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Theoretical Study of an Abstract Bubble Vibration Model

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Abstract. We present the theoretical study of a hyperbolic-elliptic system of equations called the *Abstract Bubble Vibration* (ABV) model. This simplified system is derived from a model describing a diphasic low Mach number flow. It is thus aimed at providing mathematical properties of the coupling between the hyperbolic transport equation and the elliptic Poisson equation.
We prove an existence and uniqueness result including the approximation of the time interval of existence for any smooth initial condition. In particular, we obtain a global-in-time existence result for small parameters. We then focus on properties of solutions (depending of their smoothness) such as maximum principle or evenness. In particular, an explicit formula of the mean value of solutions is given.

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1. Introduction

Over the past two centuries, several systems of equations have been proposed to model motion of fluids. The most general formulation is the compressible Navier-Stokes system that consists of conservation laws for variables such as density, momentum and energy. Then, the equations may be simplified through physical considerations. For instance, in this particular study, we are interested

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in the modelling of a diphasic flow – which can be assimilated to a nonmiscible 2-fluid flow at our scale – where the Mach number relative to each phase is very small. In other words, the ratio of the fluid velocity to the sound speed is supposed to be negligible, which enables to make a formal asymptotic expansion with respect to this small parameter. The resulting system is called *Diphasic Low Mach Number* (Dlmn) [5, 18]. See also [6] for numerical simulations.

Classically, a major consequence of the low Mach number expansion is that the system turns from hyperbolic to hyperbolic-elliptic [14]. That is why we aim at focusing on couplings between hyperbolic and elliptic equations. Indeed, assuming the velocity field is potential and decoupling from temperature and pressure laws, a 2-equation system has been derived in [6]. This system – called the *Abstract Bubble Vibration* (Abv) model – consists of a Poisson equation for the velocity field and a transport equation for the mass fraction of gas, together with initial and boundary conditions. It has a similar structure to models used in different physical frameworks. We may refer to the 2D incompressible Euler equations [24], the Keller-Segel equations in biology [22], the Smoluchowski model in astrophysics [3] or the Kull-Anisimov instability [12].

Investigations of simpler models provide reliable theoretical and numerical results. They also turn out to be useful for more general studies carried out on the full sets of equations (like Dlmn). Indeed, the Abv model has been constructed in order to yield a better understanding of the overall process of the motion of bubbles. Concerning numerical aspects (which are not the topic of this paper), we refer to [19, 21]. In particular, one of the most difficult issues raised by diphasic flows is the numerical handling of interfaces. That is why an accurate resolution requires an adaptive mesh refinement technique to avoid any diffusion of the interface [21]. For diffuse interface, a scheme has been specifically derived in [19]. Both approaches provide qualitative results for bubbly flows.

This paper is devoted to the proof of different properties of the Abv model. In Section 2, we describe the derivation of this model from the compressible Navier-Stokes equations while in Section 3, we get interested in theoretical results including existence and uniqueness issues and properties that solutions satisfy. At last, we conclude with a lemma that provides an explicit expression for the mean value of solutions that can be interpreted as the volume of a bubble in the case of nonsmooth initial data (more precisely indicator functions of subdomains).

2. Derivation of the model

As bubbles may appear in an operating reactor, we deal with a compressible diphasic flow in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$. While many formulations are based on a set of equations for each phase, our model consists of a single system in which variables are global and not specific to one phase or the other. It can be assimilated to a single interfacial velocity approach.

The compressible Navier-Stokes equations for a viscous compressible diphasic flow under gravity read in conservative variables

$$\partial_t(\rho Y_1) + \nabla \cdot (\rho Y_1 \boldsymbol{u}) = 0 \tag{1a}$$
$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{1b}$$
$$\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla P + \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g} \tag{1c}$$
$$\partial_t(\rho E) + \nabla \cdot (\rho E \boldsymbol{u}) = -\nabla \cdot (P\boldsymbol{u}) + \nabla \cdot (\kappa \nabla T) + \nabla \cdot (\boldsymbol{\sigma u}) + \rho \boldsymbol{g} \cdot \boldsymbol{u}. \tag{1d}$$

Compared to the standard conservation laws involving density ρ, momentum $\rho \boldsymbol{u}$ and total energy ρE, there is an additional equation corresponding to the conservation of partial mass (1a). Y_1 denotes the mass fraction of gas. As the two phases are nonmiscible, Y_1 can be assimilated to the indicator function of the domain $\Omega_1(t)$ occupied by the vapor phase. Then, $\Omega_2(t) = \Omega \backslash \Omega_1(t)$ is the liquid domain and $\Sigma(t) = \overline{\Omega}_1(t) \cap \overline{\Omega}_2(t)$ is the location of the interface between liquid and gas. It corresponds to the discontinuity of the function Y_1.

Here and in the sequel, $\boldsymbol{g}$ denotes the gravity field, κ the thermal conductivity, T the temperature and P the pressure. We note $\boldsymbol{\sigma}$ the linearized Cauchy stress tensor that reads under the linear elasticity assumption

$$\boldsymbol{\sigma} = \mu \left(\nabla \boldsymbol{u} + {}^t\nabla \boldsymbol{u} \right) + \lambda (\nabla \cdot \boldsymbol{u}) \mathcal{I}_d.$$

λ and μ are the Lamé coefficients (see [16] for example). The system is closed as soon as the physical coefficients ρ (or P), κ, λ and μ are known (through equations of state and constitutive laws).

After a singular perturbation analysis with respect to the Mach number $\mathcal{M}_* = \mathcal{U}_* \sqrt{\frac{\rho_*}{P_*}} \ll 1$ applied to the non-conservative formulation of System (1), the latter reduces to

$$\partial_t Y_1 + \boldsymbol{u} \cdot \nabla Y_1 = 0 \tag{2a}$$
$$\nabla \cdot \boldsymbol{u} = \mathcal{G}(t, \boldsymbol{x}) \tag{2b}$$
$$\rho \big[\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \big] = -\nabla \pi + \nabla \cdot \big[\mu (\nabla \boldsymbol{u} + {}^t\nabla \boldsymbol{u}) \big] + \rho \boldsymbol{g} \tag{2c}$$
$$\rho c_p (\partial_t T + \boldsymbol{u} \cdot \nabla T) = \alpha T P'(t) + \nabla \cdot (\kappa \nabla T) \tag{2d}$$
$$P'(t) = \mathcal{H}(t). \tag{2e}$$

For more details about the derivation of this system – called the *Diphasic Low Mach Number* (DLMN) system – please refer to [5,6]. The methods which lead to DLMN are based on works from Majda and Embid [10, 15]. In particular, Equation (2d) is deduced from (1) by means of the second law of thermodynamics and the Maxwell relations applied to the Gibbs potential.

As for notations, α and c_p are classical thermodynamic variables [11]. At order 0 in the asymptotic expansion, the thermodynamic pressure depends only on time t. Another variable is thus introduced to allow for dynamic effects in the momentum equation. That is why π is called the dynamic pressure. It corresponds to the second-order term in the asymptotic expansion of the pressure. Finally, $\mathcal{G}$ and $\mathcal{H}$ are nonlinear functionals of Y_1, T and P.

The elliptic equation (2b) is a reformulation of the mass conservation law to highlight the compressibility of the system despite the low Mach number. Furthermore, it leads to Equation (2e) by ensuring the compatibility with the boundary condition $\boldsymbol{u}_{|\partial\Omega} = 0$ [4] namely $\int_\Omega \mathcal{G}(t, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$. System (2) is thus closed.

Since all the coefficients appearing in System (2) depend implicitly on Y_1, T and P through the equations of state, the DLMN system is highly nonlinear. That is why as a preliminary we derived a simplified model based on the potential assumption that consists in stating that $\boldsymbol{u}$ is a gradient field. Let ϕ be the potential (known up to a constant), i.e. $\boldsymbol{u} = \nabla\phi$. Equation (2c) is overlooked and ϕ is determined by means of the Poisson equation $\Delta\phi = \mathcal{G}$ which is coupled to the mass fraction, temperature and pressure equations through the dependance of $\mathcal{G}$ w.r.t. (Y_1, T, P). This underlines the new mathematical structure of the low Mach number system which is hyperbolic-elliptic. To decouple the velocity equation from temperature and pressure evolution laws, we replace $\mathcal{G}$ by a simplified term depending only on Y_1 (linearly). The resulting model – called the *Abstract Bubble Vibration* (ABV) model – reads [7]:

$$\partial_t Y_1 + \nabla\phi \cdot \nabla Y_1 = 0 \tag{3a}$$

$$Y_1(0, \boldsymbol{x}) = Y^0(\boldsymbol{x}) \tag{3b}$$

$$\Delta\phi = \psi(t)\left[Y_1(t, \boldsymbol{x}) - \frac{1}{|\Omega|}\int_\Omega Y_1(t, \boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'\right] \tag{3c}$$

$$\nabla\phi \cdot \boldsymbol{n}_{|\partial\Omega} = 0. \tag{3d}$$

Y^0 and ψ are given functions of $\boldsymbol{x}$ and t respectively, with ψ continuous on $[0, +\infty)$. In addition, we assume that Ω is *smooth enough* to allow the existence of the normal unit vector $\boldsymbol{n}_{|\partial\Omega}$ and to provide elliptic regularity results for the Poisson equation. Note that the global system (3) is still non-linear due to the term $\nabla\phi \cdot \nabla Y_1$.

3. Theoretical results

In this section we present some results under different smoothness assumptions. In a Sobolev case, we prove existence and uniqueness of classical solutions in finite time. In particular, we provide an estimate of the time interval. The last paragraph deals with a less smooth case where we obtain an explicit formula for the mean value of weak solutions.

3.1. Preliminary. Before any investigation, we make two remarks about the potential ϕ. On the one hand, it is obvious that the potential cannot be unique, except up to a constant. We choose the following gauge for ϕ:

$$\int_{\Omega} \phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0. \tag{4}$$

On the other hand, as Equation (3c) is stationary, the potential necessarily satisfies the initial condition

$$\Delta\phi^0 = \psi(0)\left[Y^0(\boldsymbol{x}) - \frac{1}{|\Omega|}\int_{\Omega} Y^0(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'\right], \quad \nabla\phi^0\cdot\boldsymbol{n}_{|\partial\Omega} = 0. \tag{5}$$

Equations (4), (5) will be implicitly included in System (3) in the sequel even if they are not referred to. Any initial data satisfying (5) are called well-prepared.

We introduce the functional space related to this problem (and more specifically to advection problems) defined for $\mathscr{T} > 0$ and $s \in \mathbb{Z}_+$ by

$$\mathcal{W}_{s,\mathscr{T}}(\Omega) = \mathscr{C}^0\left([0,\mathscr{T}], L^2(\Omega)\right) \cap L^\infty\big([0,\mathscr{T}], H^s(\Omega)\big),$$

where

$$H^s(\Omega) = \left\{ f \in L^2(\Omega) \mid \forall\, s' \in \{0,\dots,s\},\ \forall\,\gamma \in \mathbb{Z}_+^d,\ |\gamma| = s' : D^\gamma f \in L^2(\Omega) \right\}.$$

This definition extends to non-integer s by means of the Slobodeckij seminorm [23] but this case will not be considered in the sequel. The set $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{s,\mathscr{T}} = \sup_{t\in[0,\mathscr{T}]} \|f(t,\cdot)\|_s.$$

The injection from $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ to $\mathscr{C}^0\big([0,\mathscr{T}], H^{s'}(\Omega)\big)$ is continuous for any $s' < s$. So does the injection from $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ to $\mathscr{C}^0\big([0,\mathscr{T}] \times \overline{\Omega}\big)$ when $s > \frac{d}{2}$ (see Lemma A.3 and [7]).

3.2. Short time existence theorem. Theorem 3.1 below was first published in [7]. Nevertheless, we present here a proof that enables to specify an approximation of the time interval (Theorem 3.2) and that leads to a global-in-time existence result for a certain class of initial data (Corollary 3.3). Let s_0 be the integer $s_0 = \lfloor \frac{d}{2} \rfloor + 1$.

Theorem 3.1. *Assume $Y^0 \in H^s(\Omega)$ with s an integer such that $s \geq s_0 + 1$ and $\psi \in \mathscr{C}^0(0,+\infty)$. Then there exists $\mathscr{T}_0 > 0$ depending on ψ and $\|Y^0\|_s$ such that System (3) has a unique classical solution $Y_1 \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ for $\mathscr{T}$ at least greater than $\mathscr{T}_0$.*

The time of existence $\mathscr{T}_0$ is not said to be optimal: It is prescribed by the way we prove Theorem 3.1, namely the combination of a boundedness property in $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ and a strong convergence in $\mathcal{W}_{0,\mathscr{T}}(\Omega)$. In the course of the proof, we derive the constraint (8) for $\mathscr{T}_0$ that we improve to obtain the following lower bound:

Theorem 3.2. *Under the same assumptions as in Theorem* 3.1, *we have* $Y_1 \in \mathcal{W}_{s,\mathscr{T}_0}(\Omega)$ *for any* $\mathscr{T}_0 > 0$ *such that*

$$\int_0^{\mathscr{T}_0} |\psi(\tau)| \,\mathrm{d}\tau \leq C_{abv}(s,d,\Omega) \left\| Y^0 - \mu(Y^0) \right\|_s^{-1}, \tag{6}$$

where $\mu(Y^0) = \frac{1}{|\Omega|}\int_\Omega Y^0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ *and* C_{abv} *is a universal constant.*

We note that the left hand side in (6) is monotone-increasing w.r.t. $\mathscr{T}_0$. Thus, the greater $\|Y^0 - \mu(Y^0)\|_s$, the lower $\mathscr{T}_0$.

Furthermore, if $Y^0 \equiv 1 \in H^\infty(\Omega)$ – which corresponds to a bubble occupying the whole domain – (resp. $Y^0 \equiv 0$), the unique solution is trivially given by $Y_1 \equiv 1$ (resp. $Y_1 \equiv 0$) without restriction on the time of existence. Likewise, for $\psi \equiv 0$, $Y_1 \equiv Y^0$ is a global solution. In those three cases, (6) is optimal.

We also infer that given $\mathscr{T} > 0$ and $\psi \in \mathscr{C}^0(0,\mathscr{T})$, there exists a local solution $Y_1 \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ for any Y^0 s.t. $\|Y^0 - \mu(Y^0)\|_s \leq C_{abv}\|\psi\|_{L^1(0,\mathscr{T})}^{-1}$. Consequently, if ψ also belongs to $L^1(0,+\infty)$, we have a global-in-time existence result:

Corollary 3.3. *Let* ψ *be a function in* $\mathscr{C}^0(0,+\infty) \cap L^1(0,+\infty)$. *Then there exists a unique solution* Y_1 *global in time for any* $Y^0 \in H^{s_0+1}(\Omega)$ *provided*

$$\left\| Y^0 - \mu(Y^0) \right\|_{s_0+1} \leq \frac{C_{abv}(s,d,\Omega)}{\|\psi\|_{L^1}}.$$

Proof of Theorem 3.1. For the proof of uniqueness, see Lemma 3.4 below. For the existence part, we consider the Picard iterates for System (3). More precisely, we introduce the sequences $(Y^{(k)})$ and $(\phi^{(k)})$ defined by induction as follows:

① $Y^{(k=0)} = Y^0$.

② Given $Y^{(k)}$, we compute $\phi^{(k)}$ as the solution of

$$\begin{cases} \Delta\phi^{(k)}(t,\boldsymbol{x}) = \psi(t)\left(Y^{(k)}(t,\boldsymbol{x}) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega Y^{(k)}(t,\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'\right) \\ \nabla\phi^{(k)} \cdot \boldsymbol{n}_{|\partial\Omega} = 0. \end{cases} \tag{7a}$$

③ Then, $Y^{(k+1)}$ satisfies

$$\begin{cases} \partial_t Y^{(k+1)} + \nabla\phi^{(k)} \cdot \nabla Y^{(k+1)} = 0 \\ \qquad\qquad Y^{(k+1)}(0,\cdot) = Y^0. \end{cases} \tag{7b}$$

We shall show that the sequence $(Y^{(k)})$ is bounded in $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ and converges strongly in $\mathcal{W}_{0,\mathscr{T}}(\Omega)$. Applying Lemma A.1 to Equation (7b) and Lemma A.2 to Equation (7a), we get[1]

$$\begin{aligned}\|Y^{(k+1)}\|_{s,\mathscr{T}} &\leq \|Y^0\|_s \exp\left[C_{adv}(s)\int_0^{\mathscr{T}} \|\mathrm{Hess}(\phi^{(k)})\|_{s-1}(t)\,\mathrm{d}t\right]\\ &\leq \|Y^0\|_s \exp\left[C_{adv}(s)\cdot C_{ell}(s-1)\cdot\|Y^{(k)}\|_{s-1,\mathscr{T}}\int_0^{\mathscr{T}} |\psi(t)|\,\mathrm{d}t\right]\\ &\leq \|Y^0\|_s \exp\big[\tilde{C}_{abv}(s)\cdot\|Y^{(k)}\|_{s-1,\mathscr{T}}\cdot\overline{\Psi}(\mathscr{T})\big],\end{aligned}$$

where $\tilde{C}_{abv}(s) = C_{adv}(s)\cdot C_{ell}(s-1)$ and $\overline{\Psi}$ is s.t. $\overline{\Psi}' = |\psi|$ and $\overline{\Psi}(0) = 0$. We introduce the sequence (u_k) defined by $u_0 = \tilde{C}_{abv}(s)\cdot\|Y^0\|_s\cdot\overline{\Psi}(\mathscr{T})$ and $u_{k+1} = u_0 \exp u_k$. Thus, we have $\tilde{C}_{abv}(s)\cdot\|Y^{(k)}\|_{s,\mathscr{T}}\cdot\overline{\Psi}(\mathscr{T}) \leq u_k$ by induction.

It is easy to prove that (u_k) converges iff $u_0 \leq e^{-1}$. Then, the limit is the lowest solution[2] x_0 of the equation $x\exp(-x) = u_0$ and we have $u_k \leq u_{k+1} \leq x_0$. Hence, under the assumption

$$\|Y^0\|_s\cdot\overline{\Psi}(\mathscr{T}) \leq C_{abv}(s) := \frac{1}{e\tilde{C}_{abv}(s)}, \tag{8}$$

the sequence $(Y^{(k)})$ is uniformly bounded in $\mathcal{W}_{s,\mathscr{T}}(\Omega)$. An upper bound is given by $e^{x_0}\|Y^0\|_s$. In particular, this result implies that the sequence $\big(\|Y^{(k)}(t,\cdot)\|_0\big)$ is equicontinuous and uniformly bounded in $\mathscr{C}^0\big([0,\mathscr{T}]\big)$. The Arzelà-Ascoli theorem yields the existence of a subsequence $(Y^{(k')})$ that converges strongly in $\mathscr{C}^0\big([0,\mathscr{T}],L^2(\Omega)\big)$. Likewise, the boundedness property in $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ also provides the weak-$\star$ convergence of a subsequence $(Y^{(k'')})$ of $(Y^{(k')})$ in the space $L^\infty\big([0,\mathscr{T}],H^s(\Omega)\big)$. We still note $(Y^{(k)})$ the weak-$\star$ convergent subsequence in $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ and $\tilde{Y}\in\mathcal{W}_{s,\mathscr{T}}(\Omega)$ its limit.

We shall prove that the sequence $(Y^{(k)})$ converges strongly in $\mathcal{W}_{0,\mathscr{T}}(\Omega)$ by means of a contraction inequality. Indeed, we deduce from Equation (7b)

$$\begin{cases}\big[\partial_t + \nabla\phi^{(k)}\cdot\nabla\big](Y^{(k+1)} - Y^{(k)}) = -(\nabla\phi^{(k)} - \nabla\phi^{(k-1)})\cdot\nabla Y^{(k)},\\ \qquad (Y^{(k+1)} - Y^{(k)})(0,\cdot) = 0\,.\end{cases}$$

[1]We emphasize dependencies on s for the constants appearing in the proof and we omit other dependencies but they are specified in the appendix.

[2]Equation $xe^{-x} = u_0$ has 2 solutions for $u_0 \in (0, e^{-1})$. Let x_0 be the solution in $(0,1)$. Moreover $x_0 = 1$ iff $u_0 = e^{-1}$, which means that (8) is an equality.

The energy estimate given by Lemma A.1 reads

$$
\begin{aligned}
&e^{-\chi_0^{(k)}(t)}\left\|\left(Y^{(k+1)}-Y^{(k)}\right)(t,\cdot)\right\|_0 \\
&\leq \int_0^t e^{-\chi_0^{(k)}(\tau)}\left\|\left(\nabla\phi^{(k)}-\nabla\phi^{(k-1)}\right)\cdot\nabla Y^{(k)}(\tau,\cdot)\right\|_0 \mathrm{d}\tau, \\
&\leq C_M(0,s-1,d)\cdot\|Y^{(k)}\|_{s,\mathscr{T}}\int_0^t e^{-\chi_0^{(k)}(\tau)}\left\|\left(\nabla\phi^{(k)}-\nabla\phi^{(k-1)}\right)(\tau,\cdot)\right\|_0 \mathrm{d}\tau, \\
&\leq \underbrace{C_M\cdot e^{x_0}\|Y^0\|_s\cdot C_{PW}\sup_{t\in[0,\mathscr{T}]}|\psi(t)|}_{C_{abv,2}}\int_0^t e^{-\chi_0^{(k)}(\tau)}\left\|\left(Y^{(k)}-Y^{(k-1)}\right)(\tau,\cdot)\right\|_0 \mathrm{d}\tau,
\end{aligned}
$$

using Lemma A.2 and the Moser inequality (Lemma A.3). Here, the exponent is given by $\chi_0^{(k)}(t) = \frac{1}{2}\int_0^t \|\Delta\phi^{(k)}(\tau,\cdot)\|_\infty \mathrm{d}\tau$. Using the boundedness property and the Sobolev embedding inequality (see Lemma A.3), we have

$$\chi_0^{(k)}(t) \leq \int_0^t |\psi(\tau)|\cdot\|Y^{(k)}(\tau,\cdot)\|_\infty \mathrm{d}\tau \leq \chi(t)$$

with $\chi(t) = e^{x_0}\|Y^0\|_s\cdot C_{sob}(s)\cdot\overline{\Psi}(t)$. We can thus replace $\chi_0^{(k)}$ by χ in the energy estimate[3]

$$e^{-\chi(t)}\left\|\left(Y^{(k+1)}-Y^{(k)}\right)(t,\cdot)\right\|_0 \leq C_{abv,2}\int_0^t e^{-\chi(\tau)}\left\|\left(Y^{(k)}-Y^{(k-1)}\right)(\tau,\cdot)\right\|_0 \mathrm{d}\tau.$$

Iterating the process, we obtain

$$
\begin{aligned}
e^{-\chi(t)}\left\|\left(Y^{(k+1)}-Y^{(k)}\right)(t,\cdot)\right\|_0 &\leq C_{abv,2}^k\int_0^t e^{-\chi(\tau)}\left\|\left(Y^{(1)}-Y^{(0)}\right)(\tau,\cdot)\right\|_0\frac{(t-\tau)^{k-1}}{(k-1)!}\mathrm{d}\tau \\
&\leq \frac{C_{abv,2}^k\, t^k}{k!}\|Y^{(1)}-Y^{(0)}\|_{0,\mathscr{T}}.
\end{aligned}
$$

Thus

$$\|Y^{(k+1)}-Y^{(k)}\|_{0,\mathscr{T}} \leq \frac{(C_{abv,2}\,\mathscr{T})^k}{k!}e^{\chi(\mathscr{T})}\|Y^{(1)}-Y^{(0)}\|_{0,\mathscr{T}}.$$

The series $\sum_k \|Y^{(k+1)}-Y^{(k)}\|_{0,\mathscr{T}}$ is convergent, which shows that the sequence $Y^{(k)}$ converges in the complete space $\mathcal{W}_{0,\mathscr{T}}(\Omega)$ to $Y\in\mathcal{W}_{0,\mathscr{T}}(\Omega)$. By uniqueness of the limit, the weak-$\star$ limit $\widetilde{Y}$ is necessarily equal to Y. Therefore, $Y\in\mathcal{W}_{s,\mathscr{T}}(\Omega)$ even if there is no proof that $Y^{(k)}$ tends to Y in $\mathcal{W}_{s,\mathscr{T}}(\Omega)$ (strongly). However, we can show by means of an interpolation inequality [17] that the convergence is strong in $\mathcal{W}_{s',\mathscr{T}}(\Omega)$ for any $s' < s$.

[3] See Lemma A.1: The exponent may be replaced by any upper bound (in the differential form).

Likewise, we prove that $\nabla\phi^{(k)}$ converges in $\mathcal{W}_{0,\mathscr{T}}(\Omega)$ to $\Phi \in \mathcal{W}_{s+1,\mathscr{T}}(\Omega)$. Indeed, applying Lemma A.2, we have

$$\|\nabla\phi^{(k)}\|_{s+1,\mathscr{T}} \leq C_{ell}(s) \sup_{t\in[0,\mathscr{T}]} |\psi(t)| \cdot \|Y^{(k)}\|_{s,\mathscr{T}} ,$$

$$\left\|\nabla\left(\phi^{(k)} - \phi^{(k-1)}\right)\right\|_{0,\mathscr{T}} \leq C_{PW} \sup_{t\in[0,\mathscr{T}]} |\psi(t)| \cdot \left\|\left(Y^{(k)} - Y^{(k-1)}\right)\right\|_{0,\mathscr{T}}.$$

The previous results for $Y^{(k)}$ provides the convergence for $\nabla\phi^{(k)}$. Moreover, there exists $\phi \in \mathcal{W}_{s+2,\mathscr{T}}(\Omega)$ such that $\Phi = \nabla\phi$ because the gradient field space is closed.

It remains to prove that Y and $\nabla\phi$ are solutions to System (3). To do so we rewrite (7b), (7a) as

$$Y^{(k+1)} = Y^0 - \int_0^t \nabla\phi^{(k)} \cdot \nabla Y^{(k+1)}\,\mathrm{d}\tau,$$

$$\forall\, \varphi \in H^1(\Omega):\ \int_\Omega \nabla\varphi \cdot \nabla\phi^{(k)}\,\mathrm{d}\boldsymbol{x} = -\psi(t)\int_\Omega \varphi\left(Y^{(k)} - \frac{1}{|\Omega|}\int_\Omega Y^{(k)}\,\mathrm{d}\boldsymbol{x}'\right)\mathrm{d}\boldsymbol{x}\,.$$

As each function involved in the latter relations belongs to $\mathscr{C}^0\big([0,\mathscr{T}]\times\overline{\Omega}\big)$ due to the embedding $\mathcal{W}_{s,\mathscr{T}}(\Omega) \subset \mathscr{C}^0\big([0,\mathscr{T}], H^{s'}(\Omega)\big)$ for $s' < s$ (see [7, Lemma A.1]), we apply the dominated convergence theorem to obtain the integral form of (3). In particular, we have $Y_1(t,\boldsymbol{x}) = Y^0(\boldsymbol{x}) - \int_0^t \nabla\phi\cdot\nabla Y_1(\tau,\boldsymbol{x})\,\mathrm{d}\tau$. The previous embedding results show that ∇Y_1 and $\nabla\phi$ are continuous. This fact implies that $Y_1 \in \mathscr{C}^1\big([0,\mathscr{T}]\times\overline{\Omega}\big)$ and we recover the differential form of Equation (3a). Similarly, $\nabla\phi \in \mathcal{W}_{s+1,\mathscr{T}}(\Omega) \subset \mathscr{C}^0\big([0,\mathscr{T}],\mathscr{C}^1(\overline{\Omega})\big)$, which means that the weak formulation above is equivalent to Equation (3c) in the strong sense. □

Proof of Theorem 3.2. Let $Y^0 \in H^s$ with $s > \frac{d}{2}+1$ and $\psi \in \mathscr{C}^0(0,+\infty)$. Then there exists $Y_1 \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ with $\mathscr{T}$ prescribed by (8) (as large as possible, maybe $\mathscr{T} = +\infty$). Let c_0 be the constant such that $\|Y^0 - c_0\|_s = \min_{c\in\mathbb{R}} \|Y^0 - c\|_s$, i.e.

$$c_0 = \frac{1}{|\Omega|}\int_\Omega Y^0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Hence, we have $\|Y^0 - c_0\|_s \leq \|Y^0\|_s$. We consider System (3) with initial condition $Z(0,\cdot) = Y^0 - c_0 \in H^s$ to which we apply Theorem 3.1. There exists a unique solution $Z \in \mathcal{W}_{s,\mathscr{T}'}(\Omega)$ for $\mathscr{T}'$ satisfying $\int_0^{\mathscr{T}'} |\psi(t)|\,\mathrm{d}t \leq \frac{C_{abv}}{\|Y^0-c_0\|_s}$. Hence, we can choose $\mathscr{T}' \geq \mathscr{T}$ with a strict inequality iff $c_0 \neq 0$ and $\mathscr{T} < +\infty$. Thus, $Y_1 = Z + c_0$ is a solution to System (3) on $[0,\mathscr{T}']$. □

It is worth underlining that unlike classical results as [1] which rely on smallness assumptions for initial data, our result states that existence is obtained provided initial datum is close enough to its mean value. This leads to a larger time of existence.

3.3. Other properties. In this paragraph, we give some formal lemmas about solutions under weaker assumptions than in Theorem 3.1. For some $\mathscr{T} > 0$, we set $\mathcal{Z}_{\mathscr{T}}(\Omega) = L^\infty\big([0,\mathscr{T}], W^{1,\infty}(\Omega)\big)$. Note that $\mathcal{W}_{s,\mathscr{T}}(\Omega) \subset \mathcal{Z}_{\mathscr{T}}(\Omega)$, which means that the lemmas below can be applied to the classical solution induced by Theorem 3.1. We do not state any existence result in $\mathcal{Z}_{\mathscr{T}}(\Omega)$ but any solution must satisfy the following properties.

First, we shall state whether $\mathcal{Z}_{\mathscr{T}}(\Omega)$ is a suitable functional space for solutions to System (3). Let $Y_1 \in \mathcal{Z}_{\mathscr{T}}(\Omega)$ be a solution of

$$Y_1 = Y^0 - \int_0^t \nabla\phi \cdot \nabla Y_1 \,\mathrm{d}\tau, \quad \Delta\phi = \psi(t)\big(Y_1 - \mu(Y_1)\big). \tag{9}$$

As $Y_1 \in L^\infty\big([0,\mathscr{T}] \times \Omega\big)$, elliptic regularity results guarantee that the solution $\nabla\phi$ of the Poisson equation in (9) belongs to $L^\infty\big([0,\mathscr{T}], \mathscr{C}^0(\Omega)\big)$. Knowing that $\nabla Y_1 \in L^\infty\big([0,\mathscr{T}] \times \Omega\big)$, the term $\nabla\phi \cdot \nabla Y_1$ is in $L^\infty\big([0,\mathscr{T}], L^2(\Omega)\big) \subset L^1\big([0,\mathscr{T}], L^2(\Omega)\big)$. Thus, the integral in (9) is continuous w.r.t. t and differentiable for almost all t (see [2, § II.4.1]) and Y_1 satisfies (3a) in $L^2(\Omega)$ and thus almost everywhere in Ω, which legitimates the following calculus.

Lemma 3.4. *There exists at most one solution in the space $\mathcal{Z}_{\mathscr{T}}(\Omega)$.*

Proof. Let (Y_1, ϕ_1) and (Y_2, ϕ_2) be two solutions. Combining the two equations with the notation $\delta * = *_1 - *_2$, we have $\partial_t \delta Y + \nabla\phi_1 \cdot \nabla\delta Y = -\nabla Y_2 \cdot \nabla\delta\phi$. Multiplying by δY and integrating by parts, we get by virtue of the Cauchy-Schwarz inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta Y\|_0 \le \frac{1}{2}\|\Delta\phi_1\|_\infty \|\delta Y\|_0 + \|\nabla Y_2\|_\infty \|\nabla\delta\phi\|_0.$$

We apply Lemma A.2 to the last term and the Grönwall's inequality to obtain $\|\delta Y\|_0 = 0$ due to the fact that $\|\delta Y(0,\cdot)\|_0 = 0$. □

Lemma 3.5. *Assume Y_1 is a solution in the space $\mathcal{Z}_{\mathscr{T}}(\Omega)$. Then, Y_1 keeps the same upper and lower bounds as Y^0 almost everywhere.*

Proof. We first prove that if $Y^0 \ge 0$, then $Y_1 \ge 0$. Multiplying Equation (3a) by $Y_1^- = \min(Y_1, 0) \in \mathcal{Z}_{\mathscr{T}}(\Omega)$ and integrating by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Y_1^-\|_0^2 = \frac{1}{2}\int_\Omega (Y_1^-)^2(t,\boldsymbol{x})\Delta\phi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

As $\Delta\phi \in \mathcal{Z}_{\mathscr{T}}(\Omega) \subset L^\infty\big([0,\mathscr{T}] \times \Omega\big)$, the Grönwall's inequality yields $\|Y_1^-\|_0 = 0$ allowing for the fact that $\|Y_1^-(0,\cdot)\|_0 = 0$. Thus $Y_1 \ge 0$ a.e.

If $Y^0 \le 1$, we apply the previous result to the variable $Z = 1 - Y_1$ which is a solution to (3) with initial condition $Z(0,\cdot) = 1 - Y^0 \ge 0$. Hence $Z \ge 0$ and $Y_1 \le 1$. The general case $Y^0 \in [a,b]$ can be inferred from the positivity of variables $Y_1 - a$ and $b - Y_1$, which are solutions to System (3) with suitable initial data. □

Lemma 3.6. *The system is time-reversible in* $\mathcal{Z}_{\mathscr{T}}(\Omega)$.

Proof. Let Y_1 be a solution to System (3) in the class $\mathcal{Z}_{\mathscr{T}}(\Omega)$ for a certain $\mathscr{T} > 0$. The question adressed in the lemma is to determine whether starting from $Y_1(\mathscr{T}, \cdot)$, one recovers the initial condition Y^0 by "inverting" the time scale by means of the transformation $t \mapsto \mathscr{T} - t$. With $\hat{\psi}(t) = -\psi(\mathscr{T} - t)$, we check out that $(\hat{Y}_1, \hat{\phi}) = (Y_1(\mathscr{T} - t, \boldsymbol{x}), -\phi(\mathscr{T} - t, \boldsymbol{x}))$ is a solution to the system on $[0, \mathscr{T}]$. By the Uniqueness Lemma 3.4, $(\hat{Y}_1, \hat{\phi})$ is the unique solution and $\hat{Y}_1(\mathscr{T}, \cdot) = Y_1(0, \cdot) = Y^0$. □

For the last lemma, we introduce an additional definition:

Ω is said to be symmetric if
- $\boldsymbol{x} \in \Omega \Longrightarrow (-\boldsymbol{x}) \in \Omega$
- $\forall\, \boldsymbol{x} \in \partial\Omega : \boldsymbol{n}(-\boldsymbol{x}) = -\boldsymbol{n}(\boldsymbol{x})$.

Lemma 3.7. *If* Ω *is symmetric and* Y^0 *is even, then any solution in the space* $\mathcal{Z}_{\mathscr{T}}(\Omega)$ *is also even.*

Proof. Denoting $\tilde{Y}_1(t, \boldsymbol{x}) = Y_1(t, -\boldsymbol{x})$ and $\tilde{\phi}(t, \boldsymbol{x}) = \phi(t, -\boldsymbol{x})$, we remark that $\int_\Omega \tilde{Y}_1(t, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_\Omega Y_1(t, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$ which shows that $(\tilde{Y}_1, \tilde{\phi})$ is a solution to System (3) with the same initial datum $Y^0(-\boldsymbol{x}) = Y^0(\boldsymbol{x})$ and the same boundary condition. The Uniqueness Lemma 3.4 provides $Y_1(t, \boldsymbol{x}) = \tilde{Y}_1(t, \boldsymbol{x}) = Y_1(t, -\boldsymbol{x})$. The velocity field is odd. □

3.4. Volume. We consider in this paragraph a more general case, namely $Y_1 \in L^\infty([0, \mathscr{T}] \times \Omega)$ and Y^0 bounded in $[0, 1]$: this case corresponds to the modelling of bubbles in which Y_1 is the mass fraction of gas. For miscible fluids, Y_1 takes values between 0 and 1 while in the present study (without phase change and at the scale of bubbles), Y_1 is exactly equal to 0 or 1. In the latter case, the mean value of Y_1 is equal to the volume of the bubble. We present in this paragraph a general result about mean values (Proposition 3.9) and its application to a physical case (Lemma 3.11).

Let $\mu_n(t)$ be the mean value of $Y_1^n(t, \cdot)$ over Ω. In the class $\mathcal{Z}_{\mathscr{T}}(\Omega)$, when Y^0 takes values in $[0, 1]$, so does Y_1 according to Lemma 3.5. The sequence $(\mu_n(t))_n$ is bounded (in $[0, 1]$) and monotone-decreasing (pointwise). Thus, $\mu_n(t)$ converges to $\mu_\infty(t) := \frac{|\Omega_1(t)|}{|\Omega|}$ where $\Omega_1(t) = \{\boldsymbol{x} \in \Omega : Y_1(t, \boldsymbol{x}) = 1\}$ since $(Y_1)_{|\Omega_1(t)} = (Y_1^n)_{|\Omega_1(t)} = 1$. Nonetheless, these considerations do not enable to conclude about the convergence of μ_n in the weaker case $L^\infty([0, \mathscr{T}] \times \Omega)$. This is achieved thanks to Proposition 3.9, which provides an explicit expression for μ_n and a new proof for a maximum principle restricted to $[0, 1]$ (Lemma 3.10).

We first establish an ODE to which μ_n is a solution.

Lemma 3.8. *The sequence* $(\mu_n)_n$ *satisfies the following ODE*:

$$\mu_n'(t) = \psi(t)\big(\mu_{n+1}(t) - \mu_1(t)\mu_n(t)\big). \tag{10}$$

Proof. If Y_1 is a weak solution of Equation (3a), then Y_1^n satisfies $\partial_t Y_1^n + \nabla\phi \cdot \nabla Y_1^n = 0$ according to the *renormalisation principle* [8]. That means

$$\forall\, \xi \in \mathscr{C}_0^\infty\left((0,\mathscr{T}) \times \overline{\Omega}\right) : \int_0^{\mathscr{T}} \int_\Omega Y_1^n \big(\partial_t \xi + \nabla \cdot (\xi \nabla \phi)\big)\, \mathrm{d}\boldsymbol{x}\mathrm{d}t = 0.$$

Taking $\xi(t,\boldsymbol{x}) = \zeta(t)\xi_p(\boldsymbol{x})$ with $\zeta \in \mathscr{C}_0^\infty\big(0,\mathscr{T}\big)$ and $\xi_p \in \mathscr{C}_0^\infty\left(\overline{\Omega}\right)$ converging pointwise to $\mathbf{1}_\Omega$, the last equality can be rewritten as

$$\int_0^{\mathscr{T}} \zeta' \int_\Omega Y_1^n \xi_p \, \mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_0^{\mathscr{T}} \zeta \int_\Omega Y_1^n \nabla \cdot (\xi_p \nabla \phi)\, \mathrm{d}\boldsymbol{x}\mathrm{d}t = 0.$$

In the limit as $p \to +\infty$ through the dominated convergence theorem, the equation reduces to

$$\int_0^{\mathscr{T}} \zeta'(t)\mu_n(t)\,\mathrm{d}t + \int_0^{\mathscr{T}} \frac{\zeta(t)\psi(t)}{|\Omega|} \int_\Omega Y_1^n(t,\boldsymbol{x})\big(Y_1(t,\boldsymbol{x}) - \mu_1(t)\big)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t = 0,$$

for all $\zeta \in \mathscr{C}_0^\infty\big(0,\mathscr{T}\big)$, i.e. to ODE (10) in the sense of distributions. Since ψ is continuous and μ_n bounded for all n, the right hand side in (10) is bounded. We deduce that μ_n is continuous, which provides the continuity of the right hand side. ODE (10) thus holds in a classical sense. □

The main consequence is that we can derive an explicit expression for μ_n in terms of ψ and Y^0.

Proposition 3.9. *Let Ψ be s.t. $\Psi' = \psi$ and $\Psi(0) = 0$. Then*

$$\mu_n(t) = \frac{\int_\Omega [Y^0(\boldsymbol{x})]^n \exp\big[\Psi(t) Y^0(\boldsymbol{x})\big]\,\mathrm{d}\boldsymbol{x}}{\int_\Omega \exp\big[\Psi(t)Y^0(\boldsymbol{x})\big]\,\mathrm{d}\boldsymbol{x}}. \tag{11}$$

Proof. Since $\mu_n' + \psi\mu_1\mu_n$ can be expressed as

$$\left[\mu_n(t)\exp\int_0^t \mu_1(\tau)\psi(\tau)\,\mathrm{d}\tau\right]' \exp\left(-\int_0^t \mu_1(\tau)\psi(\tau)\,\mathrm{d}\tau\right),$$

ODE (10) can be rewritten in the integral form

$$M_N(t) = \mu_N(0) + \int_0^t \psi(\tau) M_{N+1}(\tau)\,\mathrm{d}\tau, \tag{12}$$

with $M_N(t) = \mu_N(t)\exp\int_0^t \psi(\tau)\mu_1(\tau)\,\mathrm{d}\tau$. By induction, we show that

$$M_1(t) = \sum_{k=1}^N \mu_k(0)\frac{\Psi(t)^{k-1}}{(k-1)!} + \int_0^t \psi(\tau) M_{N+1}(\tau) \frac{[\Psi(t) - \Psi(\tau)]^N}{N!}\,\mathrm{d}\tau.$$

Since Ψ is continuous and the sequence $\mu_k(0)$ is uniformly bounded ($Y^0 \in [0,1]$), the series $\sum_k \mu_{k+1}(0)\frac{\Psi^k(t)}{k!}$ is normally convergent on every compact set. Furthermore, the last term reads

$$\frac{1}{|\Omega|}\int_0^t \int_\Omega \frac{\big[Y_1(\tau,\boldsymbol{x})\big(\Psi(t)-\Psi(\tau)\big)\big]^N}{N!} Y_1(\tau,\boldsymbol{x})\psi(\tau)e^{\int_0^\tau \psi(\sigma)\mu_1(\sigma)\,\mathrm{d}\sigma}\,\mathrm{d}\boldsymbol{x}\mathrm{d}\tau.$$

In the limit as $N \to +\infty$, the integral tends to 0 by virtue of the dominated convergence theorem. Thus

$$M_1(t) = \mu_1(t)\exp\left(\int_0^t \psi(\tau)\mu_1(\tau)\,\mathrm{d}\tau\right) = \sum_{k\geq 1}\mu_k(0)\frac{\Psi(t)^{k-1}}{(k-1)!}.$$

Multiplying by ψ and integrating we obtain

$$\exp\left(\int_0^t \psi(\tau)\mu_1(\tau)\,\mathrm{d}\tau\right) = 1 + \sum_{k\geq 1}\mu_k(0)\frac{\Psi(t)^k}{k!}.$$

The combination of the last two equalities leads to

$$\mu_1(t) = \frac{\sum_{k\geq 1}\mu_k(0)\frac{\Psi(t)^{k-1}}{(k-1)!}}{1+\sum_{k\geq 1}\mu_k(0)\frac{\Psi(t)^k}{k!}} = \frac{\int_\Omega Y^0(\boldsymbol{x})\exp\big[\Psi(t)Y^0(\boldsymbol{x})\big]\,\mathrm{d}\boldsymbol{x}}{\int_\Omega \exp\big[\Psi(t)Y^0(\boldsymbol{x})\big]\,\mathrm{d}\boldsymbol{x}}.$$

The last equality is obtained by inverting integral and sum symbols, as the series $\sum_k \frac{[Y^0(\boldsymbol{x})\Psi(t)]^k}{k!}$ converges normally. Then, we show (11) by induction: For $n = 2$ we differentiate the expression of M_1 as well as Equation (12), and so on. □

This result holds for any solution to System (3) given ψ and Y^0 at least bounded. Moreover, it enables to extend Lemma 3.5 (when $Y^0 \in [0,1]$) to the bounded case.

Lemma 3.10. *Let Y_1 be a weak solution to System* (3) *belonging to $L^\infty\big([0,\mathscr{T}]\times\Omega\big)$ for a certain $\mathscr{T} > 0$. If $Y^0 \in [0,1]$, then Y_1 also takes values in* $[0,1]$ (*almost everywhere*).

Proof. First note that Equation (11) shows that $\mu_n(t)$ converges for all t since $Y^0 \in [0,1]$. Considering the definition of μ_n, that is $\mu_n(t) = \frac{1}{|\Omega|}\int_\Omega Y_1^n(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, we shall prove that Y_1 cannot take values outside $[0,1]$. Indeed, assume there exists $\omega(t)\subset\Omega$ s.t. $|\omega(t)|\neq 0$ and $Y_1(t,\boldsymbol{x})>1$ for almost all $x\in\omega(t)$. Writing μ_{2n} as

$$\mu_{2n}(t) = \underbrace{\frac{1}{|\Omega|}\int_{\Omega\setminus\omega(t)} Y_1^{2n}(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}_{\geq 0} + \underbrace{\frac{1}{|\Omega|}\int_{\omega(t)} Y_1^{2n}(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}_{\xrightarrow[n\to+\infty]{}+\infty}$$

we show that μ_n cannot converge, which is contradictory to what we stated above. Thus, $Y_1 \leq 1$ a.e. Likewise, if $Y_1 < 0$ on a positive measure set, we consider the solution $Z = 1 - Y_1$ associated to the initial condition $Z^0 = 1 - Y^0$. Necessarily, $Z \leq 1$ as shown previously and $Y_1 \geq 0$ a.e. □

Although Proposition 3.9 and Lemma 3.10 have been proven for Y^0 taking values in $[0,1]$, they still hold for general bounded Y^0 in $[a,b]$ by considering $Z = \frac{Y-a}{b-a}$.

Proposition 3.9 is a generalization of [7, Lemma 1.1]. Indeed, when Y_1 is the mass fraction of gas as described in Section 2, Proposition 3.9 has the following simpler formulation:

Lemma 3.11. ([7, Lemma 1.1]) *Assume there exists a solution of the type* $Y_1(t, \boldsymbol{x}) = \mathbf{1}_{\Omega_1(t)}(\boldsymbol{x})$ *where* $\Omega_1(t) \subset \Omega$, $t \in [0, \mathscr{T}]$ *for a certain* $\mathscr{T} > 0$. *Let* V *be the "volume of the bubble", i.e.* $V(t) = |\Omega_1(t)|$. *Then* V *is explicitly known as:*

$$V(t) = \frac{1}{\left(\frac{1}{V(0)} - \frac{1}{|\Omega|}\right) \exp\left[-\Psi(t)\right] + \frac{1}{|\Omega|}}. \tag{13}$$

Proof. This lemma was first proven in [7]. Here the proof is based on Proposition 3.9. In this irregular case, $\mu_N(0) = \mu_1(0) = \frac{V(0)}{|\Omega|}$ for all N and (11) leads to (13). □

Remark 3.12. Equation (13) turns out to be of great interest from a numerical point of view. As it is an exact formula for the volume, we can compute this volume so as to compare it to numerical approximations. Thus we can check out the accuracy of numerical schemes [19, 21].

Formulae (11) and (13) are global in time, which tends to show that there is no blow-up in finite time, even if it is still an open problem. Moreover, they show the influence of ψ: If ψ is positive, the bubble grows and conversely. Likewise, if ψ is periodic and has a zero mean value over the period, the volume is periodic too.

Another remark is the dependence w.r.t. $|\Omega|$: The same bubble inside two domains of different sizes will evolve differently. It is the influence of the Poisson equation and more particularly the boundary condition. We recall that the Abv model is derived from a low Mach number system (Dlmn, [5, 6]). In that case, the acoustic waves have an infinite speed of propagation which gives an elliptic character to the Dlmn system. Nevertheless, Equation (13) shows that the bubble cannot reach the boundary in finite time.

Finally, we mention that Proposition 3.9 enabled to prove further results in the one-dimensional case [19]. In particular, explicit solutions are derived thanks to the mean value formula.

4. Conclusion

The mathematical coupling in the Abv model between a transport equation and a Poisson equation turned out to be an interesting problem balancing hyperbolic properties and elliptic effects. In the smooth case, we proved both short time existence and uniqueness of classical solutions to the Abv model. In less regular situations, we established properties of possible solutions. These are qualitative results which provide a better knowledge of the behaviour of solutions. Some of them tend to show that solutions evolve as expected especially concerning boundedness properties or influence of the pulse ψ. There are still open problems like the periodicity in time of solutions if ψ is periodic.

Another major issue is about existence of solutions in weaker functional spaces satisfying physical constraints. There exists an explicit solution in 1D for a bubble-kind initial datum [19], which tends to show that there exist solutions in the general bounded case even if we did not prove either existence or uniqueness yet in higher dimensions. Possible methods to carry out may be the parabolic approximation (see [22] for instance), the use of log-Lipschitz estimates [24] or the concept of renormalized solutions [8]. Those approaches lead to existence of weak solutions to similar systems. The latter method will be applied in future works [20]. The use of the bounded mean oscillation (BMO) space [13] may also be of crucial interest. The fact remains that this study forms a relevant starting point for the analysis of the Dlmn system [5, 6].

A. Appendix

In this part we recall some functional results about hyperbolic and elliptic regularity as well as classical inequalities.

The following lemma corresponds to [18, Lemma 2.10] and is an improvement of [7, Lemma 3.1] and [9, Lemma 2.4] for a special care is given to constants involved in the estimates.

Lemma A.1. *Assume that $Y^0 \in H^s(\Omega)$, $\boldsymbol{u} \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ such that $\boldsymbol{u} \cdot \boldsymbol{n}_{|\partial\Omega} = 0$ and $f \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ with $\mathscr{T} > 0$ and s an integer s.t. $s \geq s_0 + 1$. Then, the transport equation*

$$\partial_t Y + \boldsymbol{u} \cdot \nabla Y = f, \quad Y(0, \boldsymbol{x}) = Y^0(\boldsymbol{x})$$

has a unique classical solution $Y \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ satisfying the energy estimates

$$\|Y(t,\cdot)\|_0 \leq e^{\chi_0(t)} \left(\left\|Y^0\right\|_0 + \int_0^t e^{-\chi_0(\tau)} \|f(\tau,\cdot)\|_0 \,\mathrm{d}\tau \right)$$
$$\|Y(t,\cdot)\|_s \leq e^{\chi_s(t)} \left(\left\|Y^0\right\|_s + \int_0^t e^{-\chi_s(\tau)} \|f(\tau,\cdot)\|_s \,\mathrm{d}\tau \right)$$

for all $t \in [0, \mathscr{T}]$ *and any functions* χ_0 *and* χ_s *such that*

$$\chi_0'(t) \geq \frac{1}{2}\|\nabla \cdot \boldsymbol{u}(t,\cdot)\|_\infty \quad \textit{and} \quad \chi_s'(t) \geq C_{adv}(s,d,\Omega)\|\nabla \boldsymbol{u}(t,\cdot)\|_{s-1}.$$

Lemma A.2 ([7, Lemma 3.2], [2, Theorem III.5.3])**.** *Suppose* $\psi \in \mathscr{C}^0(0,+\infty)$ *and* $Y_1 \in \mathcal{W}_{s,\mathscr{T}}(\Omega)$ *for* $\mathscr{T} > 0$ *and* $s \in \mathbb{N}$. *There exists a unique solution to the system*

$$\Delta\phi(t,\boldsymbol{x}) = \psi(t)\left(Y_1(t,\boldsymbol{x}) - \frac{1}{|\Omega|}\int_\Omega Y_1(t,\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'\right),$$
$$\nabla\phi \cdot \boldsymbol{n}_{|\partial\Omega} = 0, \qquad \int_\Omega \phi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0.$$

This solution satisfies $\nabla\phi \in \mathcal{W}_{s+1,\mathscr{T}}(\Omega)$ *and*[4]

$$\|\nabla\phi(t,\cdot)\|_0 \leq C_{PW}(d,\Omega)\cdot|\psi(t)|\cdot\|Y_1(t,\cdot)\|_0,$$
$$\|\nabla\phi(t,\cdot)\|_{s+1} \leq C_{ell}(s,d,\Omega)\cdot|\psi(t)|\cdot\|Y_1(t,\cdot)\|_s.$$

Lemma A.3. *We recall that* $s_0 = \lfloor \frac{d}{2} \rfloor + 1$.

1. *Let* s_1 *and* s_2 *be two integers satisfying* $s_1 + s_2 \geq s_0$. *Assume* $f \in H^{s_1}$ *and* $g \in H^{s_2}$. *Then* $fg \in H^{s_3}$ *with* $s_3 = \min(s_1, s_2, s_1 + s_2 - s_0)$. *Moreover, there exists* $C_M = C_M(s_1,s_2,d)$ *s.t. for all* f *and* g *as above*

$$\|fg\|_{s_3} \leq C_M\,\|f\|_{s_1}\|g\|_{s_2}.$$

2. (*Sobolev embeddings*) s_0 *is the lowest integer* s *such that* $H^s(\Omega) \subset L^\infty(\Omega)$: $\forall\ s \geq s_0,\ \exists\ C_{sob}(s,d,\Omega) > 0,\ \forall\ f \in H^s(\Omega),\ \|f\|_\infty \leq C_{sob}\|f\|_s$. *Likewise, we have* $H^s(\Omega) \subset \mathscr{C}^m(\overline{\Omega})$ *as soon as* $s > m + \frac{d}{2}$.

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