© European Mathematical Society

Very Weak Solutions to the Boundary-Value Problem of the Homogeneous Heat Equation

Bernard Nowakowski and Wojciech M. Zajączkowski

Abstract. We consider the homogeneous heat equation in a domain Ω in $\mathbb{R}^n$ with vanishing initial data and Dirichlet boundary conditions. We are looking for solutions in $W_{p,q}^{r,s}(\Omega \times (0,T))$, where $r < 2$, $s < 1$, $1 \leq p < \infty$, $1 \leq q \leq \infty$. Since we work in the $L_{p,q}$ framework any extension of the boundary data and integration by parts are not possible. Therefore, the solution is represented in integral form and is referred as *very weak* solution. The key estimates are performed in the half-space and are restricted to $L_q(0,T;W_p^\alpha(\Omega))$, $0 \leq \alpha < \frac{1}{p}$ and $L_q(0,T;W_p^\alpha(\Omega))$, $\alpha \leq 1$. Existence and estimates in the bounded domain Ω follow from a perturbation and a fixed point arguments.

Keywords. Very weak solutions, integral equation, non-regular solutions, anisotropic Sobolev-Slobodeckiĭ spaces, boundary-value problem, heat equation

Mathematics Subject Classification (2010). 35C15, 35K05, 35K20, 45D05

1. Introduction

We examine the following initial boundary-value problem

$$u_t - \triangle u = 0 \quad \text{in } \Omega \times (0,T) =: \Omega^T$$
$$u = \varphi \quad \text{on } S \times (0,T) =: S^T \tag{1.1}$$
$$u|_{t=0} = 0 \quad \text{in } \Omega \times \{0\},$$

where Ω is a bounded subset in $\mathbb{R}^n$ with the boundary S or the entire half-space $\mathbb{R}_+^n$. We are intensely interested in the problem of maximal regularity of solutions in dependence on the boundary data. The solvability and the maximal regularity of (1.1) has been studied by many authors under various requirements

B. Nowakowski: Institute of Mathematics of the Polish Academy of Sciences, Śniadeckich 8, 00-956 Warsaw, Poland; bernard@impan.pl

W. M. Zajączkowski: Institute of Mathematics of the Polish Academy of Sciences, Śniadeckich 8, 00-965 Warsaw, Poland, and Institute of Mathematics and Cryptology, Military University of Technology, Kaliskiego 2, 00-908 Warsaw, Poland; wz@impan.pl

on the boundary data. Let us briefly outline certain results which are the closest to the intended contribution of this work. For a full summary of the research on solvability of (1.1) we refer the reader to the introduction in [12].

The classic case, when $\varphi \in W_p^{2-\frac{1}{p},1-\frac{1}{2p}}(S^T)$, $p > 1$, was widely studied in [5, Chapter 4, §3, §4] (for a different approach see also [3, Theorem 4.2]) and then extended in [8]. Recently, an analogous result (see [2, Theorem 2.1]) was obtained for vector-valued parabolic initial-boundary value problem of general type.

For anisotropic boundary data it was shown in [10, Theorem 3.1] that the maximal regularity of solutions in $W_{p,q}^{2,1}(\Omega^T)$-space (see Definition 2.5) can be achieved only when $\varphi \in L_q(0,T;W_p^{2-\frac{1}{p}}(S)) \cap F_{q,p}^{1-\frac{1}{2p}}(0,T;L_p(S))$, where $\frac{3}{2} < p \le q < \infty$ and $F_{q,p}^\alpha(0,T;L_p(S))$ is a Lizorkin-Triebel space. Not much later, this result was improved for any p and q satisfying $1 < p,q < \infty$ in the case of general equations of parabolic type (see [2, Theorem 2.3]).

The aim of this paper is to prove the existence and uniqueness of such solutions to problem (1.1) that have the maximal regularity of $L_q(0,T;L_p(\Omega))$ or $L_q(0,T;W_p^1(\Omega))$, where $1 \le p < \infty$, $1 \le q \le \infty$. In standard approach we could try to incorporate the classical regularizer technique from [9] but it requires more regularity for the boundary data than we want to assume. Therefore, we need another approach which is based on the concept of very weak solutions. Also note that the case $W_{2,2}^{1,0}(\Omega^T) = L_2(0,T;H^1(\Omega))$ corresponds to the regularity of weak solutions but the energy estimate in this space cannot be obtained in the standard way. This only confirms that we need another definition of solution to problem (1.1).

Definition 1.1. We say that a function u is a *very weak solution* to the problem (1.1) if and only if it satisfies the following integral equation

$$u(x,t) = \int_0^t \int_S n_i(\xi) \cdot \frac{\partial \Gamma(x-\xi,t-\tau)}{\partial \xi_i} \mu(\xi,\tau) \, \mathrm{d}S_\xi \mathrm{d}\tau, \qquad (1.2)$$

where μ is an unknown function called the density of double layer, which depends on the boundary condition φ and it has to be calculated separately, n is the unit outward vector and Γ is the fundamental solution to the heat equation and is given by the formula

$$\Gamma(x,t) = \begin{cases} \frac{1}{\left(4\pi t\right)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t < 0. \end{cases}$$

As mentioned, the function μ is a priori unknown but it is a solution to the Fredholm integral equation of second order

$$\varphi(\eta,t) = \int_0^1 \int_S n_i(\xi) \frac{\partial \Gamma(\eta-\zeta,t-\tau)}{\partial \xi} \mu(\xi,\tau) \mathrm{d}S_\xi \, \mathrm{d}\tau - \frac{1}{2}\mu(\eta,t), \quad \eta \in S, \quad (1.3)$$

which we obtain from (1.2) after passing with $x \to \xi \in S$ and using the second equation of (1.1). By $\mathrm{d}S_\xi$, $\xi \in S$, we denote the measure of S.

For a deeper discussion of the above definition of the solution we refer the reader to [5, Chapter 4, §1].

To prove the existence of the very weak solutions we solve equation (1.3). Subsequently, to find the estimates on solutions first we consider the model problem

$$
\begin{aligned}
u_t - \triangle u &= 0 \quad \text{in } \mathbb{R}^n_+ \times (0, T) \\
u &= \varphi \quad \text{in } \mathbb{R}^{n-1} \times (0, T) \\
u|_{t=0} &= 0 \quad \text{in } \mathbb{R}^n_+ \times \{t = 0\},
\end{aligned}
\tag{1.4}
$$

to which the solution has the form

$$
\begin{aligned}
u(x, t) &= -2 \int_0^t \!\!\! \int_{\mathbb{R}^{n-1}} \frac{\partial \Gamma(x' - y', x_n, t - \tau)}{\partial x_n} \varphi(y', \tau) \, \mathrm{d}y' \, \mathrm{d}\tau \\
&= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^t \!\!\! \int_{\mathbb{R}^{n-1}} \frac{x_n}{(t - \tau)^{\frac{n+2}{2}}} e^{-\frac{|x' - y'|^2 + x_n^2}{4(t - \tau)}} \varphi(y', \tau) \, \mathrm{d}y' \, \mathrm{d}\tau,
\end{aligned}
$$

where $x' = (x_1, \ldots, x_{n-1})$ and $y' = (y_1, \ldots, y_{n-1})$ (for further details see Lemma 2.2) and derive necessary estimates. Next, we introduce a partition of unity with respect to Ω in (1.1) and use the estimates obtained for the half-space.

Now we can formulate three results of this paper:

Theorem 1.2. *Let $1 \le p < \infty$, $1 \le q \le \infty$ and $r, s \ge 0$ be fixed. Suppose that $\varphi \in W_{p,q}^{r,s}(S^T)$. Then there exists a unique solution $\mu \in W_{p,q}^{r,s}(S^T)$ of the integral equation (1.3) (see Definition 1.1) such that $\|\mu\|_{W_{p,q}^{r,s}(S^T)} \le c \, \|\varphi\|_{W_{p,q}^{r,s}(S^T)}$.*

Theorem 1.3. *Suppose that:*

(A) *$p \in [1, \infty)$, $q \in [1, \infty]$ and $\varphi \in L_q(0, T; L_p(\mathbb{R}^{n-1}))$. Then, the solution u belongs to $L_q(0, T; W_p^\alpha(\mathbb{R}^n_+))$ for every $0 \le \alpha < \frac{1}{p}$ and*

$$
\|u\|_{L_q(0,T;W_p^\alpha(\mathbb{R}^n_+))} \le c(\alpha, p, T) \, \|\varphi\|_{L_q(0,T;L_p(\mathbb{R}^{n-1}))}
$$

holds.

(B) *$p \in [1, \infty)$, $q \in [1, \infty]$ and $\varphi \in L_q(0, T; W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1}))$. Then $\mathrm{D}_{x'} u \in L_q(0, T; L_p(\mathbb{R}^n_+))$ and*

$$
\|\mathrm{D}_{x'} u\|_{L_q(0,T;L_p(\mathbb{R}^n_+))} \le c(n, p) \, \|\varphi\|_{L_q\left(0,T;W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1})\right)},
$$

where by $\mathrm{D}_{x'}$ we mean a first order partial derivative along tangent direction.

(C) $p \in [1, \infty)$, $q \in [1, \infty]$ and $\varphi \in W_{p,q}^{1-\frac{1}{p}, \frac{1}{2} - \frac{1}{2p}}(\mathbb{R}^{n-1} \times (0, T))$. Then $\partial_{x_n} u \in L_q(0, T; L_p(\mathbb{R}_+^n))$ and

$$\|\partial_{x_n} u\|_{L_q(0,T;L_p(\mathbb{R}_+^n))} \le c(n, p) \|\varphi\|_{W_{p,q}^{1-\frac{1}{p}, \frac{1}{2} - \frac{1}{2p}}(\mathbb{R}^{n-1} \times (0,T))}.$$

Theorem 1.4. *Suppose that $p \in [1, \infty)$, $q \in [1, \infty]$ and $\varphi \in W_{p,q}^{1-\frac{1}{p}, \frac{1}{2} - \frac{1}{2p}}(S^T)$, where $S \in \mathcal{C}^2$. Then $u \in L_q(0, T; W_p^1(\Omega))$ and*

$$\|u\|_{L_q(0,T;W_p^1(\Omega))} \le c(n, p, q, \Omega, T) \|\varphi\|_{L_q(0,T;L_p(S))} + c(n, p, q, \Omega) \|\varphi\|_{W_{p,q}^{1-\frac{1}{p}, \frac{1}{2} - \frac{1}{2p}}(S^T)}.$$

Remark 1.5. In the above theorems we can put $T = \infty$ except for part (**A**) in Theorem 1.3, where we would deal with non-convergent improper integrals with respect to t (see (4.1)–(4.4)). In Theorem 1.4 the constant c also depends on time, but this dependence can be eliminated (see [15, Lemma A.1]).

The reader has surely noticed that Theorem 1.2 concerns the existence of the density of the double layer whereas Theorems 1.3 and 1.4 provide suitable estimates in the half-space and in bounded domains of solutions in the form (1.2) for $\varphi \in W_{p,q}^{1-\frac{1}{p}, \frac{1}{2} - \frac{1}{2p}}(S^T)$. The difference between the existence of solutions and their estimates by the boundary data is particularly visible when we compare the function spaces used in all three theorems. Note that Theorem 1.2 covers a whole range of anisotropic Sobolev-Slobodeckiĭ spaces $W_{p,q}^{r,s}(\Omega^T)$, whereas the claims of Theorems 1.3 and 1.4 are only restricted to $L_q(0, T; W_p^1(\Omega))$ and $L_q(0, T; L_p(\Omega))$. The reason behind our choice follows from technical difficulties which appear when r and s are non-integers. This case will be covered in forthcoming paper.

The reader can also easily recognize that Theorem 1.4 contains less results than Theorem 1.3. The motivation is not to extend the paper and only to show ideas of the proof in the case of a bounded domain.

Theorem 1.4 plays a crucial role in proofs concerning the existence of global and regular solutions to the Navier-Stokes equations in cylindrical domains in $\mathbb{R}^3$. It ensures the estimates for the third component of the vorticity equation, which has clearly the structure of the heat equation with the boundary data either from $L_\infty(0, T; L_2(S))$ or from $H^{\frac{1}{2}, \frac{1}{4}}(S^T)$ (see e.g. [13, Lemma 4.1], [14, Appendix], [7, Lemma 3.1], [15, Lemma 4.3]). These estimates contribute significantly to the improvement in the regularity of weak solutions.

This paper is divided into five sections. In Sections 1 and 2 the reader can find the description of the problem and auxiliary results required to prove all three theorems. Section 3 is devoted to the existence of solutions to problem (1.1). It contains the proof Theorem 1.2. In Section 4 we present various estimates for solutions to problem (1.4) which are stated in Theorem 1.3 and in Section 5 we give the proof of Theorem 1.4.

2. Auxiliary results

In this section we collect helpful tools for further calculations and introduce the function spaces that will be used frequently in this paper.

Lemma 2.1. *Let $\Gamma(x,t)$ be the fundamental solution of the heat equation. Then*

$$\int_{\mathbb{R}^n} \partial_t^r D_x^s \Gamma(x,t)\,\mathrm{d}x = \begin{cases} 1 & r = s = 0 \\ 0 & r + s \geq 1, \end{cases}$$

where the symbol D_x^s denotes any derivative of order s with respect to x.

   Note that the integral above does not depend on time. The integration is carried out only with respect to spatial variables.

Lemma 2.2. *Any solution to the problem* (1.4) *in the half-space $x_n > 0$ has the form*

$$\begin{aligned}
u(x,t) &= -2 \int_0^t \!\!\int_{\mathbb{R}^{n-1}} \frac{\partial \Gamma(x' - y', x_n, t - \tau)}{\partial x_n} \, \varphi(y', \tau)\,\mathrm{d}y'\,\mathrm{d}\tau \\
&= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^t \!\!\int_{\mathbb{R}^{n-1}} \frac{x_n}{(t-\tau)^{\frac{n+2}{2}}} \, e^{-\frac{|x'-y'|^2 + x_n^2}{4(t-\tau)}} \, \varphi(y', \tau)\,\mathrm{d}y'\,\mathrm{d}\tau,
\end{aligned} \tag{2.1}$$

where $x' = (x_1, \ldots, x_{n-1})$ and $y' = (y_1, \ldots, y_{n-1})$.

   The proofs of Lemmas 2.1 and 2.2 can be found in [5, Chapter 4, §1].

Lemma 2.3 (General Minkowski inequality). *Let $1 \leq p \leq \infty$ and let f be a measurable function on $\mathbb{R}^n \times \mathbb{R}^m$. Then*

$$\left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^m} f(x,y)\,\mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \leq \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x,y)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}y.$$

   For the detailed proof of Lemma 2.3 we refer the reader to [1, Chapter 1, §2].

Lemma 2.4. *We have*

$$\sup_{s \geq 0} s^r e^{-s} = r^r e^{-r}.$$

Definition 2.5. We say that a function f belongs to the space $W_{p,q}^{r,s}(\Omega^T)$, where $\Omega \subseteq \mathbb{R}^n$, $p, q > 1$, $r, s \geq 0$, if and only if

$$
\begin{aligned}
\|f\|_{W_{p,q}^{r,s}(\Omega^T)} = {}& \sum_{0 \leq r' \leq [r]} \left(\int_0^T \left(\int_\Omega \left| D_x^{r'} f(x,t) \right|^p \, dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \\
& + \left(\int_0^T \left(\int_\Omega \int_\Omega \frac{\left| D_x^{[r]} f(x,t) - D_y^{[r]} f(y,t) \right|^p}{|x - y|^{n + p(r - [r])}} \, dx \, dy \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \\
& + \sum_{0 \leq s' \leq [s]} \left(\int_0^T \left(\int_\Omega \left| \partial_t^{s'} f(x,t) \right|^p \, dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \\
& + \left(\int_0^T \int_0^T \frac{\left(\int_\Omega \left| \partial_t^{[s]} f(x,t) - \partial_w^{[s]} f(x,w) \right|^p \, dx \right)^{\frac{q}{p}}}{|t - w|^{1 + q(s - [s])}} \, dt \, dw \right)^{\frac{1}{q}} \\
& < \infty.
\end{aligned}
$$

We say that a function φ belongs to the space $W_{p,q}^{r,s}(S^T)$, where $S := \partial\Omega$ is a compact manifold (provided Ω is bounded and open), if all functions

$$
(\varphi \beta_i) \circ \alpha_i^{-1} \colon \alpha_i(U_i) \to \mathbb{R}, \quad i \in I
$$

belong to $\overset{\circ}{W}_{p,q}^{r,s}(\alpha_i(U_i))$, which is understood as the closed hull of $\mathcal{D}(S)$ in $W_{p,q}^{r,s}(S^T)$, whereas (U_i, α_i) is an admissible $\mathcal{C}^r$-atlas for $S \in \mathcal{C}^r$ and β_i is a subordinate partition of unity. In this case $S \cap U_i$ is given by the equation $x_n = f_i(x_1, \ldots, x_{n-1})$ and the derivatives $D^{r'}$ and $D^{[r]}$ are taken with respect to the variables $x_1, \ldots, x_{n-1}$.

For more details, see [11, Chapter 1, §3 and §4].

Lemma 2.6. *Suppose that $\psi \in \mathcal{C}^r(S)$ and $\mu \in W_{p,q}^{r,s}(S^T)$. Then $\psi\mu \in W_{p,q}^{r,s}(S^T)$ and there exists a constant c such that*

$$
\|\psi\mu\|_{W_{p,q}^{r,s}(S^T)} \leq c \|\mu\|_{W_{p,q}^{r,s}(S^T)}
$$

which depends on r and ψ.

For proof of this Lemma, see [11, Chapter 1, §4, Proof of Proposition 4.5]; there are slight differences, but they are related to technical details.

3. The existence of solutions – proof of Theorem 1.2

In this section we prove the existence of solutions to problem (1.1). The proof is based on the method of successive approximations. The beginning point is formula (1.3), which we rewrite in the form

$$\mu(\xi, t) - \int_0^t\!\!\int_S N(\xi, t; \eta, \tau)\mu(\eta, \tau)\,\mathrm{d}S_\eta\,\mathrm{d}\tau = g(\xi, t), \tag{3.1}$$

where $g(\xi, t) = -2\varphi(\xi, t)$ and $N(\xi, t; \eta, \tau)$ is given by

$$N(\xi, t; \eta, \tau) = 2\frac{\partial}{\partial n_\eta}\Gamma(\xi - \eta, t - \tau) = \frac{|\xi - \eta|\cos(\xi - \eta, n_\eta)}{t - \tau}\Gamma(\xi - \eta, t - \tau).$$

Therefore (3.1) can be regarded as a product of a weakly singular kernel of Volterra type and of a weakly singular kernel. By n_η we understand the unit outward normal vector in point $\eta \in S$ and $\frac{\partial}{\partial n_\eta}$ denotes the normal derivative with respect to variable η.

Next we solve equation (3.1). Since the kernel $N(\xi, t; \eta, \tau)$ is unbounded we first iterate this equation so many times that the obtained iteration equation possesses a bounded kernel. Before we describe this procedure carefully, two natural questions may arise: 1. Do the solutions of the iterated and original equations coincide? 2. How do we know that the iterated equation has a bounded kernel? The answers to this questions are guaranteed by two lemmas that we state below:

Lemma 3.1. *There exists an integer number m_0 such that for any integer $m > m_0$ every solution to the m-times iterated integral equation of the given weakly singular equation is the solution of the original equation.*

Lemma 3.2. *If the singular kernel $K(x, t; y, \tau)$ has the form*

$$K(x, t; y, \tau) = \frac{k(x, t; y, \tau)}{(t - \tau)^\alpha},$$

where $k(x, t; y, \tau)$ is bounded and continuous function, then there always exists an integer number m_0 dependent on α such that for $m > m_0$ the iterated kernels $K_m(x, t; y, \tau)$ are bounded.

Both Lemmas are proved in [6, Chapter 3, §3] and in [6, Chapter 3, §2] respectively.

After the first iteration of equation (3.1) we obtain

$$\mu(\xi, t) - \int_0^t\!\!\int_S N_2(\xi, t; \eta, \tau)\mu(\eta, \tau)\,\mathrm{d}S_\eta\,\mathrm{d}\tau = g_2(\xi, t)$$

$$g_2(\xi, t) = g(\xi, t) - \int_0^t\!\!\int_S N_1(\xi, t; \eta, \tau)g(\eta, \tau)\,\mathrm{d}S_\eta\mathrm{d}\tau$$

and $N_1(\xi, t; \eta, \tau) = N(\xi, t; \eta, \tau)$,

$$N_2(\xi, t; \eta, \tau) = \int_0^t\!\!\!\int_S N_1(\xi, t; \alpha, s) N_1(\alpha, s; \eta, \tau) \, \mathrm{d}S_\alpha \, \mathrm{d}s.$$

After l iterations we get

$$\mu(\xi, t) - (-1)^l \int_0^t\!\!\!\int_S N_l(\xi, t; \eta, \tau)\mu(\eta, \tau) \, \mathrm{d}S_\eta \, \mathrm{d}\tau = g_l(\xi, t)$$

$$g_l(\xi, t) = g(\xi, t) - \int_0^t\!\!\!\int_S N_1(\xi, t; \eta, \tau) g_{l-1}(\eta, \tau) \, \mathrm{d}S_\eta \mathrm{d}\tau,$$

(3.2)

where

$$N_l(\xi, t; \eta, \tau) = \int_0^t\!\!\!\int_S N_1(\xi, t; \alpha, s) N_{l-1}(\alpha, s; \eta, \tau) \, \mathrm{d}S_\alpha \, \mathrm{d}s$$

and

$$N_l(\xi, t; \eta, \tau) = \frac{e_l(\xi, t; \eta, \tau)}{(t - \tau)^{\frac{n-l+1}{2}}} e^{-\frac{|\xi - \eta|^2}{4(t-\tau)}},$$

where $e_l(\xi, t; \eta, \tau)$ is a bounded and continuous function for $t \geq \tau$.

Now we see that if $l \geq n + 1$ then the iterated equation has a bounded kernel. Therefore we can apply the method of successive approximations. We finally have

Lemma 3.3. *Let us rewrite equation (3.2) in the form*

$$\mu(\xi, t) = g_l(\xi, t) + N_l\mu(\xi, t),$$

(3.3)

where

$$N_l\mu(\xi, t) = (-1)^l \int_0^t\!\!\!\int_S N_l(\xi, t; \eta, \tau)\mu(\eta, \tau) \, \mathrm{d}S_\eta \mathrm{d}\tau.$$

Assume that $g_l \in W_{p,q}^{r,s}(S^T)$. Then there exists a unique solution μ to the above equation such that $\mu \in W_{p,q}^{r,s}(S^T)$.

Proof. Let $\mu_0 = 0$ be the first approximation. Then we get the following sequence for μ_n:

$$\mu_1(\xi, t) = g_l(\xi, t) + N_l\mu_0(\xi, t) \quad = g_l(\xi, t)$$
$$\mu_2(\xi, t) = g_l(\xi, t) + N_l\mu_1(\xi, t) \quad = g_l(\xi, t) + N_l g_l(\xi, t)$$
$$\vdots$$
$$\mu_n(\xi, t) = g_l(\xi, t) + N_l\mu_{n-1}(\xi, t) = g_l(\xi, t) + N_l g_l(\xi, t)$$
$$+ N_l^2 g_l(\xi, t) + \cdots + N_l^{n-1} g_l(\xi, t)$$
$$\vdots$$

(3.4)

where the N_l^k iteration of N_l is given by the formula

$$N_l^k g_l(\xi, t) = N_l(N_l^{k-1} g_l)(\xi, t) = (-1)^l \int_0^t \!\!\int_S N_l(\xi, t; \eta, \tau) N_l^{k-1} g_l(\eta, \tau) \, \mathrm{d}S_\eta \, \mathrm{d}\tau$$

for $k = 2, 3, \ldots$ and

$$N_l^1 g_l(\xi, t) = N_l g_l(\xi, t) = (-1)^l \int_0^t \!\!\int_S N_l(\xi, t; \eta, \tau) g_l(\eta, \tau) \, \mathrm{d}S_\eta \, \mathrm{d}\tau.$$

We see that (3.4) builds a sequence of partial sums of the Neumann series

$$g_l(\xi, t) + N_l g_l(\xi, t) + N_l^2 g_l(\xi, t) + N_l^3 g_l(\xi, t) + \cdots + N_l^n g_l(\xi, t) + \cdots$$

We will show that the above series converges in the norm of the space $W_{p,q}^{r,s}(S^T)$ for $\xi \in S$ and $t \in (0, T)$, $T > 0$, by checking the Cauchy condition. Since the norm of the space $W_{p,q}^{r,s}(S^T)$ consists of four different terms (see Definition 2.5), each term needs to be treated separately.

Let $m > n$ and let us introduce the quantity $|N_l|$ by the formula

$$|N_l| := \sup_{\substack{\xi, \eta \in S, \\ 0 \le t, \tau \le T}} |N_l(\xi, t; \eta, \tau)| . \tag{3.5}$$

Considering the difference

$$\|\mu_m - \mu_n\|_{W_{p,q}^{r,s}(S)} \tag{3.6}$$

and using Lemma 3.4 below we can estimate the first term ($m = r'$, $n = 0$) by

$$\sum_{0 \le r' \le [r]} \left(\int_0^T \left(\int_S \left| \sum_{k=n+1}^m \mathrm{D}_\xi^{r'} N_l^k g(\xi, t) \right|^p \mathrm{d}S_\xi \right)^{\frac{q}{p}} \mathrm{d}t \right)^{\frac{1}{q}} \le \sum_{0 \le r' \le [r]} \sum_{k=n+1}^m |\Phi_{r'}| \, |N_l|^k \, |S|^k \, \frac{T^k}{k!}$$

and the third term ($m = 0$, $n = s'$) by

$$\sum_{0 \le s' \le [s]} \left(\int_0^T \left(\int_S \left| \sum_{k=n+1}^m \partial_t^{s'} N_l^k g(\xi, t) \right|^p \mathrm{d}S_\xi \right)^{\frac{q}{p}} \mathrm{d}t \right)^{\frac{1}{q}} \le \sum_{0 \le s' \le [s]} \sum_{k=n+1}^m |\Phi_{s'}| \, |N_l|^k \, |S|^k \, \frac{T^k}{k!}.$$

Next we estimate the second and the fourth term by applying Lemma 3.5 and Remark 3.6 below. Finally the difference (3.6) is estimated by

$$\sum_{k=n+1}^m \left(\sum_{0 \le r' \le [r]} |\Phi_{r'}| \, |N_l|^k \, |S|^k \, \frac{T^k}{k!} + \sum_{0 \le s' \le [s]} |\Phi_{s'}| \, |N_l|^k \, |S|^k \, \frac{T^k}{k!} \right.$$

$$\left. + |\Psi| \, |N_l|^k \, |S|^k \, \frac{T^k}{k!} + 2 \, |\Theta| \, |N_l|^k \, |S|^k \, \frac{T^k}{k!} \right)$$

$$= \sum_{k=n+1}^m \left(\sum_{0 \le r' \le [r]} |\Phi_{r'}| + \sum_{0 \le s' \le [s]} |\Phi_{s'}| + |\Psi| + |\Theta| \right) |N_l|^k \, |S|^k \, \frac{T^k}{k!} ,$$

which can be an arbitrary small number if only n, m are large enough (for definition of $|\Psi|$ and $|\Theta|$ see (3.8) and (3.10) respectively). Therefore the Cauchy condition for the Neumann series is satisfied. The space $W_{p,q}^{r,s}(S^T)$ is complete, hence there exists a limit μ. To end the proof we must only check that μ solves the equation (3.3) and that μ is unique.

Indeed, μ solves (3.3). Consider the equality which defines the successive approximations:

$$\mu_n = g_l + N_l \mu_{n-1}$$

and let $h = g_l + N_l \mu$. Subtracting the second equation from the first and applying the norm of the space $W_{p,q}^{r,s}(S^T)$ yields

$$\|\mu_n - h\|_{W_{p,q}^{r,s}(S^T)} \leq \|N_l\|_{L_\infty(S^T)} \|\mu_{n-1} - \mu\|_{W_{p,q}^{r,s}(S^T)}.$$

Since $\lim_{n\to\infty} \mu_n = \mu$ a.e. in the space $W_{p,q}^{r,s}(S^T)$, so $h = \mu$ a.e. and in fact μ solves the equation (3.3).

To prove the uniqueness, suppose that $\bar{\mu}$ is another solution of the equation (3.3). Let $\psi = \mu - \bar{\mu}$. Then ψ satisfies the following homogeneous equation

$$\psi = N_l \psi.$$

After $n-1$ iterations we get $\psi = N_l^n \psi$. Taking the norm of the space $W_{p,q}^{r,s}(S^T)$ and using the estimates from Lemmas 3.4, 3.5 and Remark 3.6 we get

$$\|\psi\|_{W_{p,q}^{r,s}(S^T)} \leq \|\psi\|_{W_{p,q}^{r,s}(S^T)} \left(\sum_{0 \leq r' \leq [r]} |\Phi_{r'}| + \sum_{0 \leq s' \leq [s]} |\Phi_{s'}| + |\Psi| + |\Theta| \right) |N_l|^k |S|^k \frac{T^k}{k!}.$$

Since $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for $a \in \mathbb{R}$ the right-hand side tends to zero as $k \to \infty$. Hence $\|\psi\|_{W_{p,q}^{r,s}(S^T)}$ must be zero and this implies that the solution μ to the equation (3.3) is unique. This concludes the proof. $\qquad \square$

Below we demonstrate various estimates we used in the above proof.

Lemma 3.4. *Let $|N_l|$ be defined as in (3.5) and let*

$$|\Phi_{m,n}| := \left\| D_\eta^m \partial_\tau^n g_l \right\|_{L_q(0,T;L_p(S))},$$

where $m, n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Then for any $k = 1, 2, \ldots, \; 0 < \bar{t} \leq T$

$$\left(\int_0^{\bar{t}} \left(\int_S \left| D_\xi^m \partial_t^n N_l^k g_l(\xi, t) \right|^p dS_\xi \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \leq |\Phi_{m,n}| \, |N_l|^k \, |S|^k \, \frac{\bar{t}^k}{k!}. \qquad (3.7)$$

Proof. Let $k = 1$. Then

$$\left(\int_0^{\bar{t}} \left(\int_S \left| D_\xi^m \partial_t^n N_l g_l(\xi, t) \right|^p dS_\xi \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}$$

$$= \left(\int_0^{\bar{t}} \left(\int_S \left| \int_0^t \int_S D_\xi^m \partial_t^n N_l(\xi - \eta, t - \tau) g_l(\eta, \tau) \, dS_\eta \, d\tau \right|^p dS_\xi \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}$$

$$= \left(\int_0^{\bar{t}} \left(\int_S \left| \sum_\alpha \int_0^t \int_{S^\alpha} N_l(\eta, \tau) D_\xi^m \partial_t^n \left(\psi^\alpha(\xi - \eta) g_l(\xi - \eta, t - \tau) \right) dS_\eta \, d\tau \right|^p dS_\xi \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}$$

$$\leq \left\| D_\xi^m \partial_t^n g_l \right\|_{L_q(0, T; L_p(S))} \sup_{\eta \in S, 0 \leq \tau \leq T} |N_l(\eta, \tau)| \, |S| \, \bar{t}$$

$$\leq |\Phi_{m,n}| \, |N_l| \, |S| \frac{\bar{t}^1}{1!}.$$

In the last equality we applied the partition of unity, $\sum_\alpha \psi^\alpha$ on S in order to integrate by parts. In the last inequality we used the general Minkowski inequality (Lemma 2.3) and Lemma 2.6.

Suppose now that for a given k (3.7) holds. We will show that it is valid for $k + 1$. We have

$$\left(\int_0^{\bar{t}} \left(\int_S \left| D_\xi^m \partial_t^n N_l^{k+1} g(\xi, t) \right|^p dS_\xi \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}$$

$$= \left(\int_0^{\bar{t}} \left(\int_S \left| \int_0^t \int_S D_\xi^m \partial_t^n N_l(\xi - \eta, t - \tau) N_l^k g_l(\eta, \tau) \, dS_\eta \, d\tau \right|^p dS_\xi \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}$$

which, after introducing a partition of unity $\sum_\alpha \psi^\alpha$ on S, is equal to

$$\left(\int_0^{\bar{t}} \left(\int_S \left| \sum_\alpha \int_0^t \int_S N_l(\eta, \tau) D_\xi^m \partial_t^n \left(\psi^\alpha(\eta) N_l^k g_l(\xi - \eta, t - \tau) \right) dS_\eta \, d\tau \right|^p dS_\xi \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}$$

$$\leq c \int_0^{\bar{t}} \int_S N_l(\eta, \tau) \left(\int_0^{\bar{t}} \left(\int_S \left| D_\xi^m \partial_t^n N_l^k g_l(\xi - \eta, t - \tau) \right|^p dS_\xi \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} dS_\eta \, d\tau$$

$$\leq |N_l| \int_0^{\bar{t}} \int_S |\Phi_{m,n}| \, |N_l|^k \, |S|^k \frac{(\bar{t} - \tau)^k}{k!} \, dS_\eta \, d\tau$$

$$= |\Phi_{m,n}| \, |N_l|^{k+1} \, |S|^{k+1} \frac{\bar{t}^{k+1}}{(k+1)!}.$$

The first inequality above is due to the general Minkowski inequality and Lemma 2.6. The last equality ends the proof. $\qquad \square$

Lemma 3.5. *Let* $|N_l|$ *be defined as in* (3.5) *and let*

$$|\Psi| := \left(\int_0^T\left(\int_S\int_S\frac{\left|D_\xi^{[r]}g_l(\xi,t)-D_{\xi'}^{[r]}g_l(\xi',t)\right|^p}{|\xi-\xi'|^{n+p(r-[r])}}\,dS_\xi\,dS_{\xi'}\right)^{\frac{q}{p}}dt\right)^{\frac{1}{q}} + \left|\Phi_{[r],0}\right|, \qquad (3.8)$$

where $r \in \mathbb{R}^+$. *Then for any* $k = 1, 2, \ldots$ *we have*

$$\left(\int_0^T\left(\int_S\int_S\frac{\left|D_\xi^{[r]}N_l^kg_l(\xi,t)-D_{\xi'}^{[r]}N_l^kg_l(\xi',t)\right|^p}{|\xi-\xi'|^{n+p(r-[r])}}\,dS_\xi\,dS_{\xi'}\right)^{\frac{q}{p}}dt\right)^{\frac{1}{q}} \leq |\Psi||N_l|^k|S|^k\frac{t^k}{k!}. \qquad (3.9)$$

Proof. Let $k = 1$. Then

$$\left(\int_0^T\left(\int_S\int_S\frac{\left|D_\xi^{[r]}N_lg_l(\xi,t) - D_{\xi'}^{[r]}N_lg_l(\xi',t)\right|^p}{|\xi-\xi'|^{n+p(r-[r])}}\,dS_\xi\,dS_{\xi'}\right)^{\frac{q}{p}}dt\right)^{\frac{1}{q}}$$

$$= \left(\int_0^T\left(\int_S\int_S\left|\int_0^t\int_S D_\xi^{[r]}N_l(\xi-\eta,t-\tau)g_l(\eta,\tau)\,dS_\eta\,d\tau\right.\right.\right.$$

$$\left.\left.\left.-\int_0^t\int_S D_{\xi'}^{[r]}N_l(\xi'-\eta,t-\tau)g_l(\eta,\tau)\,dS_\eta\,d\tau\right|^p\frac{1}{|\xi-\xi'|^{n+p(r-[r])}}\,dS_\xi\,dS_{\xi'}\right)^{\frac{q}{p}}dt\right)^{\frac{1}{q}}.$$

Next we introduce a partition of unity $\sum_\alpha\psi^\alpha$ on S

$$\left(\int_0^T\left(\int_S\int_S\left|\sum_\alpha\int_0^t\int_S N_l(\xi-\eta,t-\tau)D_\eta^{[r]}\big(\psi^\alpha(\eta)g_l(\eta,\tau)\big)\,dS_\eta\,d\tau\right.\right.\right.$$

$$\left.\left.-\sum_\alpha\int_0^t\int_S N_l(\xi'-\eta,t-\tau)D_\eta^{[r]}\big(\psi^\alpha(\eta)g_l(\eta,\tau)\big)\,dS_\eta\,d\tau\right|^p$$

$$\left.\cdot\frac{1}{|\xi-\xi'|^{n+p(r-[r])}}\,dS_\xi\,dS_{\xi'}\right)^{\frac{q}{p}}dt\right)^{\frac{1}{q}}$$

and integrate by parts

$$\left(\int_0^T\left(\int_S\int_S\left|\sum_\alpha\int_0^t\int_S N_l(\eta,\tau)D_\xi^{[r]}\big(\psi^\alpha(\xi-\eta)g_l(\xi-\eta,t-\tau)\big)\,dS_\eta\,d\tau\right.\right.\right.$$

$$\left.\left.-\sum_\alpha\int_0^t\int_S N_l(\eta,\tau)D_{\xi'}^{[r]}\big(\psi^\alpha(\xi'-\eta)g_l(\xi'-\eta,t-\tau)\big)\,dS_\eta\,d\tau\right|^p$$

$$\left.\cdot\frac{1}{|\xi-\xi'|^{n+p(r-[r])}}\,dS_\xi\,dS_{\xi'}\right)^{\frac{q}{p}}dt\right)^{\frac{1}{q}}.$$

Applying now the general Minkowski inequality and Lemma 2.6 yields

$$\left(\int_0^T\left(\int_S\int_S \frac{\left|D_\xi^{[r]}N_lg_l(\xi,t)-D_{\xi'}^{[r]}N_lg_l(\xi',t)\right|^p}{|\xi-\xi'|^{n+p(r-[r])}}\,\mathrm{d}S_\xi\,\mathrm{d}S_{\xi'}\right)^{\frac{q}{p}}\mathrm{d}t\right)^{\frac{1}{q}}\leq|N_l|\,|\Psi|\,|S|\,t.$$

Let us now assume that for k given (3.9) holds. We will show that it is true also for $k+1$. We see

$$\left(\int_0^T\left(\int_S\int_S \frac{\left|D_\xi^{[r]}N_l^{k+1}g_l(\xi,t)-D_{\xi'}^{[r]}N_l^{k+1}g_l(\xi',t)\right|^p}{|\xi-\xi'|^{n+p(r-[r])}}\,\mathrm{d}S_\xi\,\mathrm{d}S_{\xi'}\right)^{\frac{q}{p}}\mathrm{d}t\right)^{\frac{1}{q}}$$

$$=\left(\int_0^T\left(\int_S\int_S\left|\int_0^t\int_S D_\xi^{[r]}N_l(\xi-\eta,t-\tau)N_l^k g_l(\eta,\tau)\,\mathrm{d}S_\eta\,\mathrm{d}\tau\right.\right.\right.$$

$$\left.\left.\left.-\int_0^t\int_S D_{\xi'}^{[r]}N_l(\xi'-\eta,t-\tau)N_l^k g_l(\eta,\tau)\,\mathrm{d}S_\eta\,\mathrm{d}\tau\right|^p\frac{1}{|\xi-\xi'|^{n+p(r-[r])}}\,\mathrm{d}S_\xi\,\mathrm{d}S_{\xi'}\right)^{\frac{q}{p}}\mathrm{d}t\right)^{\frac{1}{q}}.$$

Applying a partition of unity $\sum_\alpha\psi^\alpha$, integrating by parts and using the general Minkowski inequality and Lemma 2.6, as we did in case $k=1$, we conclude the proof. $\qquad\square$

Remark 3.6. Let $|N_l|$ be defined as in (3.5) and let

$$|\Theta|:=\left(\int_0^T\int_0^T\frac{\left(\int_S\left|\partial_t^{[s]}g_l(\xi,t)-\partial_w^{[s]}g_l(\xi,w)\right|^p\,\mathrm{d}x\right)^{\frac{q}{p}}}{|t-w|^{1+q(s-[s])}}\,\mathrm{d}t\,\mathrm{d}w\right)^{\frac{1}{q}}+\left|\Phi_{0,[s]}\right|.\qquad(3.10)$$

Then repeating the proof of the last Lemma we get

$$\left(\int_0^T\int_0^T\left(\int_S\frac{\left|D_t^{[s]}N_l^kg_l(\xi,t)-D_w^{[s]}N_l^kg_l(\xi,w)\right|^p}{|t-w|^{1+q(s-[s])}}\,\mathrm{d}S_\xi\right)^{\frac{q}{p}}\mathrm{d}t\,\mathrm{d}w\right)^{\frac{1}{q}}\leq|\Theta||N_l|^k|S|^k\frac{T^k}{k!}.$$

It remains to check that $\|g_l\|_{W_{p,q}^{r,s}(S^T)}\leq c\,\|\varphi\|_{W_{p,q}^{r,s}(S^T)}$. This can be easily seen if we consider the second equation in (3.2), apply any partition of unity on S and repeat the calculations from this section for each of four terms of the norm $W_{p,q}^{r,s}(S^T)$.

4. Estimates in the half space

In this section we estimate the solution u to the problem (1.4) in the half space.

Proof of Theorem 1.3. The proof is divided into four lemmas. Point (**A**) is proved in Lemma 4.1 for $\alpha = 0$ and in Lemma 4.2 for $0 < \alpha < \frac{1}{p}$. Point (**B**) is proved in Lemma 4.3 and point (**C**) in Lemma 4.4. $\square$

Lemma 4.1. *Let us assume that $\varphi \in L_q(0, T; L_p(\mathbb{R}^{n-1}))$, where $1 \leq p < \infty$, $1 \leq q \leq \infty$. If u is a solution to problem* (1.4), *then $u \in L_q(0, T; L_p(\mathbb{R}^n_+))$ and*

$$\|u\|_{L_q(0,T;L_p(\mathbb{R}^n_+))} \leq c(p,T) \|\varphi\|_{L_q(0,T;L_p(\mathbb{R}^{n-1}))},$$

where the constant c depends on p and T.

Proof. From (2.1) it follows that

$$|u(x,t)| \leq \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^t \int_{\mathbb{R}^{n-1}} \frac{x_n}{\tau^{\frac{n+2}{2}}} e^{-\frac{|y'|^2+x_n^2}{4\tau}} |\varphi(x'-y', t-\tau)| \, \mathrm{d}y' \, \mathrm{d}\tau.$$

Taking the $L_q\big(0, T; L_p(\mathbb{R}^n_+)\big)$-norm yields

$\|u\|_{L_q(0,T;L_p(\mathbb{R}^n_+))}$

$$\leq \frac{1}{(4\pi)^{\frac{n}{2}}} \left(\int_0^T \left(\int_{\mathbb{R}^n_+} \left| \int_0^t \int_{\mathbb{R}^{n-1}} \frac{x_n}{\tau^{\frac{n+2}{2}}} e^{-\frac{|y'|^2+x_n^2}{4\tau}} |\varphi(x'-y', t-\tau)| \, \mathrm{d}y' \, \mathrm{d}\tau \right|^p \mathrm{d}x \right)^{\frac{q}{p}} \mathrm{d}t \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{(4\pi)^{\frac{n}{2}}} \left(\int_{\mathbb{R}_+} \left| \int_0^T \int_{\mathbb{R}^{n-1}} \frac{x_n}{\tau^{\frac{n+2}{2}}} e^{-\frac{|y'|^2+x_n^2}{4\tau}} \, \mathrm{d}y' \, \mathrm{d}\tau \right|^p \mathrm{d}x_n \right)^{\frac{1}{p}} \|\varphi\|_{L_q(0,T;L_p(\mathbb{R}^{n-1}))},$$

where in the last inequality we used the general Minkowski inequality (Lemma 2.3) with respect to x'. The integral with respect to y' is equal to $(4\pi)^{\frac{n-1}{2}} \tau^{\frac{n-1}{2}}$, so using the general Minkowski inequality with respect to x_n gives

$$\|u\|_{L_q(0,T;L_p(\mathbb{R}^n_+))} \leq \frac{1}{\sqrt{4\pi}} \int_0^T \frac{1}{\tau^{\frac{3}{2}}} \left(\int_0^\infty x_n^p e^{-\frac{px_n^2}{4\tau}} \, \mathrm{d}x_n \right)^{\frac{1}{p}} \mathrm{d}\tau \, \|\varphi\|_{L_q(0,T;L_p(\mathbb{R}^{n-1}))}$$

$$= \frac{(\Gamma(\frac{1+p}{2}))^{\frac{1}{p}}}{\sqrt{\pi} p^{\frac{1}{2}+\frac{1}{2p}}} \int_0^T \frac{\tau^{\frac{1}{2p}+\frac{1}{2}}}{\tau^{\frac{3}{2}}} \, \mathrm{d}\tau \, \|\varphi\|_{L_q(0,T;L_p(\mathbb{R}^{n-1}))} \tag{4.1}$$

$$\leq c(p,T) \|\varphi\|_{L_q(0,T;L_p(\mathbb{R}^{n-1}))}$$

for any $1 \leq p < \infty$. This concludes the proof. $\square$

Lemma 4.2. *Suppose that $\varphi \in L_q(0, T; L_p(\mathbb{R}^{n-1}))$, where $1 \leq p < \infty$, $1 \leq q \leq \infty$. If u is a solution to problem* (1.4), *then $u \in L_q(0, T; W_p^\alpha(\mathbb{R}^n_+))$ and*

$$\|u\|_{L_q(0,T;W_p^\alpha(\mathbb{R}^n_+))} \leq c(\alpha, p, T) \|\varphi\|_{L_q(0,T;L_p(\mathbb{R}^{n-1}))},$$

where the constant c depends on p and T and α is such that $0 < \alpha < \frac{1}{p}$.

Proof. We write

$$u(x,t) - u(z,t) = u(x', x_n, t) - u(x', z_n, t) + u(x', z_n, t) - u(z', z_n, t) =: I_1 + I_2.$$

Let us first consider I_1. From Lemma 2.2 it follows that

$$\frac{u(x', x_n, t) - u(x', z_n, t)}{|x - z|^{\frac{n}{p}+\alpha}}$$

$$= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^t \int_{\mathbb{R}^{n-1}} \frac{\left(\frac{x_n}{\tau^{\frac{n+2}{2}}} e^{-\frac{|y'|^2 + x_n^2}{4\tau}} - \frac{z_n}{\tau^{\frac{n+2}{2}}} e^{-\frac{|y'|^2 + z_n^2}{4\tau}} \right) \varphi(x' - y', t - \tau)}{|x - z|^{\frac{n}{p}+\alpha}} \, \mathrm{d}y' \, \mathrm{d}\tau.$$

Taking the L_p-norm with respect to x' and z' and applying the general Minkowski inequality yields

$$\left(\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|u(x', x_n, t) - u(x', z_n, t)|^p}{|x - z|^{n+\alpha p}} \, \mathrm{d}x' \, \mathrm{d}z' \right)^{\frac{1}{p}}$$

$$\leq \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^t \int_{\mathbb{R}^{n-1}} \frac{e^{-\frac{|y'|^2}{4\tau}} \left| \frac{x_n}{\tau^{\frac{n+2}{2}}} e^{-\frac{x_n^2}{4\tau}} - \frac{z_n}{\tau^{\frac{n+2}{2}}} e^{-\frac{z_n^2}{4\tau}} \right| \|\varphi(t - \tau)\|_{L_p(\mathbb{R}^{n-1})}}{|x_n - z_n|^{\frac{1}{p}+\alpha}} \, \mathrm{d}y' \, \mathrm{d}\tau$$

$$= \frac{1}{(4\pi)^{\frac{1}{2}}} \int_0^t \frac{\left| \frac{x_n}{\tau^{\frac{3}{2}}} e^{-\frac{x_n^2}{4\tau}} - \frac{z_n}{\tau^{\frac{3}{2}}} e^{-\frac{z_n^2}{4\tau}} \right|}{|x_n - z_n|^{\frac{1}{p}+\alpha}} \|\varphi(t - \tau)\|_{L_p(\mathbb{R}^{n-1})} \, \mathrm{d}\tau.$$

Next we take the L_p-norm with respect to x_n and z_n and apply the general Minkowski inequality

$$\left(\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \frac{|u(x', x_n, t) - u(x', z_n, t)|^p}{|x - z|^{n+\alpha p}} \, \mathrm{d}x \, \mathrm{d}z \right)^{\frac{1}{p}}$$

$$\leq \frac{1}{(4\pi)^{\frac{1}{2}}} \int_0^t \left(\int_0^\infty \int_0^\infty \frac{\left| \frac{x_n}{\tau^{\frac{3}{2}}} e^{-\frac{x_n^2}{4\tau}} - \frac{z_n}{\tau^{\frac{3}{2}}} e^{-\frac{z_n^2}{4\tau}} \right|^p}{|x_n - z_n|^{1+\alpha p}} \, \mathrm{d}x_n \, \mathrm{d}z_n \right)^{\frac{1}{p}} \|\varphi(t - \tau)\|_{L_p(\mathbb{R}^{n-1})} \, \mathrm{d}\tau$$

$$= \frac{1}{(4\pi)^{\frac{1}{2}}} \int_0^t \frac{2^{\frac{1}{p}}}{\tau^{1+\frac{\alpha}{2} - \frac{1}{2p}}} \left(\int_0^\infty \int_0^\infty \frac{\left| \bar{x}_n e^{-\bar{x}_n^2} - \bar{z}_n e^{-\bar{z}_n^2} \right|^p}{|\bar{x}_n - \bar{z}_n|^{1+\alpha p}} \, \mathrm{d}\bar{x}_n \, \mathrm{d}\bar{z}_n \right)^{\frac{1}{p}} \|\varphi(t - \tau)\|_{L_p(\mathbb{R}^{n-1})} \, \mathrm{d}\tau$$

$$= c(p) \int_0^t \frac{1}{\tau^{1+\frac{\alpha}{2} - \frac{1}{2p}}} \|\varphi(t - \tau)\|_{L_p(\mathbb{R}^{n-1})} \, \mathrm{d}\tau.$$

Finally we take the L_q-norm and apply the general Minkowski inequality. This shows that

$$\|I_1\|_{L_q(0,T;W_p^\alpha(\mathbb{R}_+^n))} \leq c(p) \int_0^T \frac{1}{\tau^{1+\frac{\alpha}{2}-\frac{1}{2p}}}\, d\tau\, \|\varphi\|_{L_q(0,T;L_p(\mathbb{R}^{n-1}))} \tag{4.2}$$
$$\leq c(p,T)\, \|\varphi\|_{L_q(0,T;L_p(\mathbb{R}^{n-1}))}$$

if and only if $\alpha < \frac{1}{p}$.

Next, let us consider I_2. Dividing by $|x-z|^{\frac{n}{p}+\alpha}$ and applying the Hölder inequality with respect to y' leads to

$$\frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^t \frac{z_n}{\tau^{\frac{n+2}{2}}} e^{-\frac{z_n^2}{4\tau}} \underbrace{\left(\int_{\mathbb{R}^{n-1}} \left| e^{-\frac{|x'-y'|^2}{4\tau}} - e^{-\frac{|z'-y'|^2}{4\tau}} \right|^{\frac{p'}{2}} dy' \right)^{\frac{1}{p'}}}_{J_1}$$
$$\cdot \left(\int_{\mathbb{R}^{n-1}} \frac{\left| e^{-\frac{|x'-y'|^2}{4\tau}} - e^{-\frac{|z'-y'|^2}{4\tau}} \right|^{\frac{p}{2}}}{|x-z|^{n+\alpha p}} |\varphi(y',t-\tau)|^p\, dy' \right)^{\frac{1}{p}} d\tau. \tag{4.3}$$

Let us calculate J_1. From the Mean Value Theorem we have that, for some $\theta \in (0,1)$

$$J_1 = \left(\int_{\mathbb{R}^{n-1}} \left| \frac{|x'-z'|}{2\tau} e^{-\frac{|x'-(y'+\theta(x'-z'))|^2}{4\tau}} |x'-(y'+\theta(x'-z'))| \right|^{\frac{p'}{2}} dy' \right)^{\frac{1}{p'}}$$
$$\leq c(n,p) \frac{\tau^{\frac{n-1}{2p'}}}{\tau^{\frac{1}{4}}} |x'-z'|^{\frac{1}{2}}.$$

Taking the L_p-norm of (4.3) with respect to x and z and using the above estimate for J_1 yields

$$\left(\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \left| \int_0^t \frac{z_n}{\tau^{\frac{n+2}{2}}} e^{-\frac{z_n^2}{4\tau}} \frac{\tau^{\frac{n-1}{2p'}}}{\tau^{\frac{1}{4}}} |x'-z'|^{\frac{1}{2}} \right. \right.$$
$$\left. \left. \cdot \left(\int_{\mathbb{R}^{n-1}} \frac{\left| e^{-\frac{|x'-y'|^2}{4\tau}} - e^{-\frac{|z'-y'|^2}{4\tau}} \right|^{\frac{p}{2}}}{|x-z|^{n+\alpha p}} |\varphi(y',t-\tau)|^p\, dy' \right)^{\frac{1}{p}} d\tau \right|^p dx\, dz \right)^{\frac{1}{p}},$$

which in view of Lemma 2.4, the general Minkowski inequality and after integration with respect to x_n is estimated by

$$c(n,p) \int_0^t \frac{1}{\tau^{\frac{n+1}{2}}} \frac{\tau^{\frac{n-1}{2p'}}}{\tau^{\frac{1}{4}}}$$

$$\cdot \left(\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}_+^n} \int_{\mathbb{R}^{n-1}} e^{-\frac{z_n^2 p}{4\tau}} \frac{\left| e^{-\frac{|x'-y'|^2}{4\tau}} - e^{-\frac{|z'-y'|^2}{4\tau}} \right|^{\frac{p}{2}}}{|x'-z'|^{n-1+p(\alpha-\frac{1}{2})}} |\varphi(y',t-\tau)|^p \, \mathrm{d}x' \, \mathrm{d}z \, \mathrm{d}y' \right)^{\frac{1}{p}} \mathrm{d}\tau.$$

Changing the variables

$$\frac{x'-y'}{2\sqrt{\tau}} = \bar{x}', \quad \mathrm{d}x' = 2^{n-1}\tau^{\frac{n-1}{2}} \, \mathrm{d}\bar{x}', \quad \frac{z'-y'}{2\sqrt{\tau}} = \bar{z}', \quad \mathrm{d}z' = 2^{n-1}\tau^{\frac{n-1}{2}} \, \mathrm{d}\bar{z}'$$

gives

$$c(n,p) \int_0^t \frac{1}{\tau^{\frac{n+1}{2}}} \frac{\tau^{\frac{n-1}{2p'}}}{\tau^{\frac{1}{4}}} \frac{\tau^{\frac{n-1}{p}}}{\tau^{\frac{1}{2p}(n-1+p(\alpha-\frac{1}{2}))}}$$

$$\cdot \left(\int_{\mathbb{R}_+^n} \int_{\mathbb{R}^{n-1}} e^{-\frac{z_n^2 p}{4\tau}} \frac{\left| e^{-|\bar{x}'|^2} - e^{|\bar{z}'|^2} \right|^{\frac{p}{2}}}{|\bar{x}'-\bar{z}'|^{n-1+p(\alpha-\frac{1}{2})}} \, \mathrm{d}x' \, \mathrm{d}z \right)^{\frac{1}{p}} \|\varphi(t-\tau)\|_{L_p(\mathbb{R}^{n-1})} \, \mathrm{d}\tau$$

$$= c(n,p) \int_0^t \frac{\tau^{\frac{1}{2p}} \tau^{\frac{n-1}{2p'}}}{\tau^{\frac{n+1}{2}} \tau^{\frac{1}{4}}} \frac{\tau^{\frac{n-1}{p}}}{\tau^{\frac{1}{2p}(n-1+p(\alpha-\frac{1}{2}))}} \|\varphi(t-\tau)\|_{L_p(\mathbb{R}^{n-1})} \, \mathrm{d}\tau$$

$$= c(n,p) \int_0^t \frac{1}{\tau^{1+\frac{\alpha}{2}-\frac{1}{2p}}} \|\varphi(t-\tau)\|_{L_p(\mathbb{R}^{n-1})} \, \mathrm{d}\tau.$$

Next we apply the L_q-norm and use the general Minkowski inequality. It yields

$$\left(\int_0^T \left(\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^n} \frac{|u(x',z_n,t) - u(z',z_n,t)|^p}{|x-z|^{n+\alpha p}} \, \mathrm{d}x \, \mathrm{d}z \right)^{\frac{q}{p}} \mathrm{d}t \right)^{\frac{1}{q}}$$

$$\leq c(n,p) \|\varphi\|_{L_q(0,T;L_p(\mathbb{R}^{n-1}))} \int_0^T \frac{1}{\tau^{1+\frac{\alpha}{2}-\frac{1}{2p}}} \, \mathrm{d}\tau \tag{4.4}$$

$$\leq c(n,p,T) \|\varphi\|_{L_q(0,T;L_p(\mathbb{R}^{n-1}))}$$

if and only if $\alpha < \frac{1}{p}$. From (4.2) and the inequality above we conclude the proof. $\qquad\square$

Lemma 4.3. *Let us assume that* $\varphi \in L_q(0,T;W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1}))$, *where* $1 \leq p < \infty$, $1 \leq q \leq \infty$. *Then* $\mathrm{D}_{x'}u \in L_q(0,T;L_p(\mathbb{R}_+^n))$ *and it holds*

$$\|\mathrm{D}_{x'}u(x,t)\|_{L_q(0,T;L_p(\mathbb{R}_+^n))} \leq c(n,p) \|\varphi\|_{L_q(0,T;W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1}))}.$$

Proof. We follow and generalize [5, Section 4, §3]. In view of (2.1) and Lemma 2.1 we have

$$\mathrm{D}_{x'}u(x,t) = -2\int_0^\infty \int_{\mathbb{R}^{n-1}} \mathrm{D}_{y'}\partial_{x_n}\Gamma(y',x_n,\tau)\big(\varphi(x'-y',t-\tau)-\varphi(x',t-\tau)\big)\,\mathrm{d}y'\,\mathrm{d}\tau.$$

Using Lemma 2.4, the Hölder inequality with respect to x' and the general Minkowski inequality yields

$$\|\mathrm{D}_{x'}u(x,t)\|_{L_p(\mathbb{R}^{n-1})}$$
$$\leq c(n)\int_0^\infty \frac{1}{\tau^{\frac{n+2}{2}}}\int_{\mathbb{R}^{n-1}} e^{-\frac{|y'|^2+x_n^2}{16\tau}}\left(\int_{\mathbb{R}^{n-1}}|\mu(x'-y',t-\tau)-\mu(x',t-\tau)|^p\,\mathrm{d}x'\right)^{\frac{1}{p}}\mathrm{d}y\,\mathrm{d}\tau$$
$$=c(n)\int_{\mathbb{R}^{n-1}} \frac{\left(\int_{\mathbb{R}^{n-1}}|\varphi(x'-y',t-\tau)-\varphi(x',t-\tau)|^p\,\mathrm{d}x'\right)^{\frac{1}{p}}}{(|y'|^2+x_n^2)^{\frac{n}{2}}}\,\mathrm{d}y.$$

Applying the Hölder inequality with respect to y' gives

$$\|\mathrm{D}_{x'}u(x,t)\|_{L_p(\mathbb{R}^{n-1})} \leq c(n)\left(\int_{\mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}^{n-1}}|\varphi(x'-y',t)-\varphi(x',t)|^p\,\mathrm{d}x'}{(|y'|^2+x_n^2)^{\frac{n-1}{2}+\frac{p}{2}-\frac{1}{4}}}\mathrm{d}y'\right)^{\frac{1}{p}}$$
$$\cdot\left(\int_{\mathbb{R}^{n-1}} \frac{1}{(|y'|^2+x_n^2)^{\frac{n-1}{2}+\frac{p'}{4p}}}\,\mathrm{d}y'\right)^{\frac{1}{p'}}$$
$$=c(n,p)\frac{1}{x_n^{\frac{1}{2p}}}\left(\int_{\mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}^{n-1}}|\varphi(x'-y',t)-\varphi(x',t)|^p\,\mathrm{d}x'}{(|y'|^2+x_n^2)^{\frac{n-1}{2}+\frac{p}{2}-\frac{1}{4}}}\,\mathrm{d}y'\right)^{\frac{1}{p}}.$$

Next we take the L_p norm with respect to x_n

$$\|\mathrm{D}_{x'}u(x,t)\|_{L_p(\mathbb{R}^n_+)} \leq c(n,p)\left(\int_{\mathbb{R}^{n-1}} \frac{\int_{\mathbb{R}^{n-1}}|\varphi(x'-y',t)-\varphi(x',t)|^p\,\mathrm{d}x'}{|y'|^{n-2+p}}\,\mathrm{d}y'\right)^{\frac{1}{p}},$$

and he L_q norm with respect to t. This ends the proof.   □

Lemma 4.4. *Suppose that* $\varphi(x',t) \in W_{p,q}^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(\mathbb{R}^{n-1}\times(0,T))$, *where* $1 \leq q \leq p < \infty$. *Then* $\partial_{x_n}u(x,t) \in L_q(0,T;L_p(\mathbb{R}^n_+))$ *and it holds*

$$\|\partial_{x_n}u\|_{L_q(0,T;L_p(\mathbb{R}^n_+))} \leq c(n,p,T)\|\varphi\|_{W_{p,q}^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(\mathbb{R}^{n-1}\times(0,T))}.$$

Proof. In view of (2.1) and Lemma 2.1 we have

$$\partial_{x_n} u(x,t) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{1}{\tau^{\frac{n}{2}}} \partial_{x_n x_n}^2 e^{-\frac{|y'|^2 + x_n^2}{4\tau}} \varphi(x' - y', t - \tau) \, dy' \, d\tau$$

$$= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{1}{\tau^{\frac{n}{2}}} \partial_\tau e^{-\frac{|y'|^2 + x_n^2}{4\tau}} \left(\varphi(x' - y', t - \tau) - \varphi(x', t - \tau) \right) dy' \, d\tau$$

$$- \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{1}{\tau^{\frac{n}{2}}} D_{y'y'}^2 e^{-\frac{|y'|^2 + x_n^2}{4\tau}} \left(\varphi(x' - y', t - \tau) - \varphi(x' - y', t) \right) dy' \, d\tau$$

$$=: I_1 + I_2.$$

Let us consider I_1. Taking the absolute value of both sides and using Lemma 2.4 we get

$$|I_1| \leq c \int_0^\infty \frac{1}{\tau^{\frac{n+2}{2}}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|y'|^2 + x_n^2}{16\tau}} |\varphi(x' - y', t - \tau) - \varphi(x', t - \tau)| \, dy \, d\tau.$$

Next we repeat the calculations from the proof of Lemma 4.3. This yields

$$\|I_1\|_{L_q(0,T;L_p(\mathbb{R}_+^n))} \leq c(n,p) \|\varphi\|_{L_q(0,T;W_p^{1-\frac{1}{p}}(\mathbb{R}^{n-1}))}. \tag{4.5}$$

It remains to estimate I_2. Since

$$|I_2| \leq c \int_0^\infty \frac{1}{\tau^{\frac{n+2}{2}}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|y'|^2 + x_n^2}{16\tau}} |\varphi(x' - y', t - \tau) - \varphi(x' - y', t)| \, dy \, d\tau,$$

which follows from Lemma 2.4, we can take the L_p norm with respect to x' and integrate over y'. We obtain

$$\|I_2\|_{L_p(\mathbb{R}^{n-1})} \leq c \int_0^\infty \frac{1}{\tau^{\frac{3}{2}}} e^{-\frac{x_n^2}{16\tau}} |N(t - \tau, t)| \, d\tau, \tag{4.6}$$

where $N(t - \tau, t) := \left(\int_{\mathbb{R}^{n-1}} |\varphi(x', t - \tau) - \varphi(x', t)|^p \, dx' \right)^{\frac{1}{p}}$. Next we rewrite the right-hand side in (4.6) as follows

$$c \int_0^\infty \frac{1}{\tau^{\alpha_1}} e^{-\frac{x_n^2}{32\tau}} |N(t - \tau, t)| \frac{1}{\tau^{\alpha_2}} e^{-\frac{x_n^2}{32\tau}} \, d\tau,$$

where $\alpha_1 + \alpha_2 = \frac{3}{2}$, and apply the Hölder inequality. This yields

$$\|I_2\|_{L_p(\mathbb{R}^{n-1})} \leq c \left(\int_0^\infty \frac{1}{\tau^{\alpha_1 q}} e^{-\frac{q x_n^2}{32\tau}} |N(t - \tau, t)|^q \, d\tau \right)^{\frac{1}{q}} \left(\int_0^\infty \frac{1}{\tau^{\alpha_2 q'}} e^{-\frac{q' x_n^2}{32\tau}} \, d\tau \right)^{\frac{1}{q'}}$$

$$= \frac{c}{x_n^{2\alpha_2 - \frac{2}{q'}}} \left(\int_0^\infty \frac{1}{\tau^{\alpha_1 q}} e^{-\frac{q x_n^2}{32\tau}} |N(t - \tau, t)|^q \, d\tau \right)^{\frac{1}{q}}.$$

Taking the L_q norm with respect to x_n gives

$$\|I_2\|_{L_p(\mathbb{R}^n_+)} \le c \left(\int_0^\infty \left(\int_0^\infty \frac{1}{\tau^{\alpha_1 q}} \frac{1}{x_n^{2\alpha_2 q - \frac{2q}{q'}}} e^{-\frac{q x_n^2}{32\tau}} |N(t-\tau,t)|^q \, d\tau \right)^{\frac{p}{q}} dx_n \right)^{\frac{1}{p}}$$

$$\le c \left(\int_0^\infty \left(\int_0^\infty \frac{1}{\tau^{\alpha_1 p}} \frac{1}{x_n^{2\alpha_2 p - \frac{2p}{q'}}} e^{-\frac{p x_n^2}{32\tau}} dx_n \right)^{\frac{q}{p}} |N(t-\tau,t)|^q \, d\tau \right)^{\frac{1}{q}}$$

$$= c \left(\int_0^\infty \frac{|N(t-\tau,t)|^q}{\tau^{1+q\left(\frac{1}{2}-\frac{1}{2p}\right)}} \, d\tau \right)^{\frac{1}{q}}.$$

Integrating with respect to t in q-th power and taking into account the estimate (4.5) we conclude the proof. $\qquad\square$

5. Estimates in bounded domains

In this section we give the proof of Theorem 1.4. We start with recalling a fundamental definition which will be used in (5.6)

Definition 5.1. We will say that $S := \partial\Omega$ belongs to $\mathcal{C}^m_{loc}$, if for every $x_0 \in S$ there exist a number $r > 0$ and a function $f \colon \mathbb{R}^{n-1} \to \mathbb{R}$, $f \in \mathcal{C}^m_c(\mathbb{R}^{n-1})$ such that
$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) \colon x_n > f(x_1, \ldots, x_{n-1})\}.$$

Proof of Theorem 1.4. For any fixed $\lambda > 0$ (it will be chosen later) we cover Ω by sets $\Omega^{(k)}$, $\Omega \subset \bigcup_{k=0}^{N_\lambda} \Omega^{(k)}$, which satisfy
$$\frac{1}{2}\lambda < \operatorname{dist}(\Omega^0, \partial\Omega) < \lambda, \quad \Omega^{(k)} := B(x_0^{(k)}, r_\lambda) \cap S.$$

Next we introduce a partition of unity $\sum_{k=0}^{N_\lambda} \eta^{(k)} = 1$, which is subordinated to the covering $\Omega^{(k)}$. Then, multiplying (1.1) by $\eta^{(k)}$ we obtain

$$\eta^{(k)} u_t - \triangle(\eta^{(k)} u) = -2\nabla\eta^{(k)}\nabla u - u\triangle\eta^{(k)} =: g^{(k)} \quad \text{in } \Omega^T$$
$$\eta^{(k)} u = \eta^{(k)}\varphi \qquad\qquad\qquad\qquad\qquad \text{on } S^T \qquad (5.1)$$
$$\eta^{(k)} u|_{t=0} = 0 \qquad\qquad\qquad\qquad\qquad \text{in } \Omega \times \{0\}.$$

Let us denote $u^{(k)} := \eta^{(k)} u$. Then, for $k = 0$ the solution to problem (5.1) can be expressed in the form

$$u^{(0)}(x,t) = \int_{\mathbb{R}^n \times (0,T)} G(x-y, t-\tau) \left(-2\nabla\eta^{(0)}(y)\nabla u(y,\tau) - \triangle\eta^{(0)}(y)u(y,\tau)\right) dy \, d\tau$$

$$= \int_{\mathbb{R}^n \times (0,T)} \nabla G(x-y, t-\tau) \left(-2\nabla\eta^{(0)}(y)u(y,\tau)\right) dy \, d\tau$$

$$- \int_{\mathbb{R}^n \times (0,T)} G(x-y, t-\tau) \left(-2\nabla^2\eta^{(0)}(y) + \triangle\eta^{(0)}\right) u(y,\tau) dy \, d\tau.$$

For the last integral the estimate

$$\left\| \int_{\mathbb{R}^n \times (0,T)} G(x-y, t-\tau) \left(-2\nabla^2 \eta^{(0)}(y) + \triangle \eta^{(0)}\right) u(y, \tau) \mathrm{d}y\, \mathrm{d}\tau \right\|_{L_q(0,T;L_p(\mathbb{R}^n))}$$
$$\leq c(n, p, q, \lambda) \left\|u\right\|_{L_q(0,T;L_p(\Omega))}$$

holds, which implies that $\left\|u^{(0)}\right\|_{W_{p,q}^{1,\frac{1}{2}}(\mathbb{R}^n \times (0,T))} \leq c(n, p, q, \lambda) \left\|u\right\|_{L_q(0,T;L_p(\Omega))}$.

For $k > 0$ we introduce a local coordinate system $y = (y_1, \ldots, y_n)$ with the center at $x_0^{(k)}$, which we obtain from x through translations and rotations. Then, from assumptions on S we see that $S \cap \operatorname{supp} \eta^{(k)}$ is described by the equation $y_n = f^{(k)}(y_1, \ldots, y_{n-1})$, $f^{(k)}(0) = 0$, $\nabla f^{(k)}(0) = 0$.

Next, we straighten the set $\Omega \cap \operatorname{supp} \eta^{(k)}$ into the half-space through the mapping Φ, $z = \Phi(y)$, where

$$z' = y', \quad z_n = y_n - f^{(k)}(y').$$

Then, (5.1) in z-coordinates takes the form

$$\begin{aligned}
u_{,t}^{(k)} - \triangle u^{(k)} &= \left(\triangle_{\Phi^{-1}(z)} - \triangle\right) u^{(k)} + g^{(k)} & &\text{in } \mathbb{R}_+^n \times (0,T) \\
u^{(k)}|_{z_n=0} &= \varphi^{(k)} & &\text{on } \mathbb{R}^{n-1} \times (0,T) \\
u^{(k)}|_{t=0} &= 0 & &\text{in } \mathbb{R}_+^n \times \{0\}.
\end{aligned}$$

Consider now the problem

$$\begin{aligned}
v_{,t}^{(k)} - \triangle v^{(k)} &= 0 & &\text{in } \mathbb{R}_+^n \times (0,T) \\
v^{(k)}|_{z_n=0} &= \varphi^{(k)} & &\text{on } \mathbb{R}^{n-1} \times (0,T) \\
v^{(k)}|_{t=0} &= 0 & &\text{in } \mathbb{R}_+^n \times \{0\}.
\end{aligned}$$

Then, the function $w^{(k)} = u^{(k)} - v^{(k)}$ satisfies

$$\begin{aligned}
w_{,t}^{(k)} - \triangle w^{(k)} &= \left(\triangle_{\Phi^{-1}(z)} - \triangle\right) u^{(k)} + g^{(k)} & &\text{in } \mathbb{R}_+^n \times (0,T) \\
w^{(k)}|_{z_n=0} &= 0 & &\text{on } \mathbb{R}^{n-1} \times (0,T) \\
w^{(k)}|_{t=0} &= 0 & &\text{in } \mathbb{R}_+^n \times \{0\}.
\end{aligned} \qquad (5.2)$$

Now observe that

$$\triangle_{\Phi^{-1}(z)} - \triangle = \left.\frac{\partial \Phi}{\partial y}\right|_{y=\Phi^{-1}(z)} \nabla_z \cdot \left(\left.\frac{\partial \Phi}{\partial y}\right|_{y=\Phi^{-1}(z)} \nabla_z\right) - \nabla \cdot \nabla$$

$$= \left(\left(\left.\frac{\partial \Phi}{\partial y}\right|_{y=\Phi^{-1}(z)}\right)^2 - 1\right) \triangle + \left.\frac{\partial \Phi}{\partial y}\left(\frac{\partial y}{\partial z}\frac{\partial^2 \Phi}{\partial y^2}\right)\right|_{y=\Phi^{-1}(z)} \nabla$$

Next we use the Green representation formula and integrate by parts the term containing the Laplacian. The integration is justified since the Green function vanishes on the boundary (see [4] for details). Finally, the solution to (5.2) takes the form

$$
\begin{aligned}
w^{(k)}(s,t) =& \int_{\mathbb{R}^n_+ \times (0,T)} \nabla G(s-z,t-\tau) \left(\left(\frac{\partial \Phi}{\partial y}\Big|_{y=\Phi^{-1}(z)} \right)^2 - 1 \right) \nabla u^{(k)} \mathrm{d}z\,\mathrm{d}\tau \\
&+ \int_{\mathbb{R}^n_+ \times (0,T)} G(s-z,t-\tau) \\
&\cdot \left(\nabla \left(\left(\frac{\partial \Phi}{\partial y}\Big|_{y=\Phi^{-1}(z)} \right)^2 - 1 \right) + \frac{\partial \Phi}{\partial y}\left(\frac{\partial y}{\partial z}\frac{\partial^2 \Phi}{\partial y^2} \right) \Big|_{y=\Phi^{-1}(z)} \right) \nabla u^{(k)} \mathrm{d}z\,\mathrm{d}\tau \\
&+ \int_{\mathbb{R}^n_+ \times (0,T)} G(s-z,t-\tau)g^{(k)}(z,\tau)\,\mathrm{d}z\,\mathrm{d}\tau.
\end{aligned}
\tag{5.3}
$$

The first two integrals contain the small parameter λ. Let us examine the third integral. In y-coordinate it has the form

$$
\int_0^T \int_{\Omega \cap \mathrm{supp}\,\eta^{(k)}} G(s-y,t-\tau)\left(-2\nabla\eta^{(0)}(y)\nabla u(y,\tau) - \triangle\eta^{(0)}(y)u(y,\tau) \right)\mathrm{d}y\,\mathrm{d}\tau. \tag{5.4}
$$

We see that near the boundary we may write $\eta^{(k)}(y) = \eta^{(k)}_{tan}(y)\eta^{(k)}_{nor}(y)$, where the subscripts *tan* and *nor* denote the tangent and the normal parts. Since $\eta^{(k)}_{nor}(y) \equiv 1$ for every k, we infer that (5.4) does not involve differentiation along the normal direction. Hence we can integrate by parts which leads to

$$
\begin{aligned}
&- \int_0^T \int_{\Omega \cap \mathrm{supp}\,\eta^{(k)}} \nabla_{tan} G(s-y,t-\tau)\left(-2\nabla_{tan}\eta^{(k)}(y)u(y,\tau) \right)\mathrm{d}y\,\mathrm{d}\tau \\
&+ \int_0^T \int_{\Omega \cap \mathrm{supp}\,\eta^{(k)}} G(s-y,t-\tau)\left(-2\nabla^2_{tan}\eta^{(k)}(y) - \triangle\eta^{(k)}(y) \right)u(y,\tau)\mathrm{d}y\,\mathrm{d}\tau.
\end{aligned}
$$

Passing to z-coordinate does not introduce tangent derivative, so for the representation (5.3) we have the estimate

$$
\begin{aligned}
&\left\| w^{(k)} \right\|_{W^{1,\frac{1}{2}}_{p,q}(\mathbb{R}^n_+ \times (0,T))} \\
&\leq c(n,p,q)\lambda \left\| u^{(k)} \right\|_{L_q(0,T;W^1_p(\mathbb{R}^n_+))} + c(n,p,q,\lambda) \left\| u \right\|_{L_q(0,T;L_p(\Omega))}.
\end{aligned}
\tag{5.5}
$$

Taking into account Theorem 1.3 and (5.5) we get instantly that

$$\left\|u^{(k)}\right\|_{L_q(0,T;W_p^1(\mathbb{R}_+^n))} \leq \left\|u^{(k)}\right\|_{W_{p,q}^{1,\frac{1}{2}}(\mathbb{R}_+^n \times (0,T))}$$

$$\leq c(n,p,q)\lambda \left\|u^{(k)}\right\|_{L_q(0,T;W_p^1(\mathbb{R}_+^n))} + c(n,p,q,\lambda) \left\|u\right\|_{L_q(0,T;L_p(\Omega))}$$

$$+ \left\|\varphi^{(k)}\right\|_{W_{p,q}^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(\mathbb{R}^{n-1}\times(0,T))}.$$

Choosing λ small enough and summing over k yields

$$\left\|u\right\|_{L_q(0,T;W_p^1(\Omega))} \leq c(n,p,q,\Omega) \left\|u\right\|_{L_q(0,T;L_p(\Omega))} + c(n,p,q,\Omega) \left\|\varphi\right\|_{W_{p,q}^{1-\frac{1}{p},\frac{1}{2}-\frac{1}{2p}}(S^T)}.$$

To eliminate the first term on the right-hand side, we introduce a partition of unity $\{\psi_\alpha\}_{\alpha\in A}$ on S such that $\operatorname{supp}\psi_\alpha = \overline{B_\alpha(\xi_\alpha,r_\alpha)}$. Next we change the variable in the formula for the solution

$$u(x,t) = \int_0^t\!\!\int_S n_i(\xi) \cdot \frac{\partial\Gamma(x-\xi,t-\tau)}{\partial\xi_i}\mu(\xi,\tau)\,\mathrm{d}S_\xi\mathrm{d}\tau,$$

using the mapping $\Phi_\alpha\colon \operatorname{supp}\psi_\alpha \to \mathbb{R}^n$, $\xi \xrightarrow{\Phi_\alpha} [\xi', f^\alpha(\xi')]$. With this in mind, we see that

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\frac{1}{(4\pi)^{\frac{n}{2}}}\int_0^t\!\!\int_{E^\alpha} \frac{[\nabla f^\alpha(\xi'),-1]}{\sqrt{1+|\nabla f^\alpha|^2}} \cdot \frac{[x'-\xi', x_n - f^\alpha(\xi')]}{(t-\tau)^{\frac{n+2}{2}}} \\
&\quad \cdot e^{-\frac{|x'-\xi'|^2+|x_n-f^\alpha(\xi')|^2}{4(t-\tau)}} \mu_\alpha(\xi',f^\alpha(\xi'),\tau)\sqrt{1+|\nabla f^\alpha|^2}\,\mathrm{d}\xi'\,\mathrm{d}\tau \qquad (5.6) \\
&= c(n)\int_0^t\!\!\int_{E^\alpha} \frac{\nabla f^\alpha(\xi') \cdot [x'-\xi'] - x_n + f^\alpha(\xi')}{(t-\tau)^{\frac{n+2}{2}}} e^{-\frac{|x'-\xi'|^2+|x_n-f^\alpha(\xi')|^2}{4(t-\tau)}} \\
&\quad \cdot \mu_\alpha(\xi',f^\alpha(\xi'),\tau)\,\mathrm{d}\xi'\,\mathrm{d}\tau,
\end{aligned}$$

where $E^\alpha = \mathbb{R}^{n-1} \cap \operatorname{supp}\psi(\xi')$ and $\cdot$ denotes the standard inner product. From the Mean Value Theorem it follows that

$$x_n - f^\alpha(\xi') = \nabla f^\alpha(\xi^*) \cdot [x'-\xi'] + x_n - f^\alpha(x').$$

Since

$$\left|\nabla f^\alpha(\xi^*) \cdot [x'-\xi'] + x_n - f^\alpha(x')\right|^2 \geq \left|\nabla f^\alpha(\xi^*) \cdot [x'-\xi']\right|^2 + \left|x_n - f^\alpha(x')\right|^2$$

we may apply Lemma 2.4 in (5.6). It leads to

$$u(x,t) \leq c(n)\int_0^t\!\!\int_{E^\alpha} \frac{1}{(t-\tau)^{\frac{n+1}{2}}} e^{-\frac{c|x'-\xi'|^2+|x_n-f^\alpha(x')|^2}{4(t-\tau)}} \mu_\alpha(\xi',f^\alpha(\xi'),\tau)\mathrm{d}\xi'\,\mathrm{d}\tau.$$

Next we repeat the proof of Lemma 4.1. This concludes the proof. $\square$

Acknowledgment. The authors would like to express their thanks to the referee for the valuable suggestions that helped to improve the paper.

References

[1] Besov, O. V., Il'in, V. P. and Nikol'skiĭ, S. M., *Integral Representations of Functions and Imbedding Theorems. Vol. I* (translated from Russian). Scripta Ser. Math. (ed.: M. H. Taibleson). Washington, D.C.: Winston & Sons 1978.

[2] Denk, R., Hieber, M. and Prüss, J., Optimal L^p-L^q-estimates for parabolic boundary value problems with inhomogeneous data. *Math. Z.* 257 (2007)(1), 193 − 224.

[3] Grisvard, P., Commutativité de deux foncteurs d'interpolation et applications (in French). *J. Math. Pures Appl.* 45 (1966)(9), 207 − 290.

[4] Krzyżański, M., *Partial Differential Equations of Second Order. Vol. I* (transl. from Polish by H. Zorski). Monogr. Mat. 53, Warsaw: PWN Polish Scientific Publishers 1971.

[5] Ladyženskaja, O. A., Solonnikov, V. A. and Ural'ceva, N. N., *Linear and Quasilinear Equations of Parabolic Type* (transl. from Russian by S. Smith). Transl. Math. Monogr. 23. Providence (RI): Amer. Math. Soc. 1967.

[6] Pogorzelski, W., *Integral Equations and their Applications. Vol. I* (transl. from Polish by J. J. Schorr-Con et al.). Internat. Ser. Monogr. Pure Appl. Math. 88. Oxford: Pergamon Press 1966.

[7] Rencławowicz, J. and Zajączkowski, W. M., Large time regular solutions to the Navier-Stokes equations in cylindrical domains. *Topol. Methods Nonlin. Anal.* 32 (2008)(1), 69 − 87.

[8] Solonnikov, V. A., On boundary value problems for linear parabolic systems of differential equations of general form (in Russian). *Trudy Mat. Inst. Steklov* 83 (1965), 3 − 163; Engl. transl.: *Proc. Steklov Inst. Math.* 83 (1965), 1 − 184.

[9] Solonnikov, V. A., Overdetermined elliptic boundary-value problems (in Russian). *Zap. Nauchn. Sem. LOMI* 21 (1971), 112 − 158; Engl. transl.: *J. Mathem. Sciences* 1 (1973)(4), 477 − 512.

[10] Weidemaier, P., Maximal regularity for parabolic equations with inhomogeneous boundary conditions in Sobolev spaces with mixed L_p-norm. *Electron. Res. Announc. Amer. Math. Soc.* 8 (2002), 47 − 51.

[11] Wloka, J., *Partial Differential Equations* (transl. from German by C. B. Thomas et al.). Cambridge: Cambridge Univ. Press 1987.

[12] Zadrzyńska, E. and Zajączkowski, W. M., The Cauchy-Dirichlet problem for the heat equation in Besov spaces. In: *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 348 (2007), Kraevye Zadachi Mat. Fiziki i Smezhnye Voprosy Teorii Funktsii. 38, 40 − 97, 303 − 304.

[13] Zajączkowski, W. M., Long time existence of regular solutions to Navier-Stokes equations in cylindrical domains under boundary slip conditions. *Stud. Math.* 169 (2005)(3), 243 – 285.

[14] Zajączkowski, W. M., Global regular solutions to the Navier-Stokes equations in a cylinder. In: *Self-Similar Solutions of Nonlinear PDE.* Banach Center Publ. 74. Warsaw: Polish Acad. Sci. 2006, pp. 235 – 255.

[15] Zajączkowski, W. M., On global regular solutions to the Navier-Stokes equations in cylindrical domains. *Topol. Methods Nonlinear Anal.* 37 (2011)(1), 55 – 86.