Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 31 (2012), 427–439
DOI: 10.4171/ZAA/1467


# A Remark on Hausdorff Measure in Obstacle Problems

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**Abstract.** In this paper, we consider the identical zero obstacle problem for the second order elliptic equation

$$-\operatorname{div} a(\nabla u) = -1 \quad \text{in } \mathcal{D}'(\Omega),$$

where $\Omega$ is an open bounded domain of $\mathbb{R}^N, N \geq 2$. We prove that the free boundary has finite $(N-1)$-Hausdorff measure, which extends the previous works by Caffarelli, Lee and Shahgholian for $p$-Laplacian equations with $p = 2, p > 2$ respectively and contains the singular case of $1 < p < 2$.

**Keywords.** Elliptic equation, obstacle problem, free boundary, Hausdorff measure
**Mathematics Subject Classification (2010).** Primary 35J15, secondary 35J75, 35J87

## 1. Introduction

In this paper, we consider the identical zero obstacle problem for the second order elliptic equation

$$-\operatorname{div} a(\nabla u) = -1 \quad \text{in } \Omega, \tag{1}$$

where $\Omega$ is an open bounded domain of $\mathbb{R}^N, N \geq 2$, and the function $a = a(\eta) : \mathbb{R}^N \to \mathbb{R}^N$ is continuous differentiable in $\eta \in \mathbb{R}^N \setminus \{0\}$. Moreover, assume that

$$\sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \eta_j}(\eta)\xi_i\xi_j \geq \gamma_0 |\eta|^{p-2}|\xi|^2, \tag{2}$$

$$\left|\frac{\partial a_i}{\partial \eta_j}(\eta)\right| \leq \gamma_1 |\eta|^{p-2}, \tag{3}$$

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The authors were supported by NSFC 10971088

for some positive constants $\gamma_0, \gamma_1 > 0$, all $\eta \in \mathbb{R}^N \backslash \{0\}$, and all $\xi \in \mathbb{R}^N, i, j = 1, \ldots, N$. The structural assumptions on $a$ can be found in [6, 14, 16] etc.

Given functions $g$ and $\psi$ in the Sobolev Space $W^{1,p}(\Omega), 1 < p < \infty$, we define

$$K_{g,\psi} = \{v \in W^{1,p}(\Omega);\ v - g \in W_0^{1,p}(\Omega),\ v \geq \psi,\ \text{a.e. in } \Omega\},$$

which is nonempty provided $(\psi - g)^+ \in W_0^{1,p}(\Omega)$.

A function $u$ in $K_{g,\psi}$ is a solution to the obstacle problem

$$- \operatorname{div} a(\nabla u) = f \quad \text{in } \mathcal{D}'(\Omega), \tag{4}$$

if

$$\int_\Omega a(\nabla u) \cdot (\nabla v - \nabla u) \mathrm{d}x \geq \int_\Omega f(v - u) \mathrm{d}x, \quad \forall\ v \in K_{g,\psi},$$

where $f = f(x)$ is a given function in some $L^q(\Omega)$.

According to the known results (see [4, 5, 7, 10, 12–15]), any bounded solution $u$ to (4) is $C^{1,\tau}(\Omega)$ for some $\tau \in (0, 1)$, when $q > N$. But there is only little information regarding the free boundary. In 1998, Caffarelli proved that the free boundary has locally finite $(N - 1)$-dimensional Hausdorff measure for Laplacian equation with identical zero obstacle (see [1]). In 2000, Karp et al. obtained a porosity result of the free boundary for $p$-obstacle problem $(p > 2)$ (see [8]). According to [11], a porous set has Hausdorff dimension not exceeding $N - C\delta^N$, where $C = C(N) > 0$ is some constant, $\delta$ is porosity constant. Then Lee and Shahgholian obtained $(N - 1)$-Hausdorff measure for the $p$-obstacle problem with $p > 2$ in 2003 (see [9]). But for $p < 2$, there is no any result. We should note that an important work for $A$-Laplacian obstacle problem has been done by Challal and Lyaghfouri et al. in recent years. In 2009, Challal and Lyaghfouri showed that porosity of free boundary remains valid in $A$-obstacle problem(see [3]). Then they obtain a $(N - 1)$-Hausdorff measure result in 2010 (see [2]). Recently, in [17], the authors also obtained the porosity of free boundary for $p$-Laplacian type equations associated with the operator

$$\underline{\mathcal{A}u = - \operatorname{div} a(\nabla u) \quad \text{in } \mathcal{D}'(\Omega).}$$

In this paper, using an idea of [9] (see also [2]), we establish the $(N - 1)$-Hausdorff measure for the elliptic equations associated with the operator $Au = -\text{div } a(\nabla u)$. Our result is a natural extension of the same property for $p$-obstacle problem obtained in [1, 9]. It is also an extension for the case $1 < p < 2$.

To deal with our problem, assume that

$$\partial\Omega \in C^{1,\alpha}, \quad g \in W^{1,p}(\Omega) \cap C^{1,\alpha}(\partial\Omega) \quad \text{for some } \alpha,\ 0 < \alpha < 1.$$

We will restrict ourselves to the solution in $K_{g,0}$. According to [14], there is a unique solution $u$ to (1) and $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$.

However, we should note that for general $f \in L^{\infty}(\Omega)$ and $\psi \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$, one can obtain the same result as in this paper.

## 2. Main result

Let $u$ be the solution to (1). If $\partial\{u>0\} \cap \partial\Omega \neq \emptyset$, due to the regularity of $\partial\Omega$, it is trivial that the free boundary $\partial\{u>0\} \cap \partial\Omega$ has finite $(N-1)-$dimensional Hausdorff measure. So in this paper, we only consider the situation that $\partial\{u>0\} \cap \partial\Omega = \emptyset$. In this case, there exists a ball $B_R(y) \subset \Omega$ with $u(y)=0$. In order to describe the results obtained in this paper, we assume that $\Omega = B_1$, where $B_1 = B_1(0)$ is the unit ball in $\mathbb{R}^N$, and without loss of generality, we assume that $0 \in \partial\{u>0\}$.

For any $x \in \partial\{u>0\} \subset B_1$, since $u$ attains its infimum , $|\nabla u(x)| = 0 \leq \delta^{\frac{p}{p-1}} (\delta > 0)$, so $x \in \{|\nabla u| \leq \delta^{\frac{p}{p-1}}\}$. Basing on this, we will establish the volume of the set $\{|\nabla u| \leq \delta^{\frac{p}{p-1}}\} \cap B_r(x_0) \cap \{u>0\}$, for any $x_0$ on the free boundary $\partial\{u>0\}$, which, after then, will be used to estimate the $(N-1)$-dimensional Hausdorff measure for the free boundary. To do this, we use the same notations as [2, 3, 9, 17],

$$O_\delta = \left\{x \in B_1; |\nabla u(x)| \leq \delta^{\frac{1}{p-1}}\right\}, \quad \text{and} \quad O_{\delta_i} = \left\{x \in B_1; |u_{x_i}(x)| \leq \delta^{\frac{1}{p-1}}\right\}.$$

According to [17], if $x_0 \in \partial\{u>0\} \cap B_{1-\delta}$ then there exist $y_0 \in \{u>0\}$ and $c>0$ ($c = c(N,p)$) such that

$$B_{c\delta}(y_0) \subset B_\delta(x_0) \cap O_\delta \cap \{u>0\}. \tag{5}$$

Denote by $\mathcal{L}^N$ the $N$-dimensional Lebesgue measure and by $\mathcal{H}^{N-1}$ the $(N-1)$-Hausdorff measure. The main result obtained in this paper is the following theorem

**Theorem 2.1.** *Let $u$ be the solution to* (1)*, then for any $x_0 \in \partial\{u>0\} \cap B_{\frac{1}{2}}$, and $0<r<\frac{1}{4}$, there holds*

$$\mathcal{H}^{N-1}(\partial\{u>0\} \cap B_r(x_0)) \leq C_0 r^{N-1},$$

*for a nonnegative constant $C_0 = C_0(p, N, \gamma_0, \gamma_1, \|\nabla u\|_\infty)$, $\|\nabla u\|_\infty := \|\nabla u\|_{L^\infty(\overline{B}_1)}$.*

## 3. Main proofs

We use ideas of [2, 9] to give our proofs. First of all, to prove Theorem 2.1, we need to introduce the following approximating equation (see [14])

$$-\text{div } a(\nabla u_\epsilon) + \vartheta_\epsilon(u_\epsilon) = 0 \quad \text{in } B_1, \quad u_\epsilon = g \quad \text{on } \partial B_1. \tag{6}$$

Here, for each $\epsilon > 0$, $\vartheta_\epsilon : \mathbb{R} \to [0,1]$ is the nondecreasing Lipschitz function given by

$$\vartheta_\epsilon(t) = 0,\ t < 0, \quad \vartheta_\epsilon(t) = \frac{t}{\epsilon},\ 0 < t \le \epsilon, \quad \text{and} \quad \vartheta_\epsilon(t) = 1,\ t > \epsilon.$$

According to [14], there exists a unique solution $u_\epsilon$ to (6) which converges to the solution $u$ to (1) in $C^{1,\theta}(\overline{B}_1)$ for some $\theta,\ 0 < \theta < 1$. Then we have

**Proposition 3.1.** $\frac{|\vartheta_\epsilon(u_\epsilon)|^2}{\gamma_1^2} \le \left[|\nabla u_\epsilon|^{p-2}|D^2 u_\epsilon|\right]^2$ *in* $B_1$.

*Proof.* Indeed, $\vartheta_\epsilon(u_\epsilon) = \text{div } a(\nabla u_\epsilon) = \sum_{i,j=1}^N a^i_{u_{\epsilon x_j}} u_{\epsilon x_j x_i}$. The assumption (3) gives that

$$|\vartheta_\epsilon(u_\epsilon)|^2 \le \left(\sum_{i=1}^N \sum_{j=1}^N \gamma_1 |\nabla u_\epsilon|^{p-2} |u_{\epsilon x_i x_j}|\right)^2 = \gamma_1^2 \left[|\nabla u_\epsilon|^{p-2}|D^2 u_\epsilon|\right]^2. \qquad \square$$

In the following proofs, we consider two cases, $1 < p \le 2$ and $p > 2$.

**Case I.** Firstly, for $1 < p \le 2$, we claim

**Proposition 3.2.** *There is a positive constant* $M_0 = M_0(p, N, \gamma_0, \gamma_1, \|\nabla u\|_\infty)$ *such that for small* $\epsilon$*, there holds*

$$\int_{B_{\frac{r}{2}}} \left[|\nabla u_\epsilon(x)|^{p-2}|D^2 u_\epsilon(x)|\right]^2 \mathrm{d}x \le M_0 r^{N-2}, \quad \forall\ 0 < r < 1.$$

*Proof.* Let $G_\epsilon(t) = (\epsilon + t^2)^{\frac{p-2}{2}} t$, $t \in (-\infty, +\infty)$. Further $\Phi = G(u_{\epsilon x_i})\varphi^2$, where $\varphi \in \mathcal{D}(B_{\frac{3r}{4}})$ satisfying

$$\begin{cases} 0 \le \varphi \le 1, & \text{in } B_{\frac{3r}{4}}, \\ \varphi = 1, & \text{in } B_{\frac{r}{2}}, \\ |\nabla \varphi| \le \frac{4}{r}, & \text{in } B_{\frac{3r}{4}}. \end{cases}$$

Now differentiating equation (6) with respect to $x_i$, then multiplying it by $\Phi$ and taking integrating over $B_{\frac{3r}{4}}$, we get $\int_{B_{\frac{3r}{4}}} \left[(-\text{div } a(\nabla u_\epsilon))_{x_i} + (\vartheta_\epsilon(u_\epsilon))_{x_i}\right] \Phi \mathrm{d}x = 0$. Then we have

$$\int_{B_{\frac{3r}{4}}} a(\nabla u_\epsilon)_{x_i} \cdot \nabla \Phi \mathrm{d}x = -\int_{B_{\frac{3r}{4}}} (\vartheta_\epsilon(u_\epsilon))_{x_i} \Phi \mathrm{d}x. \tag{7}$$

The left hand of (7) becomes

$$
\begin{aligned}
I^i =& \int_{B_{\frac{3r}{4}}} \sum_{k=1}^{N} \left( \sum_{j=1}^{N} a^k_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} \right) \Phi_{x_k} \mathrm{d}x \\
=& \sum_{k=1}^{N} \int_{B_{\frac{3r}{4}}} \sum_{j=1}^{N} a^k_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} \left[ (p-2) u^2_{\epsilon x_i} (\epsilon + u^2_{\epsilon x_i})^{\frac{p-4}{2}} + (\epsilon + u^2_{\epsilon x_i})^{\frac{p-2}{2}} \right] u_{\epsilon x_i x_k} \varphi^2 \mathrm{d}x \\
&+ \sum_{k=1}^{N} \int_{B_{\frac{3r}{4}}} 2 \sum_{j=1}^{N} a^k_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} (\epsilon + u^2_{\epsilon x_i})^{\frac{p-2}{2}} u_{\epsilon x_i} \varphi \varphi_{x_k} \mathrm{d}x \\
=:& I^i_1 + I^i_2.
\end{aligned}
\tag{8}
$$

By (2) and $1 < p \leq 2$, we get

$$
\begin{aligned}
I^i_1 &\geq \sum_{k=1}^{N} \int_{B_{\frac{3r}{4}}} \sum_{j=1}^{N} a^k_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} u_{\epsilon x_i x_k} (p-1)(\epsilon + u^2_{\epsilon x_i})^{\frac{p-2}{2}} \varphi^2 \mathrm{d}x \\
&\geq (p-1)\gamma_0 \int_{B_{\frac{3r}{4}}} |\nabla u_\epsilon|^{p-2} |\nabla u_{\epsilon x_i}|^2 (\epsilon + u^2_{\epsilon x_i})^{\frac{p-2}{2}} \varphi^2 \mathrm{d}x.
\end{aligned}
\tag{9}
$$

By (3) and Cauchy's inequality with $\epsilon$, we have

$$
\begin{aligned}
|I^i_2| \leq& \int_{B_{\frac{3r}{4}}} 2N\gamma_1 |\nabla u_\epsilon|^{p-2} |\nabla u_{\epsilon x_i}| (\epsilon + u^2_{\epsilon x_i})^{\frac{p-2}{2}} |u_{\epsilon x_i}| \varphi |\nabla \varphi| \mathrm{d}x \\
\leq& \frac{\gamma_0(p-1)}{2} \int_{B_{\frac{3r}{4}}} |\nabla u_\epsilon|^{p-2} (\epsilon + u^2_{\epsilon x_i})^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}|^2 \varphi^2 \mathrm{d}x \\
&+ \frac{2N^2\gamma_1^2}{(p-1)\gamma_0} \int_{B_{\frac{3r}{4}}} |\nabla u_\epsilon|^{p-2} (\epsilon + u^2_{\epsilon x_i})^{\frac{p-2}{2}} |u_{\epsilon x_i}|^2 |\nabla \varphi|^2 \mathrm{d}x \\
\leq& \frac{\gamma_0(p-1)}{2} \int_{B_{\frac{3r}{4}}} |\nabla u_\epsilon|^{p-2} (\epsilon + u^2_{\epsilon x_i})^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}|^2 \varphi^2 \mathrm{d}x \\
&+ \frac{2N^2\gamma_1^2}{(p-1)\gamma_0} \int_{B_{\frac{3r}{4}}} |u_{\epsilon x_i}|^{p-2} |u_{\epsilon x_i}|^{p-2} |u_{\epsilon x_i}|^2 |\nabla \varphi|^2 \mathrm{d}x \\
\leq& \frac{\gamma_0(p-1)}{2} \int_{B_{\frac{3r}{4}}} |\nabla u_\epsilon|^{p-2} (\epsilon + u^2_{\epsilon x_i})^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}|^2 \varphi^2 \mathrm{d}x \\
&+ \frac{2N^2\gamma_1^2}{(p-1)\gamma_0} \int_{B_{\frac{3r}{4}}} |\nabla u_\epsilon|^{2(p-1)} |\nabla \varphi|^2 \mathrm{d}x.
\end{aligned}
\tag{10}
$$

The right hand of (7) becomes

$$
I^i = -\int_{B_{\frac{3r}{4}}} \vartheta'_\epsilon(u_\epsilon) u_{\epsilon x_i} (\epsilon + u^2_{\epsilon x_i})^{\frac{p-2}{2}} u_{\epsilon x_i} \varphi^2 \mathrm{d}x \leq 0. \tag{11}
$$

By (7)–(11) and the choice of $\varphi$, we have

$$\int_{B_{\frac{r}{2}}}(\epsilon+u_{\epsilon x_i}^2)^{\frac{p-2}{2}}|\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|^2\mathrm{d}x\leq\frac{64N^2\gamma_1^2}{\gamma_0^2r^2(p-1)^2}\int_{B_{\frac{3r}{4}}}|\nabla u_\epsilon|^{2(p-1)}\mathrm{d}x.$$

Since $p<2$, we have

$$\int_{B_{\frac{r}{2}}}(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}|\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|^2\mathrm{d}x\leq\frac{64N^2\gamma_1^2}{\gamma_0^2r^2(p-1)^2}\int_{B_{\frac{3r}{4}}}|\nabla u_\epsilon|^{2(p-1)}\mathrm{d}x. \quad (12)$$

Summing up (12) from $i=1$ to $N$, we get

$$\int_{B_{\frac{r}{2}}}\left[(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}|D^2u_\epsilon|\right]^2\mathrm{d}x\frac{64N^2\gamma_1^2}{\gamma_0^2r^2(p-1)^2}\int_{B_{\frac{3r}{4}}}|\nabla u_\epsilon|^{2(p-1)}\mathrm{d}x. \quad (13)$$

Since $u_\epsilon\to u$ in $C^{1,\theta}(\overline{B}_1)$, for small $\epsilon$, there exists a positive constant $M'=M'(\|\nabla u\|_\infty)$ such that $|\nabla u_\epsilon|\leq M'$ in $B_{\frac{3}{4}}$, which and (13) imply that $D^2u_\epsilon\in L^2(B_{\frac{r}{2}})$. Furthermore, we can deduce that $D^2u\in L^2(B_{\frac{r}{2}})$. Moreover, as $\epsilon\to 0$,

$$\int_{B_{\frac{r}{2}}}\left\{\left[(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}|D^2u_\epsilon|\right]^2-(|\nabla u|^{p-2}|D^2u|)^2\right\}\mathrm{d}x\to 0,$$
$$\int_{B_{\frac{r}{2}}}\left\{\left[|\nabla u_\epsilon|^{p-2}|D^2u_\epsilon|\right]^2-(|\nabla u|^{p-2}|D^2u|)^2\right\}\mathrm{d}x\to 0.$$

Then we have

$$\int_{B_{\frac{r}{2}}}\left\{\left[(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}|D^2u_\epsilon|\right]^2-(|\nabla u_\epsilon|^{p-2}|D^2u_\epsilon|)^2\right\}\mathrm{d}x\to 0. \quad (14)$$

So for $\epsilon$ small enough, by (13) and (14), we can obtain the desired result. □

Now we claim

**Lemma 3.3.** *For any ball $B_r(x_0)\subset B_{\frac{1}{2}}$, with $x_0\in\partial\{u>0\}\cap B_{\frac{1}{2}}$ and $r<\frac{1}{2}$, there holds*

$$\int_0^1\mathcal{L}^N(O_\delta\cap B_{rs}(x_0)\cap\{u>0\})ds\leq C_1\delta r^{N-1},$$

*where $\delta>0$ is arbitrary, $C_1=C_1(p,N,\gamma_0,\gamma_1,\|\nabla u\|_\infty)$ is a constant.*

*Proof.* Firstly define $O_\epsilon = \left\{|\nabla u_\epsilon| \le 2\delta^{\frac{1}{p-1}}\right\}$ and $O_{\epsilon_i} = \left\{|u_{\epsilon x_i}| \le 2\delta^{\frac{1}{p-1}}\right\}$. Then we have

$$O_\delta \cap B_{\frac{1}{2}} \subset O_\epsilon \cap B_{\frac{1}{2}}. \tag{15}$$

Indeed, there exists $\epsilon_0$ such that for $\epsilon \in (0,\epsilon_0)$ there holds $\|\nabla u_\epsilon - \nabla u\|_{\infty,\overline{B}_{\frac{1}{2}}} < \delta^{\frac{1}{p-1}}$. On the other hand, $|\nabla u_\epsilon| \le |\nabla u_\epsilon - \nabla u| + |\nabla u| \le \delta^{\frac{1}{p-1}} + \delta^{\frac{1}{p-1}} = 2\delta^{\frac{1}{p-1}}$.

Now differentiating equation (6) with respect to $x_i$ gives

$$-\operatorname{div}\left(\sum_{j=1}^N a_{u_{\epsilon x_j}} u_{\epsilon x_j x_i}\right) + \vartheta'_\epsilon(u_\epsilon) u_{\epsilon x_i} = 0. \tag{16}$$

Let

$$F(\eta) = \begin{cases} 2\delta^{\frac{1}{p-1}}(\epsilon + 4\delta^{\frac{2}{p-1}})^{\frac{p-2}{2}}, & \eta > 2\delta^{\frac{1}{p-1}}, \\ (\epsilon+\eta^2)^{\frac{p-2}{2}}\eta, & |\eta| \le 2\delta^{\frac{1}{p-1}}, \\ -2\delta^{\frac{1}{p-1}}(\epsilon + 4\delta^{\frac{2}{p-1}})^{\frac{p-2}{2}}, & \eta < -2\delta^{\frac{1}{p-1}}. \end{cases}$$

Then $F'(\eta) = \left[(p-2)\eta^2(\epsilon+\eta^2)^{\frac{p-4}{2}} + (\epsilon+\eta^2)^{\frac{p-2}{2}}\right]\chi_{\{|\eta|<2\delta^{\frac{1}{p-1}}\}}$. Multiplying (16) by $F(u_{\epsilon x_i})$ and taking integrating over $B_{rs}(x_0)$, we get

$$\begin{aligned} &\int_{B_{rs}(x_0)} \left(\sum_{j=1}^N a_{u_{\epsilon x_j}} u_{\epsilon x_j x_i}\right) \cdot \nabla F(u_{\epsilon x_i}) \mathrm{d}x + \int_{B_{rs}(x_0)} \vartheta'_\epsilon(u_\epsilon) u_{\epsilon x_i} F(u_{\epsilon x_i}) \mathrm{d}x \\ &= \int_{\partial B_{rs}(x_0)} \left(\sum_{j=1}^N a_{u_{\epsilon x_j}} u_{\epsilon x_j x_i}\right) F(u_{\epsilon x_i}) \nu \mathrm{d}S, \end{aligned} \tag{17}$$

where $\nu$ is the unit outward normal vector.

On one hand, by (3) and Proposition 3.2, we have

$$\begin{aligned} &\int_0^1 \int_{\partial B_{rs}(x_0)} \left(\sum_{j=1}^N a_{u_{\epsilon x_j}} u_{\epsilon x_j x_i}\right) F(u_{\epsilon x_i}) \nu \mathrm{d}S \mathrm{d}s \\ &\le \int_{B_r(x_0)} \sum_{k=1}^N \sum_{j=1}^N |a^k_{u_{\epsilon x_j}}||u_{\epsilon x_j x_i}|F(u_{\epsilon x_i})|\mathrm{d}x \\ &\le \int_{B_r(x_0)} N\gamma_1 |\nabla u_\epsilon|^{p-2} \sum_{j=1}^N |u_{\epsilon x_j x_i}||F(u_{\epsilon x_i})|\mathrm{d}x \\ &\le \int_{B_r(x_0)} N\gamma_1 |\nabla u_\epsilon|^{p-2}|D^2 u_\epsilon||F(u_{\epsilon x_i})|\mathrm{d}x \\ &\le N\gamma_1 \left(\int_{B_r(x_0)} \left[|\nabla u_\epsilon|^{p-2}|D^2 u_\epsilon|\right]^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_r(x_0)} |F(u_{\epsilon x_i})|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\le C_2 \delta r^{N-1}, \end{aligned} \tag{18}$$

where $C_2$ is a positive constant depending on $p, N, \gamma_0, \gamma_1, \|\nabla u\|_\infty$.

On the other hand, by (15) and (2) we have

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{B_{rs}(x_0)}\left(\sum_{j=1}^{N} a_{u_{\epsilon x_j}} u_{\epsilon x_j x_i}\right)\cdot \nabla F(u_{\epsilon x_i})\mathrm{d}x \\
&= \sum_{i=1}^{N}\int_{B_{rs}(x_0)\cap O_{\epsilon_i}} \sum_{k=1}^{N}\left[\sum_{j=1}^{N} a^k_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} F'(u_{\epsilon x_i}) u_{\epsilon x_i x_k}\right]\mathrm{d}x \\
&\geq (p-1)\sum_{i=1}^{N}\int_{B_{rs}(x_0)\cap O_{\epsilon_i}} \sum_{k=1}^{N}\left[\sum_{j=1}^{N} a^k_{u_{\epsilon x_j}} u_{\epsilon x_j x_i}(\epsilon + u^2_{\epsilon x_i})^{\frac{p-2}{2}} u_{\epsilon x_i x_k}\right]\mathrm{d}x \\
&\geq (p-1)\sum_{i=1}^{N}\int_{B_{rs}(x_0)\cap O_{\epsilon_i}} \gamma_0 |\nabla u_\epsilon|^{p-2}|\nabla u_{\epsilon x_i}|^2(\epsilon + u^2_{\epsilon x_i})^{\frac{p-2}{2}}\mathrm{d}x \\
&\geq (p-1)\gamma_0\sum_{i=1}^{N}\int_{B_{rs}(x_0)\cap O_{\epsilon_i}} (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}|\nabla u_{\epsilon x_i}|^2(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
&\geq (p-1)\gamma_0\int_{B_{rs}(x_0)\cap O_{\epsilon}} \left[(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}|D^2 u_\epsilon|\right]^2\mathrm{d}x \qquad (\text{by } O_\epsilon \subset O_{\epsilon_i}) \\
&\geq (p-1)\gamma_0\int_{B_{rs}(x_0)\cap O_{\delta}} \left[(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}|D^2 u_\epsilon|\right]^2\mathrm{d}x.
\end{aligned}
\tag{19}
$$

Moreover,

$$
\int_{B_{rs}(x_0)} \vartheta'_\epsilon(u_\epsilon) u_{\epsilon x_i} F(u_{\epsilon x_i})\mathrm{d}x \geq 0. \tag{20}
$$

For $\epsilon$ small enough, by (14), (17)–(20), we get

$$
\int_0^1\int_{B_{rs}(x_0)\cap O_\delta} |\nabla u_\epsilon|^{2(p-2)}|D^2 u_\epsilon|^2\mathrm{d}x\mathrm{d}s \leq C_1\delta r^{N-1},
$$

where $C_1$ is a constant depending on $p, N, \gamma_0, \gamma_1, \|\nabla u\|_\infty$. By Proposition 3.1, we get $\int_0^1\int_{B_{rs}(x_0)\cap O_\delta}|\vartheta_\epsilon(u_\epsilon)|^2\mathrm{d}x\mathrm{d}s \leq C_1\delta r^{N-1}$ Furthermore, we have

$$
\int_0^1\int_{B_{rs}(x_0)\cap O_\delta\cap\{u\geq\epsilon\}} |\vartheta_\epsilon(u_\epsilon)|^2\mathrm{d}x\mathrm{d}s \leq \int_0^1\int_{B_{rs}(x_0)\cap O_\delta} |\vartheta_\epsilon(u_\epsilon)|^2\mathrm{d}x\mathrm{d}s \leq C_1\delta r^{N-1}.
$$

According to [14, Theorem 2], $u_\epsilon \geq u$. By the definition of $\vartheta_\epsilon$, we have

$$
\int_0^1\int_{B_{rs}(x_0)\cap O_\delta\cap\{u\geq\epsilon\}} \mathrm{d}x\mathrm{d}s \leq C_1\delta r^{N-1}.
$$

Now letting $\epsilon \to 0$ implies that $\int_0^1 \mathcal{L}^N(B_{rs}(x_0)\cap O_\delta\cap\{u>0\})\mathrm{d}s \leq C_1\delta r^{N-1}$. This completes the proof of Lemma 3.3. □

**Case II.** Secondly, when $p > 2$, as Proposition 3.2, we can prove

**Proposition 3.4.** *There is a positive constant* $M_1 = M_1(p, N, \gamma_0, \gamma_1, \|\nabla u\|_\infty)$ *such that for small* $\epsilon$*, there holds*

$$\int_{B_{\frac{r}{2}}} \left[|\nabla u_\epsilon(x)|^{\frac{p-2}{2}} |D^2 u_\epsilon(x)|\right]^2 \mathrm{d}x \le M_1 r^{N-2}, \quad \forall\ 0 < r < 1.$$

*Proof.* Let $\Phi = u_{\epsilon x_i}\varphi^2$, where $\varphi \in \mathcal{D}(B_{\frac{3r}{4}})$ satisfying

$$\begin{cases} 0 \le \varphi \le 1, & \text{in } B_{\frac{3r}{4}}, \\ \varphi = 1, & \text{in } B_{\frac{r}{2}}, \\ |\nabla\varphi| \le \dfrac{4}{r}, & \text{in } B_{\frac{3r}{4}}. \end{cases}$$

Now differentiating equation (6) with respect to $x_i$, then multiplying it by $\Phi$ and integrating over $B_{\frac{3r}{4}}$, we get $\int_{B_{\frac{3r}{4}}} \left[(-\operatorname{div} a(\nabla u_\epsilon))_{x_i} + (\vartheta_\epsilon(u_\epsilon))_{x_i}\right] \Phi \mathrm{d}x = 0$. So we have

$$\int_{B_{\frac{3r}{4}}} a(\nabla u_\epsilon)_{x_i} \cdot \nabla\Phi \mathrm{d}x = -\int_{B_{\frac{3r}{4}}} (\vartheta_\epsilon(u_\epsilon))_{x_i} \Phi \mathrm{d}x. \tag{21}$$

The left hand of (21) becomes

$$\begin{aligned} I^i &= \int_{B_{\frac{3r}{4}}} \sum_{k=1}^N \left(\sum_{j=1}^N a^k_{u_{\epsilon x_j}} u_{\epsilon x_j x_i}\right) \Phi_{x_k} \mathrm{d}x \\ &= \sum_{k=1}^N \int_{B_{\frac{3r}{4}}} \sum_{j=1}^N a^k_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} (u_{\epsilon x_i x_k}\varphi^2 + 2u_{\epsilon x_i}\varphi\varphi_{x_k}) \mathrm{d}x \\ &= \sum_{k=1}^N \int_{B_{\frac{3r}{4}}} \sum_{j=1}^N a^k_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} u_{\epsilon x_i x_k} \varphi^2 \mathrm{d}x + 2\sum_{k=1}^N \int_{B_{\frac{3r}{4}}} \sum_{j=1}^N a^k_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} u_{\epsilon x_i} \varphi\varphi_{x_k} \mathrm{d}x \\ &=: I^i_1 + I^i_2. \end{aligned} \tag{22}$$

By (2), we have

$$I^i_1 \ge \int_{B_{\frac{3r}{4}}} \gamma_0 |\nabla u_\epsilon|^{p-2} |\nabla u_{\epsilon x_i}|^2 \varphi^2 \mathrm{d}x = \gamma_0 \int_{B_{\frac{3r}{4}}} \left[|\nabla u_\epsilon|^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}|\varphi\right]^2 \mathrm{d}x. \tag{23}$$

By (3) and Cauchy's inequality with $\epsilon$, we have

$$\begin{aligned} |I^i_2| &\le \int_{B_{\frac{3r}{4}}} \sum_{k=1}^N \sum_{j=1}^N 2\gamma_1 |\nabla u_\epsilon|^{p-2} |u_{\epsilon x_j x_i}| |\nabla u_\epsilon| \varphi |\nabla\varphi| \mathrm{d}x \\ &\le \int_{B_{\frac{3r}{4}}} 2N\gamma_1 |\nabla u_\epsilon|^{p-2} |\nabla u_{\epsilon x_i}| |\nabla u_\epsilon| \varphi |\nabla\varphi| \mathrm{d}x \\ &\le \frac{\gamma_0}{2} \int_{B_{\frac{3r}{4}}} \left[|\nabla u_\epsilon|^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}|\varphi\right]^2 \mathrm{d}x + \frac{2N^2\gamma_1^2}{\gamma_0} \int_{B_{\frac{3r}{4}}} \left[|\nabla u_\epsilon|^{\frac{p}{2}} |\nabla\varphi|\right]^2 \mathrm{d}x. \end{aligned} \tag{24}$$

The right hand of (21) becomes

$$I^i = -\int_{B_{\frac{3r}{4}}} \vartheta_\epsilon'(u_\epsilon) u_{\epsilon x_i} u_{\epsilon x_i} \varphi^2 \mathrm{d}x \le 0. \tag{25}$$

By (21)–(25) and the choice of $\varphi$, we have

$$\frac{\gamma_0}{2} \int_{B_{\frac{r}{2}}} \left[ |\nabla u_\epsilon|^{\frac{p-2}{2}} |\nabla u_{\epsilon x_i}| \right]^2 \mathrm{d}x \le \frac{32 N^2 \gamma_1^2}{\gamma_0 r^2} \int_{B_{\frac{3r}{4}}} |\nabla u_\epsilon|^p \mathrm{d}x. \tag{26}$$

Since $u_\epsilon \to u$ in $C^{1,\theta}(\overline{B}_{\frac{3}{4}})$, for small $\epsilon$, there exists a positive constant $M' = M'(\|\nabla u\|_\infty)$ such that $|\nabla u_\epsilon| \le M'$ in $B_{\frac{3}{4}}$. Summing up (26) from $i = 1$ to $N$, we can obtain the desired result. □

Now we claim

**Lemma 3.5.** *For any ball $B_r(x_0) \subset B_{\frac{1}{2}}$, with $x_0 \in \partial\{u > 0\} \cap B_{\frac{1}{2}}$ and $r < \frac{1}{2}$, there holds*

$$\int_0^1 \mathcal{L}^N(O_\delta \cap B_{rs}(x_0) \cap \{u > 0\}) ds \le C_1' \delta r^{N-1},$$

*where $\delta > 0$ is arbitrary, $C_1' = C_1'(p, N, \gamma_0, \gamma_1, \|\nabla u\|_\infty)$ is a constant.*

*Proof.* Let $F$ given by

$$F(\eta) = \begin{cases} 2^{p-1}\delta, & \eta > 2\delta^{\frac{1}{p-1}}, \\ 2^{p-2}\delta^{\frac{p-2}{p-1}}\eta, & |\eta| \le 2\delta^{\frac{1}{p-1}}, \\ -2^{p-1}\delta, & \eta < -2\delta^{\frac{1}{p-1}}. \end{cases}$$

For small $\epsilon$, as Lemma 3.3, we have

$$\begin{aligned} &\int_{B_{rs}(x_0)} \left( \sum_{j=1}^N a_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} \right) \cdot \nabla F(u_{\epsilon x_i}) \mathrm{d}x + \int_{B_{rs}(x_0)} \vartheta_\epsilon'(u_\epsilon) u_{\epsilon x_i} F(u_{\epsilon x_i}) \mathrm{d}x \\ &= \int_{\partial B_{rs}(x_0)} \left( \sum_{j=1}^N a_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} \right) F(u_{\epsilon x_i}) \nu \mathrm{d}S, \end{aligned} \tag{27}$$

where $\nu$ is the unit outward normal vector.

On one hand, by (3) and Proposition 3.4, we have

$$
\begin{aligned}
&\int_0^1 \int_{\partial B_{rs}(x_0)} \left( \sum_{j=1}^N a_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} \right) F(u_{\epsilon x_i}) \nu \mathrm{d}S \mathrm{d}s \\
&\leq \int_{B_r(x_0)} \sum_{k=1}^N \sum_{j=1}^N |a^k_{u_{\epsilon x_j}}||u_{\epsilon x_j x_i}||F(u_{\epsilon x_i})| \mathrm{d}x \\
&\leq \int_{B_r(x_0)} N\gamma_1 |\nabla u_\epsilon|^{p-2} \sum_{j=1}^N |u_{\epsilon x_j x_i}||F(u_{\epsilon x_i})| \mathrm{d}x \qquad (28)\\
&\leq \int_{B_r(x_0)} N\gamma_1 |\nabla u_\epsilon|^{p-2} |D^2 u_\epsilon||F(u_{\epsilon x_i})| \mathrm{d}x \\
&\leq 2^{p-1}\delta N\gamma_1 \left( \int_{B_r(x_0)} \left[ |\nabla u_\epsilon|^{\frac{p-2}{2}} |D^2 u_\epsilon| \right]^2 \mathrm{d}x \right)^{\frac{1}{2}} \left( \int_{B_r(x_0)} |\nabla u_\epsilon|^{p-2} \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq C_2' \delta r^{N-1},
\end{aligned}
$$

where $C_2'$ is a positive constant depending on $p, N, \gamma_0, \gamma_1, \|\nabla u\|_\infty$.

On the other hand, by (2), (16) and the fact that $\{|\nabla u_\epsilon| < 2\delta^{\frac{1}{p-1}}\} \subset \{|u_{\epsilon x_i}| < 2\delta^{\frac{1}{p-1}}\}$, we have

$$
\begin{aligned}
&\sum_{i=1}^N \int_{B_{rs}(x_0)} \left( \sum_{j=1}^N a_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} \right) \cdot \nabla F(u_{\epsilon x_i}) \mathrm{d}x \\
&= \sum_{i=1}^N \int_{B_{rs}(x_0)\cap O_{\epsilon_i}} \sum_{k=1}^N \left[ \sum_{j=1}^N a^k_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} F'(u_{\epsilon x_i}) u_{\epsilon x_i x_k} \right] \mathrm{d}x \\
&= \sum_{i=1}^N \int_{B_{rs}(x_0)\cap O_{\epsilon_i}} \sum_{k=1}^N \left( \sum_{j=1}^N a^k_{u_{\epsilon x_j}} u_{\epsilon x_j x_i} u_{\epsilon x_i x_k} \right) 2^{p-2} \delta^{\frac{p-2}{p-1}} \mathrm{d}x \qquad (29)\\
&\geq \sum_{i=1}^N \int_{B_{rs}(x_0)\cap O_\epsilon} \gamma_0 |\nabla u_\epsilon|^{p-2} |\nabla u_{\epsilon x_i}|^2 |\nabla u_\epsilon|^{p-2} \mathrm{d}x \qquad (\text{by } O_\epsilon \subset O_{\epsilon_i}) \\
&= \gamma_0 \int_{B_{rs}(x_0)\cap O_\epsilon} \left[ |\nabla u_\epsilon|^{p-2} |D^2 u_\epsilon| \right]^2 \mathrm{d}x \\
&\geq \gamma_0 \int_{B_{rs}(x_0)\cap O_\delta} \left[ |\nabla u_\epsilon|^{p-2} |D^2 u_\epsilon| \right]^2 \mathrm{d}x,
\end{aligned}
$$

where $O_{\epsilon_i}, O_\epsilon, O_\delta$ are defined as in Lemma 3.3. Moreover,

$$
\int_{B_{rs}(x_0)} \vartheta_\epsilon'(u_\epsilon) u_{\epsilon x_i} F(u_{\epsilon x_i}) \mathrm{d}x \geq 0. \qquad (30)
$$

For $\epsilon$ small enough, by (14), (27)–(30), we get

$$\int_0^1 \int_{B_{rs}(x_0)\cap O_\delta} |\nabla u_\epsilon|^{2(p-2)}|D^2 u_\epsilon|^2 \mathrm{d}x\mathrm{d}s \le C_1' \delta r^{N-1},$$

where $C_1'$ is a constant depending on $p, N, \gamma_0, \gamma_1, \|\nabla u\|_\infty$. As in Lemma 3.3, we can deduce that $\int_0^1 \mathcal{L}^N(B_{rs}(x_0)\cap O_\delta \cap \{u>0\})\mathrm{d}s \le C_1'\delta r^{N-1}$. This completes the proof of Lemma 3.5. □

Due to the above lemmas, we can exactly use the technique as [9] to prove Theorem 2.1 with $1<p<\infty$.

*Proof of Theorem* 2.1. Under the conditions of Lemma 3.3 (Lemma 3.5), firstly we can conclude there exists a positive constant $C_3 = C_3(p, N, \gamma_0, \gamma_1, \|\nabla u\|_\infty)$ such that

$$\mathcal{L}^N(O_\delta \cap B_r(x_0)\cap\{u>0\}) \le C_3\ \delta r^{N-1} \quad \text{for all } r<\frac{1}{4}.$$

If not, then there exists a ball $B_r(x_0)$ with center on the free boundary such that for any $k\in\mathbb{R}$, $\mathcal{L}^N(O_\delta\cap B_r(x_0)\cap\{u>0\}) \ge k\delta r^{N-1}$. But by Lemma 3.3 (Lemma 3.5) we have

$$\begin{aligned}\max\{C_1, C_1'\}\delta r^{N-1} &\ge \int_0^1 \mathcal{L}^N(O_\delta\cap B_{2rs}(x_0)\cap\{u>0\})\mathrm{d}s\\ &\ge \frac{1}{2}\mathcal{L}^N(O_\delta\cap B_r(x_0)\cap\{u>0\})\\ &\ge \frac{1}{2}k\delta r^{N-1},\end{aligned}$$

which is a contradiction for large $k$.

Secondly, due to Besicovitch covering theorem, let $\{B_\delta(x^i)\}_{i\in I}$ be finite coverings of $\partial\{u>0\}\cap B_r(x_0)$ with $x^i\in\partial\{u>0\}$, with at most $n$ overlapping at each point, where $n$ depends only on $N$. Then, by (5), we have

$$\begin{aligned}\sum_{i\in I}(C\delta)^N &\le \sum_{i\in I}\mathcal{L}^N(O_\delta\cap B_\delta(x^i)\cap\{u>0\})\\ &\le nC\mathcal{L}^N(O_\delta\cap B_r(x_0)\cap\{u>0\})\\ &\le C'\delta r^{N-1}.\end{aligned}$$

where $C, C'$ are positive constants. This proves Theorem 2.1. □

**Acknowledgement.** The authors would like to thank the referee for his/her careful reading and valuable suggestions of revising the manuscript, which made this article more readable.

## References


[1] Caffarelli, L. A., The obstacle problem revisited. *J. Fourier Anal. Appl.* 4 (1998)(4-5), 383 – 402.

[2] Challal, A., Lyaghfouri, A. and Rodrigues, J. F., On the $A$-obstacle problem and the Hausdorff measure of its free boundary. *Ann. Mat. Pura Appl.* 191 (2012)(1), 113 – 165.

[3] Challal, S. and Lyaghfouri, A., Porosity of free boundaries in $A$-obstacle problems. *Nonlin. Anal.* 70 (2009)(7), 2772 – 2778.

[4] Choe, H. J., A regularity theory for a general class of quasilinear elliptic partial differential equations and obstacle problems. *Arch. Ration. Mech. Anal.* 114 (1991), 383 – 394.

[5] Choe, H. J. and Lewis, J. L., On the obstacle problem for quasilinear elliptic equations of $p$-Laplacian type. *SIAM J. Math. Anal.* 22 (1991)(3), 623 – 638.

[6] DiBenedetto, E., $C^{1,\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlin. Anal.* 7 (1983), 827 – 850.

[7] Fuchs, M., Hölder continuity of the gradient for degenerate variational inequalities. *Nonlin. Anal.* 15 (1990), 85 – 100.

[8] Karp, L., Kilpeläinen, T., Petrosyan, A. and Shahgholian, H., On the porosity of free boundaries in degenerate variational inequalities. *J. Diff. Equ.* 164 (2000), 110 – 117.

[9] Karp, L. and Shahgholian, H., Hausdorff measure and stability for the $p$-obstacle problem ($2 < p < \infty$). *J. Diff. Equ.* 195 (2003), 14 – 24.

[10] Lindqvist, P., Regularity for the gradient of the solution to a nonlinear obstacle problem with degenerate ellipticity. *Nonlin. Anal.* 12 (1988), 1245 – 1255.

[11] Martio, O. and Vuorinen, M., Whitney cubes, $p$-capacity, and Minkowski content, *Exposition. Math.* 5 (1987), 17 – 40.

[12] Mu, J., Higher regularity of the solution to the $p$-Laplacian obstacle problem. *J. Diff. Equ.* 95 (1992), 370 – 384.

[13] Norando, T., $C^{1,\alpha}$ local regularity for a class of quasilinear elliptic variational inequalities. *Boll. Un. Mat. Ital. C(6)* 5 (1987), 281 – 292.

[14] Rodrigues, J. F., Stability remarks to the obstacle problem for $p$-Laplacian type equations. *Calc. Var.* 23 (2005), 51 – 65.

[15] Rodrigues, J. F., Sanchón, M. and Urbano, J. M., The obstacle problem for nonlinear elliptic equations with variable growth and $L^1$-data. *Monatsh. Math.* 154 (2008), 303 – 322.

[16] Tolksdorf, P., Regularity for a more general class of quasilinear elliptic equations. *J. Diff. Equ.* 51 (1984), 126 – 150.

[17] Zheng, J. and Zhao, P., Porosity of the free boundary in the obstacle problems. *J. Shaanxi Normal Univ.* 40 (2012), 15 – 17, 22.


Received October 13, 2011; revised February 20, 2012