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Model of Point-Like Window for Electromagnetic Helmholtz Resonator

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Abstract. Point-like interaction between internal and external operators as a model of electromagnetic Helmholtz resonator is suggested. It is based on the theory of self-adjoint extensions of symmetric operators in Pontryagin space. Formula for the resonance close to the eigenvalue of the internal operator is obtained.

Keywords. Operator extensions theory, singular perturbation, Maxwell operator, Pontryagin space

Mathematics Subject Classification (2010). Primary 47B25, secondary 35Q61, 78A45

1. Introduction

The problem of the Helmholtz resonator, i.e. the resonator with small boundary window, has long history, starting from the lord Rayleigh work [27]. It was studied by analytical, asymptotical, variational, numerical methods (see, e.g., [5, 12, 17, 18, 20]). The main question is about the real and imaginary parts of resonances (quasi-bound states) for this open system. Due to the complexity of the problem, it is useful to construct simple models allowing one to study the problem analytically. One can construct the model in the framework of the operator extensions theory (see, e.g., [20, 23]). Such models are widely used in the theory of Schrödinger operator (see, e.g., [1, 6, 18, 20]). One can mention also works about singular boundary perturbations (see, e.g., [16]). The idea of the model for acoustical Helmholtz resonator is as follows. Let Ω^{in} be bounded domain with smooth boundary in $\mathbb{R}^3$, $\Omega^{ex} = \mathbb{R}^3 \setminus \Omega^{in}$. Consider the Laplace operator $-(\Delta^{in} \oplus \Delta^{ex})$ with the Neumann boundary condition ($\frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0$) in $L_2(\Omega^{in} \oplus \Omega^{ex})$. Restrict the operator onto the set of smooth functions vanishing at some point $x_0 \in \partial\Omega$. The closure of this operator is symmetric non-self-adjoint operator with deficiency indices (2,2). It has self-adjoint extensions

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which gives us the model in question. Of course, there is fitting problem, i.e. a question how to choose the proper extension among the whole set of extensions (see, e.g., [7, 22]). The attempt to construct the corresponding model in L_2 space fails (it is necessary to extend the initial state) when one introduces zero-range interaction for the Laplacian in $\mathbb{R}^n$, $n > 3$ (see [9, 10, 26]) or constructs the Helmholtz resonator with point-like opening having the Dirichlet boundary condition (see [23]). In these papers a space with indefinite metrics is used. There is also an approach based on using of weighted spaces with definite metrics (see [2, 15]).

As for the corresponding model for electromagnetic field, there are only few works concerning to point-like interaction for the Maxwell operator (see [11]). It is related with some additional difficulties which will be described below. To construct the δ-interaction model for EM field it is necessary to extend the initial space. It should be mentioned that in some particular cases the Maxwell equations can be reduced to the Helmholtz one, i.e. to the Schrödinger case (see, e.g., [13, 24]). As for the operator extensions descriptions, boundary triples approach is often used (see [3, 14]).

In the present paper we construct the operator extensions theory model for the Maxwell operator in the domain of trap type. To introduce point-like window we use the technique analogous to that applied in [11] to introduce generalized point interaction. The main result is full description of the operator extension theory model of the electromagnetic Helmholtz resonator (Theorem 2). As an application of the model, formula for resonance (quasi-bound state) close to the eigenvalue of the internal problem is obtained. The suggested model can be, in particular, useful for computations in shape optimization (see, e.g., [19] where the analogous approach to shape optimization is fully justified for the elasticity boundary value problems).

2. Point-like boundary window for the Maxwell operator

The self-adjoint Maxwell operator in the space L_2 for arbitrary domain was correctly described by Birman and Solomyak [4]. Later this approach was successfully used in various electromagnetic problems (see, e.g., [8]). We will use Birman-Solomyak definition. Let Ω^{in} be bounded domain with smooth boundary in $\mathbb{R}^3$, $\Omega^{ex} = \mathbb{R}^3 \setminus \Omega^{in}$.

The definition of the self-adjoint Maxwell operator $M^{in,ex}$ in $\Omega^{in,ex}$ is as follows. Let $\mathbf{E}$ and $\mathbf{B}$ be, correspondingly, the electric and magnetic fields. These vector fields should satisfy the conditions

$$\nabla \cdot (\varepsilon \mathbf{E}) = 0, \qquad \nabla \cdot \mathbf{B} = 0,$$
$$\gamma_\tau \mathbf{E} = 0, \qquad \gamma_\nu \mathbf{B} = 0,$$

where $\gamma_\tau \mathbf{E}$ and $\gamma_\nu \mathbf{B}$ are, correspondingly, tangential and normal components of the corresponding fields at the boundary. Then, the Maxwell operator acts on six-dimensional vector field as

$$M^{in,ex}\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} = -i\begin{pmatrix}0 & \varepsilon^{-1}\mu^{-1}\epsilon\cdot\mathbf{p}\\ -\epsilon\cdot\mathbf{p} & 0\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix}.$$

Here $\varepsilon(x), \mu(x)$ (electric and magnetic susceptibilities) are smooth, strictly positive, bounded functions of x ($\varepsilon, \mu \in C^1(\mathbb{R}^3)$, the bounded continuous functions with bounded continuous derivatives), ϵ is the Levi-Chivita tensor density ($\epsilon_{123} = 1$ and ϵ is antisymmetric in all indices), $\mathbf{p} = -i\nabla$ is the momentum operator. In the present paper we consider the vacuum case and normalize values of ε and μ to unity. The operator $M^{in,ex}$ defined on smooth functions is essentially self-adjoint. We denote its closure by the same letter ($M^{in,ex}$). Let $M = M^{in} \oplus M^{ex}$. To construct generalized point interaction for the Maxwell operator it is necessary to extend the initial state L_2 (see [11]). Namely, let $R_M(z_0)$ be the resolvent of the operator M , corresponding to regular point $z_0 \in \mathbb{C}$. Our underlying Hilbert space is $\mathcal{H}_0 = L_2(\Omega^{in} \oplus \Omega^{ex}, dx, \mathbb{C}^6)$. Construct the scale of Hilbert spaces

$$\cdots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0 \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2} \subset \cdots, \tag{1}$$

where $\mathcal{H}_k = (R_M(z_0))^k\mathcal{H}_0$, $k \in \mathbb{N}$, $\mathcal{H}_{-k}$ is the dual of $\mathcal{H}_k$ with respect to the inner product of $\mathcal{H}_0$, $\mathcal{H}_{-k} = (R_M(z_0))^{-k}\mathcal{H}_0$. Let us construct our extended space for a simple case. Taking $\chi_h \in \mathcal{H}_{-3} \setminus \mathcal{H}_{-2}, h = 1, 2, \ldots, 6$ we construct a chain of elements $\chi_{hk} = (R_M(z_0))^{3-k}\chi_h, k = -2, -1, 0, 1$, and consider the following elements of the pre-Pontryagin space:

$$F = \begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} = F_2 + \sum_{h=1}^{6}\sum_{k=-2}^{1} F_{hk}\chi_{hk}, \quad F_2 \in \mathcal{H}_2,\ F_{hk} \in \mathbb{C},$$

the inner product being

$$[F, G] = (F_2, G_2) + \sum_{h=1}^{6}\sum_{k=-2}^{1}(F_{hk}(\chi_{hk}, G_2) + \overline{G_{hk}}(F_2, \chi_{hk}))$$
$$+ \sum_{j,h=1}^{6}\sum_{k,s=-2}^{1} F_{jk}\overline{G_{hs}}[\chi_{jk}, \chi_{hs}],$$

where

$$[\chi_{jk}, \chi_{hs}] = \begin{cases}(\chi_{jk}, \chi_{hs}), & k + s \geq 0\\ g_{ks}^{(jh)}, & k + s < 0.\end{cases}$$

Here $(\cdot,\cdot)$ is the L_2-inner product. Tensor $g_{ks}^{(jh)}$ may be chosen rather arbitrarily, but it should satisfy some conditions. The general property of inner product leads to the correlation $g_{ks}^{(jh)} = \overline{g_{sk}^{(hj)}}$. Due to the specific way of constructing of the chain χ_{hk} with using of the resolvent $R_M(z_0)$, one has

$$g_{k+1,s}^{(jh)} - g_{k,s+1}^{(jh)} = (z_0 - \overline{z}_0)\, g_{ks}^{(jh)}.$$

The obtained space $\mathcal{P}$ isn't satisfactory, both physically and mathematically since it doesn't contain the whole physical space $\mathcal{H}_0$. and the norm topology is still lacking on $\mathcal{P}$. To remove the mathematical objections, $\mathcal{P}$ will be completed into a Pontryagin space Π. Let

$$\Pi = \left\{(\phi_0, b, a) : \phi_0 \in \mathcal{H}_2, a, b \in \mathbb{C}^{12}\right\}$$

equipped with indefinite inner product:

$$(\phi, \phi') = (\phi_0, \phi_0') + a \cdot \overline{b'} + b \cdot \overline{a'} + a \cdot g \cdot \overline{a'},$$

where g is the Hermitian matrix which was introduced earlier.

Theorem 2.1. *$\mathcal{P}$ is a pre-Pontryagin space with topological completion* Π.

Proof. The embedding of $\mathcal{P}$ in Π is given by the following isometric identification mapping $\mathcal{J}$:

$$F_2 + \sum_{h=1}^{6} \sum_{k=-2}^{1} F_{hk}\chi_{hk} \to$$
$$\left(F_2 + \sum_{h=1}^{6} \sum_{k=0}^{1} F_{hk}\chi_{hk}, \left[(\chi_{hk}, F_2) + \sum_{i=0}^{1} a_{hi} g_{ik}\right]_{k=-1,h=1}^{k=-2,h=6}, [a_{hi}]_{i=-1,h=1}^{i=-2,h=6}\right). \tag{2}$$

Then, we restrict the mapping $\mathcal{J}$ to the submanifold $\mathcal{P}'$ of $\mathcal{P}$ which consists of the elements $F_2 + \sum_{h=1}^{6} \sum_{k=-2}^{-1} F_{hk}\chi_{hk}$. and obtain that $\mathcal{P}'$ and, correspondingly, $\mathcal{P}$ is topologically dense in Π because $\mathcal{H}_2$ is dense in $\mathcal{H}_0$. □

Now it is necessary to extend our self-adjoint operator H_M on Π. We shall do this by constructing of the corresponding resolvent operator $R_M(z)$. Consider the set $\mathcal{P}'$ which is dense in Π. To define the resolvent on $\mathcal{P}'$ we use the iteration of the resolvent identity (using of this trick in analogous problem was suggested in [21]).

$$R_M(z) - R_M(z_0) = (z - z_0)R_M^2(z_0) + (z - z_0)^2 R_M(z) R_M^2(z_0)$$

Of course, one can do so much number of iteration as necessary. This formula allows one to separate terms with different singularities (due to resolvent action the order of singularity decreases). Taking into account (2), one obtains

$$\begin{aligned} &R_M(z)\left(F_2, [(\chi_{hj}, F_2)]_{j=-1,h=1}^{j=-2,h=6}, [a_{hj}]_{j=-1,h=1}^{j=-2,h=6}\right) \\ &= \left(\widetilde{F}_0, \left[\widetilde{b}_{hj}\right]_{j=-1,h=1}^{j=-2,h=6}, [\widetilde{a}_{hj}]_{j=-1,h=1}^{j=-2,h=6}\right) \end{aligned} \tag{3}$$

with

$$\widetilde{F}_0 = R_M(z)F_2 + R_M(z)\chi_{-1}((z-z_0)a_{-2} + a_{-1}), \tag{4}$$

$$\widetilde{b}_{hj} = \begin{cases} \sum_{k=j+1}^{-1} (z-\overline{z}_0)^{j+1-k}(\chi_{hk}, F_2) + (z-\overline{z}_0)^{-(j+1)}(R_M(z)F_2, \chi_{h,-1}) \\ + (R_M(z)\chi_{h,-1}, \chi_{hj}) \sum_{k=-2}^{-1} (z-z_0)^{k+1} a_{hk}, \end{cases} \tag{5}$$

$$\widetilde{a}_{hj} = \sum_{k=-2}^{j-1} (z-z_0)^{j-(k+1)} a_{hk}. \tag{6}$$

This expression is easily closed in the norm topology yielding $\overline{R}_M(z)$ acting on Π:

$$\overline{R}_M(z)\Big(F_0, [b_{hj}]_{j=-1,h=1}^{j=-2,h=6}, [a_{hj}]_{j=-1,h=1}^{j=-2,h=6}\Big) = \Big(\widetilde{F}_0, \Big[\widetilde{b}_{hj}\Big]_{j=-1,h=1}^{j=-2,h=6}, [\widetilde{a}_{hj}]_{j=-1,h=1}^{j=-2,h=6}\Big) \tag{7}$$

with $\widetilde{F}_0, \widetilde{b}_{hj}, \widetilde{a}_{hj}$ are given by Equations (4)–(6) provided that $F_2, (\chi_{hj}, F_2)$ are replaced by F_0, b_{hj} respectively. Moreover, it is evident that $\overline{R}_M(z)$ is bounded on Π since $R_M(z)$ is bounded on $\mathcal{H}_0$. Hence, $\overline{R}_M(z)$ inherits from $R_M(z)$ the resolvent identity and the relation $\overline{R}^*_M(z) = \overline{R}_M(\overline{z})$.

Note that the obtained expression shows that there is a nontrivial subspace $\mathcal{N}(\overline{R}_M(z)) = \mathrm{span}(0, [0, \ldots, 0, a_{h,-2}], 0), h = 1, \ldots, 6$. It means that $\overline{R}_M(z)$ is not the resolvent of self-adjoint operator in Π. It is to be expected since $R_M(z)F$ does not explicitly depend on $(\chi_{h,-2}, F)$. The resolvent of a self-adjoint extension $\overline{R}_{A,M}(z)$ can be obtained by Krein resolvent formula:

$$\overline{R}_{A,M}(z) = \overline{R}_M(z) - \overline{R}_M(z)\chi\Gamma^{-1}(z, A)(\overline{R}_M(z)\chi, \cdot), \tag{8}$$

with

$$\begin{aligned} \Gamma(z, A) &= A^{-1} + 2^{-1}(z-z_0)(\overline{R}_M(z)\chi, \overline{R}_M(z_0)\chi) \\ &\quad + 2^{-1}(z-\overline{z}_0)(\overline{R}_M(z)\chi, \overline{R}_M(\overline{z}_0)\chi), \\ \chi &= (\chi_h)_{h=1,\ldots,6}. \end{aligned} \tag{9}$$

Nondegenerate matrix A parameterizes the extension. Thus, we come to the following theorem.

Theorem 2.2. *Let M be an unbounded self-adjoint Maxwell operator in a Hilbert space $\mathcal{H}_0$ and $(\cdot, \chi)$ be a functional in $\mathcal{H}_{-3}$ of the scale* (1). *Then $\overline{R}_{A,M}(z)$ defined by* (8), (9) *(with real non-degenerate matrix A such that $\Gamma(z_0, A)$, $\Gamma(\overline{z}_0, A)$ are non-degenerate) is a resolvent of self-adjoint operator $H_{A,M}$ in the Pontryagin space* Π.

Remark 2.3. Operator $H_{A,M}$ coincides with M on the submanifold of $\mathcal{H}_3$ on which functional $(\cdot, \chi)$ vanishes.

3. Discussion

The initial Maxwell operator is simply the orthogonal sum of the corresponding operators for the internal and the external domains. There is no interaction between these two parts. The constructed operator describes the interaction (of course, there are extensions corresponding to absence of interaction, but its are not interesting). The constructed operator can be represented in the block form by a natural way. It corresponds to the block matrix

$$\Gamma = \begin{pmatrix} \Gamma_{in,in} & \Gamma_{in,ex} \\ \Gamma_{ex,in} & \Gamma_{ex,ex} \end{pmatrix}.$$

The block $\Gamma_{in,ex}$ is responsible for the interaction of the internal and the external parts. If we choose the simplest extension among those corresponding to exactly one deficiency element for the internal and exactly one for the external parts, then $\Gamma_{in,ex}$ contains only one non-zero element, say γ. In this case the Krein's resolvent formula lead to the dispersion equation in the form

$$(\gamma_1 - D_{in}(z))(\gamma_2 - D_{ex}(z)) - |\gamma|^2 = 0. \tag{10}$$

Here γ_1, γ_2 correspond to diagonal terms of the matrix Γ parameterizing the extension,

$$D_{in,ex}(z) = \lim_{x\to 0}(R_z^{in,ex}(x,0) - (4\pi|x|)^{-1}), \tag{11}$$

where $R_z^{in} = (M^{in} - zI)^{-1}, R_z^{ex} = (M^{ex} - zI)^{-1}$, and $R_z^{in,ex}(x,0)$ is the kernel of the integral operator $R_z^{in,ex}$. Note that $\gamma \to \infty$ corresponding to absence of interaction.

To find the resonance (quasi-bound state) of the model operator in a neighborhood of an eigenvalue λ_n of the internal operator, it is necessary to represent D_{in} (11) as an eigenfunctions expansion series. Then (10) gives us an approximation for the resonance z (for the case of weak interaction ($\gamma << 1$)) near λ_n :

$$z = \lambda_n + \frac{|\phi_n(x_0)|^2(\gamma_1 - D_{in}(z))}{|\gamma|^2} + \frac{(\gamma_1 - D_{in}(z))(\gamma_2 - D_{ex}(z))}{|\gamma|^2} + o(|\gamma|^{-2}).$$

Here ϕ_n is the eigenfunction of the internal operator corresponding to λ_n, x_0 is the position of the point-like opening. Note that only the third term in the right hand side has non-trivial imaginary part (due to D_{ex}). From physical point of view the imaginary part of the resonance is related with the life time of the corresponding state.

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