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Approximation Results with Respect to Multidimensional φ-Variation for Nonlinear Integral Operators

Laura Angeloni

Abstract. In this paper we study approximation problems for functions belonging to BV^φ-spaces (spaces of functions of bounded φ-variation) in multidimensional setting. In particular, using a multidimensional concept of φ-variation in the sense of Tonelli introduced in [4], we obtain estimates, convergence results and, by means of suitable Lipschitz classes, results about the order of approximation for a family of nonlinear convolution integral operators.

Keywords. Nonlinear convolution integral operators, multidimensional φ-variation, rate of approximation, Lipschitz classes, φ-modulus of smoothness.

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1. Introduction

In [4] approximation problems for a family of linear integral operators of the form

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}^N} K_w(\mathtt{t}) f(\mathtt{s}-\mathtt{t})\, d\mathtt{t}, \quad w > 0,\ \mathtt{s} \in \mathbb{R}^N, \tag{I}$$

are studied in the frame of BV^φ-spaces. In particular, a multidimensional concept of φ-variation is introduced, with the purpose to extend to the multidimensional frame the classical notion of φ-variation. The φ-variation was introduced for the first time by L. C. Young ([37]) as a generalization of Wiener's quadratic variation ([35]), later extended to p-variation ([21,36]). This concept, however, was mainly developed by J. Musielak and W. Orlicz ([27]), so that it is known as the Musielak-Orlicz φ-variation, and later extensively studied by the Orlicz school. The φ-variation extends the notion of Jordan variation preserving a lot of its properties. However, a crucial difference with the classical variation

L. Angeloni: Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Via Vanvitelli 1, 06123, Perugia (Italy); angeloni@dmi.unipg.it

is that there is not any integral representation in terms of φ-absolutely continuous functions. As a consequence, in order to solve this problem and to get approximation theorems in the frame of BV^φ-spaces, several preliminary results are necessary (see Sections 3 and 4). Further results about φ-variation can be found, for example, in [3, 8–11, 14, 16, 22–24, 28–30, 32].

The multidimensional concept of φ-variation introduced in [4] follows the idea of Tonelli's variation ([33]) for functions of two variables, later generalized to the frame of $\mathbb{R}^N$ by C. Vinti ([34]). We recall that a different generalization of φ-variation in the multidimensional setting in the sense of Vitali was introduced by Lenze ([19, 20]), but the φ-variation in the sense of Tonelli seems to be more suitable in order to study this kind of approximation problems.

In this paper, working with the multidimensional notion of φ-variation introduced in [4] we study the nonlinear version of the operators (I), hence we obtain results about convergence and order of approximation for the following family of nonlinear convolution integral operators

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}^N} K_w(\mathtt{t}, f(\mathtt{s}-\mathtt{t}))\, d\mathtt{t}, \quad w > 0,\ \mathtt{s} \in \mathbb{R}^N, \tag{II}$$

where $\{K_w\}_{w>0}$ is a family of kernels of the form $K_w(\mathtt{s}, u) = L_w(\mathtt{s})H_w(u)$, $\mathtt{s} \in \mathbb{R}^N$, $u \in \mathbb{R}$. Here, $\{L_w\}_{w>0}$ satisfies classical singularity conditions, while on $\{H_w\}_{w>0}$ we make assumptions which are quite natural in a nonlinear setting, as pointed out in Section 2. It is not difficult to find examples of kernels which fulfill all the assumptions of our theory: An example is furnished in Section 2.

Besides some estimates (Section 3), in particular about the error of approximation $(T_w f - f)$, the main result that we obtain proves that

$$\lim_{w\to+\infty} V^\varphi[\lambda(T_w f - f)] = 0,$$

for some $\lambda > 0$, provided that, mainly, f is locally φ-absolutely continuous ($f \in AC^\varphi_{loc}(\mathbb{R}^N)$). Analogous results were obtained, for the classical variation, in [7] (linear case) and in [2] (nonlinear case) while, in the frame of φ-variation, our results generalize to the multidimensional setting those ones obtained in [3]. As in the one-dimensional case, the assumption that f is locally φ-absolutely continuous is crucial: Indeed, a fundamental step in order to achieve the convergence is to prove that

$$\lim_{\delta\to 0^+} \omega^\varphi(\lambda(H_w \circ f), \delta) = 0, \tag{1}$$

uniformly with respect to $w \geq \bar{w}$, for some $\bar{w} > 0$ and $\lambda > 0$ (see [3, 11] and, in different frames, [5, 9, 10, 29]), where $\omega^\varphi(g, \delta) := \sup_{|\mathtt{t}|\leq\delta} V^\varphi[\tau_{\mathtt{t}} g - g]$ is the φ-modulus of smoothness of g and $\tau_{\mathtt{t}} g(\mathtt{s}) := g(\mathtt{s}-\mathtt{t})$ is the translation operator ([8, 27]). Here we prove that, as in the one-dimensional case ([3]), (1) holds if

$f \in AC^{\varphi}(\mathbb{R}^N) \cap BV^{\eta}(\mathbb{R}^N)$ (Theorem 4.2), where φ and η are two φ-functions linked by a suitable growth condition (see Section 2). This situation reproduces exactly what happens in the case of linear operators (i.e., $H_w(u) = u$, see [1]) and in the case of classical variation, even in multidimensional setting ([7]).

Besides convergence, we also face the problem of the order of approximation with respect to φ-variation (Section 5) for our integral operators introducing, as it is usual in this kind of problems, suitable Lipschitz classes which depend on the φ-variational functional. In the final section we present some further results in order to complete the theory. In particular, we first provide some regularity results which prove that, if the kernels are φ-absolutely continuous, as happens in the most common cases, so are the corresponding integral operators. This implies (see Remark 6.3) that, in this case, the assumption that the function is φ-absolutely continuous is not only sufficient, but also necessary for convergence in φ-variation of our integral operators. We finally prove that all the theory can be extended to the case of $\mathcal{F}^{\varphi}$-variation (see [4]), which is a concept of multidimensional φ-variation filtered by a functional $\mathcal{F} : \mathbb{R}^N \longrightarrow \mathbb{R}^+_0$, more general than a norm-functional.

2. Notations and assumptions

A function $\varphi : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is said to be a *φ-function* if it is continuous, nondecreasing on $\mathbb{R}^+_0$, such that $\varphi(0)=0$, $\varphi(u)>0$ for $u>0$ and $\lim_{u\to+\infty} \varphi(u)=+\infty$. We will denote by Φ the class of all the φ-functions and by $\widetilde{\Phi}$ the class of all the *convex* φ-functions. In the following we will assume that $\varphi \in \widetilde{\Phi}$.

For the main results of the paper we shall need the following further property on the φ-function φ:

$$u^{-1}\varphi(u) \to 0, \quad \text{as } u \to 0^+. \tag{+}$$

This assumption, which is typical working in the frame of BV^{φ}-spaces, is essentially due to the lack of an integral representation of φ-variation in terms of absolutely continuous functions. For further comments on assumption (+) see Remark 4.5.

We shall work with the multidimensional φ-variation introduced in [4], which is a generalization of the classical concept of Musielak-Orlicz φ-variation ([27]) in the multidimensional frame, following the approach due to Tonelli and C. Vinti (see [33, 34]).

Since we work in the multidimensional setting, we now recall some notations that we will use in the following (see [4, 7]).

Given $f : \mathbb{R}^N \to \mathbb{R}$ and $\mathtt{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $N \in \mathbb{N}$, if we are interested in the j-th coordinate we will write $x'_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \in \mathbb{R}^{N-1}$, so that $\mathtt{x} = (x'_j, x_j)$, $f(\mathtt{x}) = f(x'_j, x_j)$. For an ($N$-dimensional) interval $I = \prod_{i=1}^{N}[a_i, b_i]$, $I'_j = [a'_j, b'_j]$ will denote the $(N-1)$-dimensional interval obtained by deleting the j-th coordinate from I, so that $I = [a'_j, b'_j] \times [a_j, b_j]$, $j = 1, \ldots, N$. For $j = 1, \ldots, N$, $g_j(x_j) := f(x'_j, x_j)$ will denote the j-section of f. Let us consider the $(N-1)$-dimensional integrals

$$\Phi_j^\varphi(f, I) := \int_{a'_j}^{b'_j} V^\varphi_{[a_j,b_j]}[f(x'_j, \cdot)]dx'_j,$$

where $V^\varphi_{[a_j,b_j]}[f(x'_j, \cdot)]$ is the (one-dimensional) Musielak-Orlicz φ-variation of the j-section of f. We recall that the φ-variation of a function $g : [a, b] \to \mathbb{R}$ is defined as

$$V^\varphi_{[a,b]}[g] := \sup_D \sum_{i=1}^n \varphi(|g(s_i) - g(s_{i-1})|),$$

where $D = \{s_0 = a < s_1 < \ldots < s_n = b\}$ denotes a partition of the interval $[a, b]$ ([27]), and $g : [a, b] \to \mathbb{R}$ is said to be of bounded φ-variation ($g \in BV^\varphi([a, b])$) if there exists $\lambda > 0$ such that $V^\varphi_{[a,b]}[\lambda g] < +\infty$. We refer to [27] for the properties of the classical one-dimensional φ-variation: Here we just recall that, for every $f_1, \ldots, f_n \in L^1(\mathbb{R})$,

$$V^\varphi\left[\sum_{i=1}^n f_i\right] \leq \frac{1}{n}\sum_{i=1}^n V^\varphi[nf_i]. \tag{$*$}$$

Now, let $\Phi^\varphi(f, I)$ be the euclidean norm of the vector $(\Phi_1^\varphi(f, I), \ldots, \Phi_N^\varphi(f, I))$, namely

$$\Phi^\varphi(f, I) := \left\{\sum_{j=1}^N [\Phi_j^\varphi(f, I)]^2\right\}^{\frac{1}{2}},$$

where we put $\Phi^\varphi(f, I) = +\infty$ if $\Phi_j^\varphi(f, I) = +\infty$ for some $j = 1, \ldots, N$.

The multidimensional φ-variation of f on an interval $I \subset \mathbb{R}^N$ is then defined as

$$V_I^\varphi[f] := \sup \sum_{k=1}^m \Phi^\varphi(f, J_k),$$

where the supremum is taken over all the finite families of N-dimensional intervals $\{J_1, \ldots, J_m\}$ which form partitions of I ([4]). The φ-variation of f over the whole space $\mathbb{R}^N$ is defined as

$$V^\varphi[f] := \sup_{I \subset \mathbb{R}^N} V_I^\varphi[f],$$

where the supremum is taken over all the intervals $I \subset \mathbb{R}^N$.

By $BV^\varphi(\mathbb{R}^N)$ we will denote the *space of functions of bounded φ-variation* over $\mathbb{R}^N$, i.e.,

$$BV^\varphi(\mathbb{R}^N) = \left\{f \in L^1(\mathbb{R}^N) : \exists \lambda > 0 \text{ s.t. } V^\varphi[\lambda f] < +\infty\right\}.$$

We also define, for every $k = 1, \dots, N$, the "separated" variations

$$V_k^\varphi[f, I] := \sup \left\{ \sum_{i=1}^m \Phi_k^\varphi(f, J_i) \right\},$$

where the supremum is taken over all the partitions $\{J_1, \dots, J_m\}$ of I (see [1]). $V_k^\varphi[f, I]$ is a kind of variation with respect to just the k-th direction, while $V_I^\varphi[f]$ takes into account of all the N directions. Obviously we have, for every $k = 1, \dots, N$, $V_k^\varphi[f, I] \le V_I^\varphi[f] \le \sum_{k=1}^N V_k^\varphi[f, I]$.

In [4], a multidimensional concept of φ-absolute continuity is also introduced: A function $f : \mathbb{R}^N \to \mathbb{R}$ is *locally φ-absolutely continuous* ($AC_{loc}^\varphi(\mathbb{R}^N)$) if it is (uniformly) φ-absolutely continuous in the Tonelli sense, i.e., for every interval $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}^N$ and for every $j = 1, 2, \dots, N$, the j-th section of f, $g_j : [a_j, b_j] \to \mathbb{R}$, is (uniformly) φ-absolutely continuous for almost every $x'_j \in [a'_j, b'_j]$. We recall that a function $g : [a, b] \to \mathbb{R}$ is φ-absolutely continuous if there exists $\lambda > 0$ such that the following property holds:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^n \varphi(\lambda|g(\beta_i) - g(\alpha_i)|) < \varepsilon$, for all finite sets of non-overlapping intervals $[\alpha_i, \beta_i] \subset [a, b]$, $i = 1, \dots, n$, such that $\sum_{i=1}^n \varphi(\beta_i - \alpha_i) < \delta$.

If $\varphi \in \widetilde{\Phi}$ satisfies (+), then the above definition is equivalent to the following (see [27]):

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^m \varphi(\lambda|g(t_i) - g(t_{i-1})|) < \varepsilon$, for all finite partitions $a = t_0 < t_1 < \ldots < t_m = b$ of $[a, b]$, such that $t_i - t_{i-1} < \delta$, for every $i = 1, \dots, m$.

By $AC^\varphi(\mathbb{R}^N)$ we will denote the space of functions $f \in BV^\varphi(\mathbb{R}^N) \cap AC_{loc}^\varphi(\mathbb{R}^N)$ (φ-absolutely continuous functions).

We now introduce a family of kernels. Let $\{K_w\}_{w>0}$ be a family of functions $K_w : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that

$$K_w(\mathtt{t}, u) = L_w(\mathtt{t}) H_w(u),$$

for every $\mathtt{t} \in \mathbb{R}^N$, $u \in \mathbb{R}$, where $L_w : \mathbb{R}^N \to \mathbb{R}$ and $H_w : \mathbb{R} \to \mathbb{R}$ is such that $H_w(0) = 0$. We assume that H_w is a *ψ-Lipschitz kernel* for every $w > 0$, i.e., there exists a constant $K > 0$ such that

$$|H_w(u) - H_w(v)| \le K \psi(|u - v|) \tag{$\star$}$$

for every $u, v \in \mathbb{R}$, where ψ is a φ-function.

Moreover the following conditions are satisfied:

$\mathbf{K_w.1}$) $L_w : \mathbb{R}^N \to \mathbb{R}$ is a measurable function such that $L_w \in L^1(\mathbb{R}^N)$, $\|L_w\|_1 \le A$, for an absolute constant $A > 0$, and $\int_{\mathbb{R}^N} L_w(\mathtt{t})\, d\mathtt{t} = 1$, for every $w > 0$.

$\mathbf{K_w}$.2) For every fixed $\delta > 0$, $\int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| \, d\mathtt{t} \to 0$, as $w \to +\infty$.

$\mathbf{K_w}$.3) Denoted by $G_w(u) := H_w(u) - u$, $u \in \mathbb{R}$, $w > 0$, for every $\gamma > 0$ there exists $\lambda > 0$ such that

$$\frac{V_J^{\varphi}[\lambda G_w]}{\varphi(\gamma m(J))} \to 0, \quad \text{as } w \to +\infty,$$

uniformly with respect to every (proper) bounded interval $J \subset \mathbb{R}$, i.e., in correspondence to $\gamma > 0$ there exists $\lambda > 0$ such that, for every $\varepsilon > 0$, there exists $\overline{w} > 0$ (depending only on ε) for which $\frac{V_J^{\varphi}[\lambda G_w]}{\varphi(\gamma m(J))} \leq \varepsilon$, if $w \geq \overline{w}$, for every (proper) bounded interval $J \subset \mathbb{R}$ ($m(J)$ denotes the length of J).

In the following we will say that $\{K_w\}_{w>0} \subset \mathcal{K}_w$ if the above properties hold.

We remark that assumption ($\star$) is a generalization of a Hölder-type condition, K_w.1) and K_w.2) mean that $\{L_w\}_{w>0}$ is an approximate identity, while K_w.3) is quite natural, working in a nonlinear setting (see [2,3]). Moreover it is not difficult to provide examples of kernel functions which satisfy all the above assumptions, and to which our theory can be applied. We now give an example. Other examples can be found in [3].

Example 2.1. Let $K_w : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be defined as $K_w(\mathtt{t}, u) = L_w(\mathtt{t}) H_w(u)$, where $\{L_w\}_{w>0}$ is an approximate identity, $H_w(u)$ is defined as

$$H_w(u) = \begin{cases} \dfrac{e^w u}{e^w + 1}, & 0 \leq u < 1, \\ \dfrac{e^w u^2}{e^w u + 1}, & u \geq 1, \end{cases}$$

and the definition of $H_w(u)$ is extended in odd-way for $u < 0$. Then $\{H_w\}_{w>0}$ satisfies ($\star$) with $\psi(|u|) = |u|$, $u \in \mathbb{R}$, and $K = 1$. Moreover

$$G_w(u) = \begin{cases} -\dfrac{u}{e^w + 1}, & 0 \leq u < 1, \\ -\dfrac{u}{e^w u + 1}, & u \geq 1, \end{cases}$$

and so G_w is decreasing in $\mathbb{R}_0^+$. Hence it is not difficult to see that, for every convex φ-function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $\gamma > 0$ and $[a,b] \subset [0,1]$, $V_{[a,b]}^{\varphi}[\gamma G_w] = \varphi\big(\gamma |G_w(b) - G_w(a)|\big) = \varphi\left(\gamma \frac{b-a}{e^w+1}\right) \leq \frac{1}{e^w+1} \varphi(\gamma(b-a))$, and so

$$\frac{V_{[a,b]}^{\varphi}[\gamma G_w]}{\varphi(\gamma(b-a))} \leq \frac{1}{e^w + 1} \to 0,$$

as $w \to +\infty$, while, if $[a,b] \subset [1, +\infty)$, $V_{[a,b]}^{\varphi}[\gamma G_w] = \varphi\left(\gamma\left(\frac{b}{e^w b+1} - \frac{a}{e^w a+1}\right)\right) \leq \frac{1}{(e^w+1)^2} \varphi(\gamma(b-a))$, and hence

$$\frac{V_{[a,b]}^{\varphi}[\gamma G_w]}{\varphi(\gamma(b-a))} \leq \frac{1}{(e^w + 1)^2} \to 0,$$

as $w \to +\infty$. Finally, if $[a,b]$ is such that $a < 1 < b$, then it is sufficient to notice that, by the properties of φ-variation,

$$V^{\varphi}_{[a,b]}[\gamma G_w] \leq \frac{1}{2}\left\{V^{\varphi}_{[a,1]}[2\gamma G_w] + V^{\varphi}_{[1,b]}[2\gamma G_w]\right\}.$$

Therefore $K_w.3)$ holds with $\lambda = \frac{\gamma}{2}$.

Finally, we assume the following growth condition on the function ψ of the ψ-Lipschitz assumption (see e.g. [8]). From now on we will assume that $\eta, \psi \in \Phi$.

Definition 2.2. We say that the triple (φ, η, ψ) is properly directed if for every $\gamma \in]0,1[$ there exists a constant $C_\gamma \in]0,1[$ such that

$$\varphi\big(C_\gamma \psi(|g|)\big) \leq \eta\big(\gamma|g|\big), \qquad (\star\star)$$

for every measurable function $g : \mathbb{R}^N \to \mathbb{R}$.

The above condition is the natural formulation, in the multidimensional frame, of the analogous assumption in the one-dimensional case, which is usual in convergence problems in BV^{φ}-spaces by means of nonlinear integral operators (see [2, 3, 8, 9, 23, 30]). Moreover it is not difficult to find triples of functions satisfying $(\star\star)$: As an example, it is sufficient to take $\varphi(u) = e^{u^a} - 1$, $a \geq 1$, $\psi \in \Phi$ and $\eta(u) := e^{\psi^a(u)} - 1$, $u \geq 0$.

We will now consider the following family of nonlinear integral operators

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}^N} K_w(\mathtt{t}, f(\mathtt{s}-\mathtt{t}))d\mathtt{t}, \quad w > 0,\ \mathtt{s} \in \mathbb{R}^N,$$

for every $f \in L^1(\mathbb{R}^N)$.

3. Estimates

We first prove an estimate for the φ-variation of our family of operators (II).

Proposition 3.1. *Let $f \in BV^{\eta}(\mathbb{R}^N)$. If $K_w.1)$ and $(\star)$ are satisfied and the triple (φ, η, ψ) is properly directed, then there exists $\lambda > 0$ such that, for every $w > 0$,*

$$V^{\varphi}[\lambda(T_w f)] \leq V^{\eta}[\gamma f], \qquad (2)$$

where $\gamma > 0$ is the constant for which $V^{\eta}[\gamma f] < +\infty$. Therefore, (II) *maps $BV^{\eta}(\mathbb{R}^N)$ into $BV^{\varphi}(\mathbb{R}^N)$.*

Proof. Let $I = \prod_{i=1}^N [a_i, b_i]$ be an interval in $\mathbb{R}^N$ and let $\{J_1, \dots, J_m\}$ be a partition of I, with $J_k = \prod_{j=1}^N [{}^{(k)}a_j, {}^{(k)}b_j]$, $k = 1, \dots, m$. For every fixed $j = 1, \dots N$, $k = 1, \dots, m$, let $\{s_j^o = {}^{(k)}a_j < \dots < s_j^\nu = {}^{(k)}b_j\}$ be a partition of $[{}^{(k)}a_j, {}^{(k)}b_j]$. Then, for every $\lambda > 0$,

$$\begin{aligned} S_j := &\sum_{\mu=1}^{\nu} \varphi\Big(\lambda |(T_w f)(s_j', s_j^\mu) - (T_w f)(s_j', s_j^{\mu-1})|\Big) \\ = &\sum_{\mu=1}^{\nu} \varphi\left(\lambda \left| \int_{\mathbb{R}^N} K_w(\mathtt{t}, f(s_j' - t_j', s_j^\mu - t_j)) d\mathtt{t} - \int_{\mathbb{R}^N} K_w(\mathtt{t}, f(s_j' - t_j', s_j^{\mu-1} - t_j)) d\mathtt{t} \right|\right) \\ \leq &\sum_{\mu=1}^{\nu} \varphi\left(\lambda \int_{\mathbb{R}^N} |L_w(\mathtt{t})| |H_w(f(s_j' - t_j', s_j^\mu - t_j)) - H_w(f(s_j' - t_j', s_j^{\mu-1} - t_j))| d\mathtt{t}\right). \end{aligned}$$

Now, since by $K_w.1)$ $\|L_w\|_1 \leq A$, for every $w > 0$, and φ is convex, by Jensen's inequality,

$$S_j \leq A^{-1} \int_{\mathbb{R}^N} |L_w(\mathtt{t})| \sum_{\mu=1}^{\nu} \varphi\Big(\lambda A \big| H_w(f(s_j' - t_j', s_j^\mu - t_j)) - H_w(f(s_j' - t_j', s_j^{\mu-1} - t_j))\big|\Big) d\mathtt{t},$$

and so, by $(\star)$,

$$S_j \leq A^{-1} \int_{\mathbb{R}^N} |L_w(\mathtt{t})| \sum_{\mu=1}^{\nu} \varphi\Big(\lambda A K \psi\big(|f(s_j' - t_j', s_j^\mu - t_j) - f(s_j' - t_j', s_j^{\mu-1} - t_j)|\big)\Big) d\mathtt{t}.$$

If $\lambda > 0$ is such that $\lambda A K < C_\gamma$, where C_γ is the constant of assumption $(\star\star)$, then

$$\begin{aligned} S_j &\leq A^{-1} \int_{\mathbb{R}^N} |L_w(\mathtt{t})| \sum_{\mu=1}^{\nu} \eta\Big(\gamma \big| f(s_j' - t_j', s_j^\mu - t_j) - f(s_j' - t_j', s_j^{\mu-1} - t_j)\big|\Big) d\mathtt{t} \\ &\leq A^{-1} \int_{\mathbb{R}^N} |L_w(\mathtt{t})| V^\eta_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\gamma f(s_j' - t_j', \cdot - t_j)] d\mathtt{t}. \end{aligned}$$

Then, by the Fubini-Tonelli theorem, for every $j = 1, \dots, N$,

$$\begin{aligned} \Phi_j^\varphi(\lambda(T_w f), J_k) &:= \int_{{}^{(k)}a_j'}^{{}^{(k)}b_j'} V^\varphi_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\lambda(T_w f)(s_j', \cdot)] ds_j' \\ &\leq A^{-1} \int_{{}^{(k)}a_j'}^{{}^{(k)}b_j'} \left\{ \int_{\mathbb{R}^N} |L_w(\mathtt{t})| V^\eta_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\gamma f(s_j' - t_j', \cdot - t_j)] d\mathtt{t} \right\} ds_j' \\ &= A^{-1} \int_{\mathbb{R}^N} \left\{ \int_{{}^{(k)}a_j'}^{{}^{(k)}b_j'} V^\eta_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\gamma f(s_j' - t_j', \cdot - t_j)] ds_j' \right\} |L_w(\mathtt{t})| d\mathtt{t} \\ &= A^{-1} \int_{\mathbb{R}^N} \Phi_j^\eta(\gamma f(\cdot - \mathtt{t}), J_k) |L_w(\mathtt{t})| d\mathtt{t}. \end{aligned}$$

Now, using a Minkowski-type inequality, for every $k = 1, \dots, m$ there holds

$$\begin{aligned}\Phi^{\varphi}(\lambda(T_w f), J_k) :&= \left\{\sum_{j=1}^{N}[\Phi_j^{\varphi}(\lambda(T_w f), J_k)]^2\right\}^{\frac{1}{2}} \\ &\leq A^{-1}\left\{\sum_{j=1}^{N}\left(\int_{\mathbb{R}^N}\Phi_j^{\eta}(\gamma f(\cdot - \mathtt{t}), J_k)|L_w(\mathtt{t})|d\mathtt{t}\right)^2\right\}^{\frac{1}{2}} \\ &\leq A^{-1}\int_{\mathbb{R}^N}\left\{\sum_{j=1}^{N}\left[\Phi_j^{\eta}(\gamma f(\cdot - \mathtt{t}), J_k)\right]^2\right\}^{\frac{1}{2}}|L_w(\mathtt{t})|d\mathtt{t} \\ &= A^{-1}\int_{\mathbb{R}^N}\Phi^{\eta}(\gamma f(\cdot - \mathtt{t}), J_k)|L_w(\mathtt{t})|d\mathtt{t}.\end{aligned}$$

Summing over $k = 1, \dots, m$ and passing to the supremum over all the possible partitions $\{J_1, \dots, J_m\}$ of the interval I, we conclude that

$$V_I^{\varphi}[\lambda(T_w f)] \leq A^{-1}\int_{\mathbb{R}^N}V_I^{\eta}[\gamma f(\cdot - \mathtt{t})]|L_w(\mathtt{t})|d\mathtt{t}$$

and so, by the arbitrariness of $I \subset \mathbb{R}^N$,

$$V^{\varphi}[\lambda(T_w f)] \leq A^{-1}\|L_w\|_1 V^{\eta}[\gamma f] \leq V^{\eta}[\gamma f]. \qquad \square$$

Remark 3.2. In case of $\varphi(u) = u = \eta(u)$, $u \in \mathbb{R}_0^+$, the previous inequality gives, for non-negative kernels $\{K_w\}_{w>0}$, the "variation non-augmenting property" for the operators $T_w f$ (see [7]); indeed in this case in (2) we may take $\lambda = \gamma = 1$, since $A = \|L_w\|_1 = 1$.

We now prove that, if $f \in BV^{\eta}(\mathbb{R}^N)$, then $\{H_w \circ f\}_{w>0}$ are equibounded in φ-variation.

Proposition 3.3. *Let $f \in BV^{\eta}(\mathbb{R}^N)$. If $(\star)$ is satisfied and the triple (φ, η, ψ) is properly directed, then there exists $\mu > 0$ such that, for every $w > 0$,*

$$V^{\varphi}[\mu(H_w \circ f)] \leq V^{\eta}[\gamma f],$$

where $\gamma > 0$ is the constant for which $V^{\eta}[\gamma f] < +\infty$. Hence the family $\{H_w \circ f\}_{w>0}$ is equibounded in φ-variation.

Proof. Let $I = \prod_{i=1}^{N}[a_i, b_i] \subset \mathbb{R}^N$ and let $\{J_1, \dots, J_m\}$ be a partition of I, with $J_k = \prod_{j=1}^{N}[{}^{(k)}a_j, {}^{(k)}b_j]$, $k = 1, \dots m$. For every $j = 1, \dots N$, $k = 1, \dots m$, let us consider a partition $\{s_j^o = {}^{(k)}a_j < \dots < s_j^{\nu} = {}^{(k)}b_j\}$ of $[{}^{(k)}a_j, {}^{(k)}b_j]$. Let $\gamma > 0$

be such that $V^{\eta}[\gamma f] < +\infty$. Then, by $(\star)$, if $0 < \mu < \frac{C_\gamma}{K}$, there holds

$$S_j := \sum_{\mu=1}^{\nu} \varphi\Big(\mu |(H_w \circ f)(s'_j, s_j^{\mu}) - (H_w \circ f)(s'_j, s_j^{\mu-1})|\Big)$$
$$\leq \sum_{\mu=1}^{\nu} \varphi\Big(C_\gamma \psi\left(|f(s'_j, s_j^{\mu}) - f(s'_j, s_j^{\mu-1})|\right)\Big)$$

and so, by $(\star\star)$,

$$S_j \leq \sum_{\mu=1}^{\nu} \eta\Big(\gamma |f(s'_j, s_j^{\mu}) - f(s'_j, s_j^{\mu-1})|\Big) \leq V^{\eta}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\gamma f(s'_j, \cdot)].$$

This implies that

$$\Phi_j^{\varphi}(\mu(H_w \circ f), J_k) = \int_{{}^{(k)}a'_j}^{{}^{(k)}b'_j} V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\mu(H_w \circ f)(s'_j, \cdot)]\, ds'_j$$
$$\leq \int_{{}^{(k)}a'_j}^{{}^{(k)}b'_j} V^{\eta}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\gamma f(s'_j, \cdot)]\, ds'_j = \Phi_j^{\eta}(\gamma f, J_k)$$

and so $\Phi^{\varphi}(\mu(H_w \circ f), J_k) \leq \Phi^{\eta}(\gamma f, J_k)$. Now summing over $k = 1, \ldots, m$, and passing to the supremum over all the partitions of I, by the arbitrariness of I, we conclude that

$$V^{\varphi}[\mu(H_w \circ f)] \leq V^{\eta}[\gamma f],$$

for every $w > 0$. □

We now recall the notion of V^{φ}-modulus of continuity ([1, 4]), which is the natural generalization, in the frame of $BV^{\varphi}(\mathbb{R}^N)$-spaces, of the classical modulus of smoothness (see, for example, [8, 25, 26]). The *V^{φ}-modulus of continuity* of a function $f \in BV^{\varphi}(\mathbb{R}^N)$ will be denoted by $\omega^{\varphi}(f, \delta)$, $\delta > 0$, and it is defined as

$$\omega^{\varphi}(f, \delta) := \sup_{|\mathtt{t}| \leq \delta} V^{\varphi}[\tau_{\mathtt{t}} f - f],$$

where $(\tau_{\mathtt{t}} f)(\mathtt{s}) := f(\mathtt{s} - \mathtt{t})$ for every $\mathtt{s}, \mathtt{t} \in \mathbb{R}^N$ is the translation operator.

The following proposition gives an estimate for $(T_w f - f)$, which will be crucial for the main convergence result of Section 4.

Proposition 3.4. *Let $f \in BV^{\eta}(\mathbb{R}^N)$. If $K_w.1)$ is satisfied, then, for every $\lambda, \delta > 0$ and for every $w > 0$, there holds*

$$V^{\varphi}[\lambda(T_w f - f)] \leq \frac{1}{2}\Big\{\omega^{\varphi}(2\lambda A(H_w \circ f), \delta) + A^{-1} V^{\varphi}[4\lambda A(H_w \circ f)]$$
$$\times \int_{|\mathtt{t}| > \delta} |L_w(\mathtt{t})|\, d\mathtt{t} + V^{\varphi}[2\lambda A(H_w \circ f - f)]\Big\}.$$

Proof. Using similar reasonings to Proposition 3.1, with the same notations it is possible to write, for every $\lambda > 0$,

$$\begin{aligned} S_j :&= \sum_{\mu=1}^{\nu} \varphi\Big(\lambda\Big|(T_w f)(s'_j, s^\mu_j) - f(s'_j, s^\mu_j) - [(T_w f)(s'_j, s^{\mu-1}_j) - f(s'_j, s^{\mu-1}_j)]\Big|\Big) \\ &= \sum_{\mu=1}^{\nu} \varphi\Big(\lambda\Big|\int_{\mathbb{R}^N} L_w(\mathtt{t})\Big[H_w(f(s'_j - t'_j, s^\mu_j - t_j)) - H_w(f(s'_j - t'_j, s^{\mu-1}_j - t_j)) \\ &\quad - f(s'_j, s^\mu_j) + f(s'_j, s^{\mu-1}_j)\Big] d\mathtt{t}\Big|\Big). \end{aligned}$$

Now, by $K_w.1$), Jensen's inequality and by convexity of φ, for every $\lambda > 0$,

$$\begin{aligned} S_j &\le A^{-1} \int_{\mathbb{R}^N} |L_w(\mathtt{t})| \sum_{\mu=1}^{\nu} \varphi\left(\lambda A\Big|[H_w(f(s'_j - t'_j, s^\mu_j - t_j)) - H_w(f(s'_j, s^\mu_j))\right. \\ &\quad - H_w(f(s'_j - t'_j, s^{\mu-1}_j - t_j)) + H_w(f(s'_j, s^{\mu-1}_j)) + H_w(f(s'_j, s^\mu_j)) \\ &\quad \left. - H_w(f(s'_j, s^{\mu-1}_j)) - f(s'_j, s^\mu_j) + f(s'_j, s^{\mu-1}_j)]\Big|\right) d\mathtt{t} \\ &\le \frac{A^{-1}}{2} \int_{\mathbb{R}^N} |L_w(\mathtt{t})| \sum_{\mu=1}^{\nu} \varphi\left(2\lambda A\Big|\big[H_w(f(s'_j - t'_j, s^\mu_j - t_j)) - H_w(f(s'_j, s^\mu_j))\big]\right. \\ &\quad \left. - \big[H_w(f(s'_j - t'_j, s^{\mu-1}_j - t_j)) - H_w(f(s'_j, s^{\mu-1}_j))\big]\Big|\right) d\mathtt{t} \\ &\quad + \frac{A^{-1}}{2} \int_{\mathbb{R}^N} |L_w(\mathtt{t})| \sum_{\mu=1}^{\nu} \varphi\left(2\lambda A\Big|\big[H_w(f(s'_j, s^\mu_j)) - f(s'_j, s^\mu_j)\big]\right. \\ &\quad \left. - \big[H_w(f(s'_j, s^{\mu-1}_j)) - f(s'_j, s^{\mu-1}_j)\big]\Big|\right) d\mathtt{t} =: \frac{1}{2}(I_1 + I_2). \end{aligned}$$

Let us estimate I_1 and I_2. For every fixed $\delta > 0$, using property $(*)$ of φ-variation,

$$\begin{aligned} I_1 &\le A^{-1} \int_{\mathbb{R}^N} |L_w(\mathtt{t})| V^\varphi_{[{}^{(k)}a_j, {}^{(k)}b_j]}\Big[2\lambda A\big((H_w \circ f)(s'_j - t'_j, \cdot - t_j) - (H_w \circ f)(s'_j, \cdot)\big)\Big] d\mathtt{t} \\ &= A^{-1}\left\{\int_{|\mathtt{t}|\le\delta} + \int_{|\mathtt{t}|>\delta}\right\} \\ &\quad |L_w(\mathtt{t})| V^\varphi_{[{}^{(k)}a_j, {}^{(k)}b_j]}\Big[2\lambda A\big((H_w \circ f)(s'_j - t'_j, \cdot - t_j) - (H_w \circ f)(s'_j, \cdot)\big)\Big] d\mathtt{t} \\ &\le A^{-1} \int_{|\mathtt{t}|\le\delta} |L_w(\mathtt{t})| V^\varphi_{[{}^{(k)}a_j, {}^{(k)}b_j]}\Big[2\lambda A\big((H_w \circ f)(s'_j - t'_j, \cdot - t_j) - (H_w \circ f)(s'_j, \cdot)\big)\Big] d\mathtt{t} \\ &\quad + \frac{A^{-1}}{2} \int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| V^\varphi_{[{}^{(k)}a_j, {}^{(k)}b_j]}[4\lambda A (H_w \circ f)(s'_j - t'_j, \cdot - t_j)]\, d\mathtt{t} \\ &\quad + \frac{A^{-1}}{2} \int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| V^\varphi_{[{}^{(k)}a_j, {}^{(k)}b_j]}[4\lambda A (H_w \circ f)(s'_j, \cdot)]\, d\mathtt{t}, \end{aligned}$$

and

$$I_2 \le A^{-1} \int_{\mathbb{R}^N} |L_w(\mathtt{t})| V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[2\lambda A(H_w \circ f - f)(s'_j, \cdot)]\, d\mathtt{t}$$
$$\le V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[2\lambda A(H_w \circ f - f)(s'_j, \cdot)].$$

Hence there holds

$$V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\lambda(T_w f - f)(s'_j, \cdot)]$$
$$\le \frac{A^{-1}}{2} \int_{|\mathtt{t}|\le\delta} |L_w(\mathtt{t})| V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}\Big[2\lambda A\big((H_w \circ f)(s'_j - t'_j, \cdot - t_j) - (H_w \circ f)(s'_j, \cdot)\big)\Big]\, d\mathtt{t}$$
$$+ \frac{A^{-1}}{4} \int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[4\lambda A(H_w \circ f)(s'_j - t'_j, \cdot - t_j)]\, d\mathtt{t}$$
$$+ \frac{A^{-1}}{4} \int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[4\lambda A(H_w \circ f)(s'_j, \cdot)]\, d\mathtt{t}$$
$$+ \frac{1}{2} V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[2\lambda A(H_w \circ f - f)(s'_j, \cdot)].$$

Now, applying the Fubini-Tonelli theorem,

$$\Phi^{\varphi}_j\big(\lambda(T_w f - f), J_k\big)$$
$$:= \int_{{}^{(k)}a'_j}^{{}^{(k)}b'_j} V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\lambda(T_w f - f)(s'_j, \cdot)]\, ds'_j$$
$$\le \frac{A^{-1}}{2} \int_{{}^{(k)}a'_j}^{{}^{(k)}b'_j} \Bigg(\int_{|\mathtt{t}|\le\delta} |L_w(\mathtt{t})|$$
$$V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}\Big[2\lambda A\big((H_w \circ f)(s'_j - t'_j, \cdot - t_j) - (H_w \circ f)(s'_j, \cdot)\big)\Big]\, d\mathtt{t}\Bigg)\, ds'_j$$
$$+ \frac{A^{-1}}{4} \int_{{}^{(k)}a'_j}^{{}^{(k)}b'_j} \left(\int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[4\lambda A(H_w \circ f)(s'_j - t'_j, \cdot - t_j)]\, d\mathtt{t} \right) ds'_j$$
$$+ \frac{A^{-1}}{4} \int_{{}^{(k)}a'_j}^{{}^{(k)}b'_j} \left(\int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[4\lambda A(H_w \circ f)(s'_j, \cdot)]\, d\mathtt{t} \right) ds'_j$$
$$+ \frac{1}{2} \int_{{}^{(k)}a'_j}^{{}^{(k)}b'_j} V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[2\lambda A(H_w \circ f - f)(s'_j, \cdot)]\, ds'_j$$
$$= \frac{A^{-1}}{2} \int_{|\mathtt{t}|\le\delta} |L_w(\mathtt{t})| \Phi^{\varphi}_j\big(2\lambda A((H_w \circ f)(\cdot - \mathtt{t}) - (H_w \circ f)(\cdot)), J_k\big)\, d\mathtt{t}$$
$$+ \frac{A^{-1}}{4} \int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| \Big[\Phi^{\varphi}_j\left(4\lambda A(H_w \circ f)(\cdot - \mathtt{t}), J_k\right)$$
$$+ \Phi^{\varphi}_j\left(4\lambda A(H_w \circ f), J_k\right) \Big]\, d\mathtt{t} + \frac{1}{2} \Phi^{\varphi}_j(2\lambda A(H_w \circ f - f), J_k).$$

Similarly to Proposition 3.1, we now apply a Minkowski-type inequality, and so

$$
\begin{aligned}
&\Phi^\varphi(\lambda(T_w f - f), J_k)\\
&:= \left\{\sum_{j=1}^N [\Phi_j^\varphi(\lambda(T_w f - f), J_k)]^2\right\}^{\frac{1}{2}}\\
&\leq \left\{\sum_{j=1}^N \left[\frac{A^{-1}}{2}\int_{|\mathtt{t}|\leq\delta} |L_w(\mathtt{t})| \Phi_j^\varphi\Big(2\lambda A\big((H_w \circ f)(\cdot - \mathtt{t}) - (H_w \circ f)(\cdot)\big), J_k\Big)\, d\mathtt{t}\right.\right.\\
&\quad + \frac{A^{-1}}{4}\int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| \Big[\Phi_j^\varphi\big(4\lambda A(H_w \circ f)(\cdot - \mathtt{t}), J_k\big)\Phi_j^\varphi\big(4\lambda A(H_w \circ f), J_k\big)\Big]\, d\mathtt{t}\\
&\quad \left.\left. + \frac{1}{2}\Phi_j^\varphi\big(2\lambda A(H_w \circ f - f), J_k\big)\right]^2\right\}^{\frac{1}{2}}\\
&\leq \frac{A^{-1}}{2}\int_{|\mathtt{t}|\leq\delta} |L_w(\mathtt{t})| \Phi^\varphi\Big(2\lambda A\big((H_w \circ f)(\cdot - \mathtt{t}) - (H_w \circ f)(\cdot)\big), J_k\Big)\, d\mathtt{t}\\
&\quad + \frac{A^{-1}}{4}\int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| \left[\Phi^\varphi\Big(4\lambda A(H_w \circ f)(\cdot - \mathtt{t}), J_k\Big) + \Phi^\varphi\Big(4\lambda A(H_w \circ f), J_k\Big)\right] d\mathtt{t}\\
&\quad + \frac{1}{2}\Phi^\varphi\Big(2\lambda A(H_w \circ f - f), J_k\Big).
\end{aligned}
$$

Now, summing over $k = 1, \ldots, m$ and passing to the supremum over all the possible partitions $\{J_1, \ldots, J_m\}$ of the interval I, we obtain that

$$
\begin{aligned}
&V_I^\varphi[\lambda(T_w f - f)]\\
&\leq \frac{A^{-1}}{2}\int_{|\mathtt{t}|\leq\delta} |L_w(\mathtt{t})| V_I^\varphi\Big[2\lambda A\big((H_w \circ f)(\cdot - \mathtt{t}) - (H_w \circ f)(\cdot)\big)\Big]\, d\mathtt{t}\\
&\quad + \frac{A^{-1}}{4}\int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| \Big(V_I^\varphi[4\lambda A(H_w \circ f)(\cdot - \mathtt{t})] + V_I^\varphi[4\lambda A(H_w \circ f)]\Big)\, d\mathtt{t}\\
&\quad + \frac{1}{2}V_I^\varphi[2\lambda A(H_w \circ f - f)].
\end{aligned}
$$

Hence, by the arbitrariness of $I \subset \mathbb{R}^N$ and $K_w.1)$, we conclude that

$$
\begin{aligned}
&V^\varphi[\lambda(T_w f - f)]\\
&\leq \frac{1}{2}\left\{\omega^\varphi(2\lambda A(H_w \circ f), \delta)\right.\\
&\quad \left. + A^{-1}V^\varphi[4\lambda A(H_w \circ f)]\int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})|\, d\mathtt{t} + V^\varphi[2\lambda A(H_w \circ f - f)]\right\}. \qquad \square
\end{aligned}
$$

4. Convergence results

In order to prove the main theorem of this section we need to establish some preliminary convergence results. We first prove a result of convergence in φ-variation for $(H_w \circ f - f)$ which is the generalization to the multidimensional frame of an analogous one in [3].

Proposition 4.1. *Let $f \in BV^\varphi(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$, where $C^0(\mathbb{R}^N)$ denotes the space of continuous functions on $\mathbb{R}^N$. If $K_w.3)$ holds, then there exists $\xi > 0$ such that*

$$\lim_{w\to+\infty} V^\varphi[\xi(H_w \circ f - f)] = 0.$$

Proof. Let $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}^N$ and let $\{J_1, \dots, J_m\}$ be a partition of I, with $J_k = \prod_{j=1}^N [{}^{(k)}a_j, {}^{(k)}b_j]$, $k = 1, \dots m$. Since f is continuous on $\mathbb{R}^N$, in particular $f(x'_j, \cdot)$ is continuous for every $x'_j \in [{}^{(k)}a'_j, {}^{(k)}b'_j]$ and for every $j = 1, \dots, N$. Then $f(x'_j, \cdot)$ has at most countably infinitely many proper points of maximum/minimum ([17]). Let us consider the most general case in which $f(x'_j, \cdot)$ has exactly countably infinitely many proper points of maximum/minimum, say $\overline{x}_i$, $i = 0, 1, 2, \dots$. Let $D = \{x_0 = {}^{(k)}a_j, x_1, \dots, x_n = {}^{(k)}b_j\}$ be a division of the interval $[{}^{(k)}a_j, {}^{(k)}b_j]$ and let $\widetilde{D} = \{y_0, y_1, \dots\}$ be the (infinite) division obtained adding the points $\overline{x}_i$ to D. Without any loss of generality let us assume that $\lim_{i\to+\infty} \overline{x}_i = {}^{(k)}b_j$ (in the other cases it is sufficient to split the interval) and that $\overline{x}_0 \equiv x_0 = {}^{(k)}a_j$. Indeed, if $\overline{x}_0 > {}^{(k)}a_j$, f is constant in $[{}^{(k)}a_j, \overline{x}_0]$ and so $V^\varphi_{[{}^{(k)}a_j, {}^{(k)}b_j]}[f(x'_j, \cdot)] = V^\varphi_{[\overline{x}_0, {}^{(k)}b_j]}[f(x'_j, \cdot)]$. Hence in each of the intervals $A_i := [\overline{x}_{i-1}, \overline{x}_i]$, $i = 1, 2, \dots,$ $f(x'_j, \cdot)$ is monotone, and so, for every $\mu > 0$, $w > 0$,

$$V^\varphi_{A_i}[\mu(H_w \circ f - f)(x'_j, \cdot)] \le V^\varphi_{I_i}[\mu G_w], \; i = 1, 2, \dots,$$

where $I_i := \big[\min\{f(x'_j, \overline{x}_{i-1}), f(x'_j, \overline{x}_i)\}, \max\{f(x'_j, \overline{x}_{i-1}), f(x'_j, \overline{x}_i)\}\big]$. Since by assumption $f \in BV^\varphi(\mathbb{R}^N)$, there exists $\gamma > 0$ such that $V^\varphi[\gamma f] < +\infty$, and so in particular $V^\varphi_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\gamma f(x'_j, \cdot)] < +\infty$ for a.e. $x'_j \in [{}^{(k)}a'_j, {}^{(k)}b'_j]$. Then, by assumption $K_w.3)$, in correspondence to γ there exists $\xi > 0$ such that, for a fixed $\varepsilon > 0$, there exists $\overline{w} > 0$ (depending only on ε) for which $V^\varphi_{I_i}[\xi G_w] \le \varepsilon\varphi(\gamma m(I_i))$, for every $w \ge \overline{w}$ and $i = 1, 2, \dots$. Hence

$$\begin{aligned}
\sum_{i=1}^{+\infty} \varphi\Big(\xi|(H_w \circ f - f)(x'_j, y_i) - (H_w \circ f - f)(x'_j, y_{i-1})|\Big) &\le \sum_{i=1}^{+\infty} V^\varphi_{A_i}[\xi(H_w \circ f - f)(x'_j, \cdot)] \\
&\le \sum_{i=1}^{+\infty} V^\varphi_{I_i}[\xi G_w] \\
&\le \varepsilon V^\varphi_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\gamma f(x'_j, \cdot)].
\end{aligned}$$

Then, passing to the supremum over all the possible divisions of $[{}^{(k)}a_j, {}^{(k)}b_j]$, we have that, for every $\varepsilon > 0$, there exists $\overline{w} > 0$ such that $V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\xi(H_w \circ f - f)(x'_j, \cdot)] \leq \varepsilon V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\gamma f(x'_j, \cdot)]$ for every $w \geq \overline{w}$. Then, for $w \geq \overline{w}$,

$$\begin{aligned}\Phi^{\varphi}_j(\xi(H_w \circ f - f), J_k) &= \int_{{}^{(k)}a'_j}^{{}^{(k)}b'_j} V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\xi(H_w \circ f - f)(x'_j, \cdot)]\, dx'_j \\ &\leq \varepsilon \int_{{}^{(k)}a'_j}^{{}^{(k)}b'_j} V^{\varphi}_{[{}^{(k)}a_j, {}^{(k)}b_j]}[\gamma f(x'_j, \cdot)]\, dx'_j \\ &= \varepsilon \Phi^{\varphi}_j(\gamma f, J_k),\end{aligned}$$

for every $j = 1, \ldots, N$, and so $\Phi^{\varphi}(\xi(H_w \circ f - f), J_k) \leq \varepsilon \Phi^{\varphi}(\gamma f, J_k)$. Hence, passing to the supremum over all the possible partitions of I, by the arbitrariness of $I \subset \mathbb{R}^N$, we have that

$$V^{\varphi}[\xi(H_w \circ f - f)] \leq \varepsilon V^{\varphi}[\gamma f],$$

for every $w \geq \overline{w}$, which completes the proof recalling that $V^{\varphi}[\gamma f] < +\infty$. □

In [1] it is proved that, if $\varphi \in \widetilde{\Phi}$ satisfies (+), the φ-modulus of smoothness of a function $f \in AC^{\varphi}(\mathbb{R}^N)$, $\omega^{\varphi}(\lambda f, \delta)$, converges to zero, as $\delta \to 0^+$, for some $\lambda > 0$. We now prove an analogous result for the φ-modulus of smoothness of $(H_w \circ f)$.

Theorem 4.2. *Let $f \in AC^{\varphi}(\mathbb{R}^N) \cap BV^{\eta}(\mathbb{R}^N)$ and let $\varphi \in \widetilde{\Phi}$ be such that (+) holds. If $K_w.3)$ and $(\star)$ are satisfied and the triple (φ, η, ψ) is properly directed, then there exist $\overline{\lambda} > 0$ and $\overline{w} > 0$ such that*

$$\lim_{\delta \to 0^+} \omega^{\varphi}(\overline{\lambda}(H_w \circ f), \delta) = 0,$$

uniformly with respect to $w \geq \overline{w}$.

Proof. We will prove that there exist $\overline{\lambda} > 0$ and $\overline{w} > 0$ such that, for every $\varepsilon > 0$ there is $\delta > 0$ for which

$$V^{\varphi}[\overline{\lambda}(\tau_{\mathtt{t}} g_w - g_w)] < \varepsilon,$$

if $|\mathtt{t}| \leq \delta$, for every $w \geq \overline{w}$, where $g_w := H_w \circ f$. The proof will be adapted, in the present nonlinear frame, from the proof of [1, Theorem 5]; hence, as in [1], for the sake of simplicity we will prove the result in the particular case $N = 2$, since all the reasonings can be easily adapted to the general case of $\mathbb{R}^N$, $N \geq 2$. Moreover we will consider the particular case $\mathtt{t} = (t, 0)$, $t < 0$, since all the other cases can be treated in a similar way, as pointed out in the proof of [1, Theorem 5].

Since $f \in AC^{\varphi}(\mathbb{R}^2) \cap BV^{\eta}(\mathbb{R}^2)$, as in [1, Proposition 6] it can be proved that, if $\gamma > 0$ is such that $V^{\eta}[\gamma f] < +\infty$, there exists $\overline{\lambda} > 0$ such that, for every fixed $\varepsilon > 0$, there exist $I = [a, \alpha] \times [b, \beta]$ and $\delta > 0$ for which

$$V^{\eta}_{\mathbb{R}\times(-\infty,b]}[\gamma f] + V^{\eta}_{\mathbb{R}\times[\beta,+\infty)}[\gamma f] + V^{\eta}_{(-\infty,a]\times[b,\beta]}[\gamma f] + V^{\eta}_{[\alpha,+\infty)\times[b,\beta]}[\gamma f] < \varepsilon, \quad (3)$$

and

$$V^{\varphi}_{k}[\overline{\lambda}(f - \nu_k), I_\delta] < \frac{\varepsilon}{2}, \ k = 1, 2, \quad (4)$$

where ν_k are the step functions used in [1] to approximate f and $I_\delta := [a - \delta, \alpha + \delta] \times [b - \delta, \beta + \delta]$.

Following the proof of [1, Theorem 5] one can prove that, for every $w > 0$, if $(t, 0)$ is such that $-\frac{\delta}{2} < t < 0$, and $0 < 32\lambda < \overline{\lambda}$, then

$$\begin{aligned} & V^{\varphi}[\lambda(\tau_{\mathbf{t}} g_w - g_w)] \\ & \leq \frac{1}{2} V^{\varphi}_{\mathbb{R}\times(-\infty,b]}[4\lambda g_w] + \frac{1}{4} V^{\varphi}_{\mathbb{R}\times[\beta,+\infty)}[8\lambda g_w] + \frac{1}{8} V^{\varphi}_{(-\infty,a]\times[b,\beta]}[16\lambda g_w] \\ & \quad + \frac{1}{16} V^{\varphi}_{[\alpha,+\infty)\times[b,\beta]}[32\lambda g_w] + \frac{1}{16} V^{\varphi}_{\left[a-\frac{\delta}{2},\alpha+\frac{\delta}{2}\right]\times[b,\beta]}[16\lambda(\tau_{\mathbf{t}} g_w - g_w)]. \end{aligned}$$

Now, using Proposition 3.3 there exists $\mu > 0$ such that, if $0 < 32\lambda < \mu$,

$$\begin{aligned} V^{\varphi}[\lambda(\tau_{\mathbf{t}} g_w - g_w)] & \leq \frac{1}{2} V^{\eta}_{\mathbb{R}\times(-\infty,b]}[\gamma f] + \frac{1}{4} V^{\eta}_{\mathbb{R}\times[\beta,+\infty)}[\gamma f] + \frac{1}{8} V^{\eta}_{(-\infty,a]\times[b,\beta]}[\gamma f] \\ & \quad + \frac{1}{16} V^{\eta}_{[\alpha,+\infty)\times[b,\beta]}[\gamma f] + \frac{1}{16} V^{\varphi}_{\left[a-\frac{\delta}{2},\alpha+\frac{\delta}{2}\right]\times[b,\beta]}[16\lambda(\tau_{\mathbf{t}} g_w - g_w)], \end{aligned}$$

for every $w > 0$ and so, by (3),

$$V^{\varphi}[\lambda(\tau_{\mathbf{t}} g_w - g_w)] < \frac{15}{16}\varepsilon + \frac{1}{16} V^{\varphi}_{\left[a-\frac{\delta}{2},\alpha+\frac{\delta}{2}\right]\times[b,\beta]}[16\lambda(\tau_{\mathbf{t}} g_w - g_w)].$$

About the last term, obviously there holds

$$\begin{aligned} & V^{\varphi}_{\left[a-\frac{\delta}{2},\alpha+\frac{\delta}{2}\right]\times[b,\beta]}[16\lambda(\tau_{\mathbf{t}} g_w - g_w)] \\ & \leq V^{\varphi}_{1}\left[16\lambda(\tau_{\mathbf{t}} g_w - g_w), \left[a - \frac{\delta}{2}, \alpha + \frac{\delta}{2}\right] \times [b, \beta]\right] \\ & \quad + V^{\varphi}_{2}\left[16\lambda(\tau_{\mathbf{t}} g_w - g_w), \left[a - \frac{\delta}{2}, \alpha + \frac{\delta}{2}\right] \times [b, \beta]\right] \\ & =: J_1 + J_2. \end{aligned}$$

Concerning J_1, for $-\frac{\delta}{2} < t < 0$, by the properties of φ-variation (see [1, Proposition 1]) there holds

$$\begin{aligned} J_1 &\leq \frac{1}{3}\left\{V_1^\varphi\left[48\lambda(\tau_{\mathtt{t}} g_w - \tau_{\mathtt{t}}\nu_1), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right]\right. \\ &\quad + V_1^\varphi\left[48\lambda(\tau_{\mathtt{t}}\nu_1 - \nu_1), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right] \\ &\quad \left. + V_1^\varphi\left[48\lambda(\nu_1 - g_w), \left[a-\frac{\delta}{2}, \alpha+\frac{\delta}{2}\right]\times[b,\beta]\right]\right\} \\ &\leq \frac{1}{3}\left\{2V_1^\varphi[48\lambda(\nu_1 - g_w), I_\delta] + V_1^\varphi[48\lambda(\tau_{\mathtt{t}}\nu_1 - \nu_1), I_{\frac{\delta}{2}}]\right\} \\ &\leq \frac{1}{3}\left\{V_1^\varphi[96\lambda(\nu_1 - f), I_\delta] + V_1^\varphi[96\lambda(f-g_w), I_\delta] + V_1^\varphi[48\lambda(\tau_{\mathtt{t}}\nu_1 - \nu_1), I_{\frac{\delta}{2}}]\right\}. \end{aligned}$$

Now, by (4), $V_1^\varphi[96\lambda(\nu_1 - f), I_\delta] < \frac{\varepsilon}{2}$ while, by [1, Theorem 3], $V_1^\varphi[48\lambda(\tau_{\mathtt{t}}\nu_1 - \nu_1), I_{\frac{\delta}{2}}] < \frac{\varepsilon}{2}$, if $\lambda > 0$ is small enough. Finally, by Proposition 4.1, there exist $\xi > 0$ and $\overline{w} > 0$ such that $V_1^\varphi[\xi(f-g_w), I_\delta] \leq V^\varphi[\xi(f-g_w)] < \frac{\varepsilon}{2}$, for every $w \geq \overline{w}$; hence, if λ is such that $96\lambda < \xi$, then for every $w \geq \overline{w}$, $V_1^\varphi[96\lambda(f-g_w), I_\delta] < \frac{\varepsilon}{2}$, and so $J_1 < \frac{\varepsilon}{2}$.

In analogous way it can be proved (replacing ν_1 with ν_2) that $J_2 < \frac{\varepsilon}{2}$, and hence we conclude that, if $(t,0)$ is such that $-\frac{\delta}{2} < t < 0$, $V^\varphi[\lambda(\tau_{\mathtt{t}} g_w - g_w)] < \varepsilon$, uniformly with respect to $w \geq \overline{w}$, for some $\lambda > 0$, and so the thesis follows.

The other cases for $\mathtt{t} = (t_1, t_2)$ can be treated similarly, taking into account of the same remarks pointed out in [1, Theorem 5]. □

We are now ready to prove the main result of this section.

Theorem 4.3. *Let $f \in AC^\varphi(\mathbb{R}^N) \cap BV^\eta(\mathbb{R}^N)$, where $\varphi \in \widetilde{\Phi}$ is such that $(+)$ holds. If $\{K_w\}_{w>0} \subset \mathcal{K}_w$ and (φ, η, ψ) is properly directed, then there exists $\lambda > 0$ such that*

$$\lim_{w\to+\infty} V^\varphi[\lambda(T_w f - f)] = 0.$$

Proof. By Proposition 3.4 we have that

$$\begin{aligned} V^\varphi[\lambda(T_w f - f)] \leq \frac{1}{2}\Big\{&\omega^\varphi(2\lambda A(H_w\circ f), \delta) + A^{-1}V^\varphi[4\lambda A(H_w \circ f)] \\ &\times \int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})|\, d\mathtt{t} + V^\varphi[2\lambda A(H_w\circ f - f)]\Big\}. \end{aligned}$$

for every $\lambda, \delta > 0$ and $w > 0$. Let us fix $\varepsilon > 0$. By Theorem 4.2 there exist $\overline{\lambda} > 0$, $\overline{w} > 0$ such that $\omega^\varphi(\overline{\lambda}(H_w \circ f), \delta) \to 0$, as $\delta \to 0^+$, uniformly with respect to $w \geq \overline{w}$. Hence in correspondence to ε there is $\overline{\delta} > 0$ for which

$$\omega^\varphi(2\lambda A(H_w\circ f), \delta) < \varepsilon,$$

if $0 < \delta \leq \overline{\delta}$, $2\lambda A < \overline{\lambda}$ and $w \geq \overline{w}$. Since $f \in BV^\eta(\mathbb{R}^N)$, there exists $\gamma > 0$ such that $V^\eta[\gamma f] < +\infty$, and so, by Proposition 3.3, there is $\mu > 0$ such that, if $4\lambda A < \mu$, $V^\varphi[4\lambda A(H_w \circ f)] \leq V^\eta[\gamma f]$, for every $w > 0$. Hence, for sufficiently large $w > 0$, by assumption K_w.2),

$$V^\varphi[4\lambda A(H_w \circ f)] \int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| d\mathtt{t} < \varepsilon V^\eta[\gamma f],$$

for every $\delta > 0$, and so in particular for the above fixed $0 < \delta \leq \overline{\delta}$. Finally, by Proposition 4.1, there exists $\xi > 0$ such that, if $w > 0$ is large enough and $2\lambda A < \xi$, then $V^\varphi[2\lambda A(H_w \circ f - f)] < \varepsilon$. In conclusion, we have proved that for every $\varepsilon > 0$, if $0 < \lambda < \min\{\frac{\overline{\lambda}}{2A}, \frac{\mu}{4A}, \frac{\xi}{2A}\}$, then

$$V^\varphi[\lambda(T_w f - f)] < \varepsilon \left(1 + \frac{A^{-1}}{2} V^\eta[\gamma f]\right),$$

for sufficiently large $w > 0$, and so the thesis follows since $V^\eta[\gamma f] < +\infty$. □

Remark 4.4. (a) In the linear case, i.e., $H_w(u) = u$ and $\varphi(u) = \eta(u)$, Theorem 4.2 obviously reduces exactly to [1, Theorem 5], and hence Theorem 4.3 (where it is sufficient to assume $f \in AC^\varphi(\mathbb{R}^N)$) generalizes [4, Theorem 3.3].

(b) We notice that the above convergence result does not hold, in general, with the mere assumption that $f \in BV^\eta(\mathbb{R}^N)$. We refer to [3, Example 2] for an example in the one-dimensional case, with $\varphi(u) = \eta(u)$.

Remark 4.5. We remark that, as pointed out in the Introduction, assumption (+) is natural in BV^φ-spaces, since, in some sense, it replaces the integral representation of φ-variation in terms of φ-absolute continuity. Indeed, it is used to prove the convergence of the φ-modulus of continuity (Theorem 4.2), which is, in the case of classical variation, an immediate consequence of the integral representation of variation.

Moreover, since φ is a convex φ-function, the only alternative to (+) is that $\lim_{u\to 0^+} \frac{\varphi(u)}{u} = k > 0$ (as happens, for example, for $\varphi(u) = e^u - 1$, $u \geq 0$).

Let us now consider for simplicity the case $N = 1$. If φ is such that $\lim_{u\to 0^+} \frac{\varphi(u)}{u} = k > 0$, then $BV(\mathbb{R}) = BV^\varphi(\mathbb{R})$ (see [27, 1.15]). Hence, if we want to develop a theory in BV^φ-spaces, it is natural to assume (+): Indeed in this case it is easy to see that $BV(\mathbb{R}) \subset BV^\varphi(\mathbb{R})$, and the inclusion is strict, in general. For example, the function

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & 0 < x < \frac{1}{\pi} \\ 0, & \text{otherwise}, \end{cases}$$

is such that $f \in BV^{\varphi}(\mathbb{R}) \setminus BV(\mathbb{R})$, with $\varphi(u) = u^2$. Hence, in this case, condition (+) is necessary and sufficient in order to have the strict inclusion of BV into BV^{φ}.

For $N > 1$ the situation is the same if we consider just bounded functions. In particular, with (+), the inclusion of BV into BV^{φ} is strict again, in general: An example is furnished, in the case $N = 2$, by

$$g(x,y) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) y(1-y), & (x,y) \in \left]0, \dfrac{1}{\pi}\right] \times [0,1] \\ 0, & \text{otherwise,} \end{cases}$$

for $\varphi(u) = u^2$. Actually, in the multidimensional case, functions of bounded variation need not to be bounded, in general: For example the function

$$h(x,y) = \begin{cases} \dfrac{1}{\sqrt{|x|} + \sqrt{|y|}}, & \sqrt{|x|} + \sqrt{|y|} \leq 1, \ (x,y) \neq (0,0) \\ 1, & \text{otherwise,} \end{cases}$$

belongs to $BV(\mathbb{R}^2)$, although it is not bounded in $(0,0)$. In this more general case we remark that assumption (+) does not guarantee anymore the inclusion of the BV-spaces in BV^{φ}. As an example, the function $h(x,y)$ defined before does not belong to $BV^{\varphi}(\mathbb{R}^2)$, with $\varphi(u) = u^2$.

5. Order of approximation

We will now face the problem of the rate of approximation for our family of nonlinear integral operators. In order to do this, it is necessary to introduce the notion of α-singular kernels: For a fixed $\alpha > 0$, we say that $\{L_w\}_{w>0}$ is an *α-singular kernel* if, for every $w > 0$,

$$\int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| d\mathtt{t} = O(w^{-\alpha}), \quad \text{as } w \to +\infty, \tag{5}$$

for every $\delta > 0$.

Moreover, as usual in this kind of problems, we have to introduce a suitable notion of Lipschitz class $V^{\varphi}Lip_{\mathbb{R}^N}(\alpha)$, defined as

$$V^{\varphi}Lip_{\mathbb{R}^N}(\alpha) := \Big\{ f \in AC^{\varphi}(\mathbb{R}^N) : \exists \mu > 0 \text{ s.t. } V^{\varphi}[\mu \Delta_{\mathtt{t}}(H_w \circ f)] = O(|\mathtt{t}|^{\alpha}), \text{as } |\mathtt{t}| \to 0 \Big\},$$

uniformly with respect to $w > 0$, with

$$\Delta_{\mathtt{t}}(H_w \circ f)(\mathtt{x}) := \big(\tau_{\mathtt{t}}(H_w \circ f) - (H_w \circ f)\big)(\mathtt{x}) = (H_w \circ f)(\mathtt{x} - \mathtt{t}) - (H_w \circ f)(\mathtt{x}),$$

for every $\mathtt{x}, \mathtt{t} \in \mathbb{R}^N$.

About kernel functions, assumption $K_w.3)$ has to be replaced with the following:

$\widetilde{\mathbf{K}}_{\mathbf{w}}.\mathbf{3}$) denoted by $G_w(u) := H_w(u) - u$, $u \in \mathbb{R}$, $w > 0$, for every $\gamma > 0$ there exists $\lambda > 0$ such that

$$\frac{V_J^{\varphi}[\lambda G_w]}{\varphi(\gamma m(J))} = O(w^{-\alpha}), \quad \text{as } w \to +\infty,$$

uniformly with respect to every (proper) bounded interval $J \subset \mathbb{R}$, i.e., in correspondence to $\gamma > 0$ there exists $\lambda > 0$ such that, for every $\varepsilon > 0$, there exists $\overline{w} > 0$ (depending only on ε) for which $\frac{V_J^{\varphi}[\lambda G_w]}{\varphi(\gamma m(J))} \leq N w^{-\alpha}$, for some constant $N > 0$, for every $w \geq \overline{w}$ and for every (proper) bounded interval $J \subset \mathbb{R}$ (see [3]).

It is not difficult to find examples of functions which fulfill assumption $\widetilde{K}_w.3$), among them, for example, the kernel functions $\{H_w\}_{w>0}$ of the Example of Section 2.

From now on we will say that $\{K_w\}_{w>0} \subset \widetilde{\mathcal{K}}_w$ if $(\star)$, $K_w.1$), $K_w.2$) and $\widetilde{K}_w.3$) are satisfied. We now establish a preliminary result about the order of approximation of $(H_w \circ f - f)$, with respect to convergence in φ-variation.

Proposition 5.1. *Let $f \in BV^{\varphi}(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$. If $\widetilde{K}_w.3$) is satisfied, then there exists $\xi > 0$ such that*

$$V^{\varphi}[\xi(H_w \circ f - f)] = O(w^{-\alpha}), \quad w \to +\infty.$$

Proof. Similarly to Proposition 4.1, and using the same notations, it is possible to prove that, for every $\mu > 0$, $V_{A_i}^{\varphi}[\mu(H_w \circ f - f)(x'_j, \cdot)] \leq V_{I_i}^{\varphi}[\mu G_w]$, $i = 1, 2, \ldots$, where $I_i := [\min\{f(x'_j, \overline{x}_{i-1}), f(x'_j, \overline{x}_i)\}, \max\{f(x'_j, \overline{x}_{i-1}), f(x'_j, \overline{x}_i)\}]$. By $\widetilde{K}_w.3$), there exist $\overline{w} > 0$, $N > 0$ and $\xi > 0$ such that $V_{I_i}^{\varphi}[\xi G_w] \leq N w^{-\alpha} \varphi(\gamma m(I_i))$, for every $w \geq \overline{w}$, where $\gamma > 0$ is such that $V^{\varphi}[\gamma f] < +\infty$. Then, arguing as in Proposition 4.1, one can prove that this implies that

$$V^{\varphi}[\xi(H_w \circ f - f)] \leq N V^{\varphi}[\gamma f] w^{-\alpha},$$

for $w \geq \overline{w}$, and so the thesis follows. □

We are now ready to state the main result about the rate of approximation for $(T_w f - f)$.

Theorem 5.2. *Let us assume that $\{K_w\}_{w>0} \subset \widetilde{\mathcal{K}}_w$, $\{L_w\}_{w>0}$ is an α-singular kernel and there exists $\widetilde{\delta} > 0$ such that*

$$\int_{|\mathtt{t}| \leq \widetilde{\delta}} |L_w(\mathtt{t})| |\mathtt{t}|^{\alpha} d\mathtt{t} = O(w^{-\alpha}), \quad \textit{as } w \to +\infty. \tag{6}$$

Then if $f \in BV^{\eta}(\mathbb{R}^N) \cap V^{\varphi}Lip_{\mathbb{R}^N}(\alpha)$, there exists $\lambda > 0$ such that

$$V^{\varphi}[\lambda(T_w f - f)] = O(w^{-\alpha}), \tag{7}$$

for sufficiently large $w > 0$.

Proof. As in Proposition 3.4 it is possible to prove that, for every $\lambda, \delta > 0$ and for every $w > 0$,

$$V^\varphi[\lambda(T_w f - f)] \le \frac{1}{2}\left\{A^{-1}\int_{|\mathtt{t}|\le\delta} V^\varphi\Big[2\lambda A\big(\tau_{\mathtt{t}}(H_w\circ f) - (H_w\circ f)\big)\Big]|L_w(\mathtt{t})|d\mathtt{t} + A^{-1}V^\varphi[4\lambda A(H_w\circ f)]\int_{|\mathtt{t}|>\delta}|L_w(\mathtt{t})|\,d\mathtt{t} + V^\varphi[2\lambda A(H_w\circ f - f)]\right\}.$$

Since $f \in V^\varphi Lip_{\mathbb{R}^N}(\alpha)$, there exist $\mu > 0$, $N > 0$, $\bar\delta > 0$ such that, if $2\lambda A < \mu$, $V^\varphi\Big[2\lambda A\big(\tau_{\mathtt{t}}(H_w\circ f)-(H_w\circ f)\big)\Big] \le N|\mathtt{t}|^\alpha$, if $|\mathtt{t}| \le \bar\delta$. Hence if $0 < \delta < \min\{\widetilde{\delta}, \bar\delta\}$, by (6),

$$A^{-1}\int_{|\mathtt{t}|\le\delta} V^\varphi\Big[2\lambda A\big(\tau_{\mathtt{t}}(H_w\circ f) - (H_w\circ f)\big)\Big]|L_w(\mathtt{t})|d\mathtt{t} \le A^{-1}N\int_{|\mathtt{t}|\le\delta}|\mathtt{t}|^\alpha|L_w(\mathtt{t})|d\mathtt{t} = O(w^{-\alpha}),$$

as $w \to +\infty$.

Moreover, since $f \in BV^\eta(\mathbb{R}^N)$, there exists $\gamma > 0$ such that $V^\eta[\gamma f] < +\infty$. Hence, by Proposition 3.3, there exists $\nu > 0$ such that $V^\varphi[\nu(H_w\circ f)] \le V^\eta[\gamma f]$, for every $w > 0$, and so, by (5), if $4\lambda A < \nu$,

$$A^{-1}V^\varphi[4\lambda A(H_w\circ f)]\int_{|\mathtt{t}|>\delta}|L_w(\mathtt{t})|\,d\mathtt{t} \le A^{-1}V^\eta[\gamma f]\int_{|\mathtt{t}|>\delta}|L_w(\mathtt{t})|\,d\mathtt{t} = O(w^{-\alpha}),$$

for sufficiently large $w > 0$. Finally, by Proposition 5.1, there exists $\xi > 0$ such that $V^\varphi[\xi(H_w\circ f - f)] = O(w^{-\alpha})$, as $w \to +\infty$. Hence the thesis follows taking $0 < \lambda < \min\{\frac{\mu}{2A}, \frac{\nu}{4A}, \frac{\xi}{2A}\}$. □

Remark 5.3. We point out that it is possible to furnish a generalization of Theorem 5.2 replacing the power function $|\mathtt{t}|^\alpha$ with a function $\tau(\mathtt{t})$, $\mathtt{t} \in \mathbb{R}^N$, where $\tau : \mathbb{R}^N \to \mathbb{R}^+_0$ is measurable, continuous at $\mathtt{0}$ and such that $\tau(\mathtt{t}) = 0$ iff $\mathtt{t} = \mathtt{0}$. Indeed, if we define the Lipschitz class as

$$V^\varphi Lip_{\mathbb{R}^N}(\tau) := \Big\{f \in AC^\varphi(\mathbb{R}^N) : \exists\mu > 0 \text{ s.t. } V^\varphi[\mu\Delta_{\mathtt{t}}(H_w\circ f)] = O(\tau(\mathtt{t})), \text{as } |\mathtt{t}| \to 0\Big\},$$

uniformly with respect to $w > 0$ and we replace (6) with

$$\int_{|\mathtt{t}|\le\widetilde{\delta}}|K_w(\mathtt{t})|\tau(\mathtt{t})d\mathtt{t} = O(w^{-\alpha}), \quad \text{as } w \to +\infty, \tag{6'}$$

for some $\widetilde{\delta} > 0$, following the proof of Theorem 5.2 one can prove that (7) holds.

The above result can be further generalized replacing $w^{-\alpha}$ with $\zeta(w^{-1})$, being $\zeta : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ a continuous function at 0, with $\zeta(u) > 0$ iff $u > 0$.

6. Further results

We now prove some regularity results for the family of integral operators (II). In particular, we first prove that the η-absolute continuity of the function f implies the φ-absolute continuity of the integral operators $T_w f$. Then we will prove that the same holds for f simply of η-bounded variation, provided that the kernel functions $\{L_w\}_{w>0}$ are φ-absolutely continuous.

Proposition 6.1. *If $f \in AC^\eta(\mathbb{R}^N)$, $K_w.1)$ and $(\star)$ are satisfied, and the triple (φ, η, ψ) is properly directed, then $T_w f \in AC^\varphi(\mathbb{R}^N)$, for every $w > 0$.*

Proof. Let $f \in AC^\eta(\mathbb{R}^N)$; then in particular $f \in AC^\eta_{loc}(\mathbb{R}^N)$. Let us fix $\varepsilon > 0$ and let $\delta > 0$ be the number of the absolute continuity of the j-section g_j of f, for every $j = 1, 2, \ldots, N$.

Let us fix an interval $I = \prod_{i=1}^N [a_i, b_i]$ and let $\{[\alpha_j^\mu, \beta_j^\mu]\}_{\mu=1,\ldots,\nu}$ be a collection of nonoverlapping intervals of $[a_j, b_j]$ such that $\sum_{\mu=1}^\nu \eta(\beta_j^\mu - \alpha_j^\mu) < \delta$. Then, since $f \in AC^\eta_{loc}(\mathbb{R}^N)$, there exists $\gamma > 0$ (independent by $\varepsilon > 0$) such that, for a.e. $s'_j \in [a'_j, b'_j]$,

$$\sum_{\mu=1}^{\nu} \eta\left(\gamma |f(s'_j - t'_j, \beta_j^\mu - t_j) - f(s'_j - t'_j, \alpha_j^\mu - t_j)|\right) < \varepsilon.$$

Now, with similar reasonings to Proposition 3.1, one can prove that, in correspondence to $\gamma > 0$, there exists $\lambda > 0$ such that

$$\begin{aligned}
&\sum_{\mu=1}^{\nu} \varphi\left(\lambda |(T_w f)(s'_j, \beta_j^\mu) - (T_w f)(s'_j, \alpha_j^\mu)|\right) \\
&\leq A^{-1} \int_{\mathbb{R}^N} |L_w(\mathtt{t})| \sum_{\mu=1}^{\nu} \eta\left(\gamma |f(s'_j - t'_j, \beta_j^\mu - t_j) - f(s'_j - t'_j, \alpha_j^\mu - t_j)|\right) d\mathtt{t} \\
&< \varepsilon A^{-1} \int_{\mathbb{R}^N} |L_w(\mathtt{t})|\, d\mathtt{t} \\
&\leq \varepsilon.
\end{aligned}$$

This proves that the j-section $(T_w f)(s'_j, \cdot)$ is φ-absolutely continuous for a.e. $s'_j \in [a'_j, b'_j]$, and so $T_w f \in AC^\varphi_{loc}(\mathbb{R}^N)$. Hence the assertion follows since, by Proposition 3.1, $T_w f \in BV^\varphi(\mathbb{R}^N)$. □

Proposition 6.2. *Let $f \in BV^\eta(\mathbb{R}^N)$, let $K_w.1)$ and $(\star)$ be satisfied and let (φ, η, ψ) be properly directed. If $\{L_w\}_{w>0} \subset AC^\varphi(\mathbb{R}^N)$ and $(\psi \circ |f|) \in L^1(\mathbb{R}^N)$, then $T_w f \in AC^\varphi(\mathbb{R}^N)$, for every $w > 0$.*

Proof. Without any loss of generality, we can take $\|\psi \circ |f|\|_1 > 0$, for every $w > 0$. For every $\mathtt{s} \in \mathbb{R}^N$ there holds

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}^N} K_w(\mathtt{t}, f(\mathtt{s} - \mathtt{t}))d\mathtt{t} = \int_{\mathbb{R}^N} L_w(\mathtt{s} - \mathtt{z})(H_w \circ f)(\mathtt{z})\, d\mathtt{z}.$$

For a fixed $\varepsilon > 0$, let $\delta > 0$ be the number of the φ-absolute continuity of the j-section of L_w, for $j = 1, \dots, N$. Let us fix an interval $I = \prod_{i=1}^N [a_i, b_i]$ and a collection of non-overlapping intervals $\{[\alpha_j^\mu, \beta_j^\mu]\}_{\mu=1,\dots,\nu}$ of $[a_j, b_j]$ such that $\sum_{\mu=1}^\nu \varphi(\beta_j^\mu - \alpha_j^\mu) < \delta$. Then, by ($\star$) and Jensen's inequality, recalling that $H_w(0) = 0$, there holds, for every $\lambda > 0$,

$$\begin{aligned}
&\sum_{\mu=1}^{\nu} \varphi\left(\lambda|(T_w f)(s_j', \beta_j^\mu) - (T_w f)(s_j', \alpha_j^\mu)|\right) \\
&= \sum_{\mu=1}^{\nu} \varphi\left(\lambda\left|\int_{\mathbb{R}^N} [L_w(s_j' - z_j', \beta_j^\mu - z_j) - L_w(s_j' - z_j', \alpha_j^\mu - z_j)](H_w \circ f)(\mathtt{z})d\mathtt{z}\right|\right) \\
&\leq \sum_{\mu=1}^{\nu} \varphi\left(\lambda K \int_{\mathbb{R}^N} \left|L_w(s_j' - z_j', \beta_j^\mu - z_j) - L_w(s_j' - z_j', \alpha_j^\mu - z_j)\right| \psi(|f(\mathtt{z})|)d\mathtt{z}\right) \\
&\leq \|\psi \circ |f|\|_1^{-1} \int_{\mathbb{R}^N} \psi(|f(\mathtt{z})|) \sum_{\mu=1}^{\nu} \varphi\Big(\lambda K \|\psi \circ |f|\|_1 \, |L_w(s_j' - z_j', \beta_j^\mu - z_j) \\
&\quad - L_w(s_j' - z_j', \alpha_j^\mu - z_j)|\Big) d\mathtt{z}.
\end{aligned}$$

Now, since $\sum_{\mu=1}^\nu \varphi(\beta_j^\mu - z_j - (\alpha_j^\mu - z_j)) < \delta$, and $L_w \in AC^\varphi_{loc}(\mathbb{R}^N)$, there exists $\mu > 0$ such that $\sum_{\mu=1}^\nu \varphi(\mu|L_w(s_j' - z_j', \beta_j^\mu - z_j) - L_w(s_j' - z_j', \alpha_j^\mu - z_j)|) < \varepsilon$. Then, if $0 < \lambda < \|\psi \circ |f|\|_1^{-1} K^{-1} \mu$,

$$\sum_{\mu=1}^{\nu} \varphi\Big(\lambda|(T_w f)(s_j', \beta_j^\mu) - (T_w f)(s_j', \alpha_j^\mu)|\Big) \leq \|\psi \circ |f|\|_1^{-1} \int_{\mathbb{R}^N} \varepsilon \psi(|f(\mathtt{z})|)\, d\mathtt{z} = \varepsilon.$$

This implies that $T_w f \in AC^\varphi_{loc}(\mathbb{R}^N)$ and hence the thesis follows recalling that $T_w f \in BV^\varphi(\mathbb{R}^N)$, by Proposition 3.1. □

Remark 6.3. (a) Proposition 6.2 proves that, if $f \in BV^\eta(\mathbb{R}^N)$, for AC^φ-kernels, the φ-absolute continuity of the function f is also necessary for the convergence in variation of $T_w f - f$. Indeed, if the kernels are φ-absolutely continuous, as it happens in the classical cases, then, by Proposition 6.2, so are the integral operators $T_w f$. Now in [4] it is proved that $AC^\varphi(\mathbb{R}^N)$ is a closed subspace of $BV^\varphi(\mathbb{R}^N)$ with respect to convergence in φ-variation, and so $f \in AC^\varphi(\mathbb{R}^N)$. Hence in this case the convergence in φ-variation of our integral operators holds if and only if f is φ-absolutely continuous.

(b) The above propositions are the natural generalization of the analogous results in the linear case ([4]) and in the one-dimensional case ([3]). For similar results in the case of the classical variation see [2].

We finally remark that all the theory can be extended to the case of $\mathcal{F}^\varphi$-variation, namely a concept of φ-variation, introduced in [4], filtered by a functional $\mathcal{F}$ which satisfies the following properties:

(i) $\mathcal{F}(\mathtt{p}) = 0$ if and only if $\mathtt{p} = \mathtt{0}$, $\mathtt{p} \in \mathbb{R}^N$;
(ii) $\mathcal{F}(\mathtt{p}+\mathtt{q}) \le \mathcal{F}(\mathtt{p}) + \mathcal{F}(\mathtt{q})$, $\mathtt{p}, \mathtt{q} \in \mathbb{R}^N$;
(iii) $\mathcal{F}(\alpha\mathtt{p}) = \alpha\mathcal{F}(\mathtt{p})$, $\alpha \in \mathbb{R}_0^+$, $\mathtt{p} \in \mathbb{R}^N$;
(iv) $\mathcal{F}(\mathtt{p}) \le C\|\mathtt{p}\|$, $\mathtt{p} \in \mathbb{R}^N$ (C is the Lipschitz constant of $\mathcal{F}$).

The $\mathcal{F}^\varphi$-variation of f over an interval $I \subset \mathbb{R}^N$ is defined as

$$V_{\mathcal{F}}^{\varphi}[f, I] := \sup \sum_{k=1}^{m} \mathcal{F}(\Phi^\varphi(f, J_k)),$$

where the supremum is taken over all the finite families of N-dimensional intervals $\{J_1, \ldots, J_m\}$ which form a partition of I, while the $\mathcal{F}^\varphi$-variation of f over $\mathbb{R}^N$ is defined as $V_{\mathcal{F}}^{\varphi}[f] := \sup_{I \subset \mathbb{R}^N} V_{\mathcal{F}}^{\varphi}[f, I]$, where the supremum is taken over all the intervals $I \subset \mathbb{R}^N$.

The $\mathcal{F}^\varphi$-variation is a generalization of the multidimensional φ- variation, which can be obtained taking $\mathcal{F}(\mathtt{p}) = \|\mathtt{p}\|$, $\mathtt{p} \in \mathbb{R}^N$. It is inspired by the concept of $\mathcal{F}$-variation, an extension of the classical variation, which is connected with several problems of Calculus of Variations (see e.g. [6, 12, 15, 31]).

Since, by the properties of $\mathcal{F}$, it is easy to prove (see [4]) that there exists $a > 0$ such that $aV^\varphi[f] \le V_{\mathcal{F}}^{\varphi}[f] \le CV^\varphi[f]$, then convergence in $\mathcal{F}^\varphi$-variation is equivalent to convergence in φ-variation and so all the previous theory about convergence and order of approximation for the family of operators (II) can be generalized to the frame of $\mathcal{F}^\varphi$-variation.

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