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General Boundary Stabilization of Memory-Type Thermoelasticity with Second Sound

Salim A. Messaoudi and Aisha Al-Shehri

Abstract. In this paper we consider an n-dimensional system of visco-thermoelasticity with second sound, where a viscoelastic dissipation is acting on a part of the boundary. We prove some decay results for solutions with specific regular initial data. In this regard, polynomial and general decay results are established.

Keywords. Thermoelasticity, second sound, general decay, polynomial decay, relaxation function, boundary stabilization

Mathematics Subject Classification (2010). Primary 35L51, secondary 35B35, 35B40, 35L53

1. Introduction

In the classical thermoelasticity, the heat flux is given by Fourier's law. As a result, this theory predicts an infinite speed of heat propagation. This means that any thermal disturbance at one point has an instantaneous effect elsewhere in the body. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox and disturbances, which are almost entirely thermal, propagate in a finite speed. This phenomenon in dielectric crystals is called second sound (see [5]). To overcome this physical paradox, many theories have merged. One of these theories suggests that Fourier's law be replaced by so called Cattaneo's law. For results concerning existence, blow up, and asymptotic behavior of smooth, as well as weak solutions in heat conduction with second sound, we refer the reader to [5, 7–9, 11, 13, 15].

S. A. Messaoudi: Department of Mathematics and Statistics, KFUPM, Dhahran 31261, Saudi Arabia; messaoud@kfupm.edu.sa
A. Al-Shehri: Department of Mathematics, Girls' College, Dammam University, Saudi Arabia; a_karam100@hotmail.com

For thermoelasticity with second sound, many results have been established over the past three decades. Tarabek [26] treated problems related to

$$\begin{aligned} u_{tt} - a\left(u_x,\theta,q\right)u_{xx} + b\left(u_x,\theta,q\right)\theta_x &= \alpha_1\left(u_x,\theta\right)qq_x \\ \theta_t + g\left(u_x,\theta,q\right)q_x + d\left(u_x,\theta,q\right)u_{tx} &= \alpha_2\left(u_x,\theta\right)qq_t \\ \tau\left(u_x,\theta\right)q_t + q + k\left(u_x,\theta\right)\theta_x &= 0, \end{aligned} \tag{1.1}$$

in both bounded and unbounded-domain situations and established global existence results for small initial data. He also showed that these "classical" solutions tend to equilibrium as t tends to infinity; however, no rate of decay has been discussed. In his work, Tarabek used the usual energy argument and exploited some relations from the second law of thermodynamics to overcome the difficulty arising from the lack of Poincaré's inequality in the unbounded domains.

Concerning asymptotic behavior, Racke [21] discussed (1.1) and established exponential decay results for several linear and nonlinear initial boundary value problems. In particular he studied (1.1), with $\alpha_1 = \alpha_2 = 0$, and for a rigidly clamped medium with temperature hold constant on the boundary, i.e.,

$$u\left(t,0\right) = u\left(t,1\right) = 0, \quad \theta\left(t,0\right) = \theta\left(t,1\right) = \bar{\theta}, \quad t \geq 0$$

and showed that, for small enough initial data, classical solutions decay exponentially to the equilibrium state. Messaoudi and Said-Houari [14] extended the decay result of [21] to the case when $\alpha_1 \neq 0$, $\alpha_2 \neq 0$. Recently, Qin et al. [17] considered a one-dimensional nonlinear system of thermoelasticity with thermal memory and second sound and proved global existence and exponential decay of solution provided that the initial data are close to equilibrium and the relaxation function decays exponentially. Also, Racke and Wang [23] considered a nonlinear one-dimensional Cauchy problem of thermoelasticity with second sound, discussed the well-posedness and described the long-time behavior of the global small solutions, obtaining a polynomial decay rate.

For the multi-dimensional case ($n = 2, 3$), Racke [22] established an existence result for the following n-dimensional problem

$$\left\{\begin{aligned} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta &= 0 \quad \text{in } \Omega \times (0,+\infty) \\ c\theta_t + \kappa \operatorname{div} q + \beta \operatorname{div} u_t &= 0 \quad \text{in } \Omega \times (0,+\infty) \\ \tau_0 q_t + q + \kappa\nabla\theta &= 0 \quad \text{in } \Omega \times (0,+\infty) \\ u(.,0) = u_0,\ u_t(.,0) = u_1,\ \theta(.,0) = \theta_0,\ q(.,0) &= q_0 \quad \text{in } \Omega \\ u = \theta &= 0 \quad \text{on } \partial\Omega \times [0,+\infty), \end{aligned}\right. \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$, $u = u(x,t)$,

$q = q(x,t) \in \mathbb{R}^n$, and $\mu, \lambda, \beta, \gamma, \delta, \tau, \kappa$ are positive constants, where μ, λ are Lame moduli and τ_0 is the relaxation time, a small parameter compared to the others. In particular if $\tau_0 = 0$, (1.2) reduces to the system of classical thermoelasticity, in which the heat flux is given by Fourier's law instead of Cattaneo's law. He also proved, under the conditions $\operatorname{rot} u = \operatorname{rot} q = 0$, an exponential decay result for (1.2). This result applies automatically to the radially symmetric solution, since it is only a special case. Messaoudi [10] considered (1.2), in the presence of a source term in the first equation, and proved a local existence, as well as, a blow up result for solutions with negative initial energy. This result was later extended to certain solutions with positive initial energy by Messaoudi and Said-Houari [12].

Concerning Timoshenko systems of thermoelasticity with second sound, we quote the work by Messaoudi et al. [16], in which several linear and nonlinear problems have been treated and different exponential decay results have been established in the presence of an extra frictional damping. Fernández Sare and Racke [6] showed that, in the absence of the extra frictional damping, the coupling via Cattaneo's law causes loss of the exponential decay usually obtained in the case of coupling via Fourier's law [19]. This surprising property holds even for systems with history.

In this paper we are concerned with the following system

$$\begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta = 0 & \text{in } \Omega\times(0,+\infty) \\ c\theta_t + \kappa\operatorname{div} q + \beta\operatorname{div} u_t = 0 & \text{in } \Omega\times(0,+\infty) \\ \tau_0 q_t + q + \kappa\nabla\theta = 0 & \text{in } \Omega\times(0,+\infty) \\ u(.,0) = u_0,\ u_t(.,0) = u_1,\ \theta(.,0) = \theta_0,\ q(.,0) = q_0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0\times[0,+\infty) \\ u(x,t) = -\displaystyle\int_0^t g(t-s)\left(\mu\frac{\partial u}{\partial \upsilon} + (\mu+\lambda)(\operatorname{div} u)\nu\right)(s)ds & \text{on } \Gamma_1\times[0,+\infty) \\ \theta = 0 & \text{on } \partial\Omega\times[0,+\infty) \end{cases} \tag{1.3}$$

where $\{\Gamma_0, \Gamma_1\}$ is a partition of $\partial\Omega$, ν is the outward normal to $\partial\Omega$ and the kernel g is the relaxation function, which is positive and of general decay. The boundary condition on Γ_1 is the nonlocal boundary condition responsible for the memory effect.

Our goal is to obtain a general uniform stability for the solution energy of (1.3) under suitable conditions on the boundary and for kernels of general-type decay.

2. Notations and transformation

In this section we introduce some notations and prove some lemmas. In order to establish our result we shall make the following assumption:

(**H**) There exists x_0 in $\mathbb{R}^n$, for which $m(x) = x - x_0$ satisfies

$$m(x)\cdot\nu \geq \delta > 0, \quad \forall\, x \in \Gamma_1 \quad \text{and} \quad m(x)\cdot\nu \leq 0, \quad \forall\, x \in \Gamma_0.$$

First, we will use the boundary condition

$$u(x,t) = -\int_0^t g(t-s)\left(\mu\frac{\partial u}{\partial \nu} + (\mu+\lambda)(\operatorname{div} u)\nu\right)(s)ds, \quad x \in \Gamma_1,\ t \geq 0 \quad (2.1)$$

to estimate the boundary term $\mu\frac{\partial u}{\partial \nu} + (\mu+\lambda)\operatorname{div} u$. Defining the convolution product operator by

$$(g * \varphi)(t) = \int_0^t g(t-s)\varphi(s)ds,$$

and differentiating equation (2.1), we obtain

$$\mu\frac{\partial u}{\partial \upsilon} + (\mu+\lambda)(\operatorname{div} u)\nu + \frac{1}{g(0)}\left(g' * \left(\mu\frac{\partial u}{\partial \nu} + (\mu+\lambda)(\operatorname{div} u)\nu\right)\right) = -\frac{1}{g(0)}u_t$$

on $\Gamma_1 \times \mathbb{R}^+$. Applying Volterra's inverse operator, we get $\mu\frac{\partial u}{\partial \upsilon} + (\mu+\lambda)(\operatorname{div} u)\nu = -\frac{1}{g(0)}(u_t + k * u_t)$, where the resolvent kernel k satisfies $k + \frac{1}{g(0)}(g' * k) = -\frac{1}{g(0)}g'$. Taking $\eta = \frac{1}{g(0)}$, we arrive at

$$\mu\frac{\partial u}{\partial \nu} + (\mu+\lambda)(\operatorname{div} u)\nu = -\eta(u_t + k(0)u - k(t)u_0 + k' * u) \quad \text{on } \Gamma_1 \times \mathbb{R}^+. \quad (2.2)$$

Since we are interested in relaxation functions of more general decay, we would like to know if the resolvent kernel k, involved in (2.2), inherits some properties of the relaxation function g involved in $(1.3)_4$. The following Lemma answers this question.

Let $h : [0, +\infty) \to \mathbb{R}^+$ be continuous. Let k be its resolvent, that is

$$k(t) = h(t) + (k * h)(t). \quad (2.3)$$

It is well known that k is continuous and positive (see [4, 18]).

Lemma 2.1 ([17]). *Suppose that* $h(t) \leq c_0 e^{-\int_0^t \gamma(\zeta)d\zeta}$ *for* $\gamma : [0, +\infty) \to \mathbb{R}^+$*, a nonincreasing function satisfying, for some positive constant* $\varepsilon < 1$,

$$c_1 = \int_0^{+\infty} e^{-\int_0^s (1-\varepsilon)\gamma(\zeta)d\zeta}ds < \frac{1}{c_0}.$$

Then k *satisfies*

$$k(t) \leq \frac{c_0}{1 - c_0 c_1} e^{-\varepsilon\int_0^t \gamma(\zeta)d\zeta}.$$

Proof. Let $\delta(t) = \varepsilon\gamma(t)$ and denote $K(t) = k(t)e^{\int_0^t \delta(\zeta)d\zeta}$, $H(t) = h(t)e^{\int_0^t \delta(\zeta)d\zeta}$. Multiplying (2.3) by $e^{\int_0^t \delta(\zeta)d\zeta}$ we obtain

$$\begin{aligned} K(t) &= H(t) + \int_0^t [e^{\int_0^t \delta(\zeta)d\zeta} e^{-\int_0^{t-s} \delta(\zeta)d\zeta} K(t-s)h(s)]ds \\ &= H(t) + \int_0^t [e^{\int_{t-s}^t \delta(\zeta)d\zeta} e^{-\int_0^s \gamma(\zeta)d\zeta} K(t-s) e^{\int_0^s \gamma(\zeta)d\zeta} h(s)]ds \\ &\leq c_0 + c_0 \sup_{0\leq r\leq t} K(r) \int_0^t e^{-\int_0^s [\gamma(\zeta)-\varepsilon\gamma(\zeta+t-s)]d\zeta} ds. \end{aligned}$$

Using the fact that γ is nonincreasing we arrive at

$$K(t) \leq c_0 + c_0 \sup_{0\leq r\leq t} K(r) \int_0^t e^{-\int_0^s (1-\varepsilon)\gamma(\zeta)d\zeta} ds$$

which gives $\sup_{0\leq r\leq t} K(r) \leq c_0 + c_0 \sup_{0\leq r\leq t} K(r) \int_0^{+\infty} e^{-\int_0^s (1-\varepsilon)\gamma(\zeta)d\zeta} ds$, hence $K(t) \leq \sup_{0\leq r\leq t} K(r) \leq \frac{c_0}{1-c_0c_1}$. Therefore

$$k(t) \leq \frac{c_0}{1-c_0c_1} e^{-\varepsilon \int_0^t \gamma(\zeta)d\zeta}. \qquad \square$$

Remark 2.2. The result of [18] is only a special case. See also [17] for more details.

Example 2.3. If we take $\gamma(\zeta) = a\zeta^p$, $-1 < p < 0$ and assume that $h(t) \leq c_0 e^{-\frac{a}{p+1}t^{p+1}}$ then with an appropriate choice of $a > 0$, one can easily see that, for some positive constant $\varepsilon < 1$,

$$c_1 = \int_0^{+\infty} e^{-\frac{a(1-\varepsilon)}{p+1}s^{p+1}} ds < \frac{1}{c_0}.$$

Consequently, we get

$$k(t) \leq \beta e^{-\frac{\varepsilon a}{p+1}t^{p+1}}.$$

Based on Lemma 2.1, we will use the boundary relation (2.2) instead of $(1.3)_4$. Let's define

$$\begin{aligned} (g \diamond \varphi)(t) &:= \int_0^t g(t-s) \left|\varphi(t) - \varphi(s)\right|^2 ds \\ (g \circ \varphi)(t) &:= \int_0^t g(t-s)(\varphi(t) - \varphi(s))ds. \end{aligned} \tag{2.4}$$

By using Hölder's inequality, we have

$$\left|(g \circ \varphi)(t)\right|^2 \leq \left(\int_0^t |g(s)|\, ds\right) (|g| \diamond \varphi)(t). \tag{2.5}$$

Lemma 2.4 ([1, 2, 18]). *If $g, \varphi \in C^1(\mathbb{R}^+)$ then*

$$(g * \varphi)\varphi_t = -\frac{1}{2}g(t)\left|\varphi(t)\right|^2 + \frac{1}{2}g' \diamond \varphi - \frac{1}{2}\frac{d}{dt}(g \diamond \varphi - \left(\int_0^t g(s)ds)\left|\varphi(t)\right|^2\right). \quad (2.6)$$

If we define

$$V = \left\{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_0\right\},$$

the well-posedness of system (1.3) is presented in the following theorem, which can be proved, using the "standard" Galerkin method. See [1, 3] and the reference therein.

Theorem 2.5. *Let $k \in W^{2,1}(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$, $u_0 \in (H^2(\Omega) \cap V)$, $\theta_0 \in H_0^1(\Omega)$, $q_0 \in H^1(\Omega)$, and $u_1 \in V$, with*

$$\frac{\partial u_0}{\partial \nu} + \eta u_1 = 0 \quad \text{on } \Gamma_1. \quad (2.7)$$

Then there exists a unique strong solution u of system (1.3) such that

$$\begin{aligned} &u \in C(\mathbb{R}^+; H^2(\Omega) \cap V), && u_t \in C(\mathbb{R}^+; V), && u_{tt} \in C(\mathbb{R}^+; L^2(\Omega)) \\ &\theta \in C(\mathbb{R}^+; H_0^1(\Omega)), && \theta_t \in C(\mathbb{R}^+; L^2(\Omega)) \\ &q \in C(\mathbb{R}^+; H^1(\Omega)), && q_t \in C(\mathbb{R}^+; L^2(\Omega)). \end{aligned}$$

3. General decay

In this section we discuss the asymptotic behavior of the solutions of system (1.3) when the resolvent kernel k satisfies

$$k(0) > 0, \quad k(t) \geq 0, \quad k'(t) \leq 0, \quad k''(t) \geq \gamma(t)(-k'(t)), \quad (3.1)$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying the following conditions

$$\gamma(t) > 0, \quad \gamma'(t) \leq 0, \quad \text{and} \quad \int_0^{+\infty} \gamma(t)dt = +\infty. \quad (3.2)$$

Example 3.1. Let $k(t) = \frac{1}{\ln(2+t)}$, $t \geq 0$. Direct computations show that

$$k''(t) = \gamma(t)(-k'(t)) \text{ with } \gamma(t) = \frac{1}{t+2} + \frac{2}{(t+2)\ln(2+t)}.$$

It is clear that γ is decreasing (hence $\gamma'(t) \leq 0$). Moreover,

$$\int_0^t \gamma(s)ds = \ln(t+2) + 2\ln(\ln(t+2)) - \ln 2 - 2\ln(\ln 2) \to \infty \quad \text{as } t \to \infty.$$

It is a routine procedure to define the first-order energy of system (1.3) by (see Lemma 3.4 below)

$$
\begin{aligned}
E_1(t) = \frac{1}{2}\int_\Omega & \left[|u_t|^2 + \mu|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + c\,\theta^2 + \tau_0 q^2\right] \\
& - \frac{\eta}{2}\int_{\Gamma_1} k' \diamond u + \frac{\eta}{2}\int_{\Gamma_1} k(t)|u|^2 .
\end{aligned}
\tag{3.3}
$$

Now, we differentiate (1.3), with respect to t, to obtain

$$
\begin{cases}
u_{ttt} - \mu\Delta u_t - (\mu+\lambda)\nabla(\operatorname{div} u_t) + \beta\nabla\theta_t = 0 & \text{in } \Omega\times(0,+\infty) \\
c\theta_{tt} + \kappa \operatorname{div} q_t + \beta \operatorname{div} u_{tt} = 0 & \text{in } \Omega\times(0,+\infty) \\
\tau_0 q_{tt} + q_t + \kappa\nabla\theta_t = 0 & \text{in } \Omega\times(0,+\infty)
\end{cases}
\tag{3.4}
$$

and the boundary condition (2.2) to get

$$
\mu\frac{\partial u_t}{\partial\nu} + (\mu+\lambda)(\operatorname{div} u_t)\nu = -\eta(u_{tt} + k(0)u_t + k' * u_t) \quad \text{on } \Gamma_1\times\mathbb{R}^+ . \tag{3.5}
$$

Consequently, similar computations yield the second-order energy of system (1.3):

$$
\begin{aligned}
E_2(t) = \frac{1}{2}\int_\Omega & \left[|u_{tt}|^2 + \mu|\nabla u_t|^2 + (\mu+\lambda)(\operatorname{div} u_t)^2 + c\,\theta_t^2 + \tau_0 q_t^2\right] \\
& - \frac{\eta}{2}\int_{\Gamma_1} k' \diamond u_t + \frac{\eta}{2}\int_{\Gamma_1} k(t)|u_t|^2 .
\end{aligned}
$$

Theorem 3.2. *Given* $(u_0, u_1, \theta_0, q_0) \in (H^2(\Omega)\cap V)\times V\times H_0^1(\Omega)\times H^1(\Omega)$. *Assume that* (H), (3.1), *and* (3.2) *hold, with*

$$
\lim_{t\to+\infty} k(t) = 0 . \tag{3.6}
$$

Then, for some t_0 *large enough, we have*

$$
E_1(t) \le \frac{\Pi}{\int_0^t \gamma(s)ds} + \frac{\eta}{2}\int_\Gamma |u_0|^2 \int_{t_0}^t k^2(s)ds, \quad \forall t\ge t_0
$$

where Π *is a positive constant.*

Remark 3.3. Assumption (3.6) can be replaced by $||k||_\infty$ small enough as in [4].

The main idea of proof is to construct an appropriate Lyapunov functional $\mathcal{L}$, using the multiplier techniques. The proof of Theorem 3.2 will be achieved with the help of two lemmas.

Lemma 3.4. *Under the assumptions of Theorem* 3.2*, the energies of the solution of* (1.3) *satisfy*

$$E_1'(t) \leq -\int_\Omega |q|^2 - \frac{\eta}{2}\int_{\Gamma_1}|u_t|^2 + \frac{\eta}{2}k'(t)\int_{\Gamma_1}|u|^2 - \frac{\eta}{2}\int_{\Gamma_1}k''\diamond u + \frac{\eta}{2}k^2(t)\int_\Gamma |u_0|^2 \quad (3.7)$$

$$E_2'(t) \leq -\int_\Omega |q_t|^2 \leq 0. \quad (3.8)$$

Proof. By multiplying equation $(1.3)_1$ by u_t, $(1.3)_2$ by θ, and $(1.3)_3$ by q and integrating over Ω, using integration by parts, the boundary condition (2.2), and Lemma 2.4, one can easily find that

$$E_1'(t) = -\int_\Omega |q|^2 - \eta\int_{\Gamma_1}|u_t|^2 + \frac{\eta}{2}k'(t)\int_{\Gamma_1}|u|^2 - \frac{\eta}{2}\int_{\Gamma_1}k''\diamond u + \eta\int_\Gamma k(t)u_t u_0$$

holds for strong solutions. By using Young's inequality, (3.7) is obtained. Estimate (3.8) is established in a similar way using (3.4) and (3.5). □

Remark 3.5. a) If $u_0 = 0$ on Γ_1, then E is dissipative and $E(t) \leq E(0)$.

b) If $u_0 \neq 0$ on Γ_1, then

$$E(t) \leq E(0) + \frac{\eta}{2}\int_{\Gamma_1}|u_0|^2\int_0^t k^2(s) \leq A \quad (3.9)$$

for some $A > 0$.

Lemma 3.6. *Under the assumptions of Theorem* 3.2*, the solution of* (1.3) *satisfies, for any* $\varepsilon > 0$,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t\cdot[M + (n-1)u] \\
&\leq -\int_\Omega |u_t|^2 - \mu\int_\Omega|\nabla u|^2 - \frac{\mu+\lambda}{2}\int_\Omega(\operatorname{div} u)^2 + C\int_\Omega|\nabla\theta|^2 \\
&\quad - \frac{\mu\delta}{2}\int_{\Gamma_1}|\nabla u|^2 - (\mu+\lambda)\delta\int_{\Gamma_1}(\operatorname{div} u)^2 + C\left(1+\frac{1}{\varepsilon}\right)\int_{\Gamma_1}|u_t|^2 \\
&\quad + C\,k^2(t)\int_{\Gamma_1}|u|^2 - \frac{C}{\varepsilon}\int_{\Gamma_1}k'\diamond u + \varepsilon\int_{\Gamma_1}|u|^2 + C\left(1+\frac{1}{\varepsilon}\right)k^2(t)\int_\Gamma|u_0|^2,
\end{aligned} \quad (3.10)$$

where $M = (M_1, M_2, \ldots, M_n)^T$ *such that* $M_i = 2m\cdot\nabla u^i$ *and* $m = (x - x_0)$*, and* C *is a "generic" positive constant independent of* ε.

Proof. We multiply "scalarly" equation $(1.3)_1$ by $M + (n-1)u$ to obtain

$$\int_\Omega [u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta]\cdot[M + (n-1)u]dx = 0. \quad (3.11)$$

Now, we estimate the terms of (3.11) as follows

$$\int_\Omega u_{tt}\cdot[M+(n-1)u] = \frac{d}{dt}\int_\Omega u_t\cdot[M+(n-1)u] - \int_\Omega u_t\cdot M_t - (n-1)\int_\Omega |u_t|^2.$$

But $\int_\Omega u_t\cdot M_t = \sum_{i=1}^n \int_\Omega u_t^i(2m\cdot\nabla u_t^i) = \sum_{i=1}^n\int_\Omega m\cdot\nabla|u_t^i|^2 = -\sum_{i=1}^n\int_\Omega |u_t^i|^2 \operatorname{div} m$ $+\sum_{i=1}^n\int_\Gamma |u_t^i|^2(m\cdot\nu) = -n\int_\Omega|u_t|^2 + \int_{\Gamma_1}|u_t|^2(m\cdot\nu)$, hence

$$\int_\Omega u_{tt}\cdot[M+(n-1)u] = \frac{d}{dt}\int_\Omega u_t\cdot[M+(n-1)u] + \int_\Omega |u_t|^2 - \int_{\Gamma_1}|u_t|^2(m\cdot\nu). \quad (3.12)$$

To estimate the second term of (3.11), we start with

$$-\int_\Omega \Delta u\cdot M = -\sum_{i=1}^n\int_\Omega \Delta u^i(2m\cdot\nabla u^i) = \sum_{i=1}^n\int_\Omega \nabla u^i\cdot\nabla(2m\cdot\nabla u^i) - 2\sum_{i=1}^n\int_\Gamma (m\cdot\nabla u^i)\frac{\partial u^i}{\partial\nu}.$$

Using $\nabla u^i\cdot\nabla(2m\,\nabla u^i) = 2|\nabla u^i|^2 + m\cdot\nabla\left(|\nabla u^i|^2\right)$ we get

$$\begin{aligned}-\int_\Omega \Delta u\cdot M &= 2\sum_{i=1}^n\int_\Omega|\nabla u^i|^2 + \sum_{i=1}^n\int_\Omega m\cdot\nabla\left(|\nabla u^i|^2\right) - 2\sum_{i=1}^n\int_\Gamma(m\cdot\nabla u^i)\frac{\partial u^i}{\partial\nu}\\ &= -(n-2)\int_\Omega|\nabla u|^2 + \int_\Gamma|\nabla u|^2(m\cdot\nu) - 2\sum_{i=1}^n\int_\Gamma(m\cdot\nabla u^i)\frac{\partial u^i}{\partial\nu}.\end{aligned} \quad (3.13)$$

By exploiting

$$\frac{\partial u^i}{\partial x_k} = \nu_k\frac{\partial u^i}{\partial\nu} \quad \text{on } \Gamma_0, \quad (3.14)$$

we estimate the last term of (3.13)

$$\sum_{i=1}^n\int_{\Gamma_0}(m\cdot\nabla u^i)\frac{\partial u^i}{\partial\nu} = \sum_{i=1}^n\int_{\Gamma_0}(m\cdot\nu)\left(\frac{\partial u^i}{\partial\nu}\right)^2$$

Therefore, we arrive at

$$\begin{aligned}-\int_\Omega M\cdot\Delta u &= -(n-2)\int_\Omega|\nabla u|^2 + \int_\Gamma|\nabla u|^2(m\cdot\nu)\\ &\quad - 2\sum_{i=1}^n\int_{\Gamma_1}(m\cdot\nabla u^i)\frac{\partial u^i}{\partial\nu} - 2\sum_{i=1}^n\int_{\Gamma_0}\left(\frac{\partial u^i}{\partial\nu}\right)^2(m\cdot\nu)\end{aligned} \quad (3.15)$$

It is straightforward to see that

$$-\int_\Omega u\cdot\Delta u = \int_\Omega|\nabla u|^2 - \int_\Gamma u\cdot\frac{\partial u}{\partial\nu} = \int_\Omega|\nabla u|^2 - \int_{\Gamma_1}u\cdot\frac{\partial u}{\partial\nu}. \quad (3.16)$$

Combining (3.15) and (3.16) we obtain

$$-\int_\Omega \Delta u\cdot[M+(n-1)u] = \int_\Omega |\nabla u|^2 + \int_\Gamma |\nabla u|^2(m\cdot\nu) - 2\int_{\Gamma_0}|\nabla u|^2(m\cdot\nu) - \sum_{i=1}^n \int_{\Gamma_1}\frac{\partial u^i}{\partial\nu}(2m\cdot\nabla u^i + (n-1)u^i). \tag{3.17}$$

Next, we estimate the third term of (3.11) as follows

$$\begin{aligned}
&-\int_\Omega \nabla(\operatorname{div} u)\cdot[M+(n-1)u]\\
&= -(n-1)\int_\Omega u\cdot\nabla(\operatorname{div} u) - \int_\Omega M\cdot\nabla(\operatorname{div} u)\\
&= (n-1)\int_\Omega(\operatorname{div} u)^2 - (n-1)\int_\Gamma (u\cdot\nu)\operatorname{div} u - \int_\Omega M\cdot\nabla(\operatorname{div} u)
\end{aligned}$$

But,

$$\begin{aligned}
&-\int_\Omega M\cdot\nabla(\operatorname{div} u)\\
&= -\sum_{i=1}^n\int_\Omega 2(m\cdot\nabla u^i)\frac{\partial}{\partial x_i}(\operatorname{div} u)\\
&= 2\sum_{i=1}^n\int_\Omega \frac{\partial}{\partial x_i}(m\cdot\nabla u^i)\operatorname{div} u - 2\sum_{i=1}^n\int_\Gamma(\operatorname{div} u)(m\cdot\nabla u^i)\nu_i\\
&= 2\sum_{i,j=1}^n\int_\Omega \frac{\partial}{\partial x_i}\left[m_j\frac{\partial u^i}{\partial x_j}\right]\operatorname{div} u - 2\sum_{i,j=1}^n\int(\operatorname{div} u)\left(m_j\frac{\partial u^i}{\partial x_j}\right)\nu_i\\
&= 2\int_\Omega(\operatorname{div} u)^2 + 2\sum_{i,j=1}^n\int_\Omega m_j\frac{\partial^2 u^i}{\partial x_i\partial x_j}(\operatorname{div} u) - 2\sum_{i,j=1}^n\int_\Gamma(\operatorname{div} u)\left(m_j\frac{\partial u^i}{\partial x_j}\right)\nu_i\,.
\end{aligned}$$

We then use Green's formula to evaluate

$$\begin{aligned}
2\sum_{i,j=1}^n\int_\Omega m_j\frac{\partial}{\partial x_j}\left(\frac{\partial u^i}{\partial x_i}\right)\operatorname{div} u &= \sum_{j=1}^n\int_\Omega m_j\frac{\partial}{\partial x_j}\left((\operatorname{div} u)^2\right)\\
&= -\sum_{j=1}^n\int_\Omega\frac{\partial m_j}{\partial x_j}(\operatorname{div} u)^2 + \sum_{j=1}^n\int_\Gamma(\operatorname{div} u)^2(m_j\nu_j)\\
&= -n\int_\Omega(\operatorname{div} u)^2 + \int_\Gamma(\operatorname{div} u)^2(m\cdot\nu)
\end{aligned}$$

Hence $-\int_\Omega \nabla(\operatorname{div} u)M = -(n-2)\int_\Omega(\operatorname{div} u)^2 - 2\sum_{i,j=1}^n \int_\Gamma(\operatorname{div} u)\left(m_j \frac{\partial u^i}{\partial x_j}\right)\nu_i + \int_\Gamma(\operatorname{div} u)^2(m\cdot\nu)$. Again, use of (3.14) yields

$$\begin{aligned}2\sum_{i,j=1}^n \int_{\Gamma_0}(\operatorname{div} u)\left(m_j \frac{\partial u^i}{\partial x_j}\right)\nu_i &= 2\sum_{i,j=1}^n \int_{\Gamma_0}(\operatorname{div} u)\left(m_j \frac{\partial u^i}{\partial \nu}\nu_j\right)\nu_i \\ &= 2\sum_{i=1}^n \int_{\Gamma_0}(m\cdot\nu)\operatorname{div} u \frac{\partial u^i}{\partial \nu}\nu_i \\ &= 2\sum_{i=1}^n \int_{\Gamma_0}(m\cdot\nu)\operatorname{div} u \frac{\partial u^i}{\partial x_i} \\ &= 2\int_{\Gamma_0}(\operatorname{div} u)^2(m\cdot\nu).\end{aligned}$$

So, $-\int_\Omega M\cdot\nabla(\operatorname{div} u) = -(n-2)\int_\Omega(\operatorname{div} u)^2 + \int_{\Gamma_0}(\operatorname{div} u)^2(m\cdot\nu) + \int_{\Gamma_1}(\operatorname{div} u)^2(m\cdot\nu)$ $-2\int_{\Gamma_0}(\operatorname{div} u)^2(m\cdot\nu) - 2\sum_{i=1}^n \int_{\Gamma_1}(\operatorname{div} u)(m\cdot\nabla u^i)\nu_i$. Consequently, we have

$$\begin{aligned}&-\int_\Omega \nabla(\operatorname{div} u)\cdot[M+(n-1)u] \\ &= \int_\Omega(\operatorname{div} u)^2 - (n-1)\int_{\Gamma_1}\operatorname{div} u(u\cdot\nu) - \int_{\Gamma_0}(\operatorname{div} u)^2(m\cdot\nu) \\ &\quad + \int_{\Gamma_1}(\operatorname{div} u)^2(m\cdot\nu) - 2\sum_{i=1}^n \int_{\Gamma_1}(\operatorname{div} u)(m\cdot\nabla u^i)\nu_i\,.\end{aligned} \tag{3.18}$$

Finally, to estimate the fourth term of (3.11), we proceed as follows

$$\begin{aligned}\int_\Omega M\cdot\nabla\theta = \sum_{i,j=1}^n \int 2m_j \frac{\partial u^i}{\partial x_j}\frac{\partial\theta}{\partial x_i} &= -2\sum_{i,j=1}^n \int_\Omega \theta\frac{\partial}{\partial x_i}\left(m_j\frac{\partial u^i}{\partial x_j}\right) \\ &= -2\int_\Omega \theta\operatorname{div} u - 2\sum_{i,j=1}^n\int_\Omega m_j\,\theta\frac{\partial^2 u^i}{\partial x_i\,\partial x_j} \\ &= -2\int_\Omega \theta\operatorname{div} u + 2\sum_{i,j=1}^n\int_\Omega \frac{\partial}{\partial x_j}(m_j\theta)\,\frac{\partial u^i}{\partial x_i} \\ &= -2\int_\Omega \theta\operatorname{div} u + 2\sum_{j=1}^n\int_\Omega(\operatorname{div} u)\left[\frac{\partial m_j}{\partial x_j}\theta + m_j\frac{\partial\theta}{\partial x_j}\right] \\ &= 2(n-1)\int_\Omega\theta\operatorname{div} u + 2\int_\Omega(\operatorname{div} u)(m\cdot\nabla\theta)\,.\end{aligned}$$

By noting that $(n-1)\int_\Omega u\cdot\nabla\theta = -(n-1)\int_\Omega\theta\operatorname{div} u$, we easily conclude

$$\int_\Omega \nabla\theta\cdot[M+(n-1)u] = (n-1)\int_\Omega\theta\operatorname{div} u + 2\int_\Omega(\operatorname{div} u)(m\cdot\nabla\theta). \tag{3.19}$$

A combination of (3.12), (3.17)–(3.19) leads to

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t\cdot[M+(n-1)u]\\
&=-\int_\Omega |u_t|^2+\int_{\Gamma_1}|u_t|^2(m\cdot\nu)-\mu\left[\int_\Omega|\nabla\, u|^2-\int_{\Gamma_0}|\nabla u|^2(m\cdot\nu)\right.\\
&\quad\left.+\int_{\Gamma_1}|\nabla u|^2(m\cdot\nu)-\sum_{i=1}^n\int_{\Gamma_1}\frac{\partial u^i}{\partial\nu}(2m\cdot\nabla\, u^i+(n-1)u)\right]\\
&\quad-(\mu+\lambda)\left[\int_\Omega(\operatorname{div}u)^2-(n-1)\int_{\Gamma_1}(\operatorname{div}u)(\nu\cdot u)\right.\\
&\quad\left.-\int_{\Gamma_0}(\operatorname{div}u)^2(m\cdot\nu)+\int_{\Gamma_1}(\operatorname{div}u)^2(m\cdot\nu)-2\sum_{i=1}^n\int_{\Gamma_1}(\operatorname{div}u)(m\cdot\nabla\, u^i)\nu_i\right]\\
&\quad-\beta\left[(n-1)\int_\Omega\theta\operatorname{div}u+2\int_\Omega\operatorname{div}u(m\cdot\nabla\theta)\right].
\end{aligned}$$

Now, we use the fact $m\cdot\nu\le 0$ on Γ_0, to get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t\cdot[M+(n-1)u]\\
&\le-\int_\Omega|u_t|^2-\mu\int_\Omega|\nabla u|^2+\int_{\Gamma_1}|u_t|^2(m\cdot\nu)\\
&\quad-(\mu+\lambda)\int_\Omega(\operatorname{div}u)^2-\beta(n-1)\int_\Omega\theta\operatorname{div}u-2\beta\int_\Omega(\operatorname{div}u)(m\cdot\nabla\theta)\\
&\quad-\mu\int_{\Gamma_1}|\nabla u|^2(m\cdot\nu)+\mu\sum_{i=1}^n\int_{\Gamma_1}\frac{\partial u^i}{\partial\nu}(2m\cdot\nabla u^i+(n-1)u^i)\\
&\quad+(n-1)(\mu+\lambda)\int_{\Gamma_1}(\operatorname{div}u)(\nu\cdot u)-(\mu+\lambda)\int_{\Gamma_1}(\operatorname{div}u)^2(m\cdot\nu)\\
&\quad+\sum_{i=1}^n(\lambda+\mu)\int_{\Gamma_1}(\operatorname{div}u)(2m\cdot\nabla u^i)\nu_i.
\end{aligned}\tag{3.20}$$

By exploiting the boundary condition (2.2), we obtain

$$\begin{aligned}
\mu\frac{\partial u^i}{\partial\nu}+(u+\lambda)(\operatorname{div}u)\nu_i &= -\eta[u_t^i+k(0)u^i-k(t)u_0^i+k'*u^i]\\
&= -\eta\left[u_t^i+k(0)u^i+\int_0^t k'(t-s)u^i(s)-u^i(t)]ds\right]\\
&\quad-\eta\left[k(t)u_0^i+\int_0^t k'(t-s)u^i(t)ds\right]\\
&= -\eta[u_t^i-k'\circ u^i+k(t)u^i-k(t)u_0^i]\quad\text{on }\Gamma_1.
\end{aligned}$$

By using the assumption the assumption $m \cdot \nu \geq \delta > 0$ on Γ_1, estimate (3.20) reduces to

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t \cdot [M + (n-1)u] \\
&\leq -\int_\Omega |u_t|^2 - \mu\int_\Omega |\nabla u|^2 - (\mu+\lambda)\int_\Omega (\operatorname{div} u)^2 \\
&\quad - (n-1)\beta\int_\Omega \theta \operatorname{div} u - 2\beta\int_\Omega (\operatorname{div} u)(m\cdot\nabla\theta) - \mu\delta\int_{\Gamma_1}|\nabla u|^2 \\
&\quad - \eta\sum_{i=1}^n\int_{\Gamma_1}(2m\cdot\nabla u^i)\left(u_t^i - k'\circ u^i - k(t)u_0^i + k(t)u^i\right) - (\mu+\lambda)\delta\int_{\Gamma_1}(\operatorname{div} u)^2 \\
&\quad - (n-1)\eta\sum_{i=1}^n\int_{\Gamma_1} u^i\left(u_t^i - k'\circ u^i - k(t)u_0^i + k(t)u^i\right) + \int_{\Gamma_1}|u_t|^2(m\cdot\nu).
\end{aligned}$$

By using (2.5), Young's inequality, and the boundedness of $||m||_\infty$, one can easily have

$$\begin{aligned}
&\int_{\Gamma_1}(2m\cdot\nabla u^i)\left(u_t^i - k'\circ u^i - k(t)u_0^i + k(t)u^i\right) \\
&\leq \frac{\mu\delta}{2}\int_{\Gamma_1}|\nabla u^i|^2 + C\left[\int_{\Gamma_1}|u_t^i|^2 - \int_{\Gamma_1}k'\diamond u^i + k^2(t)\left(\int_{\Gamma_1}|u^i|^2 + \int_{\Gamma_1}|u_0^i|^2\right)\right].
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&-(n-1)\eta\int_{\Gamma_1}u^i\left(u_t^i - k'\circ u^i + k(t)u^i - k(t)u_0^i\right) \\
&\leq \varepsilon\int_{\Gamma_1}|u^i|^2 + \frac{C}{\varepsilon}\left[\int_{\Gamma_1}|u_t^i|^2 - \int_{\Gamma_1}k'\diamond u^i + k^2(t)\int_{\Gamma_1}|u_0^i|^2\right].
\end{aligned}$$

Also, using Poincaré's inequality, we have

$$-(n-1)\beta\int_\Omega\theta\operatorname{div} u - 2\beta\int_\Omega(\operatorname{div} u)(m\cdot\nabla\theta) \leq \frac{(\mu+\lambda)}{2}\int_\Omega(\operatorname{div} u)^2 + C\int_\Omega|\nabla\theta|^2.$$

By combining all the above, the assertion of the lemma is established. □

Proof of Theorem 3.2. Let

$$\mathcal{L}(t) = N\left(E_1(t) + E_2(t)\right) + \int_\Omega u_t\cdot[M + (n-1)u], \tag{3.21}$$

Therefore, from (3.7), (3.8), and (3.10) we obtain

$$\begin{aligned}\mathcal{L}'(t) \leq\ & -N\int_\Omega |q|^2 - N\int_\Omega |q_t|^2 - N\eta\int_{\Gamma_1}|u_t|^2 - \frac{N}{2}\eta\int_{\Gamma_1} k''\diamond u\\ & -\int_\Omega |u_t|^2 - \mu\int_\Omega |\nabla u|^2 - \frac{1}{2}(\mu+\lambda)\int_\Omega (\operatorname{div} u)^2\\ & -\frac{\mu\,\delta}{2}\int_{\Gamma_1}|\nabla u|^2 - (\mu+\lambda)\delta\int_{\Gamma_1}(\operatorname{div} u)^2 + C\int_{\Gamma_1}|u_t|^2\\ & +\frac{C}{\varepsilon}k^2(t)\int_{\Gamma_1}|u|^2 - \frac{C}{\varepsilon}\int_{\Gamma_1} k'\diamond u + \varepsilon\int_{\Gamma_1}|u|^2 + C\int_\Omega |\nabla\theta|^2\\ & + C\left(1+\frac{1}{\varepsilon}\right)k^2(t)\int_{\Gamma_1}|u_0|^2 + N\frac{\eta}{2}k^2(t)\int_{\Gamma_1}|u_0|^2.\end{aligned}$$

By using $(1.3)_3$ and $\int_{\Gamma_1}|u|^2 \leq c_0\int_\Omega |\nabla u|^2$, $\int_\Omega |\theta|^2 \leq c_p\int_\Omega |\nabla\theta|^2$, for $c_0, c_p > 0$, we arrive at

$$\begin{aligned}\mathcal{L}'(t) \leq\ & -(N-C_1)\int_\Omega |q|^2 - (N-C_1)\int_\Omega |q_t|^2 - \int_\Omega |u_t|^2 - \int_{\Gamma_1} k(t)|u|^2\\ & -\left(\mu - \varepsilon c_0 - \frac{C}{\varepsilon}k^2(t) - c_0 k(t)\right)\int_\Omega |\nabla u|^2\\ & -\frac{1}{2}(\mu+\lambda)\int_\Omega (\operatorname{div} u)^2 - \left(\frac{N}{2}\eta - C\right)\int_{\Gamma_1}|u_t|^2\\ & - C\int_{\Gamma_1} k'\diamond u - \int_\Omega |\theta|^2 + \left[N\frac{\eta}{2} + C\left(1+\frac{1}{\varepsilon}\right)\right]k^2(t)\int_{\Gamma_1}|u_0|^2.\end{aligned} \tag{3.22}$$

At this point, we choose our constants carefully. We first, fix ε so small that $\varepsilon\, c_0 = \frac{1}{2}\mu$ and pick N large enough so that

$$a_1 = \frac{N}{2}\eta - C \geq 0 \quad \text{and} \quad a_2 = N - C_1 > 0\,.$$

Thus, (3.22) simplifies to

$$\begin{aligned}\mathcal{L}'(t) \leq\ & -\int_\Omega |u_t|^2 - \left(\frac{\mu}{2} - \frac{C}{\varepsilon}k^2(t) - c_0 k(t)\right)\int_\Omega |\nabla u|^2 - \frac{1}{2}(\mu+\lambda)\int_\Omega (\operatorname{div} u)^2\\ & -\int_{\Gamma_1} k(t)|u|^2 - a_2\int_\Omega |q|^2 - \int_\Omega |\theta|^2 - C\int_{\Gamma_1} k'\diamond u + Ck^2(t)\int_{\Gamma_1}|u_0|^2\,.\end{aligned}$$

Using the fact that $\lim\limits_{t\to+\infty} k(t) = 0$ we get

$$\mathcal{L}'(t) \leq -\alpha\, E_1(t) - c_1\int_{\Gamma_1} k'\diamond u + Ck^2(t)\int_{\Gamma_1}|u_0|^2, \quad \forall\, t \geq t_0, \tag{3.23}$$

for some t_0 large enough and positive constants α and c_1.

We multiply both sides of (3.23) by $\gamma(t)$ to obtain

$$\gamma(t)\mathcal{L}'(t) \le -\alpha\gamma(t)E_1(t) - c_1\gamma(t)\int_{\Gamma_1} k' \diamond u + C\gamma(t)k^2(t)\int_{\Gamma_1} |u_0|^2, \quad \forall\, t \ge t_0 .$$

A simple calculation, using the boundedness and nonincreasingness of $\gamma(t)$, yields $\gamma(t)\mathcal{L}'(t) \le -\alpha\gamma(t)E_1(t) + c_1\int_{\Gamma_1} k'' \diamond u + Ck^2(t)\int_{\Gamma_1}|u_0|^2$. Using (3.7) we easily see that $\gamma(t)\mathcal{L}'(t) \le -\alpha\gamma(t)E_1(t) - cE_1'(t) + Ck^2(t)\int_{\Gamma_1}|u_0|^2$, for all $t \ge t_0$ where c is some positive constant. Thus, we get, using the fact that $\gamma'(t) \le 0$,

$$\frac{d}{dt}\left(\gamma(t)\mathcal{L}(t) + cE_1(t)\right) \le -\alpha\gamma(t)E_1(t) + Ck^2(t)\int_{\Gamma_1}|u_0|^2$$

or

$$\gamma(t)E_1(t) \le -\frac{1}{\alpha}\frac{d}{dt}\left(\gamma(t)\mathcal{L}(t) + cE_1(t)\right) + Ck^2(t)\int_{\Gamma_1}|u_0|^2, \quad \forall\, t \ge t_0. \tag{3.24}$$

By setting $\sigma(t) := \int_0^t \gamma(s)ds$, we easily see, using (3.7),

$$\begin{aligned}(\sigma(t)E_1(t))' &= \gamma(t)E_1(t) + \sigma(t)E_1'(t)\\ &\le -\frac{1}{\alpha}\frac{d}{dt}\left(\gamma(t)\mathcal{L}(t) + cE_1(t)\right) + Ck^2(t)\int_{\Gamma_1}|u_0|^2 + \frac{\eta}{2}\sigma(t)k^2(t)\int_{\Gamma_1}|u_0|^2\end{aligned}$$

for all $t \ge t_0$. A simple integration over (t_0, t) leads to

$$\begin{aligned}\sigma(t)E_1(t) \le\, &\sigma(t_0)E_1(t_0) + \frac{1}{\alpha}\left(\gamma(t_0)\mathcal{L}(t_0) + cE_1(t_0)\right) - \frac{1}{\alpha}\left(\gamma(t)\mathcal{L}(t) + cE_1(t)\right)\\ &+ C\int_{\Gamma_1}|u_0|^2\int_{t_0}^t k^2(s)ds + \frac{\eta}{2}\int_{\Gamma_1}|u_0|^2\int_{t_0}^t \sigma(s)k^2(s)ds\\ \le\, &\Pi + \frac{\eta}{2}\int_{\Gamma_1}|u_0|^2\int_{t_0}^t \sigma(s)k^2(s)ds.\end{aligned}$$

This implies that

$$E_1(t) \le \frac{\Pi}{\sigma(t)} + \frac{\eta}{2}\int_{\Gamma_1}|u_0|^2\int_{t_0}^t \frac{\sigma(s)}{\sigma(t)}k^2(s)ds, \quad \forall\, t \ge t_0 . \tag{3.25}$$

By noting that $\frac{\sigma(s)}{\sigma(t)} \le 1$ for all $s \le t$, the claim of theorem is established. □

Remark 3.7. Note that, if k' decays exponentially ($\gamma(t) \equiv a$) and $u_0 \in H_0^1(\Omega)$, then (3.25) reduces to

$$E_1(t) \le \frac{\Pi}{at}, \quad \forall\, t \ge t_0, \tag{3.26}$$

i.e., E_1 decays polynomially. However, if k' decays polynomially ($\gamma(t) \equiv \frac{a}{1+t}$) and $u_0 \in H_0^1(\Omega)$, then (3.25) reduces to

$$E_1(t) \le \frac{\Pi}{a\ln(1+t)}, \quad \forall\, t \ge t_0, \tag{3.27}$$

i.e., E_1 decays logarithmically.

4. Polynomial decay

In this section we show that, in the polynomial case, we can obtain a better estimate than the one derived from the general case. In other words, we will discuss the asymptotic behavior of the solutions of system (1.3) when the resolvent kernel k satisfies

$$k(0) > 0, \quad k(t) \geq 0, \quad k'(t) \leq 0, \quad k''(t) \geq a(-k'(t))^p, \quad 1 \leq p < \frac{3}{2}. \tag{4.1}$$

Remark 4.1. Condition $1 < p < \frac{3}{2}$ is made so that

$$\int_0^{+\infty} (-k'(s))^{2-p} \, ds < +\infty \quad \text{and} \quad \int_0^{+\infty} (-k'(s))^{\frac{1}{2}} \, ds < +\infty. \tag{4.2}$$

The main result in this section is

Theorem 4.2. *Given* $(u_0, u_1, \theta_0, q_0) \in (H^2(\Omega) \cap V) \times V \times H_0^1(\Omega) \times H^1(\Omega)$. *Assume that* (H) *and* (4.1) *hold with*

$$\lim_{t \to +\infty} k(t) \;= 0.$$

Then, for some t_0 *large enough, we have*

$$E_1(t) \leq \frac{\Pi}{t^{\frac{1}{2p-1}}} + \Pi \left(\int_{\Gamma_1} |u_0|^2 \int_{t_0}^t k^2(s) ds \right)^{\frac{1}{2p-1}}, \quad \forall\, t \geq t_0, \tag{4.3}$$

where Π *is a positive constant.*

In order to prove this theorem, we need the following

Lemma 4.3. *Under the assumption of Theorem* 4.2 *we have*

$$\left(\int_{\Gamma_1} -k' \diamond u \right)^{2p-1} \leq C_0 \int_{\Gamma_1} (-k')^p \diamond u, \tag{4.4}$$

for some constant $C_0 > 0$.

Proof. By using Hölder's inequality, with $q = \frac{2p-1}{2p-2}$ and $q' = 2p - 1$, and Remark 3.3, we get

$$
\begin{aligned}
&\int_{\Gamma_1} -k' \diamond u \\
&= \int_{\Gamma_1}\int_0^t -k'(t-s)\,(u(t)-u(s))^2\,dsd\sigma \\
&= \int_{\Gamma_1}\int_0^t (-k'(t-s))^{\frac{p-1}{2p-1}}(u(t)-u(s))^{\frac{4(p-1)}{2p-1}}(-k'(t-s))^{\frac{p}{2p-1}}(u(t)-u(s))^{\frac{2}{2p-1}}\,dsd\sigma \\
&\le \left(\int_{\Gamma_1}\int_0^t (-k'(t-s))^{\frac12}(u(t)-u(s))^2\,dsd\sigma\right)^{\frac{2p-2}{2p-1}} \\
&\quad\times\left(\int_{\Gamma_1}\int_0^t (-k'(t-s))^{p}(u(t)-u(s))^2\,dsd\sigma\right)^{\frac{1}{2p-1}} \\
&= \left(\int_0^t (-k'(t-s))^{\frac12}\int_{\Gamma_1}(u(t)-u(s))^2\,d\sigma ds\right)^{\frac{2p-2}{2p-1}}\left(\int_{\Gamma_1}(-k')^p\diamond u\right)^{\frac{1}{2p-1}} \\
&\le \left(2c_0\int_0^t (-k'(t-s))^{\frac12}\int_{\Omega}\left(|\nabla u(t)|^2+|\nabla u(s)|^2\right)dxds\right)^{\frac{2p-2}{2p-1}}\left(\int_{\Gamma_1}(-k')^p\diamond u\right)^{\frac{1}{2p-1}} \\
&\le \left(4c_0\int_0^t (-k'(t-s))^{\frac12}\,E_1(s)ds\right)^{\frac{2p-2}{2p-1}}\left(\int_{\Gamma_1}(-k')^p\diamond u\right)^{\frac{1}{2p-1}} \\
&\le \left(4c_0A\int_0^t (-k'(s))^{\frac12}\,ds\right)^{\frac{2p-2}{2p-1}}\left(\int_{\Gamma_1}(-k')^p\diamond u\right)^{\frac{1}{2p-1}} \\
&\le C_0^{\frac{1}{2p-1}}\left(\int_{\Gamma_1}(-k')^p\diamond u\right)^{\frac{1}{2p-1}}. \qquad \square
\end{aligned}
$$

Also, similar calculations gives

$$
|-k'\circ u|^2 \le \left(\int_0^t (-k'(s))^{2-p}\,ds\right)(|-k'|^p\diamond u). \tag{4.5}
$$

Lemma 4.4. *Under the assumptions of Theorem* 4.2, *the solution of* (1.3) *satisfies, for any* $\varepsilon > 0$,

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t\cdot[M+(n-1)u] \\
&\le -\int_\Omega |u_t|^2 - \mu\int_\Omega|\nabla u|^2 - \frac{\mu+\lambda}{2}\int_\Omega(\operatorname{div} u)^2 \\
&\quad + C\int_\Omega|\nabla\theta|^2 - \frac{\mu\delta}{2}\int_{\Gamma_1}|\nabla u|^2 - (\mu+\lambda)\delta\int_{\Gamma_1}(\operatorname{div} u)^2 + \frac{C}{\mu}\int_{\Gamma_1}|u_t|^2 \\
&\quad + C\,k^2(t)\int_{\Gamma_1}|u|^2 + \frac{C\mu}{\varepsilon}\int_{\Gamma_1}(-k')^p\diamond u + \varepsilon\int_{\Gamma_1}|u|^2 + Ck^2(t)\int_{\Gamma_1}|u_0|^2,
\end{aligned}
\tag{4.6}
$$

where C *is a "generic" positive constant.*

Proof. The proof goes exactly like that of Lemma 3.6, using (4.4) and (4.5). □

Proof of Theorem 4.2. Let $\mathcal{L}(t)$ be given as in (3.21). Similar calculations lead to

$$\begin{aligned}\mathcal{L}'(t) \leq & -\int_\Omega |u_t|^2 - \frac{\mu}{4}\int_\Omega |\nabla u|^2 - \frac{1}{2}(\mu+\lambda)\int_\Omega (\operatorname{div} u)^2 - a_1 \int_{\Gamma_1} |u_t| \\ & - a_1 \int_\Omega |q|^2 - \int_\Omega |\theta|^2 - a_3 \int_{\Gamma_1} (-k')^p \diamond u + Ck^2(t)\int_{\Gamma_1} |u_0|^2, \quad \forall\, t \geq t_0.\end{aligned}$$

for some t_0 large enough. Thus, we obtain, for a positive constant α,

$$\begin{aligned}\mathcal{L}'(t) \leq & -\alpha \left\{ \int_\Omega |u_t|^2 + \int_\Omega |\nabla u|^2 + \int_\Omega (\operatorname{div} u)^2 + \int_{\Gamma_1} k(t)|u_t| + \int_\Omega |q|^2 \right. \\ & \left. + \int_\Omega |\theta|^2 + \int_{\Gamma_1} (-k')^p \diamond u \right\} + Ck^2(t) \int_{\Gamma_1} |u_0|^2.\end{aligned} \tag{4.7}$$

Next, by using (3.3), (3.9) and (4.4), we get

$$\begin{aligned}E_1^{2p-1}(t) \leq & C \left\{ \int_\Omega |u_t|^2 + \int_\Omega |\nabla u|^2 + \int_\Omega (\operatorname{div} u)^2 + \int_{\Gamma_1} k(t)|u_t| + \int_\Omega |q|^2 \right. \\ & \left. + \int_\Omega |\theta|^2 + \right\} + C\left(\int_{\Gamma_1} -k' \diamond u\right)^{2p-1} \\ \leq & C \left\{ \int_\Omega |u_t|^2 + \int_\Omega |\nabla u|^2 + \int_\Omega (\operatorname{div} u)^2 + \int_{\Gamma_1} k(t)|u_t| + \int_\Omega |q|^2 \right. \\ & \left. + \int_\Omega |\theta|^2 + \int_{\Gamma_1} (-k')^p \diamond u \right\}.\end{aligned} \tag{4.8}$$

By combining (4.7) and (4.8), we arrive at $\mathcal{L}'(t) \leq -c_2 E_1^{2p-1}(t) + Ck^2(t)\int_{\Gamma_1} |u_0|^2$ for $c_2 > 0$. Therefore, recalling (3.7),

$$\begin{aligned}\left(tE_1^{2p-1}(t)\right)' &= E_1^{2p-1}(t) + tE_1'(t)E_1^{2p-2}(t) \\ &\leq -\frac{1}{c_2}\left[\mathcal{L}'(t) - Ck^2(t)\int_{\Gamma_1} |u_0|^2\right] + \frac{\eta}{2}A^{2p-2}tk^2(t)\int_{\Gamma_1} |u_0|^2,\end{aligned}$$

for all $t \geq t_0$. A simple integration over (t_0, t) gives

$$\begin{aligned}tE_1^{2p-1}(t) \leq & -\frac{1}{c_2}\left[\mathcal{L}'(t) - \mathcal{L}'(t_0) + C\int_{\Gamma_1} |u_0|^2 \int_{t_0}^t k^2(s)ds\right] \\ & + t_0 E_1^{2p-1}(t_0) + \frac{\eta}{2}A^{2p-2}\int_{\Gamma_1} |u_0|^2 \int_{t_0}^t sk^2(s)ds \\ \leq & \, C + \frac{\eta}{2}A^{2p-2}\int_{\Gamma_1} |u_0|^2 \int_{t_0}^t sk^2(s)ds,\end{aligned}$$

hence

$$E_1^{2p-1}(t) \le \frac{C}{t} + \frac{\eta}{2} A^{2p-2} \int_{\Gamma_1} |u_0|^2 \int_{t_0}^{t} \frac{s}{t} k^2(s) ds.$$

Therefore,

$$E_1(t) \le \frac{\Pi}{t^{\frac{1}{2p-1}}} + \Pi \left(\int_{\Gamma_1} |u_0|^2 \int_{t_0}^{t} k^2(s) ds \right)^{\frac{1}{2p-1}}, \quad \forall\, t \ge t_0. \tag{4.9}$$

This completes the proof of Theorem 4.2. □

Remark 4.5. Note that, if $p = 1$ then (3.25) and (4.9) give the same result. Moreover, E_1 decays at the rate t^{-1} when k' decays exponentially and $u_0 \in H_0^1(\Omega)$. However, if $1 < p < \frac{3}{2}$ then (4.9) gives a better decay estimate than (3.25).

Remark 4.6. If k' satisfies

$$k''(t) \ge a(-k'(t))^p, \quad \frac{3}{2} \le p < 2$$

then the best rate we may obtain is the logarithmic decay if $u_0 \in H_0^1(\Omega)$.

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