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Strong Solutions for Two-Sided Parabolic Variational Inequalities Related to an Elliptic Part of p-Laplacian Type

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Abstract. A class of parabolic variational inequalities with two obstacles related to an elliptic part of p-Laplacian type is considered. A result of existence and uniqueness of strong solutions is given. Moreover some estimates of Lewy-Stampacchia's type are obtained for these solutions, which can be used in order to get regularity results.

Keywords. Variational inequalities, two obstacles, elliptic part of p-Laplacian type, elliptic regularization method, Lewy-Stampacchia's type estimates

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1. Introduction

It is well known that in [6] J. L. Lions and G. Stampacchia studied some variational inequalities of parabolic type and prove existence and uniqueness results for solutions, in a suitable weak sense. In the following years, many celebrated authors obtained existence, uniqueness and regularity results also for strong solutions. In [2] the existence of a unique strong solution satisfying a pair of estimates of "Lewy-Stampacchia's type", which in some particular cases yield important regularity results, is proved. The main theorem in [2] is based on the use of some estimates given by Mosco in [7] for solutions of variational inequalities, in an abstract framework, which generalize the "classical" estimates of Lewy-Stampacchia (see [6] for classical solutions and [8] for weak solutions related to Dirichlet problems). The result in [2] is concerned with the linear case. In [3] this result is extended to some nonlinear case, where the elliptic part of the differential operator is of p-Laplacian type. Let us note that, either in [2] and in [3] the convex set of the variational inequality is related to a

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one-sided constraint. The aim of this paper is to consider a two-sided parabolic variational inequality (i.e. in presence of a lower and an upper obstacle), still for cases where the elliptic part is of p-Laplacian type. Even in this case, a unique solution of the problem is obtained, satisfying a suitable pair of Lewy-Stampacchia's inequalities. The basic idea of the proof is based on a suitable extension of the theorem by Mosco [7] (which is related to one-sided constraint cases) to the cases of two-sided constraints. Then, a regularization method as in [3] is used again, but it's important to underline that the consideration of a two-obstacles problem (instead of one) creates some technical difficulties, which can be suitably overtaken. It is well known that, in the case $p = 2$, that refers to the case in which the elliptic operator is given by the standard Laplace operator, the solution u of the variational inequality is obtained as the value function of a suitable stochastic control problem (see [1]). More precisely, u is given by the min-max value of a cost functional among the pairs of stopping times of a suitable Brownian motion. This fact can also be generalized to the case of a more general stochastic process where also a drift coefficient is present. A main motivation of our research is to extend this kind of result to the case of p-Laplacian type operators, in such a way to give a possible financial interpretation of the result.

2. Preliminaries on Banach lattices

Let X be a partially ordered set and let us put, for every pair of elements a, b in X with $a \geq b$ (or $a \leq b$)

$$a \vee b = \max(a, b), \quad a \wedge b = \min(a, b).$$

Obviously, if X is totally ordered then $a \vee b$, $a \wedge b$ are well defined for any $a, b \in X$.

A linear space X is said *a lattice vector space* if the following conditions are satisfied

$$\begin{aligned} &x \leq y \ \Rightarrow x + z \leq y + z && \forall\, x, y, z \in X \\ &x \leq y \ \Rightarrow \alpha x \leq \alpha y \ (\alpha x \geq \alpha y) && \forall\, \alpha \geq 0 \ (\forall\, \alpha \leq 0) \end{aligned}$$

Let us consider a Banach space X. A proper cone P of X is a subset of X s.t.

$$P + P \subset P, \quad \lambda P \subset P \quad \forall\, \lambda > 0, \quad P \cap (-P) = \{0\}$$

The partial order "$\leq$" induced by a proper cone P is defined as

$$x \leq y \ \Leftrightarrow \ y - x \in P$$

If P is closed then X is called an *ordered Banach space*. In that situation, the elements of P are called *positive*, and P is called the *positive cone* of the ordered Banach space X.

Let us remark that, if X is a Banach space and a lattice Banach vector space, it follows by definitions, that

$$\begin{cases} x + (x \vee y) = (z + x) \vee (z + y) & \forall\, x, y, z \in X \\ x + (x \wedge y) = (z + x) \wedge (z + y) & \forall\, x, y, z \in X \\ (-x) \vee (-y) = -(x \vee y) & \forall\, x, y \in X \end{cases} \tag{1}$$

so that $(x \vee y) - x - y = -(x \wedge y)$ for all $x, y \in X$. Therefore

$$x + y = (x \vee y) + (x \wedge y) \quad \forall\, x, y \in X \tag{2}$$

Then, putting $y = 0$ in (2), one gets

$$x = x^+ - x^-$$

where $x^+ = x \vee 0$ and $x^- = -(x \wedge 0)$ are called the *positive part* and the *negative part* of x respectively. Putting $z = -x$ and $z = -y$ in the first relation in (1), one easily gets

$$x \vee y = x + (y - x)^+ = y + (x - y)^+ \tag{3}$$

and, recalling the second relation in (1),

$$x \wedge y = x - (x - y)^+ = y - (y - x)^+. \tag{4}$$

A *sublattice* U of X is, by definition, a linear subspace of X which is a lattice w.r.t. the order given in X.

We shall call the *dual order space* of U, denoted by U^*, the subspace of the dual space U' of U, which is spanned by the positive cone of U', that is

$$P' = \{v' \in U' : \langle v', v \rangle \geq 0 \quad \forall\, v \in P\},$$

that is $U^* = P' - P'$. Generally, U^* is strictly contained in U'. Under the order induced on U^* by the closed positive cone P' (which we call the *dual order*) the space U^* is a vector lattice.

Finally, let X be a Banach space and a vector lattice. Then X is called a *lattice Banach space* if

$$|x| \leq |y| \ \Rightarrow \| x \| \leq \| y \| \quad \forall\, x, y \in X \tag{5}$$

where $|x| = x \vee (-x)$ is the *modulus* x and $\| \cdot \|$ denotes the norm in X.

By (5) and the lattice properties, it can be deduced that the maps $x \to |x|$, $x \to x^+$, $x \to x^-$ are all uniformly continuous from X into itself and the maps $(x, y) \to x \vee y$ and $(x, y) \to x \wedge y$ are uniformly continuous from $X \times X$ into X. As a consequence, it follows that the positive cone

$$P_X = \{x \in X : x \geq 0\} = \{x \in X : x^- = 0\}$$

is closed in X. Moreover, one can easily prove that X^*, the order dual space of X, coincides with the dual space X' in the case that X is a Banach lattice space.

3. The main result

Let us consider the following parabolic variational inequality with bilateral constraints

$$\begin{cases} u \in g + L^p(0,T;V),\ \frac{\partial u}{\partial t} \in L^{p'}(0,T;V'),\ \psi_1 \le u \le \psi_2 \text{ a.e. in Q} \\ \langle \frac{\partial u}{\partial t} + Au, v-u\rangle \ge \langle f, v-u\rangle \\ \forall\, v \in g + L^p(0,T;V),\ \psi_1 \le v \le \psi_2 \ \text{ a.e. in Q},\ u(0) = u_0 \end{cases} \tag{6}$$

where

- $Q = \Omega \times (0,T)$ with Ω open bounded subset of $\mathbb{R}^{\mathrm{N}}$ with C^∞-boundary $\partial\Omega$
- V is the subspace of the Sobolev space $H^{1,p}(\Omega)(p \ge 2)$ given by all the functions $v \in H^{1,p}(\Omega)$ s.t. $v\ |_{\partial_1\Omega}= 0$, where $\partial_1\Omega$ is a subset of $\partial\Omega$, which can possibly be empty
- $g \in L^p(0,T;H^{1,p}(\Omega))$ with $g = 0$ if $\partial_1\Omega = \varnothing$
- $u_0 \in L^2(\Omega)$
- $\psi_1, \psi_2 \in L^p(0,T;H^{1,p}(\Omega))$
- $f \in L^{p'}(0,T;V')$, the dual space of $L^p(0,T;V)$ with $p' = \frac{p}{p-1}$
- $A : L^p(0,T;H^{1,p}(\Omega)) \longrightarrow L^{p'}(0,T;V')$ is bounded, continuous, strictly T-monotone w.r.t. $L^pP(0,T;V)$ in the sense that

$$\begin{aligned} &\langle Au - Av, (u-v)^+\rangle \ge 0 && \forall\, u,v \in L^p(0,T;H^{1,p}(\Omega)) \\ &\langle Au - Av, (u-v)^+\rangle = 0 \Leftrightarrow (u-v)^+ = 0 && \text{s.t. } (u-v)^+ \in L^p(0,T;V) \end{aligned}$$

- A is coercive on $\in L^p(0,T;V)$ in the sense that

$$\exists\, \alpha > 0 :\ \langle Av, v\rangle \ge \alpha \parallel v \parallel^p \quad \forall\, v \in L^p(0,T;V)$$

As an example of an operator A satisfying the previous conditions, one can choose the differential operator defined as

$$Av(x,t) = -\sum_{i=1}^{N} (a_i(x,t) \mid v_{x_i}(x,t) \mid^{p-2} v_{x_i}(x,t))_{x_i} + c(x,t) \mid v(x,t) \mid^{p-2} v(x,t)$$

for all $v = v(x,t) \in L^p(0,T;H^{1,p}(\Omega))$, a.e. $(x,t) \in Q$ where $a_i, c \in L^\infty(Q)$, with $a_i(x,t) \ge a_0 > 0$ and $c(x,t) \ge c_0 > 0$ a.e. in Q.

The aim of this paper is to prove the following result:

Theorem. *Let the previous hypotheses and assumptions hold and let*

$$\left(\frac{\partial\psi_i}{\partial t} + A\psi_i - f\right)^+ \in L^{p'}(0,T;V'), \quad i = 1,2 \tag{7}$$

$$\frac{\partial\psi_i}{\partial t} \in L^{p'}(0,T;V'), \quad i = 1,2 \tag{8}$$

$$\begin{cases} \psi_1 \le g \le \psi_2 \text{ a.e. on } \partial_1\Omega \times (0,T), \text{ that is there exists two sequences} \\ \{\psi_{i,n}\}_{n\in\mathbb{N}} \subset L^p(0,T;C^1(\Omega)) \ (i=1,2) \text{ such that} \\ \psi_{1,n}|_{\partial_1\Omega\times(0,T)} \le 0 \le \psi_{2,n}|_{\partial_1\Omega\times(0,T)} \text{ with} \\ \psi_{i,n} \to \psi_i - g \text{ as } n \to +\infty \text{ in } L^p(0,T;H^{1,p}(\Omega)) \ (i=1,2) \end{cases} \tag{9}$$

$$\psi_1(0) \le u_0 \le \psi_2(0).$$

Then there exists a unique solution u of (6). Furthermore the following dual estimates of Lewy-Stampacchia's type hold in the space $L^{p'}(0,T;V')$

$$f - \left(\frac{\partial\psi_2}{\partial t} + A\psi_2 - f\right)^- \le \frac{\partial u}{\partial t} + Au \le f + \left(\frac{\partial\psi_1}{\partial t} + A\psi_1 - f\right)^+ \tag{10}$$

4. An extension of a theorem by Mosco

Here we state, as a preliminar result to be used in order to prove the Theorem, a suitable extension of a theorem by Mosco [7] which established some dual estimates of Lewy-Stampacchia's type for stationary variational inequalities with unilateral constraints. We extend this result to the case of bilateral constraints. The framework is the following:

X is a reflexive Banach space which is a lattice w.r.t. the order induced by a positive cone $P_X = \{x \in X : x \ge 0\}$. (11)

$\mathcal{V}$ is a sublattice closed vector subspace of X. (12)

$$\begin{cases} A : X \to \mathcal{V}' \text{ is a strictly } T\text{-monotone operator w.r.t. } \mathcal{V}, \\ \text{i.e. } \langle Au - Av, (u-v)^+\rangle \ge 0 \quad \forall\, u,v \in X \text{ s.t. } (u-v)^+ \in \mathcal{V} \\ \text{and } \langle Au - Av, (u-v)^+\rangle = 0 \text{ iff } (u-v)^+ = 0. \end{cases} \tag{13}$$

The restriction of A on $\mathcal{V}$ is coercive in the sense that $\exists\, \alpha > 0 : \ \langle Av, v\rangle \ge \alpha \parallel v \parallel^p \quad \forall\, v \in \mathcal{V}$. (14)

Furthermore, $\psi_1, \psi_2 \in X$ satisfy the following conditions

$$\psi_1 \vee v \in \mathcal{V}, \ \psi_2 \wedge v \in \mathcal{V} \quad \forall\, v \in \mathcal{V} \tag{15}$$

$$f \in \mathcal{V}'. \tag{16}$$

Then one can state the following result:

Proposition. *Let* (11)–(16) *be satisfied and let* $\psi_1 \leq \psi_2$*, then there exists one unique solution* u *of the following variational inequality*

$$\begin{cases} u \in \mathcal{V}, \quad \psi_1 \leq u \leq \psi_2 \\ \langle Au, v-u \rangle \geq \langle f, v-u \rangle \quad \forall\, v \in \mathcal{V},\ \psi_1 \leq v \leq \psi_2 \end{cases} \tag{17}$$

Moreover one has

$$\begin{cases} h \leq Au \leq h' \text{ for every pair of elements } h, h' \text{ in } \mathcal{V}' \text{ s.t.} \\ h \leq f,\ h \leq A\psi_2, \quad h' \geq f,\ h' \geq A\psi_1. \end{cases} \tag{18}$$

Proof. First of all let us note that (15) implies that the closed convex set

$$K = \{v \in \mathcal{V} : \psi_1 \leq v \leq \psi_2\}$$

is not empty, since $\overline{v} = \psi_1 \ \vee (\psi_2 \wedge 0)$ belongs to K. Then the existence and uniqueness of the solution of (17) follows from the Hartmann-Stampacchia's theorem (see [4]).

Let $h \leq f,\ h \leq A\psi_2$ and let's show that the solution z of the auxiliary problem

$$\begin{cases} z \in \mathcal{V},\ z \geq u \\ \langle Az, w-z \rangle \geq \langle h, w-z \rangle \quad \forall\, w \in \mathcal{V},\ w \geq u \end{cases} \tag{19}$$

verifies $Az \geq h$.

Indeed, putting $w = z + v$ with $v \geq 0$ in (19), on gets $\langle Az, v \rangle \geq \langle h, v \rangle$ for all $v \in \mathcal{V},\ v \geq 0$, that is $Az \geq h$ in $\mathcal{V}'$. On the other side one can prove that $z = u$. In fact, one has

$$(z - \psi_2)^+ \in \mathcal{V} \tag{20}$$

$$\langle Az - A\psi_2, (z-\psi_2)^+ \rangle \leq 0. \tag{21}$$

Actually (20) follows from the fact that z belongs to $\mathcal{V}$, $(z-\psi_2)^+ = z - \psi_2 \wedge z$, and that ψ_1 and ψ_2 satisfy (15). On the other hand (21) is a consequence of the fact that $w = \psi_2 \wedge z$ can be chosen in (19), thus $\langle Az, (z-\psi_2)^+ \rangle \leq \langle h, (z-\psi_2)^+ \rangle$, but $A\psi_2 \geq h$, so $\langle A\psi_2, (z-\psi_2)^+ \rangle \geq \langle h, (z-\psi_2)^+ \rangle$. Therefore $\langle Az - A\psi_2, (z-\psi_2)^+ \rangle \leq 0$, that is relation (21).

At this point the strict T-monotonicity of A implies $z \leq \psi_2$. Then, putting $v = z$ in (17) and taking into account that $h \leq f$ one gets $\langle Au, z-u \rangle \geq \langle h, z-u \rangle$; while, putting $w = u$ in (19) one obtains $\langle Az, z-u \rangle \geq \langle h, z-u \rangle$, then $\langle Au - Az, z-u \rangle \geq 0$ which implies $z = u$, using the strict T-monotonicity of A and the fact that $z \geq u$.

Finally, recalling (21), one gets $h \leq Au$. In order to show that $h^{'} \geq Au$ for every $h^{'}$ satisfying (18), one can argue in an analogous way. □

It is now easy to state two corollaries of the Proposition.

Corollary 1. *Under the assumptions of the Proposition putting $\mathcal{V}^*$ as the order dual space of $\mathcal{V}$, the following estimates hold for the solution u of* (17)

$$f \wedge A\psi_2 \leq Au \leq f \vee A\psi_1 \tag{22}$$

Corollary 2. *Under the assumptions of the Proposition the following estimate for the solution of* (17) *holds*:

$$\| Au \|_{\mathcal{V}'} \leq \| f \vee A\psi_1 \|_{\mathcal{V}'} + \| f \wedge A\psi_2 \|_{\mathcal{V}'} \tag{23}$$

Proof of Corollary 2. By (22) one gets

$$\langle (f \wedge A\psi_2) - Au, -v^- \rangle \ \geq 0, \quad \langle Au - (f \vee A\psi_1), v^+ \rangle \ \leq 0$$

then

$$\begin{aligned} \langle Au, v \rangle = \langle Au, v^+ - v^- \rangle &\leq \langle f \wedge A\psi_2, -v^- \rangle + \langle f \vee A\psi_1, v^+ \rangle \\ &\leq \|f \wedge A\psi_2\|_{\mathcal{V}'} \|v^-\|_{\mathcal{V}} + \|f \vee A\psi_1\|_{\mathcal{V}'} \|v^+\|_{\mathcal{V}} \\ &\leq (\|f \wedge A\psi_2\|_{\mathcal{V}'} + \|f \vee A\psi_1\|_{\mathcal{V}'}) \|v\|_{\mathcal{V}}, \end{aligned}$$

so (23) follows. □

5. Proof of the Theorem

Let us proceed by steps.

Step I. If there exists a solution (6), then it is unique.

Proof. Let u_1, u_2 be two solutions of (6). Putting $v = u_2$ in (6) (with $u = u_1$), $v = u_1$ in (6) (with $u = u_2$), and adding the two inequalities, one gets $\left\langle \frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t} + Au_1 + Au_2, u_1 - u_2 \right\rangle \leq 0$ which yields

$$\langle Au_1 - Au_2,\ u_1 - u_2 \rangle \leq 0 \tag{24}$$

due to the fact that

$$\left\langle \frac{\partial w}{\partial t}, w \right\rangle \geq 0 \quad \forall w \in L^p(0,T;V); \quad \frac{\partial w}{\partial t} \in L^{p'}(0,T;V'),\ w(0) = 0. \tag{25}$$

At this point, since the strict T-monotonicity of A w.r.t. $L^p(0,T;V)$ implies the strict monotonicity on the same space, then $u_1 = u_2$ follows from (24) □

Step II. If the assumptions of Theorem 1 are satisfied with the choices

$$u_0 = 0 \tag{26}$$
$$g = 0 \tag{27}$$

then there exist two sequences in $L^p(0,T;H^{1,p}(\Omega))$ verifying the following relations

$$\psi_{1,n} \le \psi_{2,n} \qquad \text{a.e. in } \partial_1\Omega \times (0,T) \quad \forall\, n \in \mathbb{N} \tag{28}$$
$$\psi_{1,n}(0) \le 0 \le \psi_{2,n}(0) \qquad \text{a.e. in } \Omega \quad \forall\, n \in \mathbb{N} \tag{29}$$
$$\psi_{i,n} \to \psi_i \textit{ in } L^p(0,T;H^{1,p}(\Omega)),\ i = 1,2 \tag{30}$$
$$\frac{\partial}{\partial t}\psi_{i,n} \in L^{p'}(0,T;H^{1,p}(\Omega)'), \qquad i = 1,2 \tag{31}$$

$$\frac{\partial}{\partial t}\psi_{i,n} \to \frac{\partial}{\partial t}\psi_i \in L^{p'}(0,T;V'),\ i = 1,2 \tag{32}$$
$$\frac{\partial}{\partial t}\psi_{i,n},\ \frac{\partial^2}{\partial t^2}\psi_{i,n},\ A\psi_{i,n} \in L^2(Q), \qquad i = 1,2. \tag{33}$$

As for the proof of Step II, one can argue in an analogous way as in [3].

Step III. If the assumptions of the Theorem are satisfied with (26) and (27), then any element $v \in L^p(0,T;V)$ such that $\psi_1 \le v \le \psi_2$ a.e. in Q, can be obtained as the strong limit of a sequence $\{v_n\}$ s.t.

$$\begin{cases} v_n \in L^p(0,T;V) \text{ with } \frac{\partial}{\partial t}v_n \in L^p(0,T;V) \\ \psi_{1,n} \le v_n \le \psi_{2,n} \text{ a.e. in Q} \quad \forall\, n \in \mathrm{N} \end{cases}$$

with $\psi_{1,n}, \psi_{2,n}$ verifying (28)–(33).

Proof. For any $v \in L^p(0,T;V)$ by density arguments, there exists a sequence $\{\widetilde{v_n}\}_{n\in\mathbb{N}}$ in the space

$$\left\{ v \in L^p(0,T;V) : \frac{\partial}{\partial t}v \in L^p(0,T;V) \right\} \tag{34}$$

strongly converging to v. Let us note that the sequence $v_n = \psi_{1,n} \vee (\psi_{2,n} \wedge \widetilde{v_n})$ belongs to the space defined in (34). Thus, the lattice property of $L^p(0,T;V)$ implies that for any $n \in \mathbb{N}$

$$\psi_{1,n} \le v_n = \psi_{1,n} \vee (\psi_{2,n} \vee \widetilde{v_n}) = \psi_{1,n} \vee \psi_{2,n} = \psi_{2,n}.$$

Therefore $v_n = \psi_{1,n} \vee (\psi_{2,n} \wedge \widetilde{v_n}) = \psi_{1,n} + (\psi_{2,n} \wedge \widetilde{v_n} - \psi_{1,n})^+$. From the continuity properties of the lattice operations it follows that

$$(\psi_{2,n} \wedge \widetilde{v_n} - \psi_{1,n})^+ \to (\psi_2 \wedge v - \psi_1)^+ \quad \text{in } L^p(0,T;H^{1,p}(\Omega)).$$

Then $v_n = \psi_{1,n} + (\psi_{2,n} \wedge \widetilde{v_n} - \psi_{1,n})^+ \to \psi_1 + (\psi_2 \wedge v - \psi_1)^+ = \psi_1 + (v - \psi_1)^+ = \psi_1 \vee v = v.$ □

Let us define now a differential operator A_n of "elliptic type" which approximates, when n goes to $+\infty$, the operator A. Precisely, one considers the space

$$Y = \left\{ v \in L^p(0,T;H^{1,p}(\Omega)) : \frac{\partial v}{\partial t} \in L^2(\mathrm{Q}) \right\} \tag{35}$$

$$W = \left\{ v \in L^p(0,T;V) : \frac{\partial v}{\partial t} \in L^2(\mathrm{Q}),\ v(0) = 0 \right\} \tag{36}$$

equipped with the graph-norms w.r.t. the operator $\frac{\partial}{\partial t}$ (let's note that in this case, Y and W are reflexive Banach spaces) and the operator $A_n : Y \to W'$ defined as

$$\langle A_n v, w\rangle = \varepsilon_n \left\langle \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \right\rangle + \left\langle \frac{\partial v}{\partial t}, w \right\rangle + \langle Av, w\rangle \quad \forall\, v \in Y,\ \forall\, w \in W \tag{37}$$

with

$$\varepsilon_n = \begin{cases} \min\left\{ \dfrac{1}{n},\ \min\limits_{i=1,2} \left\| \dfrac{\partial^2 \psi_{i,n}}{\partial t^2} \right\|_{L^2(\mathrm{Q})}^{-2} \right\} & \text{if } \left\| \dfrac{\partial^2 \psi_{i,n}}{\partial t^2} \right\|_{L^2(\mathrm{Q})} \neq 0,\ i = 1,2 \\ \dfrac{1}{n} & \text{otherwise} \end{cases} \tag{38}$$

where $\{\psi_{i,n}\}_{n\in\mathrm{N}}$ $(i = 1,2)$ are functions satisfying the thesis of Step II.

Step IV. Let Y, W, A_n, ε_n defined as in (35)–(38). Then, for all $n \in \mathbb{N}$, A_n is a bounded continuous coercive operator, strictly T-monotone with respect to $L^p(0,T;V)$.

Proof. It is essentially a simple consequence of the properties of A and the obvious fact that

$$\left\langle \frac{\partial^2 y^-}{\partial t^2}, y^+ \right\rangle = -\left\langle \frac{\partial y^-}{\partial t}, \frac{\partial y^+}{\partial t} \right\rangle = 0, \quad \forall\, y \in Y \text{ s.t. } y^+ \in W. \qquad \square$$

Step V. Let us consider the following elements of $L^{p'}(0,T;V')$

$$g_1 = A\psi_1 + \frac{\partial \psi_1}{\partial t} - f, \quad g_2 = A\psi_2 + \frac{\partial \psi_2}{\partial t} - f.$$

Then there exist three sequences $\{\pi_{1,n}\}, \{\eta_{1,n}\}$ and $\{\pi_{2,n}\}$ contained in the positive cone of $L^2(\mathrm{Q})$ such that

$$\pi_{1,n} \to g_1^+, \quad \eta_{1,n} \to g_1^-, \quad \pi_{2,n} \to g_2^+$$

strongly in $L^{p'}(0,T;V')$.

Furthermore, putting

$$\eta_{2,n} = A\psi_{1,n} + \frac{\partial \psi_{1,n}}{\partial t} - A\psi_{2,n} - \frac{\partial \psi_{2,n}}{\partial t} + \pi_{2,n} - (\pi_{1,n} - \eta_{1,n})$$

where $\psi_{1,n}, \psi_{2,n}$ are defined as in Step II, then $\eta_{2,n} \in L^2(Q), \eta_{2,n} \to g_2^-$ strongly in $L^{p'}(0,T;V')$ and $\eta_{2,n}^+ \to g_2^-$ strongly in $L^{p'}(0,T;V')$. Finally, putting

$$f_{1,n} = A\psi_{1,n} + \frac{\partial}{\partial t}\psi_{1,n} - (\pi_{1,n} - \eta_{1,n})\,, \quad f_{2,n} = A\psi_{2,n} + \frac{\partial}{\partial t}\psi_{2,n} - (\pi_{2,n} - \eta_{2,n})$$

one has that $f_{1,n}, f_{2,n}$ coincide and $f_n = f_{1,n} = f_{2,n}$ strongly converges to f in $L^p(0,T;V')$.

Proof. The first statement is a consequence of the density of the positive cone of $L^2(Q)$ in the positive cone of $L^{p'}(0,T;V')$. The other statements are implied by the properties of $\psi_{1,n}, \psi_{2,n}$ and some easy calculation. □

Step VI. Let us consider the following variational inequality

$$\begin{cases} u_n \in W,\ \psi_{1,n} \le u_n \le \psi_{2,n} \\ \langle A_n u_n, w - u_n\rangle \ge \langle f_n, w - u_n\rangle \quad \forall\, w \in W,\ \psi_{1,n} \le w \le \psi_{2,n} \end{cases} \tag{39}$$

Then (39) admits a unique solution u_n, which verifies the estimates in W':

$$f_n - \left(-\varepsilon_n\left(-\frac{\partial^2 \psi_{2,n}}{\partial t^2}\right)\right)^+ - \eta_{2,n}^+ \le A_n u_n \le f_n + \varepsilon_n\left(-\frac{\partial^2 \psi_{1,n}}{\partial t^2}\right)^+ + \pi_{1,n}^+. \tag{40}$$

Proof. The existence and uniqueness of u_n are consequences of Step IV and the Hartmann-Stampacchia's theorem. The estimates (40) are deduced from the very definition of the operator A_n, from some properties of Banach lattice spaces (see Section 2) and from the Proposition. □

Step VII. The sequence $\{A_n u_n\}$ is bounded in $L^{p'}(0,T;V')$.

Proof. Indeed, from the continuity of A, the definition of ε_n and the strong convergence of $\{\pi_{1,n}\}, \{\eta_{2,n}\}$ in $L^{p'}(0,T;V')$, we get that the three sequences

$$\{f_n\},\ \left\{f_n - \left(-\varepsilon_n\left(-\frac{\partial^2 \psi_{2,n}}{\partial t^2}\right)\right)^+ - \eta_{2,n}^+\right\},\ \left\{f_n + \varepsilon_n\left(-\frac{\partial^2 \psi_{1,n}}{\partial t^2}\right)^+ + \pi_{1,n}^+\right\}$$

are bounded in $L^{p'}(0,T;V')$. The estimates (40) hold also in the sense of $L^{p'}(0,T;V')$, as W is equipped with the graph norm of $\frac{\partial}{\partial t}$ and from the fact that $L^p(0,T;V)$ is dense in this space. Then it easily follows that $\{A_n u_n\}$ is a bounded sequence in $L^{p'}(0,T;V')$. □

Step VIII. Let u_n be the solution of (39) for any $n \in \mathbb{N}$. Then there exists $u_n \in W$ and a subsequence of $\{u_n\}$, still named $\{u_n\}$, such that

$$u_n \rightharpoonup u \qquad \text{in } L^p(0,T;V) \tag{41}$$

$$\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \qquad \text{in } L^{p'}(0,T;V') \tag{42}$$

$$\varepsilon_n \frac{\partial^2 u_n}{\partial t^2} \rightharpoonup 0 \qquad \text{in } L^{p'}(0,T;V') \tag{43}$$

Proof. First of all, one has the following estimates for $\{u_n\}$

$$\varepsilon_n \left\| \frac{\partial u_n}{\partial t} \right\|_{L^2(Q)} \le c, \quad c > 0 \ \forall\, n \in \mathbb{N} \tag{44}$$

$$\| u_n \|_{L^p(0,T;V)} \le c, \quad c > 0 \ \forall\, n \in \mathbb{N}. \tag{45}$$

Indeed, using the coerciveness of A, (25) and the very definition of ε_n, one gets

$$\begin{aligned} \alpha \| u_n \|^p &\le \varepsilon_n \left\| \frac{\partial u_n}{\partial t} \right\|^2_{L^2(Q)} + \langle A u_n, u_n \rangle \\ &\le \varepsilon_n \left\| \frac{\partial^2 u_n}{\partial t^2} \right\|^2_{L^2(Q)} + \left\langle \frac{\partial u_n}{\partial t}, u_n \right\rangle + \langle A u_n, u_n \rangle \\ &\le \| A_n u_n \|_{L^p(0,T;V')} \| u_n \| . \end{aligned} \tag{46}$$

Thus, (45) follows from Step VII. On the other hand, using againg (45) and Step VII one deduces $\varepsilon_n \left\| \frac{\partial u_n}{\partial t} \right\|_{L^2(Q)} \le \|A_n u_n\|_{L^{p'}(0,T;V')} \|u_n\| \le c$, so (44) follows.

Then, by a similar argument to that given in [5, Chapter 3, Theorem 7.1], it follows that the sequence $\{\frac{\partial u_n}{\partial t}\}$ is bounded in $L^p(0,T;V)$ and that $\{\frac{\partial^2 u_n}{\partial t^2}\}$ is contained in $L^{p'}(0,T;V')$. Therefore, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that (41) and (42) hold. As for (43), one notes that

$$\varepsilon_n \frac{\partial^2 u_n}{\partial t^2} = A_n u_n - \frac{\partial u_n}{\partial t} - A u_n$$

so (43) follows, for a suitable subsequence of $\{u_n\}$, from the fact that $\varepsilon_n \langle \frac{\partial^2 u_n}{\partial t^2}, v \rangle$ converges to zero, for every $v \in \left\{ v \in L^p(0,T;V) : \frac{\partial v}{\partial t} \in L^p(0,T;V),\ v(0) = 0 \right\}$ which is a dense subspace of $L^p(0,T;V)$. □

Step IX. The element $u \in W$ is the solution of problem (6) with $u_0 = 0$, $g = 0$.

Proof. Through the theory of pseudo-monotone operators (see [1]), it is easy to see that the T-monotonicity, boundedness and continuity of A imply, using (41), that

$$\liminf \langle A u_n, u_n - v \rangle \ge \langle A u, u - v \rangle \quad \forall\, v \in L^p(0,T;V). \tag{47}$$

On the other hand, the lower semicontinuity of $\frac{\partial}{\partial t}$ on the space

$$\left\{ v \in L^p(0,T;V) : \frac{\partial}{\partial t} \in L^{'}(0,T;V^{'}) \right\}$$

(deduced from its positivity) implies

$$\liminf \left\langle \frac{\partial u_n}{\partial t}, u_n - v \right\rangle \geq \left\langle \frac{\partial u}{\partial t}, u - v \right\rangle \quad \forall\, v \in L^p(0,T;V). \tag{48}$$

Moreover, choosing, for any $v \in L^p(0,T;V)$ with $\psi_1 \leq v \leq \psi_2$, the sequence $\{v_n\}$ as in Step III, then Step VIII, (47), (48) and the fact that the positive cone in $L^p(0,T;V)$ is weakly closed in this space, imply that u is the solution of (6). □

Step X. The solution u of (6) with $u_0 = 0$ and $g = 0$, verifies the estimates (10).

Proof. It is a consequence of the the passage to the limit as $n \to +\infty$ in (39), the convergences obtained in the proof of Step VII and that $\langle A_n u_n, v\rangle \to \left\langle \frac{\partial u}{\partial t} + A_u, v \right\rangle$ as it follows from Step VII and from (41). □

Step XI - Conclusions. The general case without the restrictions $u_0 = 0$, $g = 0$.

Proof. Let us consider the solution $\widetilde{u}$ of the problem

$$\begin{cases} \widetilde{u} \in g + L^p(0,T;V),\ \frac{\partial \widetilde{u}}{\partial t} \in L^{p^{'}}(0,T;V^{'}) \\ \frac{\partial \widetilde{u}}{\partial t} + A\widetilde{u} = 0,\ \widetilde{u}(0) = u_0. \end{cases}$$

For the existence and uniqueness of such a solution $\widetilde{u}$ see [1] and replace the operator A with the operator $\widetilde{A} : L^p(0,T;V) \to L^{p^{'}}(0,T;V^{'})$ defined as

$$\widetilde{A}v = A(v + \widetilde{u}) - A(\widetilde{u}).$$

It is easy to check that $\widetilde{A}$ verifies the same conditions of A. Therefore, if $\overline{u}$ is the unique solution of the problem

$$\begin{cases} \overline{u} \in L^p(0,T;V), \quad \frac{\partial \overline{u}}{\partial t} \in L^{p'}(0,T;V'), \quad \psi_1 - \widetilde{u} \leq \overline{u} \leq \psi_2 - \widetilde{u}, \quad \overline{u}(0) = 0 \\ \langle \frac{\partial \overline{u}}{\partial t} + \widetilde{A}\overline{u}, v - \overline{u} \rangle \geq \langle f, v - \overline{u} \rangle \quad \forall\, v \in L^p(0,T;V),\ \psi_1 - \widetilde{u} \leq v \leq \psi_2 - \widetilde{u} \end{cases}$$

then it easy to verify that the element $u = \widetilde{u} + \overline{u} \in g + L^p(0,T;V)$ solves (6). Furthermore, $\overline{u}$ verifies the estimates in $L^{p'}(0,T;V')$

$$\begin{aligned} f - \left(\frac{\partial \psi_2}{\partial t} - \frac{\partial \widetilde{u}}{\partial t} + \widetilde{A}(\psi_2 - \widetilde{u}) - f \right)^{-} &\leq \frac{\partial \overline{u}}{\partial t} + \widetilde{A}\overline{u} \\ &\leq f + \left(\frac{\partial \psi_1}{\partial t} - \frac{\partial \widetilde{u}}{\partial t} + \widetilde{A}(\psi_1 - \widetilde{u}) - f \right)^{+} \end{aligned}$$

Recalling that, for $i = 1, 2$,

$$\widetilde{A}(\psi_i - \widetilde{u}) = A\psi_i - A\widetilde{u}, \quad \widetilde{A}\overline{u} = Au - A\widetilde{u}, \quad \frac{\partial \widetilde{u}}{\partial t} + A\widetilde{u} = 0$$

one easily deduces the dual estimates (10). Therefore the statement of Theorem follows. □

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