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Limiting J-Spaces for General Couples

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Abstract. We investigate limiting J-interpolation methods for general Banach couples, not necessarily ordered. We also show their relationship with the interpolation methods defined by the unit square.

Keywords. Real interpolation, J-functional, limiting methods, interpolation over the unit square

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1. Introduction

As one can see in the books by Butzer and Berens [6], Bergh and Löfström [4], Triebel [24–26], Bennett and Sharpley [3], Brudnyĭ and Krugljak [5], Connes [15] or Amrein, Boutet de Monvel and Georgescu [1], the real interpolation method is a very useful tool in many areas of mathematics, including harmonic analysis, partial differential equations, approximation theory and operator theory.

Given any Banach couple (A_0, A_1), the real interpolation spaces $(A_0, A_1)_{\theta,q}$ are defined for $0 < \theta < 1$ (we review their construction in Section 2). In the limit cases $\theta = 0$ or $\theta = 1$, the definition must be modified in order to be meaningful.

Working with ordered Banach couples, that is assuming $A_0 \hookrightarrow A_1$, limiting spaces $(A_0, A_1)_{0,q;J}$ based on the Peetre's J-functional with $\theta = 0$ have been introduced in [7] by Kühn, Ullrich and two of the present authors. They also showed that these limiting spaces arise interpolating by the J-method associated to the unit square (see [13]) the diagonally equal 4-tuple (A_0, A_1, A_1, A_0).

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The assumption $A_0 \hookrightarrow A_1$ was essential for the arguments in [7], and the later papers [9] and [11] on limiting J-spaces, but for the definition of interpolation methods it is an unnecessary restriction. This is our motivation to study limiting J-spaces for arbitrary (not necessarily ordered) couples. The corresponding problem for spaces defined by using the K-functional, the dual functional to the J-functional, has been investigated by the authors in [10].

We start by recalling some basic results on the real interpolation method and on function spaces in Section 2. Then, in Section 3, we extend the definition of limiting J-spaces with $\theta = 0$ to arbitrary Banach couples. We also introduce there limiting J-spaces with $\theta = 1$, and we show their relationship with J-spaces defined by a "broken power function" and with K-spaces defined by "broken function parameter".

In Section 4, we investigate the connection between the limiting J-spaces and the J-method associated to the unit square. Given any Banach couple (A_0, A_1), we consider the 4-tuple obtained by placing A_0 on the vertices $(0,0)$ and $(1,1)$, and A_1 on $(1,0)$ and $(0,1)$. We show that if we choose for interpolating an interior point of the square laying on the diagonals, then the resulting spaces are intersections of limiting J-spaces with real interpolation spaces. In the ordered case we recover a result of [7]. In contrast to the ordered case where the spaces are all the same along the diagonal (α, α), now there is no segment where they are constant. Moreover, the results in the general case show a symmetry which cannot be observed in the simpler case studied in [7].

2. Preliminaries

Let $\bar{A} = (A_0, A_1)$ be a *Banach couple*, that is, two Banach spaces A_j $(j = 0, 1)$ which are continuously embedded in some Hausdorff topological vector space. Let $A_0 + A_1$ be their sum and $A_0 \cap A_1$ be their intersection. These spaces become Banach spaces under the norms

$$\|a\|_{A_0+A_1} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}$$

and $\|a\|_{A_0 \cap A_1} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\}$, respectively.

The Peetre's K- and J-functionals are defined by

$$\begin{aligned} K(t,a) &= K(t,a;\bar{A}) \\ &= \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}, \quad a \in A_0 + A_1, \end{aligned}$$

and

$$J(t,a) = J(t,a;\bar{A}) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \quad a \in A_0 \cap A_1.$$

Note that $\|\cdot\|_{A_0+A_1} = K(1,\cdot;\bar{A})$ and $\|\cdot\|_{A_0 \cap A_1} = J(1,\cdot;\bar{A})$.

Let $0<\theta<1$ and $1\leq q\leq\infty$. The *real interpolation space* $\bar{A}_{\theta,q}=(A_0,A_1)_{\theta,q}$, view as a K-space, is formed by all elements $a\in A_0+A_1$ for which the norm

$$\|a\|_{\bar{A}_{\theta,q}}=\left(\int_0^\infty (t^{-\theta}K(t,a))^q\frac{dt}{t}\right)^{\frac{1}{q}}$$

is finite (when $q=\infty$ the integral should be replaced by the supremum). This space coincides with the collection of all those elements $a\in A_0+A_1$ for which there is a strongly measurable function $u(t)$ with values in $A_0\cap A_1$ such that

$$a=\int_0^\infty u(t)\frac{dt}{t}\quad \text{(convergence in } A_0+A_1\text{)}\quad\text{and}\quad \left(\int_0^\infty (t^{-\theta}J(t,u(t)))^q\frac{dt}{t}\right)^{\frac{1}{q}}<\infty.$$

Moreover,

$$\|a\|_{\bar{A}_{\theta,q;J}}=\inf\left\{\left(\int_0^\infty (t^{-\theta}J(t,u(t)))^q\frac{dt}{t}\right)^{\frac{1}{q}} : a=\int_0^\infty u(t)\frac{dt}{t}\right\}$$

is an equivalent norm to $\|\cdot\|_{\bar{A}_{\theta,q}}$. This is the description of $\bar{A}_{\theta,q}$ by means of the J-functional. We refer to [3–6, 24] for full details on the real interpolation method and to [27] for properties of the Bochner integral.

Let (Ω,μ) be a σ-finite measure space and let f be a measurable function which is finite almost everywhere. The *non-increasing rearrangement* of f is defined by $f^*(t)=\inf\{s>0:\mu\{x\in\Omega:|f(x)|>s\}\leq t\}$. We put $f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)ds$ for the average function of f^*.

It turns out that

$$K(t,f;L_\infty,L_1)=f^{**}\left(\frac{1}{t}\right). \tag{2.1}$$

This yields that $(L_\infty,L_1)_{\theta,p}=L_p$ if $\frac{1}{p}=\theta$, with equivalent norms. In a more general way, if $0<\theta<1$, $\frac{1}{p}=\theta$ and $1\leq q\leq\infty$, we obtain the Lorentz spaces

$$(L_\infty,L_1)_{\theta,q}=L_{(p,q)}=\left\{f:\|f\|_{L_{(p,q)}}=\left(\int_0^\infty \left(t^{\frac{1}{p}}f^{**}(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}<\infty\right\}.$$

We shall also need the Lorentz-Zygmund spaces (see [2, 3, 16]). Let $1\leq p\leq\infty$, $1\leq q\leq\infty$ and $b\in\mathbb{R}$. We let

$$L_{p,q}(\log L)_b=\left\{f:\|f\|_{L_{p,q}(\log L)_b}=\left(\int_0^\infty \left(t^{\frac{1}{p}}(1+|\log t|)^b f^*(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}<\infty\right\}$$

and we define $L_{(p,q)}(\log L)_b$ similarly but replacing f^* by f^{**}.

Clearly $L_{(p,q)} = L_{(p,q)}(\log L)_0$. We also have $L_p = L_{p,p}(\log L)_0 = L_{(p,p)}$. Moreover, if $1 < p \leq \infty, 1 \leq q \leq \infty$ and $b \in \mathbb{R}$, it turns out that $L_{p,q}(\log L)_b = L_{(p,q)}(\log L)_b$ (see [16, Lemma 3.4.39]). Note that if $p = q$ then $L_{p,p}(\log L)_b$ is the Zygmund space $L_p(\log L)_b$ (see [17]).

As usual, $A \hookrightarrow B$ means that the space A is continuously embedded in B. Given two quantities X, Y depending on certain parameters, we write $X \lesssim Y$ if there is a constant $c > 0$ independent of the parameters involved in X and Y, such that $X \leq cY$. If $X \lesssim Y$ and $Y \lesssim X$ we put $X \backsim Y$.

3. Limiting J-spaces

In this section we study new limiting J-spaces which are defined for arbitrary Banach couples.

Definition 3.1. Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \leq q \leq \infty$. The space $\bar{A}_{0,q;J} = (A_0, A_1)_{0,q;J}$ is the collection of all $a \in A_0 + A_1$ which can be represented as

$$a = \int_0^\infty v(t)\frac{dt}{t} \quad \text{(convergence in } A_0 + A_1), \tag{3.1}$$

where $v(t)$ is a strongly measurable function with values in $A_0 \cap A_1$ such that

$$\int_0^1 J(t, v(t))\frac{dt}{t} + \left(\int_1^\infty J(t, v(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty. \tag{3.2}$$

The norm in $\bar{A}_{0,q;J}$ is given by taking the infimum in (3.2) over all representations of the type (3.1), (3.2).

The space $\bar{A}_{1,q;J} = (A_0, A_1)_{1,q;J}$ is formed by all those $a \in A_0 + A_1$ for which there is a representation of the type (3.1) but satisfying now

$$\left(\int_0^1 (t^{-1}J(t, v(t)))^q \frac{dt}{t}\right)^{\frac{1}{q}} + \int_1^\infty t^{-1}J(t, v(t))\frac{dt}{t} < \infty. \tag{3.3}$$

The norm in $\bar{A}_{1,q;J}$ is the infimum in (3.3) over all representations (3.1), (3.3).

Let $\mathfrak{F}(\bar{A}) = \bar{A}_{0,q;J}$ or $\bar{A}_{1,q;J}$. Next we show that the functor $\mathfrak{F}$ produces *intermediate spaces*. This means that $A_0 \cap A_1 \hookrightarrow \mathfrak{F}(\bar{A}) \hookrightarrow A_0 + A_1$. Moreover, $\mathfrak{F}$ has the *interpolation property for bounded linear operators*. That is to say, whenever T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$ such that its restriction $T : A_j \longrightarrow B_j$ is bounded for $j = 0, 1$, then the restriction $T : \mathfrak{F}(\bar{A}) \longrightarrow \mathfrak{F}(\bar{B})$ is also bounded.

Proposition 3.2. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and $1 \le q \le \infty$. The spaces $\bar{A}_{0,q;J}$ and $\bar{A}_{1,q;J}$ are intermediate spaces between A_0 and A_1. Furthermore, the functors $(\cdot,\cdot)_{0,q;J}$ and $(\cdot,\cdot)_{1,q;J}$ have the interpolation property for bounded linear operators.*

Proof. Let $a \in \bar{A}_{0,q;J}$ with $a = \int_0^\infty v(t)\frac{dt}{t}$. Using Hölder's inequality we get with $\frac{1}{q} + \frac{1}{q'} = 1$,

$$\begin{aligned}\|a\|_{A_0+A_1} &\le \int_0^\infty \|v(t)\|_{A_0+A_1}\frac{dt}{t}\\ &\le \int_0^\infty \min\{1, t^{-1}\} J(t, v(t))\frac{dt}{t}\\ &\le \int_0^1 J(t, v(t))\frac{dt}{t} + \left(\int_1^\infty t^{-q'}\frac{dt}{t}\right)^{\frac{1}{q'}} \left(\int_1^\infty J(t, v(t))^q\frac{dt}{t}\right)^{\frac{1}{q}}.\end{aligned}$$

This yields that $\|a\|_{A_0+A_1} \lesssim \|a\|_{\bar{A}_{0,q;J}}$.

Assume now that $a \in A_0 \cap A_1$. Since $a = \int_1^\infty a\chi_{(1,e)}\frac{dt}{t}$, we derive $\|a\|_{\bar{A}_{0,q;J}} \le \left(\int_1^e J(t, a)^q\frac{dt}{t}\right)^{\frac{1}{q}} \lesssim \|a\|_{A_0\cap A_1}$.

The interpolation property for $(\cdot,\cdot)_{0,q;J}$ follows from

$$J(t, Tw; B_0, B_1) \le \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\} J(t, w; A_0, A_1), \quad w \in A_0 \cap A_1.$$

Indeed, if $a \in \bar{A}_{0,q;J}$ with $a = \int_0^\infty v(t)\frac{dt}{t}$, then $Ta \in \bar{B}_{0,q;J}$ because $Ta = \int_0^\infty Tv(t)\frac{dt}{t}$.

The proof for $\bar{A}_{1,q;J}$ can be carried out in the same way. □

Remark 3.3. If $A_0 \hookrightarrow A_1$ and $a \in \bar{A}_{0,q;J}$, then for any representation $a = \int_0^\infty v(t)\frac{dt}{t}$ satisfying (3.1), (3.2), we have that $a_0 = \int_0^1 v(t)\frac{dt}{t}$ belongs to A_0. Indeed,

$$\int_0^1 \|v(t)\|_{A_0}\frac{dt}{t} \le \int_0^1 J(t, v(t))\frac{dt}{t} < \infty.$$

Hence, writing $u(t) = v(t) + a_0\chi_{(1,e)}(t)$ for $1 \le t < \infty$, we get that $a = \int_1^\infty u(t)\frac{dt}{t}$ (convergence in A_1). Moreover,

$$\begin{aligned}\left(\int_1^\infty J(t, u(t))^q\frac{dt}{t}\right)^{\frac{1}{q}} &\lesssim \left(\int_1^\infty J(t, v(t))^q\frac{dt}{t}\right)^{\frac{1}{q}} + \|a_0\|_{A_0}\\ &\le \left(\int_1^\infty J(t, v(t))^q\frac{dt}{t}\right)^{\frac{1}{q}} + \int_0^1 J(t, v(t))\frac{dt}{t}.\end{aligned}$$

This yields that, in the ordered case, the space $\bar{A}_{0,q;J}$ coincide with the usual $(0, q; J)$-space studied in [7, 9]. One can show that $\bar{A}_{0,q;J} \hookrightarrow \bar{A}_{\theta,q} \hookrightarrow \bar{A}_{1,q;J}$ for any $0 < \theta < 1$.

Next we introduce other kinds of related interpolation spaces.

Definition 3.4. Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let $0 \le \theta_0, \theta_1 \le 1$ and $1 \le q \le \infty$. The space $\bar{A}_{\{\theta_0,\theta_1\},q;J} = (A_0, A_1)_{\{\theta_0,\theta_1\},q;J}$ consists of all those $a \in A_0 + A_1$ for which there is a strongly measurable function $v(t)$ with values in $A_0 \cap A_1$ such that

$$a = \int_0^\infty v(t)\frac{dt}{t} \quad \text{(convergence in } A_0 + A_1), \tag{3.4}$$

and the sum

$$\Big(\int_0^1 (t^{-\theta_0} J(t, v(t)))^q \frac{dt}{t}\Big)^{\frac{1}{q}} + \Big(\int_1^\infty (t^{-\theta_1} J(t, v(t)))^q \frac{dt}{t}\Big)^{\frac{1}{q}} \tag{3.5}$$

is finite. We set

$$\|a\|_{\bar{A}_{\{\theta_0,\theta_1\},q;J}} = \inf\Big\{\Big(\int_0^1 (t^{-\theta_0} J(t, v(t)))^q \frac{dt}{t}\Big)^{\frac{1}{q}} + \Big(\int_1^\infty (t^{-\theta_1} J(t, v(t)))^q \frac{dt}{t}\Big)^{\frac{1}{q}}\Big\}$$

where the infimum is extended over all representations v satisfying (3.4) and (3.5).

Clearly, for $0<\theta<1$, the real interpolation space $\bar{A}_{\theta,q}$ realized as a J-space is equal to $\bar{A}_{\{\theta,\theta\},q;J}$ and the norms are equivalent. Notice also that in notation of [22, 23], the space $\bar{A}_{\{\theta_0,\theta_1\},q;J}$ coincides with the J-space defined by the function parameter

$$f(t) = \begin{cases} t^{\theta_0} & \text{if } 0 < t \le 1 \\ t^{\theta_1} & \text{if } 1 < t < \infty. \end{cases}$$

Remark 3.5. Assume that $A_0 \hookrightarrow A_1$. Let $1 \le q \le \infty, 0 < \theta_0 \le 1$ and take any $a = \int_0^\infty v(t)\frac{dt}{t}$ belonging to $\bar{A}_{\{\theta_0,0\},q;J}$ with

$$\Big(\int_0^1 (t^{-\theta_0} J(t, v(t)))^q \frac{dt}{t}\Big)^{\frac{1}{q}} + \Big(\int_1^\infty J(t, v(t))^q \frac{dt}{t}\Big)^{\frac{1}{q}} \lesssim \|a\|_{\bar{A}_{\{\theta_0,0\},q;J}}.$$

Then $a_0 = \int_0^1 v(t)\frac{dt}{t}$ belongs to A_0 because

$$\begin{aligned}\int_0^1 \|v(t)\|_{A_0}\frac{dt}{t} &\le \int_0^1 J(t, v(t))\frac{dt}{t} \\ &\le \Big(\int_0^1 t^{\theta_0 q'}\frac{dt}{t}\Big)^{\frac{1}{q'}} \Big(\int_0^1 (t^{-\theta_0} J(t, v(t)))^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\ &\lesssim \|a\|_{\bar{A}_{\{\theta_0,0\},q;J}}.\end{aligned}$$

By Remark 3.3, it follows that $\bar{A}_{\{\theta_0,0\},q;J}$ is equal to $\bar{A}_{0,q;J}$ and also equal to the $(0,q;J)$-space considered in [7].

A similar reasoning shows that if $A_0 \hookrightarrow A_1, 0 < \theta_1 < 1$ and $1 \le q \le \infty$, then $\bar{A}_{\{0,\theta_1\},q;J} = \bar{A}_{\theta_1,q}$.

Lemma 3.6. *Let* $\bar{A}=(A_0,A_1)$ *be a Banach couple, let* $0<\theta<1$ *and* $1\le q\le\infty$. *Then the following holds :*

$$\text{(i) } \bar{A}_{\{\theta,0\},q;J} \hookrightarrow \bar{A}_{\theta,q} \cap \bar{A}_{0,q;J}$$
$$\text{(ii) } \bar{A}_{\{1,\theta\},q;J} \hookrightarrow \bar{A}_{1,q;J} \cap \bar{A}_{\theta,q}$$
$$\text{(iii) } \bar{A}_{\{1,0\},q;J} \hookrightarrow \bar{A}_{1,q;J} \cap \bar{A}_{0,q;J}.$$

Proof. Let $a \in \bar{A}_{\{\theta,0\},q;J}$ and let $a = \int_0^\infty v(t)\frac{dt}{t}$ be a representation of a satisfying (3.5). By Hölder's inequality, we get

$$\int_0^1 J(t,v(t))\frac{dt}{t} \le \left(\int_0^1 t^{\theta q'}\frac{dt}{t}\right)^{\frac{1}{q'}}\left(\int_0^1 (t^{-\theta}J(t,v(t)))^q\frac{dt}{t}\right)^{\frac{1}{q}} \lesssim \left(\int_0^1 (t^{-\theta}J(t,v(t)))^q\frac{dt}{t}\right)^{\frac{1}{q}}.$$

On the other hand, it is clear that

$$\int_1^\infty (t^{-\theta}J(t,v(t)))^q\frac{dt}{t} \le \int_1^\infty J(t,v(t))^q\frac{dt}{t}.$$

This yields that $a = \int_0^\infty v(t)\frac{dt}{t}$ is also a representation of a in each one of the spaces $\bar{A}_{\theta,q}$ and $\bar{A}_{0,q;J}$, and (i) follows.

The proofs of (ii) and (iii) are similar. □

Definition 3.7. Let $\bar{A} = (A_0,A_1)$ be a Banach couple, let $1 \le q \le \infty$ and assume that $f_0, f_1 : (0,\infty) \longrightarrow (0,\infty)$ are continuous functions. We write $\bar{A}_{\{f_0,f_1\},q;K} = (A_0,A_1)_{\{f_0,f_1\},q;K}$ to designate the space of all $a \in A_0 + A_1$ which have a finite norm

$$\|a\|_{\bar{A}_{\{f_0,f_1\},q;K}} = \left(\int_0^1 \left(\frac{K(t,a)}{f_0(t)}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} + \left(\int_1^\infty \left(\frac{K(t,a)}{f_1(t)}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

We are interested in spaces $\bar{A}_{\{f_0,f_1\},q;K}$ when f_0 and f_1 are any of the functions

$$\mathfrak{g}(t) = 1 + |\log t|, \quad \mathfrak{f}(t) = t(1+|\log t|) \tag{3.6}$$

or a power function. If $f_0(t) = t^\theta$ (respectively, $f_1(t) = t^\theta$) with $0 \le \theta \le 1$, we simply write $\bar{A}_{\{\theta,f_1\},q;K}$ (respectively, $\bar{A}_{\{f_0,\theta\},q;K}$). Clearly, $\bar{A}_{\theta,q} = \bar{A}_{\{\theta,\theta\},q;K}$ with equivalent norms.

Spaces $\bar{A}_{\{f_0,f_1\},q;K}$ when $f_0(t) = t^\theta(1+|\log t|)^{\alpha_0}$, $f_1(t) = t^\theta(1+|\log t|)^{\alpha_\infty}$, $0 \le \theta \le 1$ and $(\alpha_0,\alpha_\infty) \in \mathbb{R}^2$ have been extensively studied in the literature (see, for example, [19,20]).

Theorem 3.8. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let $1 < q \le \infty$ and $0 < \theta < 1$. Then we have with equivalent norms*

$$\bar{A}_{\{\theta,0\},q;J} = \bar{A}_{\theta,q} \cap \bar{A}_{0,q;J} = \bar{A}_{\{\theta,\mathfrak{g}\},q;K}.$$

Proof. Suppose $1 < q < \infty$. The case $q = \infty$ can be treated in the same way. By Lemma 3.6(i), we know that $\bar{A}_{\{\theta,0\},q;J} \hookrightarrow \bar{A}_{\theta,q} \cap \bar{A}_{0,q;J}$. Let us show that

$$\bar{A}_{\theta,q} \cap \bar{A}_{0,q;J} \hookrightarrow \bar{A}_{\{\theta,\mathfrak{g}\},q;K}. \tag{3.7}$$

Let $a \in \bar{A}_{\theta,q} \cap \bar{A}_{0,q;J}$. It is clear that $\left(\int_0^1 (t^{-\theta} K(t,a))^q \frac{dt}{t}\right)^{\frac{1}{q}} \lesssim \|a\|_{\bar{A}_{\theta,q}} \le \|a\|_{\bar{A}_{\theta,q} \cap \bar{A}_{0,q;J}}$. In order to estimate $\left(\int_1^\infty \left(\frac{K(t,a)}{\mathfrak{g}(t)}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}$ from above, we make the discretization $t = 2^\nu, \nu \in \mathbb{Z}$, and we work with the equivalent discrete norms. Since $a \in \bar{A}_{0,q;J}$, we can find a representation of a as $a = \sum_{\nu=-\infty}^{\infty} u_\nu$ (convergence in $A_0 + A_1$), with $(u_\nu) \subseteq A_0 \cap A_1$ and $\sum_{\nu=-\infty}^{0} J(2^\nu, u_\nu) + (\sum_{\nu=1}^{\infty} J(2^\nu, u_\nu)^q)^{\frac{1}{q}} \lesssim \|a\|_{\bar{A}_{0,q;J}}$. Let $n = 1, 2, \ldots$. We obtain

$$\begin{aligned} K(2^n, a) &\le \Big\| \sum_{\nu=-\infty}^{n} u_\nu \Big\|_{A_0} + 2^n \Big\| \sum_{\nu=n+1}^{\infty} u_\nu \Big\|_{A_1} \\ &\le \sum_{\nu=-\infty}^{0} J(2^\nu, u_\nu) + \sum_{\nu=1}^{n} J(2^\nu, u_\nu) + 2^n \sum_{\nu=n+1}^{\infty} 2^{-\nu} J(2^\nu, u_\nu) \\ &\lesssim \|a\|_{\bar{A}_{0,q;J}} + \sum_{\nu=1}^{n} J(2^\nu, u_\nu) + 2^n \sum_{\nu=n+1}^{\infty} 2^{-\nu} J(2^\nu, u_\nu). \end{aligned}$$

The last term can be estimated using Hölder's inequality. We have

$$2^n \sum_{\nu=n+1}^{\infty} 2^{-\nu} J(2^\nu, u_\nu) \lesssim \left(\sum_{\nu=n+1}^{\infty} J(2^\nu, u_\nu)^q \right)^{\frac{1}{q}} \lesssim \|a\|_{\bar{A}_{0,q;J}}.$$

Now, proceeding as in [7, p. 2335], by Hardy's inequality we derive

$$\begin{aligned} \left(\int_1^\infty \left(\frac{K(t,a)}{\mathfrak{g}(t)} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} &\backsim \left(\sum_{n=1}^{\infty} \left(\frac{K(2^n,a)}{n} \right)^q \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{n=1}^{\infty} \frac{1}{n^q} \right)^{\frac{1}{q}} \|a\|_{\bar{A}_{0,q;J}} + \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{\nu=1}^{n} J(2^\nu, u_\nu) \right)^q \right)^{\frac{1}{q}} \\ &\lesssim \|a\|_{\bar{A}_{0,q;J}} + \left(\sum_{n=1}^{\infty} J(2^n, u_n)^q \right)^{\frac{1}{q}} \\ &\lesssim \|a\|_{\bar{A}_{0,q;J}}. \end{aligned}$$

Note that $\sum_{n=1}^{\infty} \frac{1}{n^q} < \infty$ because $q > 1$. This establishes (3.7).

To complete the proof of the theorem it is enough to prove that

$$\bar{A}_{\{\theta,\mathfrak{g}\},q;K} \hookrightarrow \bar{A}_{\{\theta,0\},q;J}. \tag{3.8}$$

Let $a \in \bar{A}_{\{\theta,\mathfrak{g}\},q;K}$. Then

$$\big(2^{-\theta\nu} K(2^\nu, a)\big)_{\nu=-\infty}^{0} \in \ell_q \quad \text{and} \quad \big(\nu^{-1} K(2^\nu, a)\big)_{\nu=1}^{\infty} \in \ell_q. \tag{3.9}$$

For $\nu = 0, -1, -2, \dots$ we can decompose $a = a_{0,\nu} + a_{1,\nu}$ with $a_{j,\nu} \in A_j$ and

$$\|a_{0,\nu}\|_{A_0} + 2^\nu \|a_{1,\nu}\|_{A_1} \le 2K(2^\nu, a).$$

By (3.9), $\|a_{0,\nu}\|_{A_0} \le [2^{1-\theta\nu} K(2^\nu, a)] 2^{\theta\nu} \longrightarrow 0$ as $\nu \to -\infty$. For the other values of ν, following [7, Theorem 4.2], we put $\lambda_0 = 1$ and $\lambda_\nu = 2^{2^{\nu-1}}$ if $\nu = 1, 2, \dots$. We decompose $a = a_{0,\nu} + a_{1,\nu}$ with $a_{j,\nu} \in A_j$ and

$$\|a_{0,\nu}\|_{A_0} + \lambda_{\nu+1} \|a_{1,\nu}\|_{A_1} \le 2K(\lambda_{\nu+1}, a).$$

So, using again (3.9)

$$\|a_{1,\nu}\|_{A_1} \le \left[\frac{2K(\lambda_{\nu+1}, a)}{\log \lambda_{\nu+1}}\right] \frac{\log \lambda_{\nu+1}}{\lambda_{\nu+1}} \longrightarrow 0 \quad \text{as } \nu \to \infty.$$

Let $u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu} \in A_0 \cap A_1$, $\nu \in \mathbb{Z}$. Since

$$\Big\| a - \sum_{\nu=N}^{M} u_\nu \Big\|_{A_0+A_1} \le \|a_{0,N-1}\|_{A_0} + \|a_{1,M}\|_{A_1} \longrightarrow 0$$

as $M \to \infty$ and $N \to -\infty$, we have that $a = \sum_{\nu=-\infty}^{\infty} u_\nu$ in $A_0 + A_1$.

Put $I_\nu = [\lambda_{\nu-1}, \lambda_\nu)$ for $\nu = 1, 2, \dots$ and consider the function

$$v(t) = \begin{cases} \frac{1}{\log 2} u_\nu & \text{if } 2^{\nu-1} \le t < 2^\nu, \nu = 0, -1, -2, \dots \\ \frac{1}{\log 2} u_1 & \text{if } t \in I_1 \\ \frac{1}{2^{\nu-2} \log 2} u_\nu & \text{if } t \in I_\nu, \nu = 2, 3, \dots . \end{cases}$$

Then

$$\begin{aligned} \int_0^\infty v(t) \frac{dt}{t} &= \sum_{\nu=-\infty}^{0} \int_{2^{\nu-1}}^{2^\nu} \frac{1}{\log 2} u_\nu \frac{dt}{t} + \int_{I_1} \frac{1}{\log 2} u_1 \frac{dt}{t} + \sum_{\nu=2}^{\infty} \int_{I_\nu} \frac{1}{2^{\nu-2} \log 2} u_\nu \frac{dt}{t} \\ &= \sum_{\nu=-\infty}^{\infty} u_\nu \\ &= a. \end{aligned}$$

Moreover, for $\nu = 0, 1, 2, \dots$ and $2^{\nu-1} \le t < 2^\nu$,

$$J(t, v(t)) \lesssim J(2^\nu, u_\nu) \lesssim K(2^\nu, a) \lesssim K(t, a).$$

For $t \in I_1$, we have $J(t, v(t)) \lesssim J(2, u_1) \lesssim K(4, a) \lesssim \frac{K(t,a)}{1+\log t}$, and for $\nu = 2, 3, \dots$ and $t \in I_\nu$, we obtain

$$J(t, v(t)) \le \frac{J(\lambda_\nu, u_\nu)}{2^{\nu-2}\log 2} \lesssim \frac{K(\lambda_{\nu+1}, a)}{2^{\nu-2}}.$$

Consequently,

$$\begin{aligned}
&\left(\int_0^1 (t^{-\theta} J(t, v(t)))^q \frac{dt}{t}\right)^{\frac{1}{q}} + \left(\int_1^\infty J(t, v(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\lesssim \left(\int_0^1 (t^{-\theta} K(t, a))^q \frac{dt}{t}\right)^{\frac{1}{q}} + \left(\sum_{\nu=1}^\infty \int_{I_\nu} J(t, v(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\lesssim \|a\|_{\bar{A}_{\{\theta,\mathfrak{g}\},q;K}} + \left(\int_{I_1} \left(\frac{K(t,a)}{1+\log t}\right)^q \frac{dt}{t} + \sum_{\nu=2}^\infty \left(\frac{K(\lambda_{\nu+1}, a)}{2^{\nu-2}}\right)^q \int_{I_\nu} \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\lesssim \|a\|_{\bar{A}_{\{\theta,\mathfrak{g}\},q;K}} + \left(\int_{I_1} \left(\frac{K(t,a)}{1+\log t}\right)^q \frac{dt}{t} + \sum_{\nu=2}^\infty \left(\frac{K(\lambda_{\nu+1}, a)}{2^{\nu+2}}\right)^q \int_{I_{\nu+2}} \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\lesssim \|a\|_{\bar{A}_{\{\theta,\mathfrak{g}\},q;K}} + \left(\sum_{\nu=1}^\infty \int_{I_\nu} \left(\frac{K(t,a)}{1+\log t}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\
&\lesssim \|a\|_{\bar{A}_{\{\theta,\mathfrak{g}\},q;K}}.
\end{aligned}$$

This yields (3.8) and completes the proof. □

The corresponding result for $\bar{A}_{\{1,\theta\},q;J}$ involves the function $\mathfrak{f}$ defined in (3.6).

Theorem 3.9. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let $1 < q \le \infty$ and $0 < \theta < 1$. Then we have with equivalent norms*

$$\bar{A}_{\{1,\theta\},q;J} = \bar{A}_{1,q;J} \cap \bar{A}_{\theta,q} = \bar{A}_{\{\mathfrak{f},\theta\},q;K}.$$

Proof. Let $\bar{B} = (A_1, A_0)$ be the couple $\bar{A}$ with reverse order and let $\hat{K}$ and $\hat{J}$ be the K- and J-functionals associated to $\bar{B}$. Using that

$$K(t, a) = t\hat{K}(t^{-1}, a) \quad \text{and} \quad J(t, a) = t\hat{J}(t^{-1}, a),$$

it is not hard to check that $\bar{A}_{\theta,q} = \bar{B}_{1-\theta,q}$, $\bar{A}_{1,q;J} = \bar{B}_{0,q;J}$, $\bar{A}_{\{1,\theta\},q;J} = \bar{B}_{\{1-\theta,0\},q;J}$ and $\bar{A}_{\{\mathfrak{f},\theta\},q;K} = \bar{B}_{\{1-\theta,\mathfrak{g}\},q;K}$. According to Theorem 3.8,

$$\bar{B}_{\{1-\theta,0\},q;J} = \bar{B}_{1-\theta,q} \cap \bar{B}_{0,q;J} = \bar{B}_{\{1-\theta,\mathfrak{g}\},q;K}.$$

Thus we conclude the result. □

The arguments used in the proofs of Theorems 3.8 and 3.9 may be modified to give the following characterization of $\bar{A}_{\{1,0\},q;J}$.

Theorem 3.10. *Let* $\bar{A} = (A_0, A_1)$ *be a Banach couple and let* $1 < q \le \infty$. *Then we have with equivalent norms*

$$\bar{A}_{\{1,0\},q;J} = \bar{A}_{1,q;J} \cap \bar{A}_{0,q;J} = \bar{A}_{\{\mathfrak{f},\mathfrak{g}\},q;K}.$$

In order to give some examples, let (Ω, μ) be a σ-finite measure space. If $\mu(\Omega) < \infty$ then we are in the ordered case with $L_\infty \hookrightarrow L_1$ and it is shown in [7, Corollary 4.3] that the Zygmund space $L_{\infty,\infty}(\log L)_{-1} = L_{exp}$ coincides with $(L_\infty, L_1)_{0,\infty;J}$. By Theorem 3.8 and Remark 3.3, it follows that

$$(L_\infty, L_1)_{\{\theta,0\},\infty;J} = L_{\infty,\infty}(\log L)_{-1} \quad \text{for any } 0 < \theta < 1.$$

As a direct consequence of Theorem 3.8 and (2.1), we can determine these spaces when $\mu(\Omega) = \infty$.

Corollary 3.11. *Let* (Ω, μ) *be a* σ*-finite measure space and* $0 < \theta < 1$. *Then*

$$\text{(i) } \|f\|_{(L_\infty,L_1)_{\{\theta,0\},\infty;J}} \backsim \sup_{0<t<1} \frac{f^{**}(t)}{1 + |\log t|} + \sup_{1<t<\infty} t^\theta f^{**}(t)$$

$$\text{(ii) } \|f\|_{(L_\infty,L_1)_{\{1,\theta\},\infty;J}} \backsim \sup_{0<t<1} t^\theta f^{**}(t) + \sup_{1<t<\infty} \frac{t f^{**}(t)}{1 + |\log t|}$$

$$\text{(iii) } \|f\|_{(L_\infty,L_1)_{\{1,0\},\infty;J}} \backsim \sup_{0<t<1} \frac{f^{**}(t)}{1 + |\log t|} + \sup_{1<t<\infty} \frac{t f^{**}(t)}{1 + |\log t|}.$$

These interpolation spaces can be described in terms of Lorentz and Lorentz-Zygmund spaces as follows.

Corollary 3.12. *Let* (Ω, μ) *be a* σ*-finite measure space and let* $0 < \theta < 1$. *We have with equivalent norms*

$$\text{(a) } (L_\infty, L_1)_{\{\theta,0\},\infty;J} = L_{\infty,\infty}(\log L)_{-1} \cap L_{(\frac{1}{\theta},\infty)}$$

$$\text{(b) } (L_\infty, L_1)_{\{1,\theta\},\infty;J} = L_{(\frac{1}{\theta},\infty)} \cap L_{(1,\infty)}(\log L)_{-1}$$

$$\text{(c) } (L_\infty, L_1)_{\{1,0\},\infty;J} = L_{\infty,\infty}(\log L)_{-1} \cap L_{(1,\infty)}(\log L)_{-1}.$$

Proof. Recall that $L_{\infty,\infty}(\log L)_{-1} = L_{(\infty,\infty)}(\log L)_{-1}$. By Corollary 3.11(i), it is clear that $L_{\infty,\infty}(\log L)_{-1} \cap L_{(\frac{1}{\theta},\infty)} \hookrightarrow (L_\infty, L_1)_{\{\theta,0\},\infty;J}$. On the other hand, using again Corollary 3.11(i), we obtain

$$\sup_{0<t<1} t^\theta f^{**}(t) \lesssim \Big(\sup_{0<t<1} t^\theta (1 + |\log t|) \Big) \|f\|_{(L_\infty,L_1)_{\{\theta,0\},\infty;J}} \lesssim \|f\|_{(L_\infty,L_1)_{\{\theta,0\},\infty;J}}.$$

Similarly,

$$\sup_{1<t<\infty} \frac{f^{**}(t)}{1+|\log t|} \lesssim \Big(\sup_{1<t<\infty} t^{-\theta}(1+|\log t|)^{-1}\Big) \|f\|_{(L_\infty,L_1)_{\{\theta,0\},\infty;J}} \lesssim \|f\|_{(L_\infty,L_1)_{\{\theta,0\},\infty;J}}.$$

This yields that

$$\|f\|_{L_{\infty,\infty}(\log L)_{-1}\cap L_{(\frac{1}{\theta},\infty)}} \lesssim \|f\|_{(L_\infty,L_1)_{\{\theta,0\},\infty;J}}$$

and establishes (a). Equalities (b) and (c) can be checked with similar arguments. □

4. Interpolation over the unit square

Let $\Pi = \overline{P_1P_2P_3P_4}$ be the unit square in $\mathbb{R}^2$ with vertices $P_1 = (0,0), P_2 = (1,0), P_3 = (0,1)$ and $P_4 = (1,1)$. Let $\bar{A} = (A_0, A_1)$ be a Banach couple and consider the 4-tuple $\bar{\mathbb{A}} = (A_0, A_1, A_1, A_0)$. We imagine A_0 sitting on P_1 and P_4, and A_1 on P_2 and P_3. Using the coordinates of the vertices of Π, we derive the following version of the J-functional with two parameters $t, s > 0$

$$\bar{J}(t,s;a) = \max\{\|a\|_{A_0},\, t\|a\|_{A_1},\, s\|a\|_{A_1},\, ts\|a\|_{A_0}\}, \quad a \in A_0 \cap A_1.$$

Let (α,β) be an interior point to Π and let $1 \le q \le \infty$. We define the J-space $\bar{\mathbb{A}}_{(\alpha,\beta),q;J} = (A_0, A_1, A_1, A_0)_{(\alpha,\beta),q;J}$ as the collection of all those $a \in A_0 + A_1$ for which there is a strongly measurable function $u(t,s)$ with values in $A_0 \cap A_1$ such that

$$a = \int_0^\infty \int_0^\infty u(t,s) \frac{dt}{t}\frac{ds}{s} \tag{4.1}$$

and

$$\Big(\int_0^\infty \int_0^\infty (t^{-\alpha} s^{-\beta} \bar{J}(t,s;u(t,s)))^q \frac{dt}{t}\frac{ds}{s} \Big)^{\frac{1}{q}} < \infty. \tag{4.2}$$

The norm in $\bar{\mathbb{A}}_{(\alpha,\beta),q;J}$ is the infimum in (4.2) over all representations of the type (4.1), (4.2).

Spaces $\bar{\mathbb{A}}_{(\alpha,\beta),q;J}$ are a special case of interpolation spaces generated by convex polygons in $\mathbb{R}^2$. They were introduced by Cobos and Peetre [13]. Besides [13], we refer to [8,12,14,18,21] and the references given there for full details on these interpolation methods. When (α,β) lies in any diagonal of Π, the results are sometimes harder and unexpected. Next we determine $\bar{\mathbb{A}}_{(\alpha,\beta),q;J}$ in those cases.

Theorem 4.1. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let $0 < \alpha < 1$ and let $1 \le q \le \infty$. Put $\bar{\mathbb{A}} = (A_0, A_1, A_1, A_0)$. Then we have with equivalent norms*

$$\bar{\mathbb{A}}_{(\alpha,\alpha),q;J} = \begin{cases} \bar{A}_{\{2\alpha,0\},q;J} & \text{if } 0 < \alpha < \frac{1}{2} \\ \bar{A}_{\{1,0\},q;J} & \text{if } \alpha = \frac{1}{2} \\ \bar{A}_{\{2-2\alpha,0\},q;J} & \text{if } \frac{1}{2} < \alpha < 1, \end{cases}$$

and

$$\bar{\mathbb{A}}_{(\alpha,1-\alpha),q;J} = \begin{cases} \bar{A}_{\{1,1-2\alpha\},q;J} & \text{if } 0 < \alpha < \frac{1}{2} \\ \bar{A}_{\{1,0\},q;J} & \text{if } \alpha = \frac{1}{2} \\ \bar{A}_{\{1,2\alpha-1\},q;J} & \text{if } \frac{1}{2} < \alpha < 1. \end{cases}$$

Proof. Using that $\bar{\mathbb{A}}$ is diagonally equal, we get

$$\bar{J}(t,s;a) = ts\bar{J}(t^{-1},s^{-1};a), \quad a \in A_0 \cap A_1.$$

This implies that $\bar{\mathbb{A}}_{(\alpha,\beta),q;J} = \bar{\mathbb{A}}_{(1-\alpha,1-\beta),q;J}$ for any (α,β) in the interior of Π. Hence, it is enough to establish the result for $0 < \alpha \le \frac{1}{2}$. Suppose also that $1 \le q < \infty$. The proof when $q = \infty$ is similar.

We consider first the point (α,α). Take any $a \in \bar{A}_{\{2\alpha,0\},q;J}$ and let $a = \int_0^\infty v(t)\frac{dt}{t}$ be any representation with

$$\left(\int_0^1 (t^{-2\alpha}J(t,v(t)))^q \frac{dt}{t}\right)^{\frac{1}{q}} + \left(\int_1^\infty J(t,v(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} \le 2\|a\|_{\bar{A}_{\{2\alpha,0\},q;J}}.$$

It is easy to check that the integrals

$$x_1 = \int_0^1 v(t)\frac{dt}{t} \quad \text{and} \quad x_2 = \int_1^\infty v(t)\frac{dt}{t}$$

are convergent in $A_0 + A_1$. Let us show that $x_j \in \bar{\mathbb{A}}_{(\alpha,\alpha),q;J}$ for $j = 0,1$. Put

$$u(t,s) = \begin{cases} v(t) & \text{if } \frac{t}{e} \le s \le t \text{ and } 0 < t < 1 \\ 0 & \text{in any other case.} \end{cases}$$

We have that

$$\int_0^\infty \int_0^\infty u(t,s)\frac{dt}{t}\frac{ds}{s} = \int_0^1 \left(\int_{\frac{t}{e}}^t \frac{ds}{s}\right) v(t)\frac{dt}{t} = x_1.$$

Moreover, for $\frac{t}{e} \le s \le t$ and $0 < t < 1$,

$$\bar{J}(t,s;u(t,s)) = \max\{\|v(t)\|_{A_0}, t\|v(t)\|_{A_1}\} = J(t,v(t)).$$

Therefore,

$$\begin{aligned}\|x_1\|_{\bar{\mathbb{A}}_{(\alpha,\alpha),q;J}} &\le \left(\int_0^1 \int_{\frac{t}{e}}^t (t^{-\alpha}s^{-\alpha}J(t,v(t)))^q \frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^1 (t^{-2\alpha}J(t,v(t)))^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\lesssim \|a\|_{\bar{A}_{\{2\alpha,0\},q;J}}.\end{aligned}$$

To deal with x_2 we put

$$w(t,s) = \begin{cases} v\left(\frac{1}{t}\right) & \text{if } \frac{1}{t} \le s \le \frac{e}{t} \text{ and } 0 < t < 1 \\ 0 & \text{in any other case.}\end{cases}$$

Then

$$\int_0^\infty \int_0^\infty w(t,s)\frac{dt}{t}\frac{ds}{s} = \int_0^1 \left(\int_{\frac{1}{t}}^{\frac{e}{t}} \frac{ds}{s}\right) v\left(\frac{1}{t}\right)\frac{dt}{t} = x_2$$

and, for $\frac{1}{t} \le s \le \frac{e}{t}$ and $0 < t < 1$, we have

$$\bar{J}(t,s;w(t,s)) \le \max\left\{e\left\|v\left(\frac{1}{t}\right)\right\|_{A_0}, \frac{e}{t}\left\|v\left(\frac{1}{t}\right)\right\|_{A_1}\right\} \lesssim J\left(\frac{1}{t}, v\left(\frac{1}{t}\right)\right).$$

Consequently,

$$\begin{aligned}\|x_2\|_{\bar{\mathbb{A}}_{(\alpha,\alpha),q;J}} &\lesssim \left(\int_0^1 \int_{\frac{1}{t}}^{\frac{e}{t}} \left(t^{-\alpha}s^{-\alpha}J\left(\frac{1}{t}, v\left(\frac{1}{t}\right)\right)\right)^q \frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{q}} \\ &\lesssim \left(\int_1^\infty J(t,v(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &\lesssim \|a\|_{\bar{A}_{\{2\alpha,0\},q;J}}.\end{aligned}$$

This implies that $\bar{A}_{\{2\alpha,0\},q;J} \hookrightarrow \bar{\mathbb{A}}_{(\alpha,\alpha),q;J}$.

In order to establish the converse embedding, take any $a \in \bar{\mathbb{A}}_{(\alpha,\alpha),q;J}$ and choose a representation $a = \int_0^\infty \int_0^\infty u(t,s)\frac{dt}{t}\frac{ds}{s}$ with

$$\left(\int_0^\infty \int_0^\infty (t^{-\alpha}s^{-\alpha}\bar{J}(t,s;u(t,s)))^q \frac{dt}{t}\frac{ds}{s}\right)^{\frac{1}{q}} \le 2\|a\|_{\bar{\mathbb{A}}_{(\alpha,\alpha),q;J}}.$$

Consider the partition of $(0,\infty)\times(0,\infty)$ given by the sets

$$\begin{aligned}
\Omega_1 &= \{(t,s)\in\mathbb{R}^2 : 0<t\le 1,\ 0<s\le t\}\\
\Omega_2 &= \{(t,s)\in\mathbb{R}^2 : 1<t<\infty,\ 0<s\le \tfrac{1}{t}\}\\
\Omega_3 &= \{(t,s)\in\mathbb{R}^2 : 0<t<1,\ t<s\le \tfrac{1}{t}\}\\
\Omega_4 &= \{(t,s)\in\mathbb{R}^2 : 0<t\le 1,\ \tfrac{1}{t}<s<\infty\}\\
\Omega_5 &= \{(t,s)\in\mathbb{R}^2 : 1<t<\infty,\ t<s<\infty\}\\
\Omega_6 &= \{(t,s)\in\mathbb{R}^2 : 1<t<\infty,\ \tfrac{1}{t}<s\le t\},
\end{aligned}$$

and write $y_j=\int\int_{\Omega_j} u(t,s)\,\frac{dt}{t}\,\frac{ds}{s}$. We have $a=\sum_{j=1}^6 y_j$. We are going to check that $y_j\in\bar{A}_{\{2\alpha,0\},q;J}$ for $1\le j\le 6$. In the argument we shall use freely that

$$\bar{J}(t,s;u(t,s))=\max\{1,ts\}J\left(\frac{\max\{t,s\}}{\max\{1,ts\}},u(t,s)\right).$$

In Ω_1 we have $\bar{J}(t,s;u(t,s))=J(t,u(t,s))$. For $0<t\le 1$, the integral $v(t)=\int_0^t u(t,s)\,\frac{ds}{s}$ is absolutely convergent in $A_0\cap A_1$. Indeed, using Höder's inequality we obtain

$$\begin{aligned}
J(t,v(t)) &\le \int_0^t J(t,u(t,s))\frac{ds}{s}\\
&= \int_0^t \bar{J}(t,s;u(t,s))\frac{ds}{s}\\
&\le \left(\int_0^t s^{\alpha q'}\frac{ds}{s}\right)^{\frac{1}{q'}}\left(\int_0^t (s^{-\alpha}\bar{J}(t,s;u(t,s)))^q\frac{ds}{s}\right)^{\frac{1}{q}}\\
&\lesssim t^{\alpha}\left(\int_0^t (s^{-\alpha}\bar{J}(t,s;u(t,s)))^q\frac{ds}{s}\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $y_1=\int_0^1 v(t)\,\frac{dt}{t}$, it follows that

$$\begin{aligned}
\|y_1\|_{\bar{A}_{\{2\alpha,0\},q;J}} &\le \left(\int_0^1 (t^{-2\alpha}J(t,v(t)))^q\frac{dt}{t}\right)^{\frac{1}{q}}\\
&\lesssim \left(\int_0^1\int_0^t (t^{-\alpha}s^{-\alpha}\bar{J}(t,s;u(t,s)))^q\frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{q}}\\
&\lesssim \|a\|_{\bar{\mathbb{A}}_{(\alpha,\alpha),q;J}}.
\end{aligned}$$

For y_2, we write $v(t)=\int_0^{\frac{1}{t}} u(t,s)\,\frac{ds}{s}$ for $1<t<\infty$. Using that $\bar{J}(t,s;u(t,s))=J(t,u(t,s))$, $(t,s)\in\Omega_2$, we derive

$$J(t,v(t))\le\int_0^{\frac{1}{t}}\bar{J}(t,s;u(t,s))\frac{ds}{s}\lesssim t^{-\alpha}\left(\int_0^{\frac{1}{t}}(s^{-\alpha}\bar{J}(t,s;u(t,s)))^q\frac{ds}{s}\right)^{\frac{1}{q}}.$$

Therefore,

$$\begin{aligned}\|y_2\|_{\bar{A}_{\{2\alpha,0\},q;J}} &\le \left(\int_1^\infty J(t,v(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}\\ &\lesssim \left(\int_1^\infty \int_0^{\frac{1}{t}} (t^{-\alpha}s^{-\alpha}\bar{J}(t,s;u(t,s)))^q \frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{q}}\\ &\lesssim \|a\|_{\bar{\mathbb{A}}_{(\alpha,\alpha),q;J}}.\end{aligned}$$

Consider now y_3. We have

$$y_3 = \iint_{\Omega_3} u(t,s)\frac{dt}{t}\frac{ds}{s} = \int_0^1\int_0^s u(t,s)\frac{dt}{t}\frac{ds}{s} + \int_1^\infty\int_0^{\frac{1}{s}} u(t,s)\frac{dt}{t}\frac{ds}{s} = z_1 + z_2.$$

Moreover, $\bar{J}(t,s;u(t,s)) = J(s,u(t,s))$, $(t,s)\in\Omega_3$. Hence, changing the role of t and s in the argument for y_1, we obtain that $z_1 \in \bar{A}_{\{2\alpha,0\},q;J}$ with $\|z_1\|_{\bar{A}_{\{2\alpha,0\},q;J}} \lesssim \|a\|_{\bar{\mathbb{A}}_{(\alpha,\alpha),q;J}}$. A similar change in the argument used for y_2 yields that $\|z_2\|_{\bar{A}_{\{2\alpha,0\},q;J}} \lesssim \|a\|_{\bar{\mathbb{A}}_{(\alpha,\alpha),q;J}}$. It follows that $y_3 \in \bar{A}_{\{2\alpha,0\},q;J}$ with the corresponding estimate for the norm.

As for y_4, put $v(t) = \int_t^\infty u\big(\frac{1}{t},s\big)\frac{ds}{s}$ for $1\le t<\infty$. This time, $\bar{J}(t,s;u(t,s)) = tsJ\big(\frac{1}{t},u(t,s)\big)$, $(t,s)\in\Omega_4$. We obtain

$$J(t,v(t)) \le \int_t^\infty ts^{-1}\bar{J}\left(\frac{1}{t},s;u\left(\frac{1}{t},s\right)\right)\frac{ds}{s} \lesssim t^\alpha\left(\int_t^\infty\left(s^{-\alpha}\bar{J}\left(\frac{1}{t},s;u\left(\frac{1}{t},s\right)\right)\right)^q\frac{ds}{s}\right)^{\frac{1}{q}}.$$

Therefore,

$$\begin{aligned}\|y_4\|_{\bar{A}_{\{2\alpha,0\},q;J}} &\le \left(\int_1^\infty J(t,v(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}\\ &\lesssim \left(\int_1^\infty\int_t^\infty \left(t^\alpha s^{-\alpha}\bar{J}\left(\frac{1}{t},s;u\left(\frac{1}{t},s\right)\right)\right)^q\frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{q}}\\ &\lesssim \|a\|_{\bar{\mathbb{A}}_{(\alpha,\alpha),q;J}}.\end{aligned}$$

In Ω_5 we have $\bar{J}\big(t,s;u(t,s)\big) = tsJ\big(\frac{1}{t},u(t,s)\big)$. To deal with y_5, we write $v(t) = \int_{\frac{1}{t}}^\infty u\left(\frac{1}{t},s\right)\frac{ds}{s}$ for $0<t<1$. We get

$$\begin{aligned}J(t,v(t)) &\le \int_{\frac{1}{t}}^\infty ts^{-1}\bar{J}\left(\frac{1}{t},s;u\left(\frac{1}{t},s\right)\right)\frac{ds}{s}\\ &\lesssim t^{2-\alpha}\left(\int_{\frac{1}{t}}^\infty\left(s^{-\alpha}\bar{J}\left(\frac{1}{t},s;u\left(\frac{1}{t},s\right)\right)\right)^q\frac{ds}{s}\right)^{\frac{1}{q}}.\end{aligned}$$

It follows that

$$\|y_5\|_{\bar{A}_{\{2\alpha,0\},q;J}} \leq \left(\int_0^1 (t^{-2\alpha} J(t, v(t)))^q \frac{dt}{t}\right)^{\frac{1}{q}}$$
$$\lesssim \left(\int_0^1 \int_{\frac{1}{t}}^{\infty} \left(t^{2-4\alpha} t^{\alpha} s^{-\alpha} \bar{J}\left(\frac{1}{t}, s; u\left(\frac{1}{t}, s\right)\right)\right)^q \frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{q}}.$$

In the integral we have that $t^{2-4\alpha} \leq 1$ because $\alpha \leq \frac{1}{2}$. This yields that $\|y_5\|_{\bar{A}_{\{2\alpha,0\},q;J}} \lesssim \|a\|_{\bar{\mathbb{A}}_{(\alpha,\alpha),q;J}}$.

Finally, for y_6, we derive

$$y_6 = \int_1^{\infty} \int_{\frac{1}{t}}^{t} u(t,s) \frac{ds}{s}\frac{dt}{t} = \int_0^1 \int_{\frac{1}{s}}^{\infty} u(t,s) \frac{dt}{t}\frac{ds}{s} + \int_1^{\infty} \int_{s}^{\infty} u(t,s) \frac{dt}{t}\frac{ds}{s} = z_4 + z_5.$$

Moreover, $\bar{J}(t, s; u(t, s)) = tsJ\big(\frac{1}{s}, u(t, s)\big)$, $(t, s) \in \Omega_6$. Consequently, changing the role of t and s, we can treat z_4 as y_4 and z_5 as y_5. This completes the proof for (α, α). For the remaining case $(\alpha, 1-\alpha)$, the proof can be carried out in the same way. □

If $A_0 \hookrightarrow A_1$ we recover [7, Theorem 5.1] as a direct consequence of Theorem 4.2 and Remark 3.5.

Having in mind Theorems 3.8, 3.9 and 3.10, we obtain the following description of $\bar{\mathbb{A}}_{(\alpha,\alpha),q;J}$ and $\bar{\mathbb{A}}_{(\alpha,1-\alpha),q;J}$ as intersections of real interpolation spaces and limiting J-spaces.

Corollary 4.2. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let $0 < \alpha < 1$ and let $1 < q \leq \infty$. Put $\bar{\mathbb{A}} = (A_0, A_1, A_1, A_0)$. Then we have with equivalent norms*

$$\bar{\mathbb{A}}_{(\alpha,\alpha),q;J} = \begin{cases} \bar{A}_{2\alpha,q} \cap \bar{A}_{0,q;J} & \text{if } 0 < \alpha < \frac{1}{2} \\ \bar{A}_{1,q;J} \cap \bar{A}_{0,q;J} & \text{if } \alpha = \frac{1}{2} \\ \bar{A}_{2-2\alpha,q} \cap \bar{A}_{0,q;J} & \text{if } \frac{1}{2} < \alpha < 1, \end{cases}$$

and

$$\bar{\mathbb{A}}_{(\alpha,1-\alpha),q;J} = \begin{cases} \bar{A}_{1-2\alpha,q} \cap \bar{A}_{1,q;J} & \text{if } 0 < \alpha < \frac{1}{2} \\ \bar{A}_{0,q;J} \cap \bar{A}_{1,q;J} & \text{if } \alpha = \frac{1}{2} \\ \bar{A}_{2\alpha-1,q} \cap \bar{A}_{1,q;J} & \text{if } \frac{1}{2} < \alpha < 1. \end{cases}$$

Theorem 4.1 and Corollary 4.2 show a symmetry which does not appear in the ordered case studies in [7]. Moreover, $\bar{\mathbb{A}}_{(\alpha,\alpha),q;J} = \bar{A}_{0,q;J}$ for any $0 < \alpha < 1$ if $A_0 \hookrightarrow A_1$. But in the general case, the J-space may change along the diagonals. We illustrate this fact in our last result which is a consequence of Theorem 4.1 and Corollary 3.12.

Corollary 4.3. *Let* (Ω, μ) *be a* σ*-finite measure space. Then*

$$(L_\infty, L_1, L_1, L_\infty)_{(\alpha,\alpha),\infty;J} = \begin{cases} L_{(\frac{1}{2\alpha},\infty)} \cap L_{\infty,\infty}(\log L)_{-1} & \text{if } 0 < \alpha < \frac{1}{2} \\ L_{(1,\infty)}(\log L)_{-1} \cap L_{\infty,\infty}(\log L)_{-1} & \text{if } \alpha = \frac{1}{2} \\ L_{(\frac{1}{2-2\alpha},\infty)} \cap L_{\infty,\infty}(\log L)_{-1} & \text{if } \frac{1}{2} < \alpha < 1, \end{cases}$$

and

$$(L_\infty, L_1, L_1, L_\infty)_{(\alpha,1-\alpha),\infty;J} = \begin{cases} L_{(\frac{1}{1-2\alpha},\infty)} \cap L_{(1,\infty)}(\log L)_{-1} & \text{if } 0 < \alpha < \frac{1}{2} \\ L_{\infty,\infty}(\log L)_{-1} \cap L_{(1,\infty)}(\log L)_{-1} & \text{if } \alpha = \frac{1}{2} \\ L_{(\frac{1}{2\alpha-1},\infty)} \cap L_{(1,\infty)}(\log L)_{-1} & \text{if } \frac{1}{2} < \alpha < 1. \end{cases}$$

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