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Existence of Solutions of Inverse Problems for Elliptic Complex Equations with Degenerate Curve in Multiple Connected Domains

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Abstract. The present paper deals with the inverse problem for degenerate elliptic systems of first order equations in multiple connected domains with Riemann-Hilbert type map. Firstly the formulation and the complex form of the problem for the degenerate elliptic systems of first order are given, and then the coefficients of the above systems are constructed by a new complex analytic method. As an application of the above results, we can derive the corresponding results of the inverse problem for degenerate elliptic equations of second order in multiple connected domains from Dirichlet to Neumann map.

Keywords. Existence theorems, inverse problems, elliptic complex equations, degeneracy, multiple connected domains.

Mathematics Subject Classification (2010). Primary 35R30, secondary 35J70

1. Formulation of the inverse problem for degenerate elliptic complex equations of first order

In [1–4, 6–8, 13, 14], the authors posed and discussed the inverse problem of second order elliptic equations without degenerate line. In this paper, by using the complex analytic method, the existence of solutions of the inverse problem for elliptic complex equations of first order with degenerate curve in multiple connected domains with Riemann-Hilbert type map is discussed.

Let D be an $N+1$-connected bounded domain in the complex plane $\mathbb{C}$ with the boundary $\partial D = \Gamma = \cup_{j=0}^{N}\Gamma_j \in C^1_\mu\,(0 < \mu < 1)$, where $\Gamma_j\,(j = 0, 1, \dots, N)$ are inside of $\Gamma_0\,(= \Gamma_{N+1})$, and we can assume that the point $z = 0 \in D$. Consider the linear degenerate elliptic systems of first order equations

$$\begin{cases} H(\hat{y})u_x - v_y = au + bv \\ H(\hat{y})v_x + u_y = cu + dv \end{cases} \text{ in } D, \tag{1.1}$$

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in which $\hat{y} = y - x^2$, $H(\hat{y}) = \sqrt{|K(\hat{y})|}$, $G(\hat{y}) = \int_0^{\hat{y}} H(t)dt$, $G'(\hat{y}) = H(\hat{y})$, $K(\hat{y}) = |\hat{y}|^m$ is continuous in $\overline{D}$, where m is a positive number, and a, b, c, d $(j = 1, 2)$ are functions of $x+iy$ $(\in D)$ satisfying the condition $a, b, c, d \in L_\infty(D)$, which is called *Condition C*. The following degenerate elliptic system is a special case of system (1.1) with $H(\hat{y}) = |\hat{y}|^{\frac{m}{2}}$:

$$\begin{cases} |\hat{y}|^{\frac{m}{2}} u_x - v_y = au + bv \\ |\hat{y}|^{\frac{m}{2}} v_x + u_y = cu + dv \end{cases} \text{in } D. \tag{1.2}$$

We can discuss equation (1.2), and equation (1.1) can be similarly discussed. From the ellipticity condition [12, Chapter I], we have

$$J = 4K_1K_4 - (K_2 + K_3)^2 = 4H^2(\hat{y}) > 0 \quad \text{in } \overline{D}\backslash\gamma$$

and $J = 0$ on $\gamma = D \cap \{\hat{y} = y - x^2 = 0\}$, hence system (1.1) or (1.2) is elliptic system of first order equations in $D\backslash\gamma$ with the parabolic degenerate curve γ (see [12]). Setting $Y = G(\hat{y}) = \int_0^{\hat{y}} H(t)dt$, $Z = x + iY$ in $\overline{D}$, if $H(\hat{y}) = |\hat{y}|^{\frac{m}{2}}$, $Y = \int_0^{\hat{y}} H(t)dt = \frac{2}{m+2}|\hat{y}|^{\frac{m+2}{2}}$, then its inverse function is $\hat{y} = \left[(m + 2)\frac{Y}{2}\right]^{\frac{2}{m+2}} = JY^{\frac{2}{m+2}}$. Denote

$$\begin{aligned} &W(z) = u + iv, \\ &W_{\bar{z}} = \frac{1}{2}[H(\hat{y})W_x + iW_y] = \frac{H(\hat{y})}{2}[W_x + iW_Y] = H(\hat{y})W_{x-iY} = H(\hat{y})W_{\overline{Z}}, \end{aligned}$$

then the system (1.1) can be written in the complex form

$$\begin{aligned} W_{\bar{z}} &= H(\hat{y})W_{\overline{Z}} = -A(z)W - B(z)\overline{W} \quad \text{in } D \\ A &= -\frac{1}{4}[a + ic - ib + d] \\ B &= -\frac{1}{4}[a + ic + ib - d], \end{aligned} \tag{1.3}$$

in which D_Z is the image domain of D with respect to the mapping $Z = Z(z) = x+iY = x+iG(\hat{y})$ in D, and denoted by D again for simplicity. For convenience we only discuss the complex equation (1.2) about the number Z replaced by z later on.

Introduce the modified Riemann-Hilbert boundary condition for the equation (1.3) as follows:

$$\operatorname{Re}[\overline{\lambda(z)}W(z)] = r(z) + f(z) = f_1(z), \quad z \in \Gamma \tag{1.4}$$

where

$$f(z)=\begin{cases}0,\quad z\in\Gamma, & \text{if } K\geq N\\ \left.\begin{array}{l} g_j,\ z\in\Gamma_j,\ j=1,\ldots,N-K\\ 0,\ \ z\in\Gamma_j,\ j=N-K+1,\ldots,N+1\end{array}\right\} & \text{if }\ 0\leq K<N\\ \left.\begin{array}{l} g_j,\ \ z\in\Gamma_j,\ \ j=1,\ldots,N\\ g_0+\operatorname{Re}\displaystyle\sum_{m=1}^{-K-1}(g_m^+ + ig_m^-)[z(\zeta)]^m, z\in\Gamma_0\end{array}\right\} & \text{if } K<0\end{cases}$$

in which $\lambda(z)\,(\neq 0)$, $r(z)\in C_\alpha(\Gamma)$, $\alpha\ \left(\leq\frac{p-2}{p}\right)$ is a positive constant, g_j $(j=0,1,\ldots,N)$, $g_m^\pm\,(m=1,\ldots,-K-1,K<0)$ are unknown real constants to be determined appropriately, and $z=z(\zeta)$ is a conformal mapping from the unit disk $|\zeta|<1$ onto the bounded domain D_0 bounded by Γ_0. In addition, for $K\geq 0$ the solution $W(z)$ is assumed to satisfy the point conditions

$$\operatorname{Im}[\overline{\lambda(z_j)}W(z_j)]=q_j,\quad j\in J=\begin{cases}1,\ldots,2K-N+1, & \text{if } K\geq N\\ N-K+1,\ldots,N+1, & \text{if }\ 0\leq K<N,\end{cases}$$

in which $z_j\in\Gamma_j\,(j=1,\ldots,N)$, $z_j\in\Gamma_0\,(j=N+1,\ldots,2K-N+1,K\geq N)$ are distinct points, and $q_j(j\in J)$ are all real constants where $K=\frac{1}{2\pi}\Delta_\Gamma\arg\lambda(z)$ is called the index of $\lambda(z)$ on Γ. The above boundary value problem is called *Problem* RH1 for equation (1.3). Under Condition C, we can find the unique solution $W(z)$ of Problem RH1 for equation (1.3) in D. In fact, we can only choose any index, for instance the index $K=N-1$, in this case, $f(z)=0$ on $\Gamma\backslash\Gamma_1$ and $f(z)=g_1$ on Γ_1, g_1 is an undetermined real constant, and there are N point conditions $\operatorname{Im}[\overline{\lambda(z_j)}W(z_j)]=q_j$, $z_j\in\Gamma_j$, $j=2,\ldots,N+1$.

It is clear that the above solution $W(z)$ satisfies the following Riemann-Hilbert type boundary condition for the equation (1.3):

$$\operatorname{Im}[\overline{\lambda(z)}W(z)]=f_2(z)\quad\text{on }\Gamma,\tag{1.5}$$

and then the boundary conditions of modified Riemann-Hilbert to Riemann-Hilbert type map can be written as follows

$$\begin{aligned}\overline{\lambda(z)}W(z)&=f_1(z)+if_2(z)\quad\text{on }\Gamma,\ \text{i.e.}\\ W(z)=h_1(z)&=\frac{f_1(z)+if_2(z)}{\overline{\lambda(z)}}\quad\text{on }\Gamma,\end{aligned}$$

which will be called *Problem* R1 for the complex equation (1.3) (or (1.1)), where $h_1(z)\in C_\alpha(\Gamma)$ is a complex function, and denote by $\{h_1(z)\}$ the set of above all functions.

For the further requirement, we give the modified Riemann-Hilbert problem (*Problem* RH2) for the equation (1.3). Herein we only choose the modified boundary conditions with the index $K = N - 1$, namely

$$\operatorname{Re}[\overline{i\lambda(z)}W(z)] = \operatorname{Im}[\overline{\lambda(z)}W(z)] = r(z) + f(z) = \tilde{f}_1(z), \quad z \in \Gamma \tag{1.6}$$

where $\lambda(z)$, $r(z)$ on Γ are the similar to before, $f(z) = \begin{cases} g_1 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma\backslash\Gamma_1, \end{cases}$ and assume that the solution $W(z)$ satisfies the N point conditions

$$\operatorname{Re}[\overline{\lambda(z_j)}W(z_j)] = q_j, \quad z_j \in \Gamma_j,\ j = 2, \ldots, N+1,$$

in which g_1 is an undetermined real constant, and $q_j\,(j = 2, \ldots, N+1)$ are N real constants. Under Condition C, Problem RH2 for equation (1.3) in D has a unique solution. It is clear that the solution $W(z)$ satisfies

$$\operatorname{Re}[\overline{\lambda(z)}W(z)] = \tilde{f}_2(z) \quad \text{on } \Gamma. \tag{1.7}$$

Thus we have

$$\overline{\lambda(z)}W(z) = \tilde{f}_2(z) + i\tilde{f}_1(z), \quad W(z) = h_2(z) = \frac{\tilde{f}_2(z) + i\tilde{f}_1(z)}{\overline{\lambda(z)}} \quad \text{on } \Gamma,$$

which will be called *Problem* R2 for the complex equation (1.3) (or (1.1)), where $h_2(z) \in C_\alpha(\Gamma)$ is a complex function. It is not difficult to see that the function $h_2(z)$ is also a function of the set $\{h_1(z)\}$.

On the basis of the above discussion, we see that for any function $f_1(z)$ (or $\tilde{f}_1(z)$) of the set $C_\alpha(\Gamma)$ in the modified Riemann-Hilbert boundary condition (1.4) (or (1.6)), there is a set $\{f_2(z)\}$ (or $\tilde{f}_2(z)$) of the functions of Riemann-Hilbert type boundary condition (1.5) (or (1.7)), furthermore we obtain $h_1(z)$ (or $h_2(z)$). Denote by R_h the set of $\{h(z)\}$ including $\{h_1(z)\}$ and $\{h_2(z)\}$, our inverse problem is to determine the coefficient a, b, c and d of equation (1.1) (or $A(z)$, $B(z)$ in (1.3)) from the set R_h, which will be verified later on.

We mention that if $A = B = 0$, $H = H(\hat{y})$ in the ε-neighborhood $D_\varepsilon = D \cap \{|\hat{y}| < \varepsilon\}$ of $D \cap \{\hat{y} = 0\}$, and the above coefficients $A(z)$, $B(z)$ weakly converge to $A(z), B(z)$ in D as $\varepsilon \to 0$, then on the basis of Lemma 4.1 below, we see the Hölder continuity of solution $W(Z)$ and $TW_{\overline{Z}} = -T\big[\frac{AW+B\overline{W}}{H}\big]$ of the complex equation (1.3) with above coefficients and $TW_{\overline{Z}} = -T\big[\frac{AW+B\overline{W}}{H}\big]$ (see [9, 10, 12]), hence from $\{W(z)\}$ and $TW_{\overline{Z}}$, we can choose the subsequences which uniformly converges the Hölder continuous functions in $\overline{D}$ respectively. From this, we can also obtain the corresponding Pompeiu and Plemelj-Sokhotzki formulas about $W(z)$ in $\overline{D}$.

2. Existence of solutions of the inverse problem for degenerate elliptic complex equations of first order

According to [9] introduce the notations

$$\tilde{T}f(z) = T\left[\frac{f}{H}\right] = -\frac{1}{\pi}\iint_D \frac{\frac{f(\zeta)}{H(\hat{y})}}{\zeta - Z}\, d\sigma_\zeta,$$

in which $|\hat{y}|^\tau f(z) \in L_\infty(D)$, where $\tau = \max\big(1-\frac{m}{2}, 0\big)$. Suppose that $f(z) = 0$ in $\mathbb{C}\backslash\overline{D}$. Then $|\hat{y}|^\tau f(z) \in L_\infty(\mathbb{C})$, from Lemma 4.1 below, it follows $(\tilde{T}f)_{\bar{z}} = \frac{f(z)}{H}$ in $\mathbb{C}$. We consider the first order complex equation with singular coefficients

$$\begin{gathered} W_{\overline{z}} - \frac{A(z)}{H}W - \frac{B(z)}{H}\overline{W} = 0, \text{ i.e.} \\ [g(z)]_{\overline{z}} - \frac{A(z)g(z) - B(z)\overline{g(z)}}{H(\hat{y})} = 0 \quad \text{in } \mathbb{C}, \end{gathered} \tag{2.1}$$

where $Z = x + iG(\hat{y})$, $G(\hat{y}) = \int_0^{\hat{y}} H(\hat{y})d\hat{y}$, $g(z) = W(z)$. Applying the Pompeiu formula (see [9, Chapters I, III]), the corresponding integral equation of the complex equation (2.1) is as follows

$$g(z) + T\left[\frac{Ag + B\overline{g}}{H}\right] = \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta \quad \text{in } D. \tag{2.2}$$

For simplicity we can only consider the following integral equation

$$g(z) + T\left[\frac{Ag + B\overline{g}}{H}\right] = 1 \text{ or } = i \quad \text{in } D$$

later on. On the basis of Lemma 4.1 below, we know that the integral in (2.2) is a completely continuous operator, hence by using the similar method as in [9, Section 5, Chapter III] and the proof of [14, Lemma 2.2], we can verify that the above integral equation has a unique solution.

We first prove the following lemma (see [2]).

Lemma 2.1. *The function $g(z) = h_j(z)$ $(j = 1, 2)$ is a solution of the integral equation*

$$g(z) + T\left(\frac{A}{H}\right)g + T\left(\frac{B}{H}\right)\overline{g} = \left\{\begin{array}{l} 1 \\ i \end{array}\right. \text{in } \overline{D}, \quad g(z) = \left\{\begin{array}{l} h_1(z) \\ h_2(z) \end{array}\right. \text{on } \Gamma, \tag{2.3}$$

if and only if it is a solution of the integral equation

$$\begin{gathered} \frac{1}{2}g(z) + \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta = \left\{\begin{array}{l} 1 \\ i \end{array}\right., \quad g(\zeta) = \left\{\begin{array}{l} h_1(\zeta) \\ h_2(\zeta) \end{array}\right., \text{ i.e.} \\ \frac{h_1(z)}{2} + \frac{1}{2\pi i}\int_\Gamma \frac{h_1(\zeta)}{\zeta - z}d\zeta = 1, \quad \frac{h_2(z)}{2} + \frac{1}{2\pi i}\int_\Gamma \frac{h_2(\zeta)}{\zeta - z}d\zeta = i \text{ on } \Gamma, \end{gathered} \tag{2.4}$$

respectively.

Proof. It is clear that we can only discuss the case of h_1. If $g(z)$ is a solution of the first integral equation in (2.3), then $g_{\bar z} = -\frac{Ag}{H} - \overline{\frac{Ag}{H}}$. On the basis of the Pompeiu formula

$$g(z)=\frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta-z}d\zeta + T[g(\zeta)]_{\bar\zeta}=\frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta-z}d\zeta - T\left[\frac{Ag}{H}+\frac{B\overline{g}}{H}\right] \quad \text{in } D \qquad (2.5)$$

(see [9, Chapters I, III]), we have

$$g(z,k) + T\left[\frac{Ag}{H}\right] + T\left[\frac{B\overline{g}}{H}\right] = 1 = \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta \quad \text{in } D,$$

where $g(\zeta) = h_1(\zeta)$ on Γ. Moreover by using the Plemelj-Sokhotzki formula for Cauchy type integral (see [5, 11])

$$1 = \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta + \frac{1}{2}g(z), \quad g(\zeta) = h_1(\zeta) \text{ on } \Gamma,$$

this is the first formula in (2.4).

Inversely if the first integral equation in (2.4) is true, then there exists a solution of equation $g_{\bar z} = -\frac{Ag}{H} - \frac{B\overline{g}}{H}$ in $\overline{D}$ with the boundary values $g(\zeta) = h_1(\zeta)$ on Γ, thus we have (2.5), where the integral $\frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta-z}d\zeta$ in D is analytic, whose boundary value on Γ is

$$\lim_{z'(\in D)\to z(\in\Gamma)} \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z'}d\zeta = \frac{1}{2}g(z) + \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta = 1,$$

hence $\frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta-z}d\zeta = 1$ in D, and the first formula in (2.3) is true. □

Lemma 2.2. *Under the above conditions, the functions $h_1(z)$, $h_2(z)$ as stated in Section 1 are the solutions of the system of integral equations*

$$\begin{aligned}
&\frac{1}{2}(1 - iS)h_1 = 1, \quad Sh_1 = \frac{1}{\pi}\int_\Gamma \frac{h_1(\zeta)}{\zeta - z}d\zeta,\\
&\frac{1}{2}(1 - iS)h_2 = i, \quad Sh_2 = \frac{1}{\pi}\int_\Gamma \frac{h_2(\zeta)}{\zeta - z}d\zeta.
\end{aligned} \qquad (2.6)$$

Proof. On the basis of the theory of integral equations (see [5, 7, 14]), we can obtain the solutions $h_1(z)$ and $h_2(z)$ of (2.6). In fact, from Lemma 2.1 we can define the functions

$$w_1(z) = \begin{cases} 1 - \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{h_1(\zeta)}{\zeta - z}\,d\zeta, & z \in \mathbb{C}\backslash\overline{D} \\[2ex] 1 + \dfrac{1}{\pi}\displaystyle\iint_{\mathbb{C}} \frac{Aw_1(\zeta) + B\overline{w_1(\zeta)}}{(\zeta - z)H}\,d\sigma_\zeta, & z \in \overline{D}, \end{cases}$$

$$w_2(z) = \begin{cases} i - \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{h_2(\zeta)}{\zeta - z}\,d\zeta, & z \in \mathbb{C}\backslash\overline{D} \\[2ex] i + \dfrac{1}{\pi}\displaystyle\iint_{\mathbb{C}} \frac{Aw_2(\zeta) + B\overline{w_2(\zeta)}}{(\zeta - z)H}\,d\sigma_\zeta, & z \in \overline{D}, \end{cases}$$

which are analytic in $\mathbb{C}\backslash\overline{D}$ with the boundary values $h_1(z)$, $h_2(z)$ on Γ respectively, and satisfy the formula (2.6). $\square$

Theorem 2.3. *For the above inverse problem of the equation* (1.2) *with Condition C, we can reconstruct the coefficients $a(z)$, $b(z)$, $c(z)$ and $d(z)$.*

Proof. We shall find two solutions $\phi_1(z) = W_1(z)$ and $i\phi_2(z) = W_2(z)$ of the complex equation

$$[\phi]_{\bar{z}} - \frac{A}{H\phi} - \frac{B\overline{\phi}}{H} = 0 \quad \text{in } \mathbb{C}$$

with the conditions $\phi_1(z) \to 1$ and $i\phi_2(z) \to i$ as $z \to \infty$. In fact the above solutions $F(z) = \phi_1(z)$, $G(z) = i\phi_2(z)$ are also the solutions of integral equations

$$\begin{cases} F(z) + T\left[\dfrac{AF + B\overline{F}}{H}\right] = 1 \\ G(z) + T\left[\dfrac{AG + B\overline{G}}{H}\right] = i \end{cases} \quad \text{in } \mathbb{C}.$$

As stated in Lemmas 2.1, 2.2, we can require that the above solutions satisfy the boundary conditions $F(z) = h_1(z)$, $G(z) = h_2(z)$ on Γ where $h_1(z)$, $h_2(z) \in R_h$.

Note that $F(z)$, $G(z)$ satisfy the complex equations

$$\begin{cases} F_{\bar{z}} - \dfrac{AF + B\overline{F}}{H} = 0 \\ G_{\bar{z}} - \dfrac{AG + B\overline{G}}{H} = 0 \end{cases} \quad \text{in } \mathbb{C}. \tag{2.7}$$

Moreover on the basis of Lemma 2.4 below, we have

$$\operatorname{Im}[F(z)\overline{G(z)}] = \frac{F(z)\overline{G(z)} - \overline{F(z)}G(z)}{2i} \neq 0 \quad \text{in } D. \tag{2.8}$$

Thus from (2.7), the coefficients $\frac{A}{H}$ and $\frac{B}{H}$ can be determined as follows

$$\begin{aligned} \frac{A}{H} &= \frac{F_{\bar{z}}\overline{G} - G_{\bar{z}}\overline{F}}{F\overline{G} - \overline{F}G}, \qquad \frac{B}{H} = -\frac{F_{\bar{z}}G - G_{\bar{z}}F}{F\overline{G} - \overline{F}G} \quad \text{in } D, \text{ i.e.} \\ A &= H\frac{F_{\bar{z}}\overline{G} - G_{\bar{z}}\overline{F}}{F\overline{G} - \overline{F}G}, \qquad B = -H\frac{F_{\bar{z}}G - G_{\bar{z}}F}{F\overline{G} - \overline{F}G} \quad \text{in } D. \end{aligned}$$

From the above formulas, the coefficients $a(z)$, $b(z)$, $c(z)$ and $d(z)$ of the equation (1.1) are obtained, i.e.

$$a(z) + ic(z) = 2[A(z) + B(z)], \quad d(z) - ib(z) = 2[A(z) - B(z)] \quad \text{in } D. \qquad \square$$

Lemma 2.4. *For the solutions $F(z), G(z)$ of equations* (2.7)*, we can get the inequality* (2.8)*.*

Proof. Suppose that (2.8) is not true, then there exists a point $z_0 \in D$ such that $\mathrm{Im}[\overline{F(z_0)}G(z_0)] = 0$, i.e.

$$\begin{vmatrix} \mathrm{Re}F(z_0) & \mathrm{Im}F(z_0) \\ \mathrm{Re}G(z_0) & \mathrm{Im}G(z_0) \end{vmatrix} = 0.$$

Thus we have two real constants c_1, c_2, which are not all equal to 0, such that $c_1F(z_0) + c_2G(z_0) = 0$.

In the following, we prove that the equality of $c_1F(z_0) + c_2G(z_0) = 0$ is not true. If $W(z_0) = c_1F(z_0) + c_2G(z_0) = 0$, then $W(z) = \Phi(z)e^{\phi(z)} = (z - z_0)\Phi_0(z)e^{\phi(z)}$, where $\Phi(z)$, $\Phi_0(z)$ are analytic functions in D, and

$$(z-z_0)\Phi_0(z)e^{\phi(z)} - \frac{1}{\pi}\iint_D \frac{(\zeta-z_0)\Phi_0(\zeta)e^{\phi(\zeta)}[AW(\zeta) + B\overline{W(\zeta)}]}{HW(\zeta)(\zeta - z)} d\sigma_\zeta = c_1 + c_2 i.$$

Letting $z \to z_0$, we have $-\frac{1}{\pi}\iint_D \Phi_0(\zeta)e^{\phi(\zeta)}\left[\frac{A}{H} + \frac{B\overline{W(\zeta)}}{HW(\zeta)}\right] d\sigma_\zeta = c_1 + c_2 i$, and then

$$\begin{aligned} c_1 + c_2 i &= (z-z_0)\Phi_0(z)e^{\phi(z)} - \frac{1}{\pi}\iint_D \frac{(\zeta-z_0)\Phi_0(\zeta)e^{\phi(\zeta)}[AW(\zeta)+B\overline{W(\zeta)}]}{HW(\zeta)(\zeta - z)} d\sigma_\zeta \\ &= (z-z_0)\left\{\Phi_0(z)e^{\phi(z)} - \frac{1}{\pi}\iint_D \frac{\Phi_0(\zeta)e^{\phi(\zeta)}[AW(\zeta) + B\overline{W(\zeta)}]}{HW(\zeta)(\zeta - z)} d\sigma_\zeta\right\} \\ &\quad - \frac{1}{\pi}\iint_D \Phi_0(\zeta)e^{\phi(\zeta)}\left[\frac{A}{H} + \frac{B\overline{W(\zeta)}}{HW(\zeta)}\right] d\sigma_\zeta. \end{aligned}$$

The above equality implies

$$\Phi_0(z)e^{\phi(z)} - \frac{1}{\pi}\iint_D \frac{\Phi_0(\zeta)e^{\phi(\zeta)}[AW(\zeta) + B\overline{W(\zeta)}]}{HW(\zeta)(\zeta - z)} d\sigma_\zeta = 0 \quad \text{in } D,$$

and the above homogeneous integral equation only have the trivial solution, namely $\Phi_0(z) = 0$ in D, thus $W(z) = \Phi(z)e^{\phi(z)} = (z - z_0)\Phi_0(z)e^{\phi(z)} \equiv 0$ in D. This is impossible. □

For the above discussion, we see that four real coefficients $a(z)$, $b(z)$, $c(z)$, $d(z)$ of the system (1.1) or two complex coefficients $A(z)$, $B(z)$ of the complex equation (1.4) can be determined by two boundary functions $h_1(z)$, $h_2(z)$ in the set R_h.

We have tried to prove the existence of solutions for above inverse problem by the inverse scatting method as stated in [7, 8, 13, 14], but it cannot be completed, hence the new complex method in the presented article is used. Besides, the global uniqueness of solutions for the inverse problem will be further investigated.

3. Existence of solutions of the inverse problem for degenerate elliptic equations of second order

In this section, by using the similar method, the corresponding results of the inverse problem for elliptic equations of second order with degenerate curve from Dirichlet to Neumann map can be obtained.

Let D be an $N+1$-connected bounded domain in the complex plane $\mathbb{C}$ with the boundary $\partial D = \Gamma = \cup_{j=0}^{N}\Gamma_j \in C^2_\mu(0<\mu<1)$, where $\Gamma_j\,(j=0,1,\ldots,N)$ are inside of Γ_0 and $0\in D$ as stated in Section 1. Consider the degenerate elliptic equation of second order

$$K(\hat{y})u_{xx}+u_{yy}+au_x+bu_y=0 \quad \text{in } D, \tag{3.1}$$

in which $\hat{y}=y-x^2$, $K(\hat{y})=|\hat{y}|h(\hat{y})$, $h(\hat{y})$ is a continuously differentiable positive function in $\overline{D}$, and a,b are real functions of $z=x+i\hat{y}\,(\in D)$ satisfying $a,b\in L_\infty(D)$. Moreover define $K(\hat{y})=1$, $a=b=0$ in $\mathbb{C}\backslash\overline{D}$. The above conditions will be called Condition C. It is clear that equation (3.1) in $D\backslash\gamma$ is elliptic with the parabolic degenerate curve $\gamma=D\cap\{\hat{y}=0\}$.

If the function $K(\hat{y})=|\hat{y}|$, $H(\hat{y})=|\hat{y}|^{\frac{1}{2}}$, then $|G(\hat{y})|=|\int_0^{\hat{y}}H(t)dt|=\frac{2}{3}|\hat{y}|^{\frac{3}{2}}$ in $\overline{D}$, and the inverse function of $Y=G(\hat{y})$ is

$$\hat{y}=\pm|G^{-1}(Y)|=\pm\left(\frac{3}{2}\right)^{\frac{2}{3}}|Y|^{\frac{2}{3}}=\pm J|Y|^{\frac{2}{3}} \quad \text{in } \overline{D}.$$

Denote

$$\begin{aligned}
W(z)=U+iV&=\frac{H(\hat{y})u_x-iu_y}{2}=u_{\tilde{z}}=H(\hat{y})\frac{u_x-iu_Y}{2}=H(\hat{y})u_Z\\
W_{\bar{\tilde{z}}}&=\frac{H(\hat{y})W_x+iW_y}{2}=H(\hat{y})[W_x+iW_Y]=H(\hat{y})W_{\overline{Z}},
\end{aligned}$$

where $\hat{y}=y-x^2$, $G(\hat{y})=|\hat{y}|h(\hat{y})$, $G'(\hat{y})=H(\hat{y})$, we can get $K(\hat{y})u_{xx}+u_{yy}=H[Hu_x-iu_y]_x+i[Hu_x-iu_y]_y-iH_yu_x=2\{H[U+iV]_x+i[U+iV]_y\}-iH_yu_x=4W_{\bar{\tilde{z}}}-i\frac{H_y}{H}Hu_x=4H(\hat{y})W_{\overline{Z}}-i\frac{H_y}{H}Hu_x=-[au_x+bu_y]$, i.e.

$$\begin{aligned}
W_{\overline{Z}}=\frac{W_{\bar{\tilde{z}}}}{H(\hat{y})}&=\frac{i\,H_yu_x-(au_x+bu_y)}{4H(\hat{y})}\\
&=\frac{\left(\frac{iH_y}{H}-\frac{a}{H}\right)(W+\overline{W})-ib(W-\overline{W})}{4H(\hat{y})}\\
&=\frac{A(z)W+B(z)\overline{W}}{H(\hat{y})} \quad \text{in } \overline{D}_Z,
\end{aligned} \tag{3.2}$$

where

$$A[z(Z)]=\frac{1}{4}\left[\frac{iH_y}{H}-\frac{a}{H}-ib\right],\quad B[z(Z)]=\frac{1}{4}\left[\frac{iH_y}{H}-\frac{a}{H}+ib\right],$$

and D_Z is the image domain of D with respect to the mapping $Z = Z(z) = x + iG(\hat{y})$. Obviously the complex equation $W_{\bar{z}} = 0$ in $\overline{D}$, i.e. $W_{\overline{Z}} = 0$ in $\overline{D_Z}$, is a special case of equation (3.2). For convenience we only discuss the complex equation (3.2) about the number Z replaced by z later on.

Introduce the Dirichlet boundary condition for the equation (3.1) as follows:

$$u = f(z) \quad \text{on } \Gamma, \text{ i.e. } u = f(z) \quad \text{on } \partial D, \tag{3.3}$$

where $f(z) \in C^1_\alpha(\Gamma)$, $\alpha\,(0 < \alpha \le \frac{p-2}{p}$ is a positive constant, which is called *Problem* D for equation (3.1). If we find the derivative of positive tangent direction with respect to the unit arc length parameter s of the boundary Γ with $s(0) = \arg(z_0 + 0) = 0$, where the point $z_0 \in \Gamma_0$, then

$$f_s = \frac{\partial f(z)}{\partial s} = u_z z_s + u_{\bar{z}} \bar{z}_s = 2\mathrm{Re}[z_s u_z] = f_1(z) \quad \text{on } \Gamma.$$

It is clear that the equivalent boundary value problem is found a solution $[W(z), u(z)]$ of the complex equation (3.2) with the boundary conditions

$$\mathrm{Re}[\overline{\lambda(z)} w(z)] = \mathrm{Re}[z_s w(z)] = \frac{f_s}{2},\ z \in \Gamma, \quad u(z_0) = f(z_0),$$

and the relation

$$u(z) = 2\mathrm{Re}\int_{z_0}^{z} w(z)dz + f(z_0) \quad \text{in } \overline{D}, \tag{3.4}$$

in which $\lambda(z) = \overline{z_s}$, $z \in \Gamma$. Taking into account the partial indexes of $K_0 = \Delta_{\Gamma_0}\arg[\lambda(z)] = -1$ and $K_j = \Delta_{\Gamma_j}\arg[\lambda(z)] = 1\,(j = 1, \ldots, N)$, thus the index of the above boundary value problem is $K = K_0 + K_1 + \cdots + K_N = N - 1$, obviously this is a special case of Riemann-Hilbert boundary value problem (*Problem* RH) as stated in Sections 1 and 2. It is easy to see that

$$2\mathrm{Re}\int_{\Gamma_j} W(z)dz = 2\mathrm{Re}\int_{\Gamma_j} u_z z_s ds = \int_0^{S_j} f_s ds = 0, \quad j = 0, 1, \ldots, N,$$

herein $S_j\,(j = 0.1\ldots, N)$ is the arc length of $\Gamma_j\,(j = 1, \ldots, N)$ and applying the Green formula, the function $u(z)$ determined by the integral in (3.4) in $\overline{D}$ is single-valued.

Under the above condition, the corresponding Neumann boundary condition is

$$u_n = \frac{\partial u}{\partial n} = u_z z_n + u_{\bar{z}} \bar{z}_n = 2\mathrm{Im}[z_s u_z] = f_2(z) \quad \text{on } \Gamma, \tag{3.5}$$

where n is the unit outwards normal vector of Γ. The boundary value problem (3.1) (or (3.2)), (3.5) will be called *Problem* N. Hence the boundary conditions of Dirichlet and Neumann problems can be written as follows

$$u_s + iu_n = 2\mathrm{Re}[z_s u_z] + 2i\mathrm{Im}[z_s u_z] = 2z_s w(z),\ z\in\Gamma,\ \text{i.e.}$$
$$w(z) = h(z) = f_1(z) + if_2(z) = \frac{u_s + iu_n}{2z_s},\quad z\in\Gamma,$$

which will be called *Problem* DN for the complex equation (3.2) (or (3.1)) with the relation (3.4), where $h(z)\in C_\alpha(\Gamma)$ is a complex function satisfying the condition $\int_{\Gamma_j}\mathrm{Re}[z_s]u_z ds = 0$, $j = 0, 1, \ldots, N$. For any function $f(z)$ of the set $C^1_\alpha(\Gamma)$ in the Dirichlet boundary condition (3.3), there is a set $\{f_2(z)\}$ of the functions of Neumann boundary condition (3.5) as stated before, which is called the Dirichlet to Neumann map. According to the method of Section 1, we can obtain the set of functions $\{h(z)\}$, which is denoted by R_h. Later on we will determine the coefficients of equation (3.1) from the set $R_h = \{h(z)\}$.

According to [9] introduce the notations

$$\tilde{T}f(z) = T\left[\frac{f}{H}\right] = -\frac{1}{\pi}\iint_D \frac{f(\zeta)}{H(\zeta - Z)}d\sigma_\zeta,$$

in which $|\hat{y}|^\tau f(z)\in L_\infty(D)$, $\tau = \max\left(1 - \frac{m}{2}, 0\right)$. Suppose that $f(z) = 0$ in $\mathbb{C}\backslash\overline{D}$. Then $(\tilde{T}f)_{\bar z} = \frac{f(z)}{H}$ in $\mathbb{C}$. We consider the first order complex equation with singular coefficients

$$\begin{gathered} HW_{\bar z} - A(z)W - B(z)\overline{W} = 0,\ \text{i.e.}\\ H(\hat{y})[g(z)]_{\bar z} - A(z)g(z) - B(z)\overline{g(z)} = 0 \quad \text{in } \mathbb{C}, \end{gathered} \tag{3.6}$$

where $Z = x + iG(\hat{y})$, $G(\hat{y}) = \int_0^{\hat y} H(\hat{y})d\hat{y}$, $g(z) = W(z)$. On the basis of the Pompeiu formula (see [9, Chapters I, III]), the corresponding integral equation of the complex equation (3.6) is as follows

$$g(z) + T\left[\frac{Ag + B\overline{g}}{H}\right] = \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta \quad \text{in } D. \tag{3.7}$$

For simplicity we can only consider the following integral equation

$$g(z) + T\left[\frac{Ag + B\overline{g}}{H}\right] = 1 \text{ or } = i \quad \text{in } D$$

later on. Similarly to Section 2, we see that the above integral equation has a unique solution.

Similarly to the proof of Lemma 2.2 as stated before, we can prove the following lemma.

Lemma 3.1. *The function $g(z) = h_j(z)\,(j = 1, 2)$ is a solution of the integral equation*

$$g(z) + T\left[\frac{Ag + B\overline{g}}{H}\right] = \begin{cases} 1 \\ i \end{cases} \text{in } \overline{D}, \quad g(z) = \begin{cases} h_1(z) \\ h_2(z) \end{cases} \text{on } \Gamma,$$

if and only if it is a solution of the integral equation

$$\frac{1}{2}g(z) + \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta = \begin{cases} 1 \\ i \end{cases}, \quad g(\zeta) = \begin{cases} h_1(\zeta) \\ h_2(\zeta) \end{cases}, \text{ i.e.}$$

$$\frac{h_1(z)}{2} + \frac{1}{2\pi i}\int_\Gamma \frac{h_1(\zeta)}{\zeta - z}d\zeta = 1, \quad \frac{h_2(z)}{2} + \frac{1}{2\pi i}\int_\Gamma \frac{h_2(\zeta)}{\zeta - z}d\zeta = i \quad \text{on } \Gamma.$$

Lemma 3.2. *Under the above conditions, the functions $h_1(z)$, $h_2(z)$ as stated in Section 1 are the solutions of the system of integral equations*

$$\begin{aligned} &\frac{1}{2}(1 - iS)h_1 = 1, \quad Sh_1 = \frac{1}{\pi}\int_\Gamma \frac{h_1(\zeta)}{\zeta - z}d\zeta, \\ &\frac{1}{2}(1 - iS)h_2 = i, \quad Sh_2 = \frac{1}{\pi}\int_\Gamma \frac{h_2(\zeta)}{\zeta - z}d\zeta. \end{aligned} \tag{3.8}$$

Proof. From Lemma 3.1, we define the functions

$$w_1(z) = \begin{cases} 1 - \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{h_1(\zeta)}{\zeta - z}d\zeta, & z \in \mathbb{C}\backslash\overline{D} \\ 1 + \dfrac{1}{\pi}\displaystyle\iint_{\mathbb{C}} \frac{Aw_1 + B\overline{w_1}}{H(\zeta - z)}d\sigma_\zeta, & z \in \overline{D}, \end{cases}$$

$$w_2(z) = \begin{cases} i - \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{h_2(\zeta)}{\zeta - z}d\zeta, & z \in \mathbb{C}\backslash\overline{D} \\ i + \dfrac{1}{\pi}\displaystyle\iint_{\mathbb{C}} \frac{Aw_2 + B\overline{w_2}}{H(\zeta - z)}d\sigma_\zeta, & z \in \overline{D}, \end{cases}$$

which are analytic in $\mathbb{C}\backslash\overline{D}$ with the boundary values $h_1(z)$, $h_2(z)$ on Γ respectively. Thus we can get the solutions $h_1(z)$ and $h_2(z)$ of (3.8) and satisfy the complex equation (3.7). □

Theorem 3.3. *For the above inverse problem of the equation* (3.1) *with Condition C, we can reconstruct the coefficients $a(z)$ and $b(z)$.*

Proof. We shall find two solutions $\phi_1(z) = W_1(z)$ and $i\phi_2(z) = W_2(z)$ of complex equation

$$[\phi]_{\bar z} - \frac{A\phi}{H} - \frac{B\overline{\phi}}{H} = 0 \quad \text{in } \mathbb{C}$$

with the conditions $\phi_1(z) \to 1$ and $i\phi_2(z) \to i$ as $z \to \infty$. In fact the above solutions $F(z) = \phi_1(z)$, $G(z) = i\phi_2(z)$ are also the solutions of integral equations

$$\begin{aligned} F(z) - T\left[\frac{AF + B\overline{F}}{H}\right] &= 1 \\ G(z) - T\left[\frac{AG + B\overline{G}}{H}\right] &= i \end{aligned} \quad \text{in } \mathbb{C}.$$

As stated in Lemmas 3.1, 3.2, we can require that the above solutions satisfy the boundary conditions $F(z) = h_1(z)$, $G(z) = h_2(z)$ on Γ, where $h_1(z), h_2(z) \in R_h$.

Note that $F(z)$, $G(z)$ satisfy the complex equations

$$\begin{cases} F_{\bar z} - \dfrac{AF + B\overline{F}}{H} = 0 \\ G_{\bar z} - \dfrac{AG + B\overline{G}}{H} = 0 \end{cases} \quad \text{in } D,$$

and similarly to Lemma 2.4, the inequality $\mathrm{Im}[F(z)\overline{G(z)}] = \frac{F(z)\overline{G(z)} - \overline{F(z)}G(z)}{2i} \neq 0$ in D can be verified. Hence we can determine the coefficients $\frac{A}{H}$ and $\frac{B}{H}$ as follows

$$\begin{aligned} \frac{A}{H} &= \frac{F_{\bar z}\overline{G} - G_{\bar z}\overline{F}}{F\overline{G} - \overline{F}G}, \qquad \frac{B}{H} = -\frac{F_{\bar z}G - G_{\bar z}F}{F\overline{G} - \overline{F}G} \quad \text{in } D, \text{ i.e.} \\ A &= H\frac{F_{\bar z}\overline{G} - G_{\bar z}\overline{F}}{F\overline{G} - \overline{F}G}, \qquad B = -H\frac{F_{\bar z}G - G_{\bar z}F}{F\overline{G} - \overline{F}G} \quad \text{in } D. \end{aligned}$$

From the above formulas, we can obtain the coefficients $a(z)$ and $b(z)$ of the equation (3.1), i.e.

$$a = iH_y - 2H[A(z) + B(z)], \quad b(z) = 2iH[A(z) - B(z)] \quad \text{in } D. \quad \square$$

4. The property of an integral operator

It is clear that the complex equation

$$w_{\overline{Z}} = 0 \quad \text{in } \overline{D_Z} \tag{4.1}$$

is a special case of equation (1.4). On the basis of Theorem 1.3, [12, Chapter I], we can find a unique solution of Problem RH for equation (4.1) in $\overline{D_Z}$.

Now we consider the function $g(Z) \in L_\infty(D_Z)$, and first extend the function $g(Z)$ to the exterior of $\overline{D_Z}$ in $\mathbb{C}$, i.e. set $g(Z) = 0$ in $\mathbb{C}\backslash\overline{D_Z}$, hence we can only discuss the domain $D_0 = \{|x| < R_0\} \cap\{\mathrm{Im}Y \neq 0\} \supset \overline{D_Z}$, here $Z = x + iY$, R_0 is a positive number. In the following we shall verify that the integral

$$\Psi(Z) = T\left[\frac{g}{H}\right] = -\frac{1}{\pi}\iint_{D_0} \frac{g(t)}{H(\mathrm{Im}t)(t - Z)} d\sigma_t \ \text{ in } D_0, \quad L_\infty[g(Z), D_0] \le k_3, \tag{4.2}$$

satisfies the estimate (4.3) below, where $H(\hat{y}) = \hat{y}^{\frac{m}{2}}$, $\hat{y} = y - x^2$, m ia a positive number. It is clear that the function $\frac{g(Z)}{H(\hat{y})}$ belongs to the space $L_1(D_0)$ and in general is not belonging to the space $L_p(D_0)\,(p > 2, m \geq 2)$, and the integral $\Psi(Z_0)$ is definite when $\mathrm{Im}Z_0 \neq 0$. If $Z_0 \in D_0$ and $\mathrm{Im}Z_0 = 0$, we can define the integral $\Psi(Z_0)$ as the limit of the corresponding integral over $D_0 \cap \{|\mathrm{Re}t - \mathrm{Re}Z_0| \geq \varepsilon\} \cap \{|\mathrm{Im}t - \mathrm{Im}Z_0| \geq \varepsilon\}$ as $\varepsilon \to 0$, where ε is a sufficiently small positive number. The Hölder continuity of the integral will be proved by the following method.

Lemma 4.1. *If the function $g(Z)$ in D_Z satisfies the condition in (4.2), and $H(\hat{y}) = \hat{y}^{\frac{m}{2}}$, where m is a positive number, then the integral in (4.2) satisfies the estimate*

$$C_\beta[\Psi(Z), \overline{D_Z}] \leq M_1, \tag{4.3}$$

where $\beta = \frac{2}{m+2} - \delta$, δ is a sufficiently small positive constant, and $M_1 = M_1(\beta,\, k_3, H, D_Z)$ is a positive constant.

Proof. We first give the estimates of $\Psi(Z)$ of (4.2) in $D \cap \{\mathrm{Im}Y \geq 0\}$, and verify the boundedness of the function in (4.2). As stated before, if $H(\hat{y}) = \hat{y}^{\frac{m}{2}}$, then $H(\hat{y}) = J^{\frac{m}{2}} Y^{\frac{m}{m+2}}$. For any two points $Z_0 = x_0 \in \gamma = D \cap \{\hat{y} = 0\}$ on x-axis and $Z_1 = x_1 + iY_1 (Y_1 > 0) \in D_0$ satisfying the condition $\frac{2\mathrm{Im}Z_1}{\sqrt{3}} \leq |Z_1 - Z_0| \leq 2\mathrm{Im}Z_1$, this means that the inner angle at Z_0 of the triangle $Z_0Z_1Z_2\,(Z_2 = x_0 + iY_1 \in D_0)$ is not less than $\frac{\pi}{6}$ and not greater than $\frac{\pi}{3}$, choose a sufficiently large positive number q, from the Hölder inequality, we have $L_1[\Psi(Z), D_0] \leq L_q[g(Z), D_0] L_p\big[\frac{1}{H(\mathrm{Im}t)(t-Z)}, D_0\big]$, where $p = \frac{q}{q-1}\,(> 1)$ is close to 1. In fact we can derive as follows

$$\begin{aligned}
|\Psi(Z_0)| &\leq \left|\frac{1}{\pi}\iint_{D_0} \frac{g(t)}{H(\mathrm{Im}t)(t - Z_0)} d\sigma_t\right| \\
&\leq \frac{1}{J^{\frac{m}{2}}\pi} L_q[g(Z), D_0] \left[\iint_{D_0} \left|\frac{1}{t^{\frac{m}{m+2}}(t - Z_0)}\right|^p d\sigma_t\right]^{\frac{1}{p}} \\
&= \frac{1}{J^{\frac{m}{2}}\pi} L_q[g(Z), D_0]\, J_1^{\frac{1}{p}},
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad J_1 &= \iint_{D_0} \left|\frac{1}{t^{\frac{m}{m+2}}(t - Z_0)}\right|^p d\sigma_t \\
&\leq \iint_{D_0} \frac{1}{|t|^{\frac{pm}{m+2}} |\mathrm{Im}(t - Z_0)|^{p\beta_0} |\mathrm{Re}(t - Z_0)|^{p(1-\beta_0)}} d\sigma_t \\
&\leq \left|\int_0^{d_0} \frac{1}{Y^{\frac{pm}{m+2}} |Y - Y_0|^{p\beta_0}}\, dY \int_{d_1}^{d_2} \frac{1}{|x - x_0|^{p(1-\beta_0)}}\, dx\right| \\
&\leq k_4,
\end{aligned}$$

in which $d_0 = \max_{Z \in \overline{D_0}} \mathrm{Im}Z$, $d_1 = \min_{Z \in \overline{D_0}} \mathrm{Re}Z$, $d_2 = \max_{Z \in \overline{D_0}} \mathrm{Re}Z$, $\beta_0 = \frac{2}{m+2} - \varepsilon$, $\varepsilon\,\big(< \frac{1}{p} - \frac{m}{m+2}\big)$ is a sufficiently small positive constant, we can choose

$\varepsilon = \frac{2(p-1)}{p}\,(\le \frac{2}{m+2})$, such that $p\,(1-\beta_0) < 1$ and $p[\frac{m}{m+2}+\beta_0] < 1$, and $k_4 = k_4(\beta, k_3, H, D_0)$ is a non-negative constant.

Next we estimate the Hölder continuity of the integral $\Psi(Z)$ in $\overline{D_0}$, i.e.

$$
\begin{aligned}
|\Psi(Z_1)-\Psi(Z_0)| &\le \frac{|Z_1-Z_0|}{\pi}\left|\iint_{D_0}\frac{g(t)}{H(\mathrm{Im}t)(t-Z_0)(t-Z_1)}d\sigma_t\right|\\
&\le \frac{|Z_1-Z_0|}{J^{\frac{m}{2}}\pi}L_q[g(Z),D_0]\left[\iint_{D_0}\left|\frac{1}{t^{\frac{m}{m+2}}(t-Z_0)(t-Z_1)}\right|^p d\sigma_t\right]^{\frac{1}{p}},
\end{aligned}
$$

$$
\begin{aligned}
\text{and}\quad J_2 &= \iint_{D_0}\left|\frac{1}{t^{\frac{m}{m+2}}(t-Z_0)(t-Z_1)}\right|^p d\sigma_t\\
&\le \iint_{D_0}\frac{|\mathrm{Re}(t-Z_0)|^{p(\frac{\beta_0}{2}-1)}|\mathrm{Re}(t-Z_1)|^{p(\frac{\beta_0}{2}-1)}}{|t|^{\frac{pm}{m+2}}|\mathrm{Im}(t-Z_0)|^{\frac{p\beta_0}{2}}|\mathrm{Im}(t-Z_1)|^{\frac{p\beta_0}{2}}}d\sigma_t\\
&\le \int_0^{d_0}\frac{1}{Y^{\frac{pm}{m+2}}|\mathrm{Im}(Y-Z_0)|^{\frac{p\beta_0}{2}}|\mathrm{Im}(Y-Z_1)|^{\frac{p\beta_0}{2}}}dY\\
&\quad\times\int_{d_1}^{d_2}\frac{1}{|\mathrm{Re}(t-Z_0)|^{p(1-\frac{\beta_0}{2})}|\mathrm{Re}(t-Z_1)|^{p(1-\frac{\beta_0}{2})}}\,d\mathrm{Re}t\\
&\le k_5\int_{d_1}^{d_2}\frac{1}{|x-x_0)|^{p(1-\frac{\beta_0}{2})}|x-x_1|^{p(1-\frac{\beta_0}{2})}}dx,
\end{aligned}
$$

where $\beta_0 = \frac{2}{m+2}-\varepsilon$ is chosen as before and

$$
k_5 = \max_{Z_0,Z_1\in D_0}\int_0^{d_0}\left[Y^{\frac{pm}{m+2}}|\mathrm{Im}(Y-Z_0)|^{\frac{p\beta_0}{2}}|\mathrm{Im}(Y-Z_1)|^{\frac{p\beta_0}{2}}\right]^{-1}dY.
$$

Denote $\rho_0 = |\mathrm{Re}(Z_1-Z_0)| = |x_1-x_0|$, $L_1 = D_0\cap\{|x-x_0|\le 2\rho_0, Y=Y_0\}$ and $L_2 = D_0\cap\{2\rho_0 < |x-x_0| \le 2\rho_1 < \infty, Y=Y_0\} \supset [d_1,d_2]\backslash L_1$, where ρ_1 is a sufficiently large positive number, we can derive

$$
\begin{aligned}
J_2 &\le k_5\Bigg[\int_{L_1}\frac{1}{|x-x_0|^{p(1-\frac{\beta_0}{2})}|x-x_1|^{p(1-\frac{\beta_0}{2})}}\,dx\\
&\quad+\int_{L_2}\frac{1}{|x-x_0|^{p(1-\frac{\beta_0}{2})}|x-x_1|^{p(1-\frac{\beta_0}{2})}}\,dx\Bigg]\\
&\le k_5\left[|x_1-x_0|^{1-2p+p\beta_0}\int_{|\xi|\le 2}\frac{1}{|\xi|^{p(1-\frac{\beta_0}{2})}|\xi\pm 1|^{p(1-\frac{\beta_0}{2})}}\,d\xi+k_6\left|\int_{2\rho_0}^{2\rho_1}\rho^{p\beta_0-2p}d\rho\right|\right]\\
&\le k_7|x_1-x_0|^{1-p(2-\beta_0)}\\
&= k_7|x_1-x_0|^{p(\frac{2}{m+2}-\varepsilon+\frac{1}{p}-2)},
\end{aligned}
$$

in which we use $|x-x_0| = \xi|x_1-x_0|$, $|x-x_1| = |x-x_0-(x_1-x_0)| = |\xi\pm 1||x_1-x_0|$ if $x\in L_1$, $|x-x_0| = \rho \le 2|x-x_1|$ if $x\in L_2$, choose that $p\,(>1)$ is close to 1 such that $1-p(2-\beta_0)<0$, and $k_j=k_j(\beta,k_3,H,D_0)$ $(j=6,7)$ are non-negative constants. Thus we get

$$|\Psi(Z_1)-\Psi(Z_0)|\le k_7|Z_1-Z_0||x_1-x_0|^{\frac{2}{m+2}-\varepsilon+\frac{1}{p}-2}\le k_8|Z_1-Z_0|^{\beta},$$

in which we use that the inner angle at Z_0 of the triangle $Z_0Z_1Z_2$ $(Z_2=x_0+iY_1$, $Z_2\in D_0)$ is not less than $\frac{\pi}{6}$ and not greater than $\frac{\pi}{3}$, and choose $\varepsilon=\frac{2(p-1)}{p}$, $\beta=\frac{2}{m+2}-\delta,\delta=\frac{3(p-1)}{p}$, $k_8=k_8(\beta,k_3,H,D_0)$ is a non-negative constant. The above points $Z_0=x_0$, $Z_1=x_1+iY_1$ can be replaced by $Z_0=x_0+iY_0$, $Z_1=x_1+iY_1\in\overline{D_0}$, $0<Y_0<Y_1$ and $\frac{2(Y_1-Y_0)}{\sqrt{3}}\le|Z_1-Z_0|\le 2(Y_1-Y_0)$.

Finally we consider any two points $Z_1=x_1+iY_1$, $Z_2=x_2+iY_1$ and $x_1<x_2$, from the above estimates, the following estimate can be derived

$$\begin{aligned}|\Psi(Z_1)-\Psi(Z_2)|&\le|\Psi(Z_1)-\Psi(Z_3)|+|\Psi(Z_3)-\Psi(Z_2)|\\&\le k_8|Z_1-Z_3|^{\beta}+k_8|Z_3-Z_2|^{\beta}\\&\le k_9|Z_1-Z_2|^{\beta},\end{aligned}\tag{4.4}$$

where $Z_3=\frac{x_1+x_2}{2}+i\big[Y_1+\frac{x_2-x_1}{2\sqrt{3}}\big]$. If $Z_1=x_1+iY_1$, $Z_2=x_1+iY_2$, $Y_1<Y_2$, and we choose $Z_3=x_1+\frac{Y_2-Y_1}{2\sqrt{3}}+\frac{i(Y_2+Y_1)}{2}$, and can also get (4.4). If $Z_1=x_1+iY_1$, $Z_2=x_2+iY_2$, $x_1<x_2, Y_1<Y_2$, and we choose $Z_3=x_2+iY_1$, obviously

$$|\Psi(Z_1)-\Psi(Z_2)|\le|\Psi(Z_1)-\Psi(Z_3)|+|\Psi(Z_3)-\Psi(Z_2)|,$$

and $|\Psi(Z_1)-\Psi(Z_3)|$, $|\Psi(Z_3)-\Psi(Z_2)|$ can be estimated by the above way, hence we can obtain the estimate of $|\Psi(Z_1)-\Psi(Z_2)|$. For the function $\Psi(Z)$ in (4.2) in $D\cap\{\mathrm{Im}Y\le 0\}$, the similar estimates can be also derived. Hence we have the estimate (4.3). □

Remark 4.2. If the condition $H(\hat{y})=\hat{y}^{\frac{m}{2}}$ in Lemma 4.1 is replaced by $H(\hat{y})=\hat{y}^{\eta}$, herein $\hat{y}=y-x^2$, η is a positive constant satisfying the inequality $\eta<\frac{m+2}{2}$, then by the same method we can prove that the integral $\Psi(Z)=T\big(\frac{g}{H}\big)$ satisfies the estimate

$$C_\beta[\Psi(Z),D_Z]\le M_1,$$

in which $\beta=1-\frac{2\eta}{m+2}-\delta$, δ is a sufficiently small positive constant, and $M_1=M_1(\beta,k_3,H,D_Z)$ is a positive constant. In particular if $H(\hat{y})=\hat{y}=y-x^2$, i.e. $\eta=1$, then we can choose $\beta=\frac{m}{m+2}-\delta$, δ is a sufficiently small positive constant.

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