the characteristic function of a Borel set $E \subset \mathbb{R}$, and therefore if h is a simple function. There exists a pointwise converging sequence of bounded simple functions with limit h with the same compact support as h. The Lebesgue dominated convergence theorem gives the result. □

Fact 2. For every $h \in C_c(\mathbb{R})$ with, say, $\mathrm{supp}(h) \subset [-k,k]$,

$$\int_{\mathbb{R}} h(\xi) d\mu_*(\xi) = \int_{Q_T} a(\nabla T_k(u)) \cdot \nabla T_k(u) h(u) dx dt. \tag{14}$$

Proof. Let (φ_n) be a nonnegative sequence of $C_c(Q_T)$ such that $\varphi_n \uparrow 1$ everywhere on Q_T. By definition of μ, we have

$$\int_{U_T} \varphi_n(x,t) h(\xi) d\mu(x,t,\xi) = \int_{Q_T} a(\nabla T_k(u)) \cdot \nabla T_k(u) \varphi_n(x,t) h(u) dx dt.$$

The Lebesgue dominated convergence theorem then gives, at the limit $n \to +\infty$,

$$\int_{U_T} h(\xi) d\mu(x,t,\xi) = \int_{Q_T} a(\nabla T_k(u)) \cdot \nabla T_k(u) h(u) dx dt.$$

We conclude by (13). □

Fact 3. The measure μ_* has no atom.

Proof. Given $k > 0$, set $v = T_k(u)$. For $\xi_* \in (-k,k)$, let (h_n) be a sequence of $C_c(-k,k)$ converging monotonically to $\mathbf{1}_{\{\xi_*\}}$ (take the h_n to be tent functions for example). For every n, we have, by (14),

$$\int_{\mathbb{R}} h_n(\xi) d\mu_*(\xi) = \int_{Q_T} a(\nabla v) \cdot \nabla v h_n(v) dx dt.$$

At the limit $n \to +\infty$, we obtain, by the Lebesgue dominated convergence theorem,

$$\mu_*(\{\xi_*\}) = \int_{Q_T} a(\nabla v) \cdot \nabla v \mathbf{1}_{\{\xi_*\}}(v) dx dt. \tag{15}$$

For a.a. t, $v(t) \in W^{1,p}(\Omega)$. For such t's, we have $\nabla v(t) = 0$ a.e. on $\{x \in \Omega, v(x,t) = \xi^*\}$. Indeed, we recall the following property of Sobolev functions (the proof goes back to Stampacchia and can be found in [13]):

Lemma 2.3 (Stampacchia)**.** *Let $w \in W^{1,1}(\Omega)$ and let $Z \subset \mathbb{R}$ be a Borel negligible set, then the set*

$$\{x \in \Omega; w(x) \in Z, \nabla w(x) \neq 0\}$$

is negligible in Ω. In particular, for all $k \in \mathbb{R}$, $\nabla w(x) = 0$ a.e. on $\{w = k\}$.

It follows therefore from (15) that $\mu_*(\{\xi_*\}) = 0$. □

Fact 4. For every $l > k$,

$$\int_{\mathbb{R}} \mathbf{1}_{(k,l)}(\xi) d\mu_*(\xi) = \int_{Q_T \cap \{k<u<l\}} a(\nabla u) \cdot \nabla u dx dt. \tag{16}$$

Proof. In the right hand-side of (16), u stands for $T_m(u)$, $m := \max(|k|, |l|)$. Let (h_n) be a nonnegative sequence of $C_c(k,l)$ such that $h_n \uparrow \mathbf{1}_{(k,l)}$. For each n, we have by (14),

$$\int_{\mathbb{R}} h_n(\xi) d\mu_*(\xi) = \int_{Q_T} a(\nabla u) \cdot \nabla u h_n(u) dx dt.$$

At the limit $n \to +\infty$, the dominated convergence theorem gives the result. □

Fact 5. For $\varphi \in C_c(Q_T)$, $\varphi \geq 0$, define

$$\mu_\varphi(A) := \int_A \varphi(x,t) d\mu(x,t,\xi), \quad \forall\, A \text{ Borel subset of } U_T.$$

The measure μ_φ has the same properties as μ and its analysis follows the same lines. In particular, $\mu_{\varphi,*}$ has no atoms and, for every $k > 0$,

$$\mu_{\varphi,*}([-k,k]) = \mu_{\varphi,*}((-k,k)) = \int_{Q_T} a(\nabla T_k(u)) \cdot \nabla T_k(u) \varphi(x,t) dx dt. \tag{17}$$

Remark 2.4. Note that the proof of the above Facts depends only on the property (5) of the truncates $T_k(u)$. Actually, we may even replace $a(\nabla u)$ by any measurable $\sigma \colon Q_T \to \mathbb{R}^N$, such that $\sigma \mathbf{1}_{|u|<k} \in L^{p'}(Q_T)^N$ for all $k > 0$. This will be used in Paragraph 2.3.3.

2.2.2. Relaxation of the definition of renormalized solution. According to the above Facts (Paragraph 2.2.1), the condition (12) may be rewritten in terms of the push-forward μ_* uniquely as

$$\lim_{k\to\pm\infty} \mu_*((k,k+1)) = 0, \tag{18}$$

where we recall that μ is defined by (11). This simplifies the statement of Theorem 2.2 somewhat. However, what really makes plainer the characterization of renormalized solutions is the fact that, to some extent, it is not necessary to specify μ. This characterization is as follows.

Theorem 2.5. *Let u be a function of $L^\infty(0,T;L^1(\Omega))$ which has the regularity of the truncates* (5) *and satisfies the condition at infinity* (7)*. Then u is a renormalized solution of Problem* (1) *if and only if there exists a nonnegative Radon measure μ on U_T satisfying* (18) *and such that*

$$\partial_t\chi_u - \mathrm{div}(a(\nabla u)\delta_{u=\xi}) = \chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi} + \partial_\xi\mu, \tag{19}$$

in the sense of distributions on U_T.

The proof of Theorem 2.5 consists in showing that $\mu = a(\nabla u)\cdot\nabla u\,\delta_{u=\xi}$. It is therefore a result of structure of μ: Under the hypotheses of Theorem 2.5 and (19), μ *has to* be the measure $a(\nabla u)\cdot\nabla u\,\delta_{u=\xi}$. Theorem 2.5 has the virtue to give a plain characterization of renormalized solutions to (1). However, to prove the convergence of a sequence of approximate solutions to (1) and the existence of solution, we will need a slight generalization of Theorem 2.5 contained in the following lemma.

Lemma 2.6. *Let u be a function of $L^\infty(0,T;L^1(\Omega))$ which has the regularity of the truncates* (5)*. Let σ be a measurable function $\Omega\times(0,T)\to\mathbb{R}^N$ such that $\sigma\,\mathbf{1}_{|u|<k}\in L^{p'}(Q_T)^N$ for all $k>0$. Suppose that there exists a nonnegative Radon measure μ on U_T such that*

$$\lim_{k\to\pm\infty} \mu_*((k,k+1)) = 0, \tag{20}$$

and such that the following equation is satisfied in $\mathcal{D}'(U_T)$

$$\partial_t\chi_u - \mathrm{div}(\sigma\delta_{u=\xi}) = \chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi} + \partial_\xi\mu. \tag{21}$$

Suppose also that either

$$u\geq 0 \ \text{a.e.} \quad \text{and} \quad \mathrm{supp}(\mu)\subset \overline{Q_T}\times[0,+\infty), \tag{22}$$

or, more generally, that the distribution $\sigma\cdot\nabla u\,\delta_{u=\xi}$ satisfies the (*sided*) *condition at infinity*

$$\liminf_{k\to-\infty}\langle \sigma\cdot\nabla u\,\delta_{u=\xi},\varphi\otimes\mathbf{1}_{(k-1,k)}\rangle \leq 0, \quad \forall\,\varphi\in C(\overline{Q_T}),\ \varphi\geq 0. \tag{23}$$

Then $\mu = \sigma\cdot\nabla u\,\delta_{u=\xi}$.

In Lemma 2.6 the definition of the distribution $\sigma \cdot \nabla u\, \delta_{u=\xi}$ is comparable to the definition of the distribution $a(\nabla u) \cdot \nabla u\, \delta_{u=\xi}$ by (9):

$$\langle \sigma \cdot \nabla u\, \delta_{u=\xi}, \alpha \rangle = \int_{Q_T} (\sigma \mathbf{1}_{|u|<k}) \cdot \nabla T_k(u) \alpha(x,t,T_k(u)) dxdt, \tag{24}$$

for all $\alpha \in \mathcal{D}_K(U_T)$, K compact subset of $Q_T \times [-k,k]$.

Equation (21) appears naturally when one considers limits of renormalized solutions, in particular of solutions of approximate equations $u^n_t - \mathrm{div}(a(\nabla u^n)) = f^n$, see Paragraphs 2.3.2 and 2.3.3.

The proof of Lemma 2.6 is given in Section 3.2.

Proof of Theorem 2.5. Lemma 2.6 is actually a generalization of Theorem 2.5. We only have to notice that (23) is satisfied where, here, $\sigma = a(\nabla u)$:

$$\lim_{k\to-\infty} \langle \sigma \cdot \nabla u\, \delta_{u=\xi}, \varphi \otimes \mathbf{1}_{(k-1,k)} \rangle = \lim_{k\to-\infty} \int_{Q_T \cap \{k-1<u<k\}} a(\nabla u) \cdot \nabla u\, \varphi dxdt = 0,$$

where we have used the condition at infinity (7). □

In the situation of Lemma 2.6, once the equality $\mu = \sigma \cdot \nabla u\, \delta_{u=\xi}$ has been proved, and thanks to Remark 2.4, we deduce the following corollary.

Corollary 2.7. *Under the hypotheses of Lemma* 2.6*, and given* $\varphi \in C_c(Q_T)$*,* $\varphi \geq 0$*, the measure* $\mu_{\varphi,*}$ *has no atom and*

$$\mu_{\varphi,*}([-k,k]) = \mu_{\varphi,*}((-k,k)) = \int_{Q_T} \sigma \cdot \nabla T_k(u) \varphi(x,t) dxdt, \tag{25}$$

for all $k > 0$.

2.3. Existence of a renormalized solution - Strong convergence of the gradient.

2.3.1. Approximation. Let (u^n_0) and (f^n) be some approximating sequences of, respectively, u_0 and f in, respectively, $L^1(\Omega)$ and $L^1(\Omega \times (0,T))$ such that $u^n_0 \in L^p \cap L^2(\Omega)$, $f^n \in L^{p'}(\Omega \times (0,T))$. For each n, the problem

$$u^n_t - \mathrm{div}(a(\nabla u^n)) = f^n \quad \text{in } \Omega \times (0,T) \tag{26a}$$
$$u^n = u^n_0 \quad \text{on } \Omega \times \{0\} \tag{26b}$$
$$u^n = 0 \quad \text{on } \Sigma, \tag{26c}$$

has a unique solution u^n in the space $\mathcal{W}_T$, where

$$\mathcal{W}_T = \left\{ v \in L^p(0,T; W^{1,p}_0(\Omega));\ v_t \in L^{p'}(0,T; W^{-1,p'}(\Omega)) \right\} \tag{27}$$

if $p \geq 2$ and

$$\mathcal{W}_T = \left\{ v \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^2(Q_T);\ v_t \in L^{p'}(0,T; W^{-1,p'}(\Omega)) + L^2(Q_T) \right\} \quad (28)$$

if $p < 2$ [25]. The function u^n is a weak solution to (26). By considering test-functions depending on u^n itself and by a chain-rule lemma (*e.g.* Lemma 3.1 below with $\varepsilon = 0$), we obtain that u^n satisfies

$$\begin{aligned} &\int_{Q_T} S(u^n)\varphi_t - S'(u^n)a(\nabla u^n) \cdot \nabla \varphi dxdt - \int_{Q_T} a(\nabla u^n) \cdot \nabla u^n S''(u^n)\varphi dxdt \\ &= \int_\Omega S(u_0^n)\varphi(x,0)dx + \int_{Q_T} f^n S'(u^n)\varphi dxdt, \end{aligned} \quad (29)$$

for all $S \in C(\mathbb{R})$ such that $S' \in W^{1,\infty}(\mathbb{R})$ and for all $\varphi \in C_c^\infty([0,T) \times \Omega)$. Equivalently, u^n satisfies the following equation

$$\partial_t \chi_{u^n} - \operatorname{div}(a(\nabla u^n)\delta_{u^n=\xi}) = \chi_{u_0^n} \otimes \delta_{t=0} + f^n \delta_{u^n=\xi} + \partial_\xi \mu^n, \quad (30)$$

where μ^n is defined by

$$\mu^n := a(\nabla u^n) \cdot \nabla u^n \delta_{u^n=\xi}. \quad (31)$$

If additionally $S'(0) = 0$, then $S'(u^n)$ vanishes on $\partial\Omega$ and the class of test-functions φ in (29) can be enlarged to the $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$. In particular, by considering a sequence (φ_j) of test-functions converging to the characteristic function $\mathbf{1}_{[0,t]}$, $0 < t < T$, it is possible to pass to the limit in (29) since the space $\mathcal{W}_T$ is embedded in $C([0,T]; L^2(\Omega))$. We then obtain the following identity:

$$\int_\Omega S(u^n)(t)dx + \int_{Q_T} a(\nabla u^n) \cdot \nabla u^n S''(u^n)dxdt = \int_\Omega S(u_0^n)dx + \int_{Q_T} f^n S'(u^n)dxdt, \quad (32)$$

valid for all $t \in [0,T]$ and all $S \in C(\mathbb{R})$ such that $S' \in W^{1,\infty}(\mathbb{R})$ satisfying $S'(0) = 0$.

2.3.2. Estimates and limit equation. Up to a subsequence (and as a consequence of the strong convergence in L^1), we can assume that there exists some functions $\overline{u}_0, \overline{f}$ in $L^1(\Omega)$ and $L^1(Q_T)$ respectively such that $|u_0^n| \leq \overline{u}_0$, $|f^n| \leq \overline{f}$ a.e. For $k \geq 0$, we define S_k by

$$\begin{aligned} S_k(u) &= \int_0^u (T_{k+1} - T_k)^+(s)ds \\ &= \frac{1}{2}(|u| - k)^2 \mathbf{1}_{k<|u|<k+1} + \left(|u| - k - \frac{1}{2}\right) \mathbf{1}_{k+1 \leq |u|}. \end{aligned} \quad (33)$$

Then $S_k \in C(\mathbb{R})$, $S_k' \in W^{1,\infty}(\mathbb{R})$ $S_k'(0) = 0$ and $S_k'' \geq 0$ a.e. Since $|u| \leq S_0(u)$, $|S_0'| \leq 1$ and $S_0(u) \leq 1 + |u|$ for all $u \in \mathbb{R}$, it follows from (32) that

$$\int_\Omega |u^n(t)|dx \leq |\Omega| + \left(\|\overline{u}_0\|_{L^1(\Omega)} + \|\overline{f}\|_{L^1(Q_T)}\right), \tag{34}$$

for all $t \in [0, T]$. In particular, we obtain a bound independent on n on $\|u^n\|_{L^\infty(0,T;L^1(\Omega))}$. Similarly, choosing $S = S_k$ for $k > 0$ in (32), we obtain

$$\int_{\Omega \cap \{u^n(t) > k+1\}} |u^n(t)|dx \leq \int_{\Omega \cap \{u^n(t) > k\}} |\overline{u}_0|dx + \int_{Q_T \cap \{u^n > k\}} |\overline{f}|dxdt.$$

In particular, (u^n) is equi-integrable in $L^1(Q_T)$. We now derive a bound on $\nabla T_k(u^n)$, $k > 0$. Let

$$\mathcal{T}_k(u) = \int_0^u T_k(s)ds = \frac{|u|^2}{2}\mathbf{1}_{0 \leq |u| < k} + \left(k|u| - \frac{k^2}{2}\right)\mathbf{1}_{k \leq |u|}.$$

Then $0 \leq \mathcal{T}_k(u) \leq k|u|$ and, taking $S = \mathcal{T}_k$ in (32) gives

$$\int_{Q_T} a(\nabla u^n) \cdot \nabla u^n T_k'(u^n)dxdt \leq k\left(\|\overline{u}_0\|_{L^1(\Omega)} + \|\overline{f}\|_{L^1(Q_T)}\right). \tag{35}$$

Recall that $\nabla u^n = 0$ a.e. on $\{u^n = k\}$ (see Lemma 2.3), so that

$$\nabla T_k(u^n) = T_k'(u^n)\nabla u^n = \mathbf{1}_{[u^n < k]}\nabla u^n = \mathbf{1}_{[u^n \leq k]}\nabla u^n \text{ a.e.}$$

In particular, $a(\nabla u^n) \cdot \nabla T_k(u^n) = a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)$ a.e. Then, using (35) and Assumption 1, we deduce the following bounds (where C_k denotes a constant depending on k , but not on n):

$$\|a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)\|_{L^1(Q_T)} \leq C_k \tag{36}$$

and

$$\|\nabla T_k(u^n)\|_{L^p(Q_T)} \leq C_k, \ \|a(\nabla T_k(u^n))\|_{L^{p'}(Q_T)} \leq C_k. \tag{37}$$

Let us now prove that (up to a subsequence), there exists $u \in L^\infty(0, T; L^1(\Omega))$ such that $u^n \to u$ in $L^1(Q_T)$: We have already proved that (u^n) is equi-integrable on Q_T and obtained a bound on (u^n) in $L^\infty(0, T; L^1(\Omega))$. It is therefore sufficient to show that there exists $u \in L^1(Q_T)$ such that $u^n \to u$ a.e. Let us fix a functions $S_m \in C(\mathbb{R})$ such that $S_m' \in W^{1,\infty}(\mathbb{R})$ has a compact support and $S_m(u) = u$ for $|u| \leq m$. By (29), we have

$$S_m(u^n)_t = \mathrm{div}(a(\nabla u^n)S_m'(u^n)) - a(\nabla u^n) \cdot \nabla u^n S_m''(u^n) + f_n S_m'(u^n)$$

in the distribution sense.

By (36) and (37), $(S_m(u^n)_t)$ is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q_T)$. It follows that $(S_m(u^n)_t)$ is bounded in $L^1(0,T;Y)$, $Y:=L^1(\Omega)+W^{-1,p'}(\Omega)$. By (34), (36) and (37), $(S_m(u^n))$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$. Using the injections $W_0^{1,p}(\Omega)\subset L^p(\Omega)$ (compact injection) and $L^p(\Omega)\subset Y$, we deduce by Aubin-Simon's compactness theorem [34] that $(S_m(u^n))$ is precompact in $L^p(Q_T)$. Consequently, there exists a subsequence of $(S_m(u^n))$ converging a.e. on Q_T and in $L^1(Q_T)$ to a $u_m\in L^1(Q_T)$. We then conclude by a diagonal arguments: We obtain $u^n\to u$ a.e., where $u=u_m$ on $\{|u|\le m\}$.

Since $T_k(u^n)$ converges to $T_k(u)$ in $L^1(Q_T)$, then $\nabla T_k(u^n)$ converges to $\nabla T_k(u)$ in the sense of distributions. Thanks to the second estimate in (36), we deduce that $\nabla T_k(u^n)$ converges weakly in $L^p(Q_T)$ to $\nabla T_k(u)$ for all $k>0$ as $n\to\infty$.

Let $\mathcal{K}\subset(0,\infty)$ be the set of points where the monotone function

$$[k\in(0,\infty)\to \mathrm{meas}([|u|<k])]$$

is continuous. Recall that $(0,\infty)\setminus\mathcal{K}$ is at most denumerable. For all $k\in\mathcal{K}$, $\mathrm{meas}([|u|=k])=0$ and $\mathbf{1}_{[|u^n|<k]}\to\mathbf{1}_{[|u|<k]}$ a.e. as $n\to\infty$.

Now, let $(k^m)_{m\in\mathbb{N}}$ be an increasing sequence of points of $\mathcal{K}$ such that $\lim_{m\to\infty}k^m=+\infty$. From (37), and using a diagonal process, we claim that there exists a subsequence $(n_q)_{q\in\mathbb{N}}$ such that, for all $m\in\mathbb{N}$, $\sigma^q_{k^m}:=a\left(\nabla T_{k^m}(u^{n_q})\right)$ converges weakly in $L^{p'}(Q_T)$ as $q\to+\infty$ to some $\sigma_{k^m}\in L^{p'}(Q_T)$.

We have, for $k\in\mathcal{K}$ and $k^m>k$, $a(\nabla T_k(u^{n_q}))=\mathbf{1}_{[|u^{n_q}|<k]}\sigma^q_{k^m}$, from which we deduce that $a(\nabla T_k(u^{n_q}))$ converges weakly in $L^{p'}(Q_T)$ to $\sigma_k:=\mathbf{1}_{[|u|<k]}\sigma_{k^m}$ as $q\to\infty$. In particular, $\sigma_{k^l}=\mathbf{1}_{[|u|<k^l]}\sigma_{k^m}$ for all $m>l$. We may define $\sigma:Q_T\to\mathbb{R}^N$ measurable such that, for all $k\in\mathcal{K}$, $\sigma_k=\mathbf{1}_{[|u|<k]}\sigma$ and

$$\forall\, k\in\mathcal{K},\quad a(\nabla T_k(u^{n_q}))\text{ converges to }\mathbf{1}_{[|u|<k]}\sigma\text{ weakly in }L^{p'}(Q_T).$$

To pass to the limit in (30), there remains now to study the measure μ^n defined by (31). The bound (36) on $a(\nabla T_k(u^n))\cdot\nabla T_k(u^n)$ in $L^1(Q_T)$ gives a uniform bound on $\mu^n(K)$ for each compact subset K of U_T. Up to subsequence, we can therefore suppose that (μ^n) converges weakly to a Radon measure μ in U_T. Note that we have then

$$\mu_*(E)\le\liminf_{n\to+\infty}\mu^n_*(E), \tag{38}$$

for each $E\subset\mathbb{R}$ open. Indeed, $\mu_*(E)=\mu(Q_T\times E)\le\liminf_{n\to+\infty}\mu^n(Q_T\times E)=\liminf_{n\to+\infty}\mu^n_*(E)$, since $Q_T\times E$ is open in U_T.

With these results of convergence at hand, we let $n\to+\infty$ in (30) to obtain the limit equation

$$\partial_t\chi_u-\mathrm{div}(\sigma\delta_{u=\xi})=\chi_{u_0}\otimes\delta_{t=0}+f\delta_{u^n=\xi}+\partial_\xi\mu. \tag{39}$$

In the remaining part of this paragraph, we will show that (u, σ, μ) satisfies the conditions at infinity (20) and (23). First, by (32) with $S = S_k$, where S_k is defined by (33), we have

$$\int_{Q_T \cap \{k<|u^n|<k+1\}} a(\nabla u^n) \cdot \nabla u^n dxdt \le \int_{\Omega \cap \{|u_0^n|>k\}} |u_0^n| dx + \int_{Q_T \cap \{|u^n|>k\}} |f^n| dxdt.$$

This is

$$\mu_*^n((k, k+1)) + \mu_*^n((-k-1, -k)) \le \int_{\Omega \cap \{|u_0^n|>k\}} |u_0^n| dx + \int_{Q_T \cap \{|u^n|>k\}} |f^n| dxdt.$$

Up to a subsequence (and as a consequence of the strong convergence in L^1), there exists a function $\overline{u} \in L^1(Q_T)$ such that $|u^n| \le \overline{u}$ a.e. Recall that we also supposed $|u_0^n| \le \overline{u}_0$, $|f^n| \le \overline{f}$ a.e. Thus μ^n satisfies the uniform estimates

$$\mu_*^n((k, k+1)) + \mu_*^n((-k-1, -k)) \le \int_{\Omega \cap \{\overline{u}_0>k\}} \overline{u}_0 dx + \int_{Q_T \cap \{\overline{u}>k\}} \overline{f} dxdt, \quad (40)$$

from which we deduce by (38):

$$\mu_*((k, k+1)) + \mu_*((-k-1, -k)) \le \int_{\Omega \cap \{\overline{u}_0>k\}} \overline{u}_0 dx + \int_{Q_T \cap \{\overline{u}>k\}} \overline{f} dxdt.$$

In particular, μ satisfies (20).

In the nonnegative case, i.e., $u_0, u_0^n \ge 0$ a.e., $f, f^n \ge 0$ a.e., the approximate solutions are nonnegative, and therefore $u \ge 0$ a.e. and μ is supported in $\overline{Q_T} \times [0, +\infty)$: Hypothesis (22) in Lemma 2.6 is satisfied. Let us show that, independently on any sign condition, Hypothesis (23) is satisfied: Let $\varphi \in C(\overline{Q_T})$, $\varphi \ge 0$. For $k < 0$, $n, m \in \mathbb{N}$, and by monotonicity of a, we have

$$0 \le \int_{Q_T} \langle a(\nabla v_k^n) - a(\nabla v_k^m), \nabla v_k^n - \nabla v_k^m \rangle \varphi dxdt, \quad v_k^n := (T_{|k|+1} - T_{|k|})^-(u^n),$$

i.e.,

$$\begin{aligned} &\int_{Q_T} (a(\nabla u^n) \cdot \nabla u^m + a(\nabla u^m) \cdot \nabla u^n) \mathbf{1}_{(k-1,k)}(u^n) \mathbf{1}_{(k-1,k)}(u^m) \varphi dxdt \\ &\le \int_{Q_T} (a(\nabla u^n) \cdot \nabla u^n \mathbf{1}_{(k-1,k)}(u^n) + a(\nabla u^m) \cdot \nabla u^m \mathbf{1}_{(k-1,k)}(u^m)) \varphi dxdt. \end{aligned}$$

Denoting by ε_k the right hand-side of (40) (with $|k|$ instead of k since $k < 0$ here), we deduce

$$\int_{Q_T} (a(\nabla u^n) \cdot \nabla u^m + a(\nabla u^m) \cdot \nabla u^n) \mathbf{1}_{(k-1,k)}(u^n) \mathbf{1}_{(k-1,k)}(u^m) \varphi dxdt \le 2\varepsilon_k \|\varphi\|_\infty. \quad (41)$$

with $\lim_{k\to-\infty}\varepsilon_k = 0$. Recall that $\mathcal{K}$ is the set of continuity points of the monotone function $[k \in (0,\infty) \to \text{meas}([|u| < k])]$. Let now (k_j) be a sequence of negative numbers such that $\lim_{j\to+\infty} k_j = -\infty$, for all $j, k_j - 1, k_j \in \mathcal{K}$. Then $\mathbf{1}_{(k_j-1,k_j)}(u^n) \to \mathbf{1}_{(k_j-1,k_j)}(u)$ a.e. in Q_T. Take $k = k_j$ in (41). Taking the limit $n \to +\infty$, then $[m \to +\infty]$ and $[j \to +\infty]$, we obtain

$$\limsup_{j\to+\infty} \langle \sigma \cdot \nabla u\, \delta_{u=\xi}, \varphi \otimes \mathbf{1}_{(k_j-1,k_j)} \rangle \leq 0,$$

which shows that Hypothesis (23) is satisfied.

2.3.3. Strong convergence of the gradient. We are now in position to apply Lemma 2.6, which gives

$$\mu = \sigma \cdot \nabla u\, \delta_{u=\xi}.$$

To conclude, we want to examine the weak convergence of the push-forward μ^n_* to μ_*. We fix a test-function $\varphi \in C_c(Q_T)$, $\varphi \geq 0$. We use the notations of Section 2.2.1, in particular

$$\mu_\varphi(A) := \int_A \varphi(x,t) d\mu(x,t,\xi), \quad \forall\, A \in \mathcal{B}(U_T).$$

Then, if $\psi \in C_c(\mathbb{R})$, we have $\int_{\mathbb{R}} \psi d\mu^n_{\varphi,*} = \int_{U_T} \varphi \otimes \psi d\mu^n$, where $\varphi \otimes \psi(x,t,\xi) = \varphi(x,t)\psi(\xi) \in C_c(U_T)$, hence

$$\int_{\mathbb{R}} \psi d\mu^n_{\varphi,*} \to \int_{U_T} \varphi \otimes \psi d\mu = \int_{\mathbb{R}} \psi d\mu_{\varphi,*},$$

and we conclude that $(\mu^n_{\varphi,*})$ converges weakly to μ_φ on $\mathbb{R}$. Let $k > 0$. By (25) the $\mu_{\varphi,*}$-measure of the boundary of $[-k,k]$ is zero and, by weak convergence, we obtain

$$\mu^n_{\varphi,*}([-k,k]) \to \mu_{\varphi,*}([-k,k]). \tag{42}$$

This identity (42) is the central result in the proof of the strong convergence of the gradient. Indeed, by (17) and (25), (42) reads

$$\int_{Q_T} a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n) \varphi dxdt \to \int_{Q_T} \sigma \cdot \nabla T_k(u) \varphi dxdt \tag{43}$$

and from (43) follows the strong convergence of the gradient

$$\nabla T_k(u^n) \to \nabla T_k(u)\ a.e. \tag{44}$$

Although the argument is classical, we give the proof of the implication (43) ⇒ (44) in Section 3.4. By (44), we have in particular $\sigma = a(\nabla u)$ a.e. on Q_T. Therefore u is solution to the level-set p.d.e. associated to Problem (1):

$$\partial_t \chi_u - \text{div}(a(\nabla u)\delta_{u=\xi}) = \chi_{u_0} \otimes \delta_{t=0} + f\delta_{u=\xi} + \partial_\xi(a(\nabla u) \cdot \nabla u\, \delta_{u=\xi}).$$

By Theorem 2.2, (u^n) converges to u, which is a renormalized solution to (1).

3. Completion of the proofs

3.1. Proof of Theorem 2.2. Since the vector space generated by

$$\{\varphi \otimes \theta; \varphi \in \mathcal{D}(Q_T), \theta \in \mathcal{D}(\mathbb{R})\}$$

is dense in $\mathcal{D}(U_T)$, (10) is equivalent to: For all $\theta \in \mathcal{D}(\mathbb{R})$,

$$\langle \partial_t \chi_u - \operatorname{div}(a(\nabla u)\delta_{u=\xi}), \theta\rangle_{\mathcal{D}'(\mathbb{R}_\xi),\mathcal{D}(\mathbb{R}_\xi)} = \langle \chi_{u_0} \otimes \delta_{t=0} + f\delta_{u=\xi} + \partial_\xi \mu, \theta\rangle_{\mathcal{D}'(\mathbb{R}_\xi),\mathcal{D}(\mathbb{R}_\xi)}$$

in $\mathcal{D}'(Q_T)$. By definition of μ, this is equivalent to: For all $\theta \in \mathcal{D}(\mathbb{R})$,

$$\partial_t \int_{\mathbb{R}} \chi_u \theta d\xi - \operatorname{div}(\theta(u)a(\nabla u)) = \left(\int_{\mathbb{R}} \chi_{u_0}\theta d\xi\right) \otimes \delta_{t=0} + \theta(u)f - \theta'(u)a(\nabla u)\cdot\nabla u \quad (45)$$

in $\mathcal{D}'(Q_T)$. The correspondence between (6) and (10) is obtained by taking $\theta = S'$ in (45), by the identity $\int_{\mathbb{R}} \chi_u(\xi)S'(\xi)d\xi = S(u)$, satisfied for all $S \in W^{2,\infty}(\mathbb{R})$ such that $S(0) = 0$, and by a standard argument of density.

3.2. Proof of Lemma 2.6. Set $\nu := \sigma\cdot\nabla u\, \delta_{u=\xi}$ (see (24) for the definition of ν). We have to check that $\langle \mu, \varphi \otimes \psi\rangle = \langle \nu, \varphi \otimes \psi\rangle$ for all $\varphi \in \mathcal{D}(Q_T)$, $\psi \in \mathcal{D}(\mathbb{R})$. We first suppose that $\psi = \partial_\xi \theta$ with $\theta \in \mathcal{D}(\mathbb{R})$, so that $\langle \mu, \varphi \otimes \psi\rangle = -\langle \partial_\xi \mu, \varphi \otimes \theta\rangle$. By (21), $\langle \mu, \varphi \otimes \psi\rangle = \langle \nu, \varphi \otimes \psi\rangle$ is then equivalent to the following identity

$$\begin{aligned} &-\int_0^T \int_\Omega \left(\int_{u_0}^u \theta(\xi)d\xi\right)\varphi_t + \int_0^T \int_\Omega (\sigma \cdot \nabla\varphi)\theta(u) - \int_0^T \int_\Omega f\varphi\theta(u) \\ &= -\int_0^T \int_\Omega (\sigma \cdot \nabla u)\varphi\theta'(u). \end{aligned} \quad (46)$$

By use of the rule of derivation of a product of functions in $W^{1,p} \cap L^\infty$, we obtain the equivalent, more compact form of (46):

$$-\int_0^T \int_\Omega \left(\int_{u_0}^u \theta(\xi)d\xi\right)\varphi_t + \int_0^T \int_\Omega \sigma \cdot \nabla(\varphi\theta(u)) - \int_0^T \int_\Omega f\varphi\theta(u) = 0. \quad (47)$$

Equation (47) can be *formally* deduced from the chain-rule formula and from the equation

$$0 = \partial_t u - \operatorname{div}(\sigma) - u_0 \otimes \delta_{t=0} - f. \quad (48)$$

Let us also remark that, formally, the equation (48) can be deduced from Equation (21) by integrating with respect to $\xi \in \mathbb{R}$. Indeed, that $\mu(\xi) \to 0$ when $\xi \to \pm\infty$ is, still at the formal level, a consequence of the condition $\mu_*((k, k+1)) \to 0$ when $k \to \pm\infty$. Therefore, we begin with the derivation of an approximate form of Equation (48): Fix $k > 0$, let $(\rho_n)_n$ be an approximation of the unit on $\mathbb{R}$ (ρ_n having compact support in $[-\frac{1}{n},\frac{1}{n}]$), set $\alpha_k := \rho_k * \mathbf{1}_{[k,k+1]}$, and define

$$r^k = r^k(u) = \int_{|u|}^{\infty} \alpha_k, \quad v^k := \int_{\mathbb{R}} \chi_u(\xi) r^k(\xi) d\xi, \quad v_0^k := \int_{\mathbb{R}} \chi_{u_0}(\xi) r^k(\xi) d\xi.$$

We have $v^k \in L^p(-1,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$, $v_0^k \in L^\infty(\Omega)$ and, for $l > 0$,

$$r^k \to 1 \text{ a.e.}, \quad T_l(v^k) \to T_l(u) \text{ in } L^p(-1,T;W_0^{1,p}(\Omega)), \quad v_0^k \to u_0 \text{ a.e.},$$

when k tends to $+\infty$. Test Equation (21) against $\varphi(t,x)r_k(\xi)$ to obtain

$$-\int_0^T \int_\Omega (v^k - v_0^k)\varphi_t + \int_0^T \int_\Omega \sigma \cdot \nabla\varphi r^k - \int_0^T \int_\Omega f\varphi r^k = \int_0^T \int_\Omega \int_{\mathbb{R}} \varphi \alpha_k d\mu. \quad (49)$$

This is the approximate form of (48). Now we want to use a kind of chain-rule formula to obtain an approximation of (47). To this purpose, we first infer from (49) the inequality

$$\left| \int_{Q_T} \varphi_t (v^k - v_0^k) - \int_0^T \langle G^k, \varphi \rangle dt \right| \le \|\varphi\|_{L^\infty} \varepsilon_k, \quad (50)$$

where $G^k := -(\mathrm{div}(\sigma r^k(u)) + f r^k(u)) \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ and $\varepsilon_k := \mu_*((k-1,k+2)) + \mu_*((-k-2,-k+1)) \to 0$ when $k \to +\infty$. We then consider the following lemma.

Lemma 3.1. *Let $\varepsilon > 0$, $v \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^1(Q_T)$, $v_0 \in L^\infty(\Omega)$ and $G \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T)$ satisfy*

$$\left| \int_0^T \int_\Omega \varphi_t (v - v_0) - \int_0^T \langle G, \varphi \rangle dt \right| \le \|\varphi\|_{L^\infty} \varepsilon, \quad (51)$$

for all $\varphi \in \mathcal{D}(Q_T)$. Then, for all $\varphi \in \mathcal{D}(\mathbb{R}^N \times (-1,T))$, for all $h \in W^{1,\infty}(\mathbb{R})$ such that

$$(h(v)\varphi)(t) = 0 \text{ on } \partial\Omega, \text{ for a.e. } t \in (0,T), \quad (52)$$

we have

$$\left| \int_0^T \int_\Omega \varphi_t \int_{v_0}^{v} h(\xi) d\xi - \int_0^T \langle G, h(v)\varphi \rangle dt \right| \le \|\varphi\|_{L^\infty} \|h\|_{L^\infty} \varepsilon. \quad (53)$$

The Dirichlet condition (52) makes sense since $h(v)\varphi \in L^p(0,T;W^{1,p}(\Omega))$. The proof of Lemma 3.1 is given in the following section. We apply Lemma 3.1 to (50), with $\varphi \in \mathcal{D}(Q_T)$, $h(v) = \theta(v)$ to deduce

$$\left| \int_0^T \int_\Omega \left(\int_{v_0^k}^{v^k} \theta(\xi) d\xi \right) \varphi_t - \int_0^T \int_\Omega \sigma r_k(u) \cdot \nabla \left(\varphi\theta(v^k) \right) + \int_0^T \int_\Omega f r_k(u) \varphi \theta(v^k) \right|$$
$$\le \|\varphi \otimes \theta\|_{L^\infty} \varepsilon_k.$$

By use of the Lebesgue dominated convergence theorem, we obtain (47) at the limit $k \to +\infty$. Recall that $\psi = \partial_\xi \theta$, so that we actually proved that $\partial_\xi(\mu - \nu) = 0$. By a classical lemma in the theory of distributions, this shows that $\mu - \nu$ is constant with respect to ξ, or, more precisely, that for every $\kappa \in \mathcal{D}(Q_T)$ the distribution on $\mathbb{R}$ defined by $\psi \mapsto \langle \mu - \nu, \kappa \otimes \psi \rangle$ is represented by a constant c_κ. There remains to show that $c_\kappa = 0$.

In the case of nonnegative solution, i.e., under Hypothesis (22), this is straightforward since both μ and ν vanish on $Q_T \times (-\infty, 0)_\xi$. In the general case, i.e., under Hypothesis (23), we show that ν actually satisfies the following conditions at infinity: For all non-negative $\varphi \in C_c(Q_T)$,

$$\limsup_{k \to -\infty} \langle \nu, \varphi \otimes \mathbf{1}_{(k-1,k)} \rangle \geq 0, \quad \liminf_{k \to -\infty} \langle \nu, \varphi \otimes \mathbf{1}_{(k-1,k)} \rangle \leq 0. \tag{54}$$

Since $c_\kappa = \langle \mu - \nu, \kappa \otimes \mathbf{1}_{(k-1,k)} \rangle$ for all k, it follows then from (20) that c_κ is both non-negative and non-positive, i.e., $c_\kappa = 0$.

To prove (54), we first observe that it is sufficient to obtain (54) for regular test-functions φ in the multiplicative form

$$\varphi(x,t) = \varphi_1(t)\varphi_2(x), \quad \varphi_1 \in C_c^1(-1,T), \ \varphi_2 \in C_c^1(\Omega), \ \varphi_i \geq 0.$$

We then apply Lemma 3.1 to Equation (49) with $\varphi(x,t) = \varphi_1(t)\|\varphi_2\|_\infty$, $h(v) = (T_{|l|+1} - T_{|l|})^-(v)$, $l < 0$ (observe that $h \in W^{1,\infty}(\mathbb{R})$, and $h(0) = 0$ so that (52) is satisfied) and let $k \to +\infty$ to obtain Equation (47) as above with $\varphi = \varphi_1(t)\|\varphi_2\|_\infty$, i.e.,

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega (\sigma \cdot \nabla u)\varphi_1(t)\|\varphi_2\|_\infty \mathbf{1}_{(l-1,l)}(u) \\
&= \int_0^T\!\!\int_\Omega \left(\int_{u_0}^u (T_{|l|+1} - T_{|l|})(\xi)d\xi \right) \varphi_1'(t)\|\varphi_2\|_\infty + \int_0^T\!\!\int_\Omega f\varphi_1(t)\|\varphi_2\|_\infty (T_{|l|+1} - T_{|l|})(u)d\xi.
\end{aligned}$$

Relabel l by k and take the limit $k \to -\infty$ to obtain

$$\lim_{k \to -\infty} \langle \nu, \varphi_1\|\varphi_2\|_\infty \otimes \mathbf{1}_{(k-1,k)} \rangle = 0. \tag{55}$$

Since $\pm\varphi + \varphi_1\|\varphi_2\|_\infty \in C(\overline{Q_T})$ is nonnegative, we also have, by (23): $\liminf_{k \to -\infty} \langle \nu, (\pm\varphi + \varphi_1\|\varphi_2\|_\infty) \otimes \mathbf{1}_{(k-1,k)} \rangle \leq 0$. This, combined with (55), gives (54).

3.3. Proof of Lemma 3.1. It is a variation on the proof of [17, Lemma 4.3] ([17, Lemma 4.3] corresponds to the case $\varepsilon = 0$).

Step 1. Suppose that v_0 additionally satisfies $v_0 \in W_0^{1,p}(\Omega)$. For $t < 0$, set $v(t) = v_0$. Also first suppose h is non-increasing and φ nonnegative or h is non-decreasing and φ non-positive. We have

$$-\|\varphi\|_{L^\infty}\varepsilon \leq \int_0^T \int_\Omega \varphi_t(v - v_0) - \int_0^T \langle G, \varphi \rangle dt \leq \|\varphi\|_{L^\infty}\varepsilon \tag{56}$$

for all $\varphi \in \mathcal{D}(Q_T)$ and thus, by regularity of v, G, for all φ satisfying $\varphi \in L^p(-1,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ and $\varphi_t \in L^{p'}(Q_T)$. To use the function $h(v)$ as a test-function in (56), we have first to regularize its dependence on t: For fixed $\varphi \in \mathcal{D}^+(Q_T)$ and for $\eta > 0$ small enough (such that $\text{supp}(\varphi) \subset \Omega\times(-1,T-2\eta]$), we set $\zeta := \varphi h(v)$,

$$\zeta_\eta \colon (x,t) \to \frac{1}{\eta}\int_{t-\eta}^t \zeta(x,s)ds.$$

In (56), this gives

$$\begin{aligned}\int_0^T \langle G,\zeta_\eta\rangle dt &\le \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_0^T\int_\Omega (\varphi_\eta)_t(v-v_0)\\ &= \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_0^T\int_\Omega \frac{1}{\eta}(\zeta(x,t)-\zeta(x,t-\eta))(v-v_0)(x,t)dxdt\\ &= \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_{\mathbb{R}}\int_\Omega \frac{1}{\eta}(v(x,t)-v(x,t+\eta))\zeta(x,t)dxdt\\ &= \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_{\mathbb{R}}\int_\Omega \frac{1}{\eta}(v(t)-v(t+\eta))h(v(t))\varphi(t)dxdt.\end{aligned}$$

Since h is non-increasing and φ nonnegative or h is non-decreasing and φ non-positive, we have the inequality

$$(v(t)-v(t+\eta))h(v(t))\varphi(t) \le \int_{v(t)}^{v(t+\eta)} h(r)dr\varphi(t), \quad t<T,$$

hence

$$\begin{aligned}\int_0^T \langle G,\zeta_\eta\rangle dt &\le \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_{\mathbb{R}}\int_\Omega \varphi(t)\frac{1}{\eta}\int_{v(t)}^{v(t+\eta)} h(r)dr\\ &= \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_{\mathbb{R}}\int_\Omega \frac{1}{\eta}(\varphi(t)-\varphi(t-\eta))\int_{v_0}^{v(t)} h(r)dr.\end{aligned}$$

At the limit $\eta \to 0$, a first inequality is obtained

$$\int_0^T \langle G, h(v)\varphi\rangle dt \le \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_0^T\int_\Omega \varphi_t\int_{v_0}^{v(t)} h(r)dr.$$

By use of $\zeta_\eta \colon (x,t) \to \frac{1}{\eta}\int_t^{t+\eta}\zeta(x,s)ds$ as a test-function, we derive in a similar way the second inequality

$$\int_0^T \langle G, h(v)\varphi\rangle dt \ge -\|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon + \int_0^T\int_\Omega \varphi_t\int_{v_0}^{v(t)} h(r)dr,$$

which gives (53). In case h is non-decreasing and φ nonnegative or h is non-increasing and φ non-positive, proceed similarly (just exchanging the order of the different time-regularizations) to prove (53), then decompose h as the sum of two monotone functions and φ as the sum of two signed functions to deduce the result in the general case.

Step 2. In the general case where $v_0 \in L^\infty(Q)$, regularize v_0 by v_0^n, $v_0^n \in W_0^{1,p}(\Omega)$, $\|v_0 - v_0^n\|_{L^1(\Omega)} \leq \frac{1}{n}$. Observe that, from (51), we deduce

$$\left|\int_0^T \int_\Omega \varphi_t(v - v_0^n) - \int_0^T \langle G, \varphi\rangle dt\right| \leq \|\varphi\|_{L^\infty}\left(\varepsilon + \frac{1}{n}\right).$$

Apply Step 1 to get $\left|\int_0^T\int_\Omega \varphi_t \int_{v_0^n}^v h(\xi)d\xi - \int_0^T \langle G, h(v)\varphi\rangle dt\right| \leq \|\varphi\|_{L^\infty}\|h\|_{L^\infty}(\varepsilon + \frac{1}{n})$, then pass to the limit $n \to +\infty$ to achieve the proof of Lemma 3.1.

3.4. Proof of the strong convergence of the gradient. We start from (43) and prove the strong convergence of the gradient by the arguments of Minty, Browder and Leray, Lions [16, 24, 28]. Let $\varphi \in C_c(\Omega \times (0,T))$, $\varphi \geq 0$ be given. Consider the sum

$$\int_{Q_T} (a(\nabla T_k(u^n)) - a(\nabla T_k(u))) \cdot (\nabla T_k(u^n) - \nabla T_k(u))\varphi dxdt. \tag{57}$$

We develop the product in this last term. The result (43) yields precisely the convergence of the term $\int_{Q_T} a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)dxdt$. The other terms, which are linear with respect to $\nabla T_k(u^n)$ or $a(\nabla T_k(u^n))$, converge by weak convergence. At the limit $n \to +\infty$ in (57), we obtain

$$\lim_{n\to+\infty}\int_{Q_T} (a(\nabla T_k(u^n)) - a(\nabla T_k(u))) \cdot (\nabla T_k(u^n) - \nabla T_k(u))\varphi dxdt = 0.$$

Since $F_n := (a(\nabla T_k(u^n)) - a(\nabla T_k(u))) \cdot (\nabla T_k(u^n) - \nabla T_k(u))\varphi$ is nonnegative (by monotonicity of a), this shows that $F_n \to 0$ in $L^1(Q_T)$. A subsequence of (F_n) (still denoted (F_n)) therefore converges to 0 on a set A of full measure in Q_T. Let $(x,t) \in A$ and let q be an adherence value of $(\nabla T_k(u^n))$ in $\overline{\mathbb{R}}^N$. Without loss of generality, we can suppose that $\varphi(x,t) > 0$. The vector q has finite-valued components as a consequence of the growth of $a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)$, which gives

$$(\alpha|\nabla T_k(u^n)(x,t)|^p - C|\nabla T_k(u^n)(x,t)|)\varphi(x,t) \leq F_n(x,t) \to 0.$$

At the limit $n \to +\infty$ in $F_n(x,t) \to 0$, we thus obtain

$$(a(q) - a(\nabla T_k(u)(x,t))) \cdot (q - \nabla T_k(u)(x,t))\varphi(x,t) = 0.$$

By strict monotonicity of the flux a, $q = \nabla T_k(u)(x,t)$. Thus $(\nabla T_k(u^n)(x,t))$ has only one possible adherence value and is therefore convergent: $\nabla T_k(u^n) \to \nabla T_k(u)$ a.e. on Q_T. Together with the uniform bound on $\nabla T_k(u^n)$ in $L^p(Q_T)$, this shows the strong convergence of $\nabla T_k(u^n)$ to $\nabla T_k(u)$ in any $L^r(Q_T)$, $r < p$. Similarly, $a(\nabla T_k(u^n))$ converges to $a(\nabla T_k(u))$ a.e. and in $L^r(Q_T)$, $r < p'$. In particular, $\sigma = a(\nabla u)$ a.e.

To conclude, notice that we can recover the strong convergence $\nabla T_k(u^n) \to \nabla T_k(u)$ in $L^p_{\text{loc}}(Q_T)$. To this purpose we will use the following lemma.

Lemma 3.2 (Variation on the dominated convergence)**.** *Let* $(X, \mathcal{A}, \mu)$ *be a measure space, let* $w, v \colon X \to \mathbb{R}$ *be some measurable functions and let* (v_n), (w_n) *be some sequences in* $L^1(X)$ *such that* $|w_n| \leq v_n$, $w_n \to w$ *a.e.,* $v_n \to v$ *a.e. and* $\int_X v_n d\mu \to \int_X v d\mu$. *Then* $w_n \to w$ *in* $L^1(X)$.

Let K be compact subset of Q_T. By the weak convergence of $(a(\nabla T_k(u^n)))$ and $(\nabla T_k(u^n))$ to $a(\nabla T_k(u))$ and $\nabla T_k(u)$ respectively, and by the convergence

$$(a(\nabla T_k(u^n)) - a(\nabla T_k(u))) \cdot (\nabla T_k(u^n) - \nabla T_k(u)) \to 0$$

in $L^1(K)$, $(a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n))$ converges to $a(\nabla T_k(u)) \cdot \nabla T_k(u))$ in $L^1(K)$-weak. We also have $a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n) \to a(\nabla T_k(u)) \cdot \nabla T_k(u)$ a.e. in K. By Lemma 3.2 applied to $v_n = w_n = a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n)$, it follows that

$$a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n) \to a(\nabla T_k(u)) \cdot \nabla T_k(u) \quad \text{in } L^1(K).$$

Then, by Hypothesis (2a), $\tilde{w}_n := |\nabla T_k(u^n) - \nabla T_k(u)|^p$ is dominated by

$$\tilde{v}_n := 2^p \alpha^{-1}(a(\nabla T_k(u^n)) \cdot \nabla T_k(u^n) + a(\nabla T_k(u)) \cdot \nabla T_k(u)).$$

Since $\tilde{w}_n \to 0$ a.e. in K, using again Lemma 3.2, we obtain the strong convergence $\nabla T_k(u^n) \to \nabla T_k(u)$ in $L^p(K)$.

Proof of Lemma 3.2. By Fatou's lemma $w, v \in L^1(X)$. By applying Fatou's lemma also to the non-negative function $v_n + |w| - |w - w_n|$, we obtain

$$\int_X (v + |w|) d\mu \leq \int_X (v + |w|) d\mu - \limsup_{n \to +\infty} \int_X |w - w_n| d\mu,$$

i.e., $w_n \to w$ in $L^1(X)$. □

Remark 3.3 (Minty's trick)**.** As announced in the introduction, we study Problem (1) as a prototype of more elaborated parabolic equations with L^1 data, in particular problems of the type of (1) where a depends also on u, or Problem (58) below, and for this last class of problems, the proof of the strong convergence $\nabla T_k(u^n) \to \nabla T_k(u)$ in L^p is necessary in the proof of existence by approximation. Consequently, it appears to be necessary to do the hypothesis of strict monotonicity (2c). However, let us emphasize here that, for the special case of Problem (1), this hypothesis can be relaxed to the mere monotony of a:

$$(a(X) - a(Y)) \cdot (X - Y) \geq 0, \quad \forall\, X, Y \in \mathbb{R}^N.$$

Indeed, since a is continuous, a is then maximal monotone; let us recall the proof of this fact: If $X, W \in \mathbb{R}^N$ and $0 \leq (W - a(Y)) \cdot (X - Y)$, for all $Y \in \mathbb{R}^N$, then by taking $Y = X + \varepsilon Z$, $\varepsilon \neq 0$, $Z \in \mathbb{R}^N$, and by dividing by ε, we obtain

$$0 \leq -\mathrm{sgn}(\varepsilon)(W - a(X + \varepsilon Z)) \cdot Z.$$

At the limit $\varepsilon \to 0$, this gives $0 = (W - a(X)) \cdot Z$ for all $Z \in \mathbb{R}^N$, hence $W = a(X)$. Now we come back to (57) (what follows is the Minty's trick [28]). Instead of (57), we write that $\int_{Q_T} (a(\nabla T_k(u^n)) - a(X)) \cdot (\nabla T_k(u^n) - X)\varphi dxdt \geq 0$, for all $X \in \mathbb{R}^N$. At the limit $n \to +\infty$, by (43) and weak convergence, we obtain

$$\int_{Q_T} (\sigma_k - a(X)) \cdot (\nabla T_k(u) - X)\varphi dxdt \geq 0, \quad \sigma_k := \sigma \mathbf{1}_{|u| \leq k}.$$

Since φ is arbitrary, it follows that $(\sigma_k - a(X)) \cdot (\nabla T_k(u) - X) = 0$ a.e. in Q_T, hence $\sigma_k = a(\nabla T_k(u))$ a.e. in Q_T since a is maximal monotone. This identification is then sufficient (cf. (39)) to conclude that u is a renormalized solution to (1).

4. Parabolic equation with a term with natural growth

In this section, we briefly indicate how to adapt the arguments and proofs given above to solve the question of the strong convergence of the gradient (and, therefore, prove the existence of a renormalized solution) in the approximation by regularization and truncation of the following problem:

$$u_t - \operatorname{div}(a(\nabla u)) + \gamma(u)|\nabla u|^p = f \quad \text{in } \Omega \times (0, T) \tag{58a}$$

$$u = u_0 \quad \text{on } \Omega \times \{0\} \tag{58b}$$

$$u = 0 \quad \text{on } \Sigma. \tag{58c}$$

We keep the same assumptions on a and on the data: Assumptions 1 and 2. The function $\gamma \in C(\mathbb{R})$ is supposed to satisfies the sign condition

$$u\gamma(u) \geq 0, \quad \forall\, u \in \mathbb{R}.$$

This sign condition ensures *a priori* estimates for the additional term $\gamma(u)|\nabla u|^p$, with a bound in $L^1(Q_T)$. More generally, we may consider a term $\gamma(u)|\nabla u|^r$ with a power $r \in [1, p]$, instead of the term $\gamma(u)|\nabla u|^p$.

Numerous works have been devoted to the study of Problem (58) (or to its elliptic version). Let us cite in particular [9, 10, 14, 15, 30, 33] and references therein.

In case $p = 2$, $a = \mathrm{Id}$, there is a change of variables that transforms the equation in a classical heat equation:

$$v_t - \Delta v = g, \quad v = \int_0^u e^{-\int_0^\xi \gamma} d\xi, \quad g = f e^{-\int_0^u \gamma}.$$

It is this change of variables that we will adapt to the nonlinear case by use of the kinetic formulation (or level-set PDE).

A renormalized solution to (58) is defined as follows.

Definition 4.1. A function $u \in L^\infty(0,T;L^1(\Omega))$ is a renormalized solution to (58) if

$$T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega)), \quad \forall\, k>0,$$

and, for every function $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support and $S(0)=0$,

$$S(u)_t - \mathrm{div}(S'(u)a(\nabla u)) + S'(u)\gamma(u)|\nabla u|^p = S(u_0)\otimes\delta_{t=0} + S'(u)f - S''(u)a(\nabla u)\cdot\nabla u$$

and

$$\lim_{k\to+\infty}\int_{Q_T\cap\{k<u<k+1\}} a(\nabla u)\cdot\nabla u dxdt = 0.$$

We can also use directly the level-set PDE and define a renormalized solution to (58) as a function $u \in L^\infty(0,T;L^1(\Omega))$ having the regularity of the truncates $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$, $\forall\, k>0$, which satisfies the equation:

$$\partial_t\chi_u - \mathrm{div}(a(\nabla u)\delta_{u=\xi}) + \gamma(\xi)|\nabla u|^p\delta_{u=\xi} = \chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi} + \partial_\xi\mu,$$

where $\mu := a(\nabla u)\cdot\nabla u\,\delta_{u=\xi}$ satisfies the condition at infinity $\lim_{k\to\pm\infty}\mu_*((k,k+1))=0$.

We now explain how to prove the existence of a renormalized solution to Problem (58). For the sake of simplicity, we will suppose that the solution has a sign: We assume

$$u_0 \geq 0 \text{ a.e.} \quad \text{and} \quad f \geq 0 \text{ a.e.}$$

Step 1. (Approximation) Let (u_0^n) and (f^n) be some nonnegative approximating sequences of, respectively, u_0 and f in, respectively, $L^1(\Omega)$ and $L^1(\Omega\times(0,T))$ such that $u_0^n \in L^p\cap L^2(\Omega)$, $f^n \in L^{p'}(\Omega\times(0,T))$. For each n, the problem

$$u_t^n - \mathrm{div}(a(\nabla u^n)) + \gamma(u^n)|\nabla u^n|^p = f^n \quad \text{in } \Omega\times(0,T) \tag{59a}$$

$$u^n = u_0^n \quad \text{on } \Omega\times\{0\} \tag{59b}$$

$$u^n = 0 \quad \text{on } \Sigma, \tag{59c}$$

has a unique solution u^n in the space $\mathcal{W}_T$ defined in (27), (28). The function u^n is a weak solution to (59), hence a renormalized solution and therefore satisfies the equation

$$\partial_t\chi_{u^n} - \mathrm{div}(a(\nabla u^n)\delta_{u^n=\xi}) + \gamma(\xi)|\nabla u^n|^p\delta_{u^n=\xi} = \chi_{u_0^n}\otimes\delta_{t=0} + f^n\delta_{u^n=\xi} + \partial_\xi\mu^n, \tag{60}$$

where μ^n is defined by $\mu^n := a(\nabla u^n)\cdot\nabla u^n\delta_{u^n=\xi}$.

Step 2. (Estimates) As in Section 2.3.2, we show that, up to a subsequence, $u_n \to u \in L^\infty(0,T;L^1(\Omega))$ in $L^1(Q_T)$, $a(\nabla T_k(u^n)) \to \sigma\mathbf{1}_{|u|<k}$, $k\in\mathcal{K}$ and $\mu^n \to \mu$ weakly. We also prove, by the same technique as in Section 2.3.2, the conditions at infinity

$$\lim_{k\to\pm\infty}\mu_*((k,k+1)) = 0. \tag{61}$$

Since $u^n \geq 0$ a.e., we also have $u \geq 0$ a.e. and μ is supported in $\overline{Q_T}\times[0,+\infty)$.

Step 3. (Limit of the equation) To pass to the limit of Equation (60), there is a difficulty in the fact that the term $\gamma(\xi)|\nabla u^n|^p\delta_{u^n=\xi}$ is uniformly bounded in L^1 and that no stronger *a priori* bound is available. We define

$$\Gamma_+(\xi) = \begin{cases} \frac{1}{\alpha}\int_0^\xi \gamma & \text{if } \xi > 0 \\ -\frac{1}{\beta}\int_\xi^0 \gamma & \text{if } \xi < 0. \end{cases}$$

The function Γ_+ is continuous, not C^1, on $\mathbb{R}$, but a step of regularization shows that we have

$$\begin{aligned} &\partial_t e^{-\Gamma_+(\xi)}\chi_{u^n} - \operatorname{div}(e^{-\Gamma_+(\xi)}a(\nabla u^n)\delta_{u^n=\xi}) \\ &= e^{-\Gamma_+(\xi)}(\chi_{u_0^n}\otimes\delta_{t=0} + f^n\delta_{u^n=\xi}) + \partial_\xi(e^{-\Gamma_+(\xi)}\mu^n) + R, \end{aligned}$$

where $R := \gamma(\xi)e^{-\Gamma_+(\xi)}\{(\alpha^{-1}\mathbf{1}_{\xi>0} + \beta^{-1}\mathbf{1}_{\xi<0})\mu^n - |\nabla u^n|^p\delta_{u^n=\xi}\}$ (observe that the function $\xi \mapsto \gamma(\xi)(\alpha^{-1}\mathbf{1}_{\xi>0} + \beta^{-1}\mathbf{1}_{\xi<0})$ is continuous since $\gamma(0) = 0$). Since $\mu^n = a(\nabla u^n)\cdot\nabla u^n\delta_{u^n=\xi}$, the Hypotheses (2a) and (2b) on the flux a ensure that $R \geq 0$ and, therefore, that

$$\begin{aligned} &\partial_t e^{-\Gamma_+(\xi)}\chi_{u^n} - \operatorname{div}(e^{-\Gamma_+(\xi)}a(\nabla u^n)\delta_{u^n=\xi}) \\ &\geq e^{-\Gamma_+(\xi)}(\chi_{u_0^n}\otimes\delta_{t=0} + f^n\delta_{u^n=\xi}) + \partial_\xi(e^{-\Gamma_+(\xi)}\mu^n). \end{aligned} \tag{62}$$

Similarly, we define

$$\Gamma_-(\xi) = \begin{cases} \frac{1}{\beta}\int_0^\xi \gamma & \text{if } \xi > 0 \\ -\frac{1}{\alpha}\int_\xi^0 \gamma & \text{if } \xi < 0 \end{cases}$$

and show the inequality

$$\begin{aligned} &\partial_t e^{-\Gamma_-(\xi)}\chi_{u^n} - \operatorname{div}(e^{-\Gamma_-(\xi)}a(\nabla u^n)\delta_{u^n=\xi}) \\ &\leq e^{-\Gamma_-(\xi)}(\chi_{u_0^n}\otimes\delta_{t=0} + f^n\delta_{u^n=\xi}) + \partial_\xi(e^{-\Gamma_-(\xi)}\mu^n). \end{aligned} \tag{63}$$

It is then possible to pass to the limit $n \to +\infty$ in (62) and (63) to obtain

$$\begin{aligned} &\partial_t e^{-\Gamma_+(\xi)}\chi_u - \operatorname{div}(e^{-\Gamma_+(\xi)}\sigma\delta_{u=\xi}) \\ &\geq e^{-\Gamma_+(\xi)}(\chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi}) + \partial_\xi(e^{-\Gamma_+(\xi)}\mu), \end{aligned} \tag{64}$$

and

$$\begin{aligned} &\partial_t e^{-\Gamma_-(\xi)}\chi_u - \operatorname{div}(e^{-\Gamma_-(\xi)}\sigma\delta_{u=\xi}) \\ &\leq e^{-\Gamma_-(\xi)}(\chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi}) + \partial_\xi(e^{-\Gamma_-(\xi)}\mu). \end{aligned} \tag{65}$$

What information do we extract from (64) and (65)? At a *formal* level, we can do the following computations: Sum each inequality with respect to $\xi \in \mathbb{R}$ and use the condition at infinity (61) to obtain the (formal) weak equations

$$\partial_t \int_{\mathbb{R}} e^{-\Gamma_+(\xi)}\chi_u d\xi - \operatorname{div}(e^{-\Gamma_+(u)}\sigma) \geq \int_{\mathbb{R}} e^{-\Gamma_+(\xi)}\chi_{u_0} d\xi \otimes \delta_{t=0} + e^{-\Gamma_+(u)} f,$$
$$\partial_t \int_{\mathbb{R}} e^{-\Gamma_-(\xi)}\chi_u d\xi - \operatorname{div}(e^{-\Gamma_-(u)}\sigma) \leq \int_{\mathbb{R}} e^{-\Gamma_l(\xi)}\chi_{u_0} d\xi \otimes \delta_{t=0} + e^{-\Gamma_-(u)} f.$$

Multiply the first inequality by $e^{\Gamma_+(\xi)-\Gamma_-(\xi)}\delta_{u=\xi}$ and the second inequality by $e^{-\Gamma_+(\xi)+\Gamma_-(\xi)}\delta_{u=\xi}$ to obtain (still after formal computations)

$$\begin{aligned}&\partial_t e^{-\Gamma_-(\xi)}\chi_u - \operatorname{div}(e^{-\Gamma_-(\xi)}a(\nabla u)\delta_{u=\xi})\\ &\geq e^{-\Gamma_-(\xi)}(\chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi}) - e^{-\Gamma_-(\xi)}\sigma\cdot\nabla\delta_{u=\xi},\end{aligned}$$

and

$$\begin{aligned}&\partial_t e^{-\Gamma_+(\xi)}\chi_u - \operatorname{div}(e^{-\Gamma_+(\xi)}\sigma\delta_{u=\xi})\\ &\leq e^{-\Gamma_+(\xi)}(\chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi}) - e^{-\Gamma_+(\xi)}\sigma\cdot\nabla\delta_{u=\xi}.\end{aligned}$$

At last, use the identity $e^{-\Gamma_\pm(\xi)}\sigma\cdot\nabla\delta_{u=\xi} = -\partial_\xi(e^{-\Gamma_\pm(\xi)}\nu)$, where $\nu := \sigma\cdot\nabla u\,\delta_{u=\xi}$, (this is also a very *formal* identity) to obtain

$$\begin{aligned}&\partial_t e^{-\Gamma_-(\xi)}\chi_u - \operatorname{div}(e^{-\Gamma_-(\xi)}a(\nabla u)\delta_{u=\xi})\\ &\geq e^{-\Gamma_-(\xi)}(\chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi}) + \partial_\xi(e^{-\Gamma_-(\xi)}\nu),\end{aligned} \tag{66}$$

and

$$\begin{aligned}&\partial_t e^{-\Gamma_+(\xi)}\chi_u - \operatorname{div}(e^{-\Gamma_+(\xi)}\sigma\delta_{u=\xi})\\ &\leq e^{-\Gamma_+(\xi)}(\chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi}) + \partial_\xi(e^{-\Gamma_+(\xi)}\nu).\end{aligned} \tag{67}$$

Come back to the starting point (64), (65) to deduce the inequalities

$$\partial_\xi(e^{-\Gamma_+(\xi)}\mu) \leq \partial_\xi(e^{-\Gamma_+(\xi)}\nu), \quad \partial_\xi(e^{-\Gamma_-(\xi)}\nu) \leq \partial_\xi(e^{-\Gamma_-(\xi)}\mu). \tag{68}$$

Assume for the moment that (68) is satisfied in $\mathcal{D}'(U_T)$. A test-function $\varphi \in \mathcal{D}^+(Q_T)$ being fixed, we consider the distributions on $\mathbb{R}$ defined by

$$\mu_\varphi\colon \psi \mapsto \langle \mu, \varphi\otimes\psi\rangle, \quad \nu_\varphi\colon \psi \mapsto \langle \nu, \varphi\otimes\psi\rangle.$$

They satisfy the inequalities

$$\partial_\xi(e^{-\Gamma_+(\xi)}\mu_\varphi) \leq \partial_\xi(e^{-\Gamma_+(\xi)}\nu_\varphi), \quad \partial_\xi(e^{-\Gamma_-(\xi)}\nu_\varphi) \leq \partial_\xi(e^{-\Gamma_-(\xi)}\mu_\varphi)$$

in $\mathcal{D}'(\mathbb{R})$. Consider the first of these inequalities. Since $\mu_\varphi = \nu_\varphi = 0$ on $(-\infty, 0)$, we have $e^{-\Gamma_+(\xi)}\mu_\varphi \leq e^{-\Gamma_+(\xi)}\nu_\varphi$ in $\mathcal{D}'(\mathbb{R})$. Similarly, using the second inequality, we obtain $e^{-\Gamma_-(\xi)}\mu_\varphi \geq e^{-\Gamma_-(\xi)}\nu_\varphi$ and conclude that $\mu_\varphi = \nu_\varphi$. This being true for every $\varphi \in \mathcal{D}^+(Q_T)$, we have the desired result $\mu = \nu$.

Step 4. (Strong convergence of the gradient) The identity $\mu = \nu$ is the key point in the proof of the strong convergence of the gradient. Once this has been proved, we proceed as in Paragraph 2.3.3. We prove in particular that $\nabla T_k(u^n) \to \nabla T_k(u)$ in $L^p_{\rm loc}(Q_T)$, and this allows to pass to the limit in Equation (60) to obtain

$$\partial_t\chi_u - {\rm div}(a(\nabla u)\delta_{u=\xi}) + \gamma(\xi)|\nabla u|^p\delta_{u=\xi} = \chi_{u_0}\otimes\delta_{t=0} + f\delta_{u=\xi} + \partial_\xi(a(\nabla u)\cdot\nabla u\,\delta_{u=\xi}),$$

i.e., the fact that u is a renormalized solution.

Step 5. (Rigorous proof of (68)) This is a variation on the proof of Lemma 2.6 given in Section 3.2. Let us explain the main arguments. Introduce $\alpha_k := \rho_k * \mathbf{1}_{[k,k+1]}$, and define $r^k = r^k(u) = \int_{|u|}^\infty \alpha_k$, and

$$v^k := \int_{\mathbb{R}} e^{-\Gamma_+(\xi)}\chi_u(\xi) r_k(\xi)d\xi, \quad v_0^k := \int_{\mathbb{R}} e^{-\Gamma_+(\xi)}\chi_{u_0}(\xi) r_k(\xi)d\xi.$$

Set also $\tilde{r}^k = e^{-\Gamma_+(u)}r^k$ and

$$v := \int_{\mathbb{R}} e^{-\Gamma_+(\xi)}\chi_u(\xi)d\xi, \quad v_0 := \int_{\mathbb{R}} e^{-\Gamma_+(\xi)}\chi_{u_0}(\xi)d\xi.$$

We have $v^k \in L^p(-1,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$, $v_0^k \in L^\infty(\Omega)$ and $T_l(v^k) \to T_l(v)$ in $L^p(-1,T;W_0^{1,p}(\Omega))$ $(l>0)$, $v_0^k \to v_0$, $r^k \to 1$ a.e. when k tends to $+\infty$. Test Equation (64) against $\varphi(t,x)r^k(\xi)$ (with $\varphi \in \mathcal{D}^+(Q_T)$), to obtain

$$-\int_0^T\int_\Omega (v^k - v_0^k)\varphi_t + \int_0^T\int_\Omega \sigma\cdot\nabla\varphi\tilde{r}^k - \int_0^T\int_\Omega f\varphi\tilde{r}^k \geq \int_0^T\int_\Omega\int_{\mathbb{R}} \varphi e^{-\Gamma_+(\xi)}\alpha_k d\mu.$$

We deduce the inequality $\int_{Q_T}\varphi_t(v^k - v_0^k) - \int_0^T\langle G^k,\varphi\rangle dt \leq \|\varphi\|_{L^\infty}\varepsilon_k$, where $G^k := -({\rm div}(\sigma\tilde{r}^k(u)) + f\tilde{r}^k(u)) \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ and $\varepsilon_k := \mu_*((k-1,k+2)) \to 0$ when $k \to +\infty$. The analogue of Lemma 3.1 then shows that, for every $h \in W^{1,\infty}(\mathbb{R})$, v^k satisfies the following inequality:

$$\int_{Q_T}\varphi_t\int_{v_0^k}^{v^k} h(\zeta)d\zeta - \int_0^T \langle G^k, \varphi h(v^k)\rangle dt \leq \|\varphi\|_{L^\infty}\|h\|_{L^\infty}\varepsilon_k.$$

Taking h with compact support, we obtain at the limit $k \to +\infty$ the inequality $\int_{Q_T}\varphi_t\int_{v_0}^v h(\zeta)d\zeta - \int_0^T\langle G,\varphi h(v)\rangle dt \leq 0$, i.e.,

$$\int_{Q_T}\varphi_t\int_{v_0}^v h(\zeta)d\zeta - \int_{Q_T} e^{-\Gamma_+(u)}\sigma\cdot\nabla(\varphi h(v))dt + \int_{Q_T} e^{-\Gamma_+(u)} f\varphi h(v) \leq 0. \quad (69)$$

We then fix $\theta \in \mathcal{D}(\mathbb{R})$ and apply (69) with

$$h(\zeta) := e^{-(\Gamma_- - \Gamma_+)(\phi^{-1}(\zeta))}\theta(\phi^{-1}(\zeta)), \quad \phi(\xi) := \int_0^\xi e^{-\Gamma_+},$$

in such a way that $\int_{v_0}^v h(\zeta)d\zeta = \int_{u_0}^u e^{-\Gamma_-(\xi)}\theta(\xi)d\xi$, $\quad h(v) = e^{-(\Gamma_- - \Gamma_+)(u)}\theta(u)$, to obtain the weak form of (66). Similarly, we prove (67). As explained in Step 3, these two inequalities combined with (64) and (65) imply (68).

References

[1] Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M. and Vázquez, J.-L., An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 22 (1995)(2), 241 – 273.

[2] Bénilan, P, Carrillo, J. and Wittbold, P., Renormalized entropy solutions of scalar conservation laws. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 29 (2000)(2), 313 – 327.

[3] Bénilan, P. and Wittbold, P., On mild and weak solutions of elliptic-parabolic problems. *Adv. Diff. Equs.* 1 (1996)(6), 1053 – 1073.

[4] Blanchard, D., Truncations and monotonicity methods for parabolic equations. *Nonlinear Anal.* 21 (1993)(10), 725 – 743.

[5] Blanchard, D., Murat, F. and Redwane, H., Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems. *J. Diff. Equs.* 177 (2001)(2), 331 – 374.

[6] Blanchard, D. and Porretta, A., Stefan problems with nonlinear diffusion and convection. *J. Diff. Equs.* 210 (2005)(2), 383 – 428.

[7] Boccardo, L., Dall'Aglio, A., Gallouët, T. and Orsina, L., Existence and regularity results for some nonlinear parabolic equations. *Adv. Math. Sci. Appl.* 9 (1999)(2), 1017 – 1031.

[8] Boccardo, L. and Gallouët, T., Nonlinear elliptic equations with right-hand side measures. *Comm. Partial Diff. Equs.* 17 (1992)(3-4), 641 – 655.

[9] Boccardo, L. and Gallouët, T., Strongly nonlinear elliptic equations having natural growth terms and L^1 data. *Nonlinear Anal.* 19 (1992)(6), 573 – 579.

[10] Boccardo, L., Gallouët, T. and Murat, F., A unified presentation of two existence results for problems with natural growth. In: *Progress in Partial Differential Equations: the Metz Surveys, 2 (1992).* Pitman Res. Notes Math. Ser. 296. Harlow: Longman Sci. Tech. 1993, pp. 127 – 137.

[11] Boccardo, L., Giachetti, D., Diaz, J. I. and Murat, F., Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms. *J. Diff. Equs.* 106 (1993)(2), 215 – 237.

[12] Boccardo, L. and Murat, F., Remarques sur l'homogénéisation de certains problèmes quasi-linéaires (in French). *Portugal. Math.* 41 (1982)(1-4), 535 – 562.

[13] Boccardo, L. and Murat, F., Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Anal.* 19 (1992)(6), 581 – 597.

[14] Boccardo, L., Murat, F. and Puel, J.-P., Existence de solutions faibles pour des équations elliptiques quasi-linéaires à croissance quadratique (in French). In: *Nonlinear Partial Differential Equations and their Applications. Collège de France Seminar, Vol. IV (Paris, 1981/1982).* Res. Notes Math. 84., Boston (MA): Pitman 1983, pp. 19–73.

[15] Boccardo, L., Murat, F. and Puel, J.-P., Existence results for some quasilinear parabolic equations. *Nonlinear Anal.* 13 (1989)(4), 373 – 392.

[16] Browder, F. E., Variational boundary value problems for quasi-linear elliptic equations of arbitrary order. *Proc. Nat. Acad. Sci. U.S.A.* 50 (1963), 31 – 37.

[17] Carrillo, J. and Wittbold, P., Uniqueness of renormalized solutions of degenerate elliptic-parabolic problems. *J. Diff. Equs.* 156 (1999), 93 – 121.

[18] Chen, G.-Q. and Perthame, B., Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 20 (2003)(4), 645 – 668.

[19] Dal Maso, G., Murat, F., Orsina, L. and Prignet, A., Definition and existence of renormalized solutions of elliptic equations with general measure data. *C. R. Acad. Sci. Paris Sér. I Math.* 325 (1997)(5), 481 – 486.

[20] Dal Maso, G., Murat, F., Orsina, L. and Prignet, A., Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 28 (1999)(4), 741 – 808.

[21] DiPerna, R. J. and Lions, P.-L., On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)* 130 (1989)(2), 321 – 366.

[22] DiPerna, R. J. and Lions, P.-L., Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* 98 (1989)(3), 511 – 547.

[23] Evans, L. C., *Partial Differential Equations.* Graduate Studies Math. 19. Providence (RI): American Mathematical Society 1998.

[24] Leray, J. and Lions, J.-L., Quelques résultats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder (in French). *Bull. Soc. Math. France* 93 (1965), 97 – 107.

[25] Lions, J.-L., *Quelques Méthodes de Résolution des Problèmes aux Limites Nonlinéaires* (in French). Paris: Dunod et Gauthier-Villars 1969.

[26] Lions, P.-L., *Mathematical Topics in Fluid Mechanics. Vol. 1. Incompressible Models.* Oxford Lecture Ser. Math. Appl. 3. New York: Oxford Univ. Press 1996.

[27] Lions, P.-L., Perthame, B. and Tadmor, E., A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Amer. Math. Soc.* 7 (1994), 169 – 191.

[28] Minty, G. J., On a "monotonicity" method for the solution of nonlinear equations in Banach spaces. *Proc. Nat. Acad. Sci. U.S.A.* 50 (1963), 1038 – 1041.

[29] Perthame, B., *Kinetic Formulation of Conservation Laws.* Oxford Lecture Ser. Math. Appl. 21. Oxford: Oxford Univ. Press 2002.

[30] Porretta, A., Existence for elliptic equations in L^1 having lower order terms with natural growth. *Portugal. Math.* 57 (2000)(2), 179 – 190.

[31] Porretta, A. and Vovelle, J., L^1 solutions to first order hyperbolic equations in bounded domains. *Comm. Part. Diff. Equs.* 28 (2003)(1-2), 381 – 408.

[32] Prignet, A., Existence and uniqueness of “entropy” solutions of parabolic problems with L^1 data. *Nonlinear Anal.* 28 (1997)(12), 1943 – 1954.

[33] Segura de León, S., Existence and uniqueness for L^1 data of some elliptic equations with natural growth. *Adv. Diff. Equs.* 8 (2003)(11), 1377 – 1408.

[34] Simon, J., Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl. (4)* 146 (1987), 65 – 96.

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