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Identification of an Unknown Parameter Function in the Main Part of an Elliptic Partial Differential Equation

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Abstract. The identification of an unknown parameter function in the main part of an elliptic partial differential equation is studied. We use a Tichonov regularization with an H^s-norm and $s > 0$. Moreover, pointwise bounds for the unknown parameter are assumed. Existence of solutions is shown and necessary optimality conditions are established. The main contribution is the discussion of second-order sufficient optimality conditions. Here, we get a size condition of the parameter s.

Keywords. Parameter identification, inverse problems, Tichonov regularization, optimal control, inequality constraints, necessary and sufficient optimality conditions

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1. Introduction

In this paper, we study the following minimization problem:

$$\left.\begin{array}{lrl} \text{minimize} & J(y,a) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|a\|^2_{H^s(\Omega)} & \\ \text{subject to} & -\nabla\cdot(a\nabla y) = g & \text{in } \Omega \\ & y = 0 & \text{on } \Gamma \\ & 0 < a_{\min} \le a(x) \le a_{\max} & \text{a.e. in } \Omega \\ & y \in H_0^1(\Omega),\ a \in H^s(\Omega),\ s > 0 & \end{array}\right\} \qquad (1)$$

where y_d, $g \in L^2(\Omega)$ and $\alpha > 0$ are given and the constants $a_{\min}, a_{\max}$ satisfy $a_{\min} < a_{\max}$. Moreover, we require a bounded Lipschitz-domain $\Omega \in \mathbb{R}^N$ with boundary $\Gamma = \partial\Omega$.

Our aim is to identify the unknown parameter function a in the main part of the elliptic operator. The quantities $a_{\min}, a_{\max}$ describe maximal and minimal

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values due to physical properties of the material. We will study this problem for a fixed parameter $\alpha > 0$, whereas we want to choose the second regularisation parameter s in an appropriate way. Therefore, our main focus will be on this smoothness parameter in the space underlying the Tichonov regularization. Motivated by applications, we are interested in the identification of discontinuous functions a. Consequently, $s \in (0, \frac{1}{2})$ seems to be an attractive choice.

We will see that existence of solutions of (1) can be guaranteed for arbitrary $s > 0$. The derivation of necessary optimality conditions is also possible for arbitrary positive s. In contrast to this, we will obtain a size condition for s for second-order sufficient conditions. Let us mention that such conditions lead to Lipschitz stability, see for instance [1], which is the main ingredient in the convergence analysis of SQP-Methods [7]. Furthermore, second order optimality conditions play an important role for sensitivity analysis [8] and for discretization error estimates [2].

Let us give a short overview on literature about this type of parameter identification problems by referring to the following papers and the references therein. Parameter identification problems are generally studied in [3,6]. In [5] the problem is discussed with state constraints, [10] deals with matrix-valued parameter identification. The identification of smooth parameters is studied in [14] and extended to discontinuous parameters by introducing the space of bounded variation functions. Let us mention some papers dealing with numerical aspects of parameter identification problems. An augmented Lagrangian method is studied in [11], while [12] is considering proper orthogonal decomposition and [13] is solving the problem by minimization of an associated functional and applying a conjugate gradient algorithm.

The paper is organized as follows: In Section 2 we will introduce notations and prove the existence of solutions of the minimization problem. Optimality conditions as the main result are investigated in Section 3.

2. Existence of solutions

Let us start this section by introducing some helpful notations. We split the objective into two functionals depending on y and a, respectively, $F(y) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)}$, $Q(a) := \frac{\alpha}{2}\|a\|^2_{H^s(\Omega)}$, hence $J(y,a) = F(y) + Q(a)$.

We define the set of admissible parameter functions as

$$A_{ad} = \{a \in H^s(\Omega) : 0 < a_{\min} \le a(x) \le a_{\max} \text{ a.e. in } \Omega\}.$$

Let us remark that $a \in L^\infty(\Omega)$ holds for all admissible parameter functions by the bounds $a_{\min}$ and $a_{\max}$. We denote the weak solution of the given PDE by

$y = y(a)$ and obtain the variational form for $y \in H_0^1(\Omega)$, by a test function $v \in H_0^1(\Omega)$ and $a \in A_{ad}$ as follows

$$\int_\Omega a\,\nabla y \cdot \nabla v\,dx = \int_\Omega g\,v\,dx =: G(v) \quad \forall\, v \in H_0^1(\Omega).$$

From the lemma of Lax and Milgram we obtain the existence of a unique solution $y \in H^1(\Omega)$ for every $a \in A_{ad}$. Thus there exists a *parameter-to-state-mapping* $S : L^\infty(\Omega) \to H^1(\Omega)$ with $y = S(a)$ for all admissible a. Furthermore we get

$$\|y\|_{H^1(\Omega)} \leq c_2 \|G\|_{(H^1(\Omega))^*} \tag{2}$$

where the constant c_2 only depends on $a_{\min}$ and $a_{\max}$. Hence, we have uniformly boundedness of $\|y\|_{H^1(\Omega)}$ for all $a \in A_{ad}$.

Let us now introduce the dual space of $W_0^{k,p}(\Omega)$ with $k > 0$, where $p \in (1, \infty)$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$ as

$$W^{-k,q}(\Omega) = \Big(W_0^{k,p}(\Omega)\Big)^*$$

with the norm

$$\|u\|_{W^{-k,q}(\Omega)} := \sup_{v \in W_0^{k,p}(\Omega),\, v \neq 0} \frac{|\int_\Omega u(x)v(x)dx|}{\|v\|_{W^{k,p}(\Omega)}}.$$

These spaces are Banach spaces as well.

For the proof of the following imbedding theorem we refer to [4, Theorem 6.5.1].

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. Let $s > 0$ and $1 < p < N$.*

1. *If $N > sp$, then $W^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ for $p \leq r \leq \frac{Np}{N-sp}$.*
2. *If $N = sp$, then $W^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ for $p \leq r < \infty$.*

We extract from [9, Theorem 1] the existence of a constant $\bar{q} = \bar{q}(\Omega)$, depending only on the domain Ω, such that

$$\|y\|_{W^{1,p}(\Omega)} \leq c_{\bar{q}} \|G\|_{W^{-1,p}(\Omega)}, \tag{3}$$

with $p \in [2, \bar{q}]$ and $c_{\bar{q}} = c_{\bar{q}}(a_{\min}, a_{\max}, \Omega, p)$ depending on the parameter bounds $a_{\min}$ and $a_{\max}$, the domain Ω and p. G is determined by the right hand side g of the PDE. We need to ensure that the right hand sides of the original PDE and later the adjoint equation belong to $W^{-1,p}(\Omega)$. Therefore we use the imbedding theorem. In two dimensions g and y_d being out of $L^2(\Omega)$ is sufficient. For simplicity we choose $g, y_d \in L^\infty(\Omega)$ in order to ensure them to be in $W^{-1,p}(\Omega)$, for $p \in [2, \bar{q}]$, too.

Lemma 2.2. *The admissible states $y = S(a)$, $a \in A_{ad}$ are uniformly bounded in $W^{1,p}$, $p \in [2, \bar{q}]$, i.e.*

$$\|y\|_{W^{1,p}(\Omega)} \leq K \quad \forall y = S(a), \text{ where } a \in A_{ad},\ p \in [2, \bar{q}].$$

The estimation holds since $\|G\|_{W^{-1,p}}$ is fixed and the constant $c_{\bar{q}}$ in (3) only depends on $a_{\min}, a_{\max}, p$ and Ω.

We underline the importance of these results to this paper. Let us now apply this result to the imbedding theorem. Choosing the parameter s as $s \geq \frac{N}{\bar{q}}$ we obtain

$$\begin{aligned} &\bullet \text{ If } \bar{q} > 2 \text{ then } H^{\frac{N}{\bar{q}}}(\Omega) \hookrightarrow L^p(\Omega) \text{ for } 2 \leq p \leq \tfrac{1}{\frac{1}{2} - \frac{1}{\bar{q}}}. \\ &\bullet \text{ If } \bar{q} = 2 \text{ then } H^{\frac{N}{\bar{q}}}(\Omega) \hookrightarrow L^p(\Omega) \text{ for } 2 \leq p < \infty. \end{aligned} \tag{4}$$

The Hilbert space $H^s(\Omega)$ with $s \geq \frac{N}{\bar{q}}$ is compactly imbedded into $L^p(\Omega)$ for $2 \leq p \leq \frac{1}{\frac{1}{2} - \frac{1}{\bar{q}}}$, thus we get

$$\forall\, a \in H^s(\Omega)\ \exists\, c_i > 0 \ :\ \|a\|_{L^p(\Omega)} \leq c_i \|a\|_{H^s(\Omega)}.$$

Furthermore $H^s(\Omega)$ is compactly imbedded into $L^2(\Omega)$ for all $s > 0$. Thus a bounded set in $H^s(\Omega)$ is pre-compact in $L^2(\Omega)$, for all $s > 0$. Throughout the paper c will be a generic constant.

The proof of existence of an optimal solution differs slightly from the common proof we find for example in [15]. The main difficulty lies in the link of the parameter and the state in the same term on the left hand side of the PDE.

Theorem 2.3. *The problem* (1) *admits at least one solution $\bar{a} \in A_{ad}$ with the optimal state $\bar{y} = S(\bar{a})$, i.e.*

$$J(\bar{y}, \bar{a}) \leq J(y, a) \quad \forall a \in A_{ad},\ y = S(a).$$

Proof. (i) *Boundedness of the objective functional.* As we mentioned before the Lax-Milgram lemma ensures the existence of a unique weak solution $y = S(a) \in H_0^1(\Omega)$ for every $a \in A_{ad}$ and given $g \in L^2(\Omega)$. We can easily see that the objective functional $J(y, a)$ is bounded below, thus we conclude the existence of a nonnegative real number j defined by $j := \inf_{a \in A_{ad}} J(y, a)$. Let (y_n, a_n), with $a_n \in A_{ad}$ and $y_n = S(a_n)$ be a sequence minimizing the objective functional, i.e. $J(y_n, a_n) \to j$ for $n \to \infty$. By definition y_n and a_n satisfy $(\nabla y_n, a_n \nabla v)_{(L^2(\Omega))^N} = (g, v)_{L^2(\Omega)}$ for all $v \in H_0^1(\Omega)$. Let us now examine the behavior of $(\nabla y_n, a_n \nabla v)_{(L^2(\Omega))^N}$ for $n \to \infty$. We show the existence of $\bar{a}$ and $\bar{y}$, such that

$$\begin{aligned} &1. \quad a_n \nabla v \to \bar{a} \nabla v \quad \text{in } (L^q(\Omega))^N \\ &2. \quad \nabla y_n \rightharpoonup \nabla \bar{y} \quad \text{in } (L^p(\Omega))^N \end{aligned}$$

for a certain subsequence, where $p > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, in order to obtain convergence of $(\nabla y_n, a_n \nabla v)_{(L^2(\Omega))^N}$ in the following way

$$(\nabla y_n, a_n \nabla v)_{(L^2(\Omega))^N} \to (\nabla \bar{y}, \bar{a} \nabla v)_{(L^2(\Omega))^N}.$$

(ii) *Convergence of* $a_n \nabla v$ *towards* $\bar{a} \nabla v$. Let (y_1, a_1) be the first element of the minimizing sequence introduced before. Without loss of generality $J(y_1, a_1)$ is an upper bound for the functional values of all elements of the sequence. We obtain in particular that $\{a_n\}_{n=1}^{\infty}$ is uniformly bounded in $H^s(\Omega)$, i.e. $\frac{\alpha}{2}\|a_n\|^2_{H^s(\Omega)} \leq J(y_1, a_1)\ \forall n \geq 1$. According to the imbedding theorem a bounded set in $H^s(\Omega)$ is pre-compact in $L^2(\Omega)$. Hence, there exists a subsequence of $\{a_n\}_{n=1}^{\infty}$, again denoted by $\{a_n\}_{n=1}^{\infty}$ that is a Cauchy-sequence. Thus $\{a_n\}_{n=1}^{\infty}$ converges to a limit $\bar{a} \in L^2(\Omega)$, i.e.

$$\|a_n - \bar{a}\|_{L^2(\Omega)} \to 0, \quad \text{as } n \to \infty.$$

Furthermore we note that $\|a_n - \bar{a}\|_{L^\infty(\Omega)} \leq a_{\max} - a_{\min}$, for all $n \in \mathbb{N}$ and consequently $\|a_n - \bar{a}\|^r_{L^r(\Omega)} \leq \int_\Omega |a_n - \bar{a}|^2 |a_n - \bar{a}|^{r-2} dx \leq \|a_n - \bar{a}\|^2_{L^2(\Omega)} \|a_n - \bar{a}\|^{r-2}_{L^\infty(\Omega)}$, e.g.

$$\|a_n - a\|_{L^r(\Omega)} \to 0, \quad \text{as } n \to \infty \tag{5}$$

for an arbitrary real number $2 \leq r < \infty$. After these considerations we can show that $a_n \nabla v \to \bar{a} \nabla v$, $n \to \infty$ in $(L^q(\Omega))^N$. We obtain by Hölder's Inequality for $\frac{1}{r} + \frac{1}{t} = 1$, $\|(a_n - \bar{a}) \nabla v\|_{(L^q(\Omega))^N} \leq \|a_n - \bar{a}\|_{L^{rq}(\Omega)} \|\nabla v\|_{(L^{tq}(\Omega))^N}$. Next, we choose $t := \frac{2}{q} \Rightarrow r = \frac{2}{2-q} < \infty$ and obtain

$$\|(a_n - \bar{a}) \nabla v\|_{(L^q(\Omega))^N} \leq \|a_n - \bar{a}\|_{L^{\frac{2q}{2-q}}(\Omega)} \|\nabla v\|_{(L^2(\Omega))^N}$$

for arbitrary $q \in [1, 2)$. With (5) and because $v \in H_0^1(\Omega)$ is fixed this term converges to zero for $n \to \infty$.

(iii) *Weak convergence of* ∇y_n *towards* $\nabla \bar{y}$. From Lemma 2.2 we know that $\{y_n\}_{n=1}^{\infty}$ is uniformly bounded in $W^{1,p}(\Omega)$. Consequently, there exists a weakly convergent subsequence of $\{y_n\}_{n=1}^{\infty}$, without loss fo generality we take the sequence itself, satisfying $y_n \rightharpoonup \bar{y}$ in $W^{1,p}(\Omega)$ as $n \to \infty$. Hence we get $\nabla y_n \rightharpoonup \nabla \bar{y}$ in $(L^p(\Omega))^N$, $n \to \infty$, because the mapping $y \to \nabla y$ is linear and continuous from $W^{1,p}(\Omega)$ to $(L^p(\Omega))^N$.

(iv) $(\bar{y}, \bar{a})$ *is optimal.* Now we easily see that $(\bar{y}, \bar{a})$ satisfies the variational form. Furthermore $\bar{a} \in A_{ad}$, since the admissible set is weakly sequentially closed. At last we need to show that $(\bar{y}, \bar{a})$ is optimal, i.e., its objective functional value $J(\bar{y}, \bar{a})$ is equal to j. The objective functional is divided into two parts, $J(y, a) = F(y) + Q(a)$. Given that the functional $F(y) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)}$ is continuous, we directly see that $\lim_{n\to\infty} F(y_n) = F(\bar{y})$, because of $y_n \to \bar{y}$ in $L^2(\Omega)$. In order to arrive at a similar conclusion for the functional $Q(a) = \frac{\alpha}{2}\|a\|^2_{H^s(\Omega)}$ we additionally need the fact that $Q(a)$ is convex and consequently weakly lower semi-continuous, which means

$$a_n \rightharpoonup \bar{a} \text{ in } H^s(\Omega) \ \Rightarrow\ \liminf_{n\to\infty} Q(a_n) \geq Q(\bar{a}).$$

Now we conclude

$$j = \lim_{n\to\infty} J(y_n, a_n) \geq \lim_{n\to\infty} F(y_n) + \liminf_{n\to\infty} Q(a_n) \geq F(\bar{y}) + Q(\bar{a}) = J(\bar{y}, \bar{a}).$$

On the other hand we know that j satisfies $j \leq J(\bar{y}, \bar{a})$, because j was defined to be the infimum of all values of the objective functional $j = \inf_{a\in A_{ad}} J(y, a)$. Hence, we get $j = J(\bar{y}, \bar{a})$. □

We have seen that $s > 0$ is sufficient to guarantee existence of an optimal solution.

3. Optimality conditions

In this section we consider necessary and sufficient optimality conditions. First order necessary conditions are needed to construct algorithms. Second order sufficient conditions are important in order to show stability with respect to perturbations and convergence results of optimization methods and they play an important role for discretization error estimates.

3.1. First-order necessary optimality condition. First of all we would like to restate the partial differential equation underlying the parameter-to-state mapping $S : L^\infty(\Omega) \to H^1(\Omega)$, $S(a) = y$

$$\begin{aligned} -\nabla \cdot (a\nabla y) &= g \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{6}$$

Next we show that this mapping is Fréchet-differentiable.

Lemma 3.1. *The parameter-to-state-mapping $S : L^\infty(\Omega) \to H^1(\Omega)$ is Fréchet-differentiable. Its derivative can be described by $S'(a)a_1 = y_1'$, where $y_1' \in H^1(\Omega)$ is the weak solution of the following problem*

$$\begin{aligned} -\nabla \cdot (a\nabla y_1') &= \nabla \cdot (a_1 \nabla y) \ \ \text{in } \Omega \\ y_1' &= 0 \qquad\qquad\quad \text{on } \Gamma. \end{aligned} \tag{7}$$

Here, a is an admissible parameter function with respect to (1) *and y is the corresponding state $y = S(a)$.*

With the lemma of Lax and Milgram we see that $y_1' \in H^1(\Omega)$ is well-defined, because $\nabla \cdot (a_1 \nabla y)$ is an element of $H^{-1}(\Omega)$.

Proof. We have to show the existence of a linear continuous operator $D : L^\infty(\Omega) \to H^1(\Omega)$, such that $S(a+a_1) - S(a) = Da_1 + r(a, a_1)$ holds for all $a_1 \in L^\infty(\Omega)$ satisfying the equation

$$\frac{\|r(a,a_1)\|_{H^1(\Omega)}}{\|a_1\|_{L^\infty(\Omega)}} \to 0, \quad \text{for } \|a_1\|_{L^\infty(\Omega)} \to 0.$$

Then D is the Fréchet-derivative of S. Let us assume

$$-\nabla \cdot (a \nabla y_1') = \nabla \cdot (a_1 \nabla y) \tag{8}$$

to be the PDE associated with Da_1. We easily verify linearity and continuity of D. Next, we want to examine the term $r(a, a_1)$, thus $S(a+a_1) - S(a) - Da_1$. In order to do so, we subtract the associated PDEs of $S(a)$ and Da_1, i.e. (6) and (8), from the PDE of $S(a+a_1) = y_1'$, which is given as $-\nabla \cdot ((a+a_1)\nabla y_1) = g$. All partial differential equations mentioned before have homogeneous Dirichlet boundary conditions. A short computation gives

$$-\nabla \cdot (a \nabla (\underbrace{y_1 - y - y_1'}_{=:y_\delta})) = \nabla \cdot (a_1 \nabla (y_1 - y)).$$

Next, we show

$$\frac{\|y_\delta\|_{H^1(\Omega)}}{\|a_1\|_{L^\infty(\Omega)}} \to 0, \quad \text{for } \|a_1\|_{L^\infty(\Omega)} \to 0.$$

With (2) and G denominating the right hand side of the PDE stated above we obtain the inequality chain

$$\begin{aligned}
\|y_\delta\|_{H^1(\Omega)} &\le c_2 \|G\|_{(H^1(\Omega))^*} \\
&= c \sup_{v \in H_0^1(\Omega),\, v \neq 0} \frac{\left|\int_\Omega a_1 \nabla (y_1 - y) \nabla v \, dx\right|}{\|v\|_{H^1(\Omega)}} \\
&\le c \sup_{v \in H_0^1(\Omega),\, v \neq 0} \|a_1 \nabla (y_1 - y)\|_{L^2(\Omega)} \frac{\|\nabla v\|_{L^2(\Omega)}}{\|v\|_{H^1(\Omega)}} \\
&\le c \|a_1 \nabla (y_1 - y)\|_{L^2(\Omega)} \\
&\le c \|a_1\|_{L^\infty(\Omega)} \|y_1 - y\|_{H^1(\Omega)}.
\end{aligned}$$

We still have to prove that $\|y_1 - y\|_{H^1(\Omega)} \to 0$, for $\|a_1\|_{L^\infty(\Omega)} \to 0$.

Let us consider the variational forms to y and y_1, taking in both cases $y - y_1$ as test function, and subtract them. We get

$$(a \nabla (y - y_1), \nabla (y - y_1))_{(L^2(\Omega))^N} = (a_1 \nabla y_1, \nabla (y - y_1))_{(L^2(\Omega))^N}.$$

Employing Friedrichs' inequality we estimate the left hand side of the equation as follows $(a\nabla(y-y_1), \nabla(y-y_1))_{(L^2(\Omega))^N} \geq c\|y-y_1\|^2_{H^1(\Omega)}$, where c is a constant depending on $a_{\min}$ and Ω. The right hand side can be transformed using Cauchy-Schwarz inequality $(a_1\nabla y_1, \nabla(y-y_1))_{(L^2(\Omega))^N} \leq \|a_1\|_{L^\infty(\Omega)}\|y_1\|_{H^1(\Omega)}\|y-y_1\|_{H^1(\Omega)}$. Hence, we obtain

$$\begin{aligned} c\|y-y_1\|^2_{H^1(\Omega)} &\leq (a\nabla(y-y_1), \nabla(y-y_1))_{(L^2(\Omega))^N} \\ &= (a_1\nabla y_1, \nabla(y-y_1))_{(L^2(\Omega))^N} \\ &\leq \|a_1\|_{L^\infty(\Omega)}\|y_1\|_{H^1(\Omega)}\|y-y_1\|_{H^1(\Omega)}. \end{aligned}$$

This leads to $\|y-y_1\|_{H^1(\Omega)} \leq c\|a_1\|_{L^\infty(\Omega)}\|y_1\|_{H^1(\Omega)}$. As we montioned before, $\|y_1\|_{H^1(\Omega)}$ is uniformly bounded. Thus we finally obtain

$$\|y_\delta\|_{H^1(\Omega)} \leq c\|a_1\|^2_{L^\infty(\Omega)}\|y_1\|_{H^1(\Omega)} \leq cK\|a_1\|^2_{L^\infty(\Omega)} \leq c\|a_1\|^2_{L^\infty(\Omega)}.$$

The remainder term condition holds and D, described by (7), is the derivative of the *parameter-to-state-mapping* S. □

An optimal parameter function $\bar{a} \in A_{ad}$ has to fulfill the following variational inequality

$$f'(\bar{a})(a-\bar{a}) \geq 0 \quad \forall a \in A_{ad}.$$

Let us now compute the derivative of the objective in $\bar{a}$ which was given as $f(\bar{a}) := J(y(\bar{a}), \bar{a}) = F(S(\bar{a})) + Q(\bar{a})$:

$$\begin{aligned} f'(\bar{a})(a-\bar{a}) &= F'(S(\bar{a}))S'(\bar{a})(a-\bar{a}) + Q'(\bar{a})(a-\bar{a}) \\ &= F'(S(\bar{a}))y_1' + Q'(\bar{a})(a-\bar{a}) \\ &= (S(\bar{a})-y_d, y_1')_{L^2(\Omega)} + (\alpha\bar{a}, (a-\bar{a}))_{H^s(\Omega)}. \end{aligned}$$

We introduce an adjoint state in order to transform the variational equation into the desired form. The weak solution $p \in H^1_0(\Omega)$ of the adjoint equation

$$\begin{aligned} -\nabla\cdot(a\nabla p) &= y - y_d \text{ in } \Omega \\ p &= 0 \qquad \text{on } \Gamma \end{aligned}$$

is called adjoint state. We denote by $\bar{p}$ the adjoint state belonging to the optimal pairing $\bar{a}$, $\bar{y}$. Considering the weak formulations of the adjoint equation and (7) with y_1' and $\bar{p}$ as test functions, respectively, we easily see, that

$$-\int_\Omega (a-\bar{a})\nabla\bar{y}\nabla\bar{p}\,dx = \int_\Omega (\bar{y}-y_d)y_1'\,dx$$

holds. Thus we obtain a first order necessary optimality condition:

Lemma 3.2. *An optimal parameter $\bar{a}$ together with the optimal state $\bar{y} = S(\bar{a})$ and the optimal adjoint state p necessarily fulfills the following condition*

$$-((a-\bar{a})\nabla\bar{y}, \nabla\bar{p})_{(L^2(\Omega))^N} + (\alpha\bar{a}, a-\bar{a})_{H^s(\Omega)} \geq 0, \tag{9}$$

for all $a \in A_{ad}$.

3.2. Second-order sufficient optimality condition. In order to prove the second order Fréchet-differentiability of the operator S we show that the mapping $a \to S'(a)a_1$ is Fréchet-differentiable for all $a_1 \in L^\infty(\Omega)$.

Lemma 3.3. *The mapping $a \to S'(a)a_1$ is Fréchet-differentiable from $L^\infty(\Omega)$ onto $H^1(\Omega)$ for all $a_1 \in L^\infty(\Omega)$. Its derivative is given by $S''(a)[a_1, a_2] = y''$ where y'' is the weak solution of the following problem*

$$\begin{aligned} -\nabla \cdot (a\nabla y'') &= \nabla \cdot (a_1 \nabla y_2') + \nabla \cdot (a_2 \nabla y_1') && \text{in } \Omega \\ y'' &= 0 && \text{on } \Gamma. \end{aligned} \tag{10}$$

With y_i', $i = 1, 2$ being defined as the weak solution of $-\nabla \cdot (a\nabla y_i') = \nabla \cdot (a_i \nabla y)$ and $y = S(a)$ being the solution of $-\nabla \cdot (a\nabla y) = g$.

Proof. We define the operator $D(a; a_1) : L^\infty(\Omega) \to H^1(\Omega)$, with $D(a; a_1)a_2 = S''(a)[a_1, a_2]$. It is determined by (10), thus we easily see its linearity and continuity. The remainder term $r(a, a_1, a_2)$ is defined by

$$S'(a + a_2)a_1 - S'(a)a_1 = D(a; a_1)a_2 + r(a, a_1, a_2) \quad \forall a_2 \in L^\infty(\Omega).$$

It remains to show the remainder term property

$$\frac{\|r(a, a_1, a_2)\|_{H^1(\Omega)}}{\|a_2\|_{L^\infty(\Omega)}} \to 0, \quad \text{for } \|a_2\|_{L^\infty(\Omega)} \to 0.$$

The terms $S'(a + a_2)a_1, S'(a)a_1$ and $D(a; a_2)a_1$ are defined as solutions of the partial differential equations (7) and (10), respectively. Consequently, we obtain that the remainder term itself solves a partial differential equation. The expressions on the right hand side of the resulting partial differential equations can again be estimated by the techniques presented in the proof of Lemma 4. Hence, we get the desired remainder property. □

Next, we calculate the second order derivative of the objective

$$J(y, a) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|a\|^2_{H^s(\Omega)} = J(S(a), a) = F(S(a)) + Q(a) = f(a)$$

in $\bar{a}$. For F and Q we get

$$F''(\bar{y})[y_1, y_2] = (y_2, y_1)_{L^2(\Omega)}, \quad Q''(\bar{a})[a_1, a_2] = \alpha(a_2, a_1)_{H^s(\Omega)}.$$

Thus we obtain for $f''(\bar{a})[a_1, a_2]$:

$$\begin{aligned} f''(\bar{a})[a_1, a_2] &= F''(S(\bar{a}))[S'(\bar{a})a_1, S'(\bar{a})a_2] + F'(S(\bar{a}))S''(\bar{a})[a_1, a_2] + Q''(\bar{a})[a_1, a_2] \\ &= (S'(\bar{a})a_2, S'(\bar{a})a_1)_{L^2(\Omega)} + \alpha(a_2, a_1)_{H^s(\Omega)} \\ &\quad + (S(\bar{a}) - y_d, S''(\bar{a})[a_1, a_2])_{L^2(\Omega)} \end{aligned}$$

and consequently

$$\begin{aligned} f''(\bar{a})[a_1, a_2] &= (y_2', y_1')_{L^2(\Omega)} + \alpha(a_2, a_1)_{H^s(\Omega)} + (S(\bar{a}) - y_d, y'')_{L^2(\Omega)} \\ &= (y_2', y_1')_{L^2(\Omega)} + \alpha(a_2, a_1)_{H^s(\Omega)} \\ &\quad - (a_1 \nabla y_2', \nabla \bar{p})_{(L^2(\Omega))^N} - (a_2 \nabla y_1', \nabla \bar{p})_{(L^2(\Omega))^N}. \end{aligned}$$

Again we obtain the last terms by comparing the weak formulations of the adjoint equation and (10) with y'' and $\bar{p}$ as test function, respectively.

In the next theorem we provide a second order sufficient condition. During the proof we will use the following estimation which we are going to prove afterwards. For the moment we assume it to hold.

$$[f''(\bar{a} + \theta(a - \bar{a})) - f''(\bar{a})](a - \bar{a})^2 \leq L\|a - \bar{a}\|_{L^\infty(\Omega)} \|a - \bar{a}\|^2_{H^s(\Omega)} \tag{11}$$

for all $s \geq \frac{N}{q}$, $\theta \in (0, 1)$ and some positive constant L.

Theorem 3.4. *Let* (11) *be valid and let the parameter $\bar{a} \in A_{ad}$, the associated state $\bar{y} = S(\bar{a})$ and the adjoint state p fulfill the necessary condition* (9). *If in addition $\bar{a}$ and $\bar{y}$ satisfy the second-order-sufficient-condition*

$$f''(\bar{a})(a - \bar{a})^2 \geq \delta \|a - \bar{a}\|^2_{H^s(\Omega)} \tag{12}$$

for some constant $\delta > 0$ and for all $a \in A_{ad}$, then there are constants $\varepsilon > 0$ und $\sigma > 0$, such that the quadratic condition for growth

$$f(a) \geq f(\bar{a}) + \sigma \|a - \bar{a}\|^2_{H^s(\Omega)}$$

holds for all $a \in A_{ad}$ with $\|a - \bar{a}\|_{L^\infty(\Omega)} \leq \varepsilon$ and the belonging state $y = S(a)$. Thus $\bar{a}$ is a locally optimal parameter.

Proof. We develop the Taylor expansion up to the term of second order

$$f(a) = f(\bar{a}) + f'(\bar{a})(a - \bar{a}) + \frac{1}{2} f''(\bar{a} + \theta(a - \bar{a}))(a - \bar{a})^2$$

with $\theta \in (0, 1)$. The first order term is nonnegative due to the necessary condition. We now estimate the second order term

$$\begin{aligned} f''(\bar{a} + \theta(a - \bar{a}))(a - \bar{a})^2 &= f''(\bar{a})(a - \bar{a})^2 + [f''(\bar{a} + \theta(a - \bar{a})) - f''(\bar{a})](a - \bar{a})^2 \\ &\geq \delta \|a - \bar{a}\|^2_{H^s(\Omega)} - L\|a - \bar{a}\|_{L^\infty(\Omega)} \|a - \bar{a}\|^2_{H^s(\Omega)} \\ &\geq \frac{\delta}{2} \|a - \bar{a}\|^2_{H^s(\Omega)}. \end{aligned}$$

This is valid if ε is sufficiently small, namely $\varepsilon \leq \frac{\delta}{2L}$. For these estimates we used (11) and the sufficient optimality condition (12). At last we obtain

$$f(a) \geq f(\bar{a}) + \frac{\delta}{4} \|a - \bar{a}\|^2_{H^s(\Omega)} = f(\bar{a}) + \sigma \|a - \bar{a}\|^2_{H^s(\Omega)}$$

with $\sigma = \frac{\delta}{4}$, if $\|a - \bar{a}\|^2_{L^\infty(\Omega)} \leq \varepsilon$ and $\varepsilon \leq \frac{\delta}{2L}$. $\square$

We are going to prove the estimate (11) now, that was essential to the proof of the second order sufficient condition. For simplicity we do the proof for a, y and p instead of $\bar{a}, \bar{y}$ and $\bar{p}$. It turns out that a size condition for s is needed. In the following we choose $s \geq \frac{N}{\bar{q}}$ and we will use (4). Let us first of all clarify some notation. The functions $y_h = S(a+h)$, p_h, $y_i' = S'(a)a_i$ and $y_{i,h}' = S'(a+h)a_i$ are the weak solutions of the following partial differential equations

$$\begin{aligned}
-\nabla \cdot ((a+h)\nabla y_h) &= g \\
-\nabla \cdot ((a+h)\nabla p_h) &= y_h - y_d \\
-\nabla \cdot (a\nabla y_i') &= \nabla \cdot (a_i \nabla y) \\
-\nabla \cdot ((a+h)\nabla y_{i,h}') &= \nabla \cdot (a_i \nabla y_h),
\end{aligned}$$

each with homogeneous Dirichlet boundary conditions, $i \in \{1, 2\}$.

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz-domain. Then there is a constant L belonging to the objective functional*

$$f(a) = J(y,a) = J(S(a),a) = \frac{1}{2}\|S(a) - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|a\|_{H^s(\Omega)}^2,$$

that is independent from a, h, a_1, a_2, such that

$$|f''(a+h)[a_1,a_2] - f''(a)[a_1,a_2]| \leq \; L\|h\|_{L^\infty(\Omega)}\|a_1\|_{H^s(\Omega)}\|a_2\|_{H^s(\Omega)}$$

for all $a, h, a_1, a_2 \in L^\infty(\Omega)$.

For the proof we split the left hand side of the last inequality into six terms

$$\begin{aligned}
&|f''(a+h)[a_1,a_1] - f''(a)[a_1,a_2]| \\
&= |\alpha(a_1,a_2)_{H^s(\Omega)} + (y_{1,h}', y_{2,h}')_{L^2(\Omega)} - (a_2\nabla y_{1,h}', \nabla p_h)_{(L^2(\Omega))^N} \\
&\quad - (a_1\nabla y_{2,h}', \nabla p_h)_{(L^2(\Omega))^N} - \alpha(a_1,a_2)_{H^s(\Omega)} - (y_1', y_2')_{L^2(\Omega)} \\
&\quad + (a_2\nabla y_1', \nabla p)_{(L^2(\Omega))^N} - (a_1\nabla y_2', \nabla p)_{(L^2(\Omega))^N}| \\
&\leq |(y_{1,h}', y_{2,h}' - y_2')_{L^2(\Omega)}| + |(y_{1,h}' - y_1', y_2')_{L^2(\Omega)}| \\
&\quad + |(a_2\nabla y_{1,h}', \nabla(p_h - p))_{(L^2(\Omega))^N}| + |(a_2\nabla(y_{1,h}' - y_1'), \nabla p)_{(L^2(\Omega))^N}| \\
&\quad + |(a_1\nabla y_{2,h}', \nabla(p_h - p))_{(L^2(\Omega))^N}| + |(a_1\nabla(y_{2,h}' - y_2'), \nabla p)_{(L^2(\Omega))^N}| \\
&= T_1 + T_2 + T_3 + T_4 + T_5 + T_6. \qquad (13)
\end{aligned}$$

In Lemma 3.6 and Lemma 3.7 we will prove auxiliary results. T_1 and T_2 will be estimated in Lemma 3.8. Lemma 3.9 contains the estimate of T_3 and T_5. The remaining terms T_4 and T_6 are considered in Lemma 3.10. The occuring partial differential equations have homogeneous Dirichlet boundary conditions.

Lemma 3.6. *The following estimate holds true*

$$\|y_h - y\|_{W^{1,\tilde{q}}(\Omega)} \leq c\|h\|_{L^\infty(\Omega)} \qquad (14)$$

for arbitrary $\tilde{q} \in [2, \bar{q}]$.

Proof. We consider the equations for y_h and y respectively and subtract them.

$$-\nabla\cdot((a+h)\nabla y_h)+\nabla\cdot(a\nabla y)=g-g \;\Leftrightarrow\; -\nabla\cdot(a\nabla(y_h-y))=\nabla\cdot(h\nabla y_h)$$

With (3) and again F denoting the right hand side of the given PDE we get $\|y_h-y\|_{W^{1,\tilde q}(\Omega)}\le c_{\bar q}\|\nabla\cdot(h\nabla y_h)\|_{W^{-1,\tilde q}(\Omega)}\le c\|h\nabla y_h\|_{(L^{\tilde q}(\Omega))^N}\le c\|h\|_{L^\infty(\Omega)}\|y_h\|_{W^{1,\tilde q}(\Omega)}$ $\le c\|h\|_{L^\infty(\Omega)}$ since y_h is uniformly bounded in $W^{1,\tilde q}(\Omega)$ for $\tilde q\in[2,\bar q]$ for all $h\in L^\infty(\Omega)$ due to (3) that we obtained from [9, Theorem 1]. □

Lemma 3.7. *The estimate*

$$\|y'_{i,h}\|_{H^1(\Omega)}\le c\|a_i\|_{H^s(\Omega)} \tag{15}$$

is satisfied for $i\in\{1,2\}$.

Proof. We consider the partial differential equations $-\nabla\cdot(a\nabla y_i')=\nabla\cdot(a_i\nabla y)$ with the weak solutions $y'_{i,h}$, $i\in\{1,2\}$ and use (2). Hence, we obtain after some calculation $\|y'_{i,h}\|_{H^1(\Omega)}\le c\|a_i\nabla y_h\|_{(L^2(\Omega))^N}$. In the next estimate we use Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$

$$\|y'_{i,h}\|_{H^1(\Omega)}\le c\|a_i\|_{L^{2p}(\Omega)}\|\nabla y_h\|_{(L^{2p'}(\Omega))^N}.$$

With the choice $2p':=\bar q$, the states y_h are uniformly bounded in $W^{1,2p'}(\Omega)$ for all $h\in L^\infty(\Omega)$ due to (3). This choice yields $2p=\frac{1}{\frac{1}{2}-\frac{1}{\bar q}}$ and with (4) we obtain $\|a_i\|_{L^{2p}(\Omega)}\le\|a_i\|_{H^s(\Omega)}$, and thus

$$\|y'_{i,h}\|_{H^1(\Omega)}\le c\|a_i\|_{H^s(\Omega)}\|y_h\|_{W^{1,2p'}(\Omega)}\le c\|a_i\|_{H^s(\Omega)}$$

again with a generic constant c. □

Lemma 3.8. *The terms T_1 and T_2 can be bounded by*

$$\begin{aligned}|(y'_{1,h},y'_{2,h}-y'_2)_{L^2(\Omega)}|&\le c\|a_1\|_{H^s(\Omega)}\|a_2\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)},\\ |(y'_{1,h}-y'_1,y'_2)_{L^2(\Omega)}|&\le c\|a_1\|_{H^s(\Omega)}\|a_2\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)}.\end{aligned}$$

Proof. With (3.7) we obtain in both cases

$$\begin{aligned}|(y'_{1,h},y'_{2,h}-y'_2)_{L^2(\Omega)}|&\le\|y'_{1,h}\|_{H^1(\Omega)}\|y'_{2,h}-y'_2\|_{H^1(\Omega)}\le c\|a_1\|_{H^s(\Omega)}\|y'_{2,h}-y'_2\|_{H^1(\Omega)}\\ |(y'_{1,h}-y'_1,y'_2)_{L^2(\Omega)}|&\le\|y'_{1,h}-y'_1\|_{H^1(\Omega)}\|y'_2\|_{H^1(\Omega)}\;\le c\|a_2\|_{H^s(\Omega)}\|y'_{1,h}-y'_1\|_{H^1(\Omega)}.\end{aligned}$$

It remains to show

$$\|y'_{i,h}-y'_i\|_{H^1(\Omega)}\le c\|a_i\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)},\quad i\in\{1,2\}. \tag{16}$$

We consider the equations belonging to $y'_{i,h}$ and y'_i and subtract them

$$-\nabla\cdot((a+h)\nabla y'_{i,h})+\nabla\cdot(a\nabla y'_i)=\nabla\cdot(a_i\nabla y_h)-\nabla\cdot(a_i\nabla y)$$
$$\Leftrightarrow -\nabla\cdot(a\nabla(y'_{i,h}-y'_i))=\nabla\cdot(a_i\nabla(y_h-y))+\nabla\cdot(h\nabla y'_{i,h}).$$

With (2) for the first estimate and Hölder's inequality for the second estimate we get

$$\begin{aligned}\|y'_{i,h}-y'_i\|_{H^1(\Omega)} &\le c_2\left(\|a_i\nabla(y_h-y)\|_{(L^2(\Omega))^N}+\|h\nabla y'_{i,h}\|_{(L^2(\Omega))^N}\right)\\ &\le c\left(\|a_i\|_{L^{2p}(\Omega)}\|y_h-y\|_{L^{2p'}(\Omega)}+\|h\|_{L^\infty(\Omega)}\|\nabla y'_{i,h}\|_{(L^2(\Omega))^N}\right)\end{aligned}$$

with $p,p'>1,\ \frac{1}{p}+\frac{1}{p'}=1$. Set $2p':=\bar q \Leftrightarrow 2p=\frac{1}{\frac{1}{2}-\frac{1}{\bar q}}$ and apply (4):

$$\|y'_{i,h}-y'_i\|_{H^1(\Omega)}\le c\left(\|a_i\|_{H^s(\Omega)}\|y_h-y\|_{W^{1,\bar q}(\Omega)}+\|h\|_{L^\infty(\Omega)}\|y'_{i,h}\|_{H^1(\Omega)}\right).$$

Furthermore, we obtain with (14) and (15)

$$\begin{aligned}\|y'_{i,h}-y'_i\|_{H^1(\Omega)} &\le c\|a_i\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)}+c\|h\|_{L^\infty(\Omega)}\|a_i\|_{H^s(\Omega)}\\ &\le c\|a_i\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)}.\end{aligned}$$ □

Lemma 3.9. *The following estimates for the terms T_3 and T_5 are valid for $i,j\in\{1,2\}$, $i\neq j$:*

$$|(a_i\nabla y'_{j,h},\nabla(p_h-p))_{(L^2(\Omega))^N}|\le c\|a_i\|_{H^s(\Omega)}\|a_j\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)}.$$

The terms T_3 and T_5 are given as

$$|(a_2\nabla y'_{1,h},\nabla(p_h-p))_{(L^2(\Omega))^N}|\quad \textit{and}\quad |(a_1\nabla y'_{2,h},\nabla(p_h-p))_{(L^2(\Omega))^N}|,$$

respectively.

Proof. For $i,j\in\{1,2\}$, $i\neq j$ and $p,p'>1,\frac{1}{p}+\frac{1}{p'}=1$ we get

$$|(a_i\nabla y'_{j,h},\nabla(p_h-p))_{(L^2(\Omega))^N}|\le\|a_i\nabla y'_{j,h}\|_{(L^p(\Omega))^N}\|\nabla(p_h-p)\|_{(L^{p'}(\Omega))^N}$$

We set $p':=\bar q$ and obtain with $q,q'>1,\frac{1}{q}+\frac{1}{q'}$:

$$\begin{aligned}|(a_i\nabla y'_{j,h},\nabla(p_h-p))_{(L^2(\Omega))^N}| &\le\|a_i\|_{L^{pq}(\Omega)}\|\nabla y'_{j,h}\|_{(L^{pq'}(\Omega))^N}\|p_h-p\|_{W^{1,\bar q}(\Omega)}\\ &\le\|a_i\|_{L^{pq}(\Omega)}\|y'_{j,h}\|_{W^{1,pq'}(\Omega)}\|p_h-p\|_{W^{1,\bar q}(\Omega)}\\ &\le\|a_i\|_{L^{pq}(\Omega)}\|a_j\|_{H^s(\Omega)}\|p_h-p\|_{W^{1,\bar q}(\Omega)}\end{aligned}$$

Where we chose $pq':=2$, thus $pq=\frac{1}{\frac{1}{2}-\frac{1}{\bar q}}$ to obtain the last term for $s\ge\frac{N}{\bar q}$ with (4). We accomplish the proof by showing $\|p_h-p\|_{W^{1,\bar q}(\Omega)}\le c\|h\|_{L^\infty(\Omega)}$. Therefore we consider the equations belonging to p and p_h and subtract them

$$-\nabla\cdot(a\nabla(p_h-p))=y_h-y+\nabla\cdot(h\nabla p_h).$$

We estimate this term with (3) in the usual way and arrive at the following estimation

$$\begin{aligned}\|p_h - p\|_{W^{1,\bar q}(\Omega)} &\le c\left(\|y_h - y\|_{L^{\bar q}(\Omega)} + \|h\nabla p_h\|_{(L^{\bar q}(\Omega))^N}\right)\\ &\le c\left(\|y_h - y\|_{W^{1,\bar q}(\Omega)} + \|h\|_{L^\infty(\Omega)}\|\nabla p_h\|_{(L^{\bar q}(\Omega))^N}\right)\\ &\le c\left(\|h\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega)}\|p_h\|_{H^1(\Omega)}\right)\\ &\le c\|h\|_{L^\infty(\Omega)}.\end{aligned}$$

We used the uniform boundedness of $\|p_h\|_{H^1(\Omega)}$ and $\|y_h\|_{H^1(\Omega)}$ due to (2). □

Lemma 3.10. *The terms T_4 and T_6 can be estimated as*

$$|(a_i\nabla(y'_{j,h} - y'_j), \nabla p)_{(L^2(\Omega))^N}| \le c\|a_i\|_{H^s(\Omega)}\|a_j\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)}$$

with $i, j \in \{1, 2\}$, $i \ne j$.

Proof. For $i, j \in \{1, 2\}$, $i \ne j$ and $p, p' > 1, \frac{1}{p} + \frac{1}{p'} = 1$ we get

$$|(a_i\nabla(y'_{j,h} - y'_j), \nabla p)_{(L^2(\Omega))^N}| \le \|a_i\nabla(y'_{j,h} - y'_j)\|_{(L^p(\Omega))^N}\|\nabla p\|_{(L^{p'}(\Omega))^N}.$$

As before we set $p' := \bar q$. With $q, q' > 1, \frac{1}{q} + \frac{1}{q'} = 1$ we get

$$\begin{aligned}|(a_i\nabla(y'_{j,h} - y'_j), \nabla p)_{(L^2(\Omega))^N}| &\le \|a_i\|_{L^{pq}(\Omega)}\|\nabla(y'_{j,h} - y'_j)\|_{(L^{pq'}(\Omega))^N}\|p\|_{W^{1,\bar q}(\Omega)}\\ &\le c\|a_i\|_{H^s(\Omega)}\|y'_{j,h} - y'_j\|_{H^1(\Omega)}\|p\|_{W^{1,\bar q}(\Omega)}\end{aligned}$$

Again we set $pq' := 2$, thus $pq = \frac{1}{\frac{1}{2} - \frac{1}{\bar q}}$ and applied (4) to obtain the last estimate. We use the uniform boundedness of p in $W^{1,\bar q}(\Omega)$ and (16) to complete the proof. □

Finally we have all components for the proof of Theorem 3.5.

Proof of Theorem 3.5. In Lemmata 3.8–3.10 we showed how the different terms in (13) can be estimated. All in all this yields to

$$\begin{aligned}|f''(a+h)[a_1, a_1] - f''(a)[a_1, a_2]| &= T_1 + T_2 + T_3 + T_4 + T_5 + T_6\\ &\le c\|a_i\|_{H^s(\Omega)}\|a_j\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)}\end{aligned}$$

for all $a, h, a_1, a_2 \in L^\infty(\Omega)$. □

Corollary 3.11. *Condition* (11) *is satisfied for $s \ge \frac{N}{\bar q}$. Consequently, Theorem* 3.4 *is valid under this assumption.*

Remark 3.12. [9, Theorem 1] enables us to to be more precise about the size of the constant $\bar{q}$. In particular for domains with smooth boundaries we find the asymptotic properties

$$\frac{a_{\min}}{a_{\max}} \to 0 \;\Rightarrow\; \bar{q} \to 2,$$
$$\frac{a_{\min}}{a_{\max}} \to 1 \;\Rightarrow\; \bar{q} \to \infty.$$

If the quantity $\frac{a_{\min}}{a_{\max}}$ is small, then we have to choose $s \sim \frac{N}{2}$. Otherwise, if the quantity $\frac{a_{\min}}{a_{\max}}$ is close to 1, then we can use a small parameter s. The size of s influences the regularity of the optimal solution $\bar{a}$. Small parameters s are of practical interest for situations with jumping coefficients.

4. Conclusion

We proved existence of an optimal solution for arbitrary parameters $s > 0$. Furthermore we derived first order necessary optimality conditions for arbitrary parameters $s > 0$ as well, whereas we required further restrictions to this parameter, namely $s \geq \frac{N}{\bar{q}}$, in order to derive second order sufficient conditions.

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