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Universal Singular Sets and Unrectifiability

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Abstract. The geometry of universal singular sets has recently been studied by M. Csörnyei et al. [Arch. Ration. Mech. Anal. 190 (2008)(3), 371–424]. In particular they proved that given a purely unrectifiable compact set $S \subseteq \mathbb{R}^2$, one can construct a C^∞-Lagrangian with a given superlinearity such that the universal singular set of L contains S. We show the natural generalization: That given an F_σ purely unrectifiable subset of the plane, one can construct a C^∞-Lagrangian, of arbitrary superlinearity, with universal singular set covering this subset.

Keywords. Partial regularity, universal singular set, purely unrectifiable set

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1. Introduction

The basic problem of the (one-dimensional) calculus of variations is that of minimizing the functional

$$\mathscr{L}(u) = \int_a^b L(x, u(x), u'(x))\, dx \tag{1}$$

for a given function $L\colon [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ known as the *Lagrangian*, over the class of absolutely continuous functions $u\colon [a,b] \to \mathbb{R}$ with prescribed boundary conditions $u(a) = A$, $u(b) = B$. Tonelli [8,9] performed the first rigorous analysis of this problem, giving conditions on the Lagrangian $L = L(x, y, p)$ to guarantee existence and C^k-regularity of a minimizer, see also the book [3] for a good summary of these results. The key assumptions on L for existence are some continuity, and convexity and superlinearity in the third variable p.

Tonelli also gave the following *partial regularity* statement: For $L \in C^3(\mathbb{R}^3)$ satisfying $L_{pp} > 0$, any minimizer has a (possibly infinite) classical derivative everywhere in $[a,b]$, and this derivative is continuous as a map into the extended real line. Thus the *singular set*, defined to be the set of points in $[a,b]$ where the derivative is infinite, is closed (and necessarily Lebesgue null, since u' is integrable, see for example [3, Theorem 2.17]).

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Tonelli gave various conditions to ensure the singular set is empty, but it has been known since the work of Lavrentiev [6] that minimizers can have infinite derivative. Ball and Mizel [1] gave the first examples of problems satisfying the exact requirements of the partial regularity theorem which have minimizers with non-empty singular set. They also studied the properties of the singular set, and Davie [5] completed this work, showing that nothing more can be said about it except the immediate information that it is closed and Lebesgue null.

The superlinearity condition (see subsection 2.1 below) required for existence of minimizers would seem to prevent infinite derivatives from occurring too often in solutions to minimization problems, since large derivatives imply high energies. That this is not the case must be due to dramatic behaviour of the Lagrangian in the first two coordinates around the point in the plane where the graph of a singular minimizer has a vertical tangent. To make this observation more precise, Ball and Nadirashvili [2] introduced the *universal singular set* of a Lagrangian, which records for which points (x, y) in the plane there exists an interval $[a, b]$ and a choice of boundary conditions $a = A$, $b = B$, such that the problem (1) has a minimizer with graph passing through (x, y) with infinite derivative. They showed that this set is of the first Baire category, and Sychëv [7] showed that it has zero two-dimensional measure. Csörnyei et al. [4] showed that universal singular sets intersect most absolutely continuous curves in sets of zero linear measure, the exceptions being some curves with vertical tangents. They also showed that any set covered by universal singular sets of Lagrangians with arbitrary superlinearity is purely unrectifiable. The ability to choose the superlinearity arbitrarily seems here to be the key. To prove this pure unrectifiability result, a superlinearity is constructed to deal with a given test C^1 curve. On the other hand the universal singular set of a Lagrangian with some fixed superlinearity can contain non-trivial rectifiable curves, see [4, Theorem 11].

To complement these results, they give examples of Lagrangians with large universal singular sets, in particular showing that any compact purely unrectifiable set can lie inside the universal singular set of a smooth Lagrangian with arbitrary superlinearity. In this paper we show that this is true for F_σ purely unrectifiable sets. A natural converse to this result would be that any set E which can be covered by universal singular sets of smooth Lagrangians with arbitrary superlinearity must admit a purely unrectifiable F_σ cover. That this might be true seems plausible: By [4, Theorem 8] (see below) E is purely unrectifiable, and moreover, for any fixed superlinearity, since the Lagrangians are smooth, each universal singular set is F_σ, as proved in [2]. However, as mentioned above, it is not true in general that for a fixed superlinearity these universal singular sets are purely unrectifiable. It is not currently known whether E must in fact admit an F_σ purely unrectifiable cover.

2. Preliminaries

2.1. Notation. We denote by $\|\cdot\|_\infty$ the usual supremum norm on $\mathbb{R}^2$, which shall be the norm we use throughout the paper and for the following definitions. For $x \in \mathbb{R}^2$ and $r > 0$, we write $B_r(x)$ for the open ball of radius r around x; analogously for $A \subseteq \mathbb{R}^2$ we write $B_r(A)$ for the open r-neighbourhood around A. The distance between two non-empty subsets $A, B \subseteq \mathbb{R}^2$ shall be denoted $\mathrm{dist}(A, B)$; in the case that $A = \{x\}$ for some $x \in \mathbb{R}^2$ then we write $\mathrm{dist}(x, B)$. The notation $\|\cdot\|$ will be used for the supremum norm of a function on $\mathbb{R}^2$.

Given $(a, A), (b, B) \in \mathbb{R}^2$, we let $Q(a, A; b, B)$ denote the smallest closed rectangle in $\mathbb{R}^2$ with two vertices at (a, A) and (b, B) and sides parallel to the coordinate axes (we admit the possibility that this contains zero area). For any function $u\colon \mathbb{R} \to \mathbb{R}$ we let $U\colon \mathbb{R} \to \mathbb{R}^2$ be given by $U(x) = (x, u(x))$.

For a bounded interval $[a, b]$ in $\mathbb{R}$, we shall write $\mathrm{AC}(a, b)$ for the class of absolutely continuous functions on $[a, b]$. We denote by λ the one-dimensional Lebesgue measure on $\mathbb{R}$. Partial derivatives shall be denoted by subscripts, e.g. Φ_x, Φ_y, and L_p for functions $\Phi = \Phi(x, y)\colon \mathbb{R}^2 \to \mathbb{R}$ and $L = L(x, y, p)\colon \mathbb{R}^3 \to \mathbb{R}$.

We recall that a set $S \subseteq \mathbb{R}^2$ is *purely unrectifiable* if it meets every Lipschitz curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ in a set of linear measure zero.

We shall call a function $\omega \in C^\infty(\mathbb{R})$ a *superlinearity* if

- $\omega(p) \geq \omega(0) = 0$ for all $p \in \mathbb{R}$;
- ω is strictly convex; and
- (superlinearity) $\frac{\omega(p)}{|p|} \to \infty$ as $|p| \to \infty$.

A *Lagrangian* shall be a function $L = L(x, y, p)\colon \mathbb{R}^3 \to \mathbb{R}$, of class C^∞, superlinear and strictly convex in p, where here superlinear means that for some superlinearity ω, $L(x, y, p) \geq \omega(p)$ for all $(x, y, p) \in \mathbb{R}^3$. These assumptions suffice to guarantee existence and partial regularity of a solution to the minimization problem (1) over those $u \in \mathrm{AC}(a, b)$ satisfying $u(a) = A$ and $u(b) = B$ (see for example [3, Theorem 3.7]).

All of our Lagrangians will be of the form $L(x, y, p) = F(x, y, p) + \omega(p)$, for functions $F\colon \mathbb{R}^3 \to \mathbb{R}$ satisfying the following conditions, which we shall refer to as $(\star_F)$:

$(\star_1)$ $F \in C^\infty(\mathbb{R}^3)$;
$(\star_2)$ $F \geq 0$ on $\mathbb{R}^3$ and $F(x, y, 0) = 0$ for all $(x, y) \in \mathbb{R}^2$; and
$(\star_3)$ $p \mapsto F(x, y, p)$ is convex for each fixed $(x, y) \in \mathbb{R}^2$.

We shall say a Lagrangian L of this form is of form $(\star)$ (so this terminology agrees with that of [4]).

2.2. Universal singular sets. Ball and Nadirashvili [2] give the following definition.

Definition 2.1. The universal singular set of a Lagrangian $L\colon \mathbb{R}^3 \to \mathbb{R}$, which we shall write $\mathrm{uss}(L)$, is defined as those points $(x_0, y_0) \in \mathbb{R}^2$ where one can find an interval $[a, b]$ in $\mathbb{R}$ containing x_0 and a choice of boundary conditions $u(a)=A$, $u(b)=B$, such that there is a minimizer u of the associated variational problem (1) with $u(x_0) = y_0$ and $|u'(x_0)| = \infty$.

In [4], Csörnyei et al. prove the following theorems, their theorems 8 and 10 respectively:

Theorem 2.2. *Let $E \subseteq \mathbb{R}^2$ be such that for any superlinearity ω there is a Lagrangian $L\colon \mathbb{R}^3 \to \mathbb{R}$ with this prescribed superlinearity such that the universal singular set of the Lagrangian contains E. Then E is purely unrectifiable.*

Theorem 2.3. *Let ω be a given superlinearity, and let $S \subseteq \mathbb{R}^2$ be a compact purely unrectifiable set.*
Then there exists a Lagrangian L of form $(\star)$ with the prescribed superlinearity ω such that the universal singular set of L contains S.

The result of this paper is the natural generalization of this latter result to F_σ purely unrectifiable sets:

Theorem 2.4. *Let ω be a given superlinearity, and let $S \subseteq \mathbb{R}^2$ be an F_σ purely unrectifiable set.*
Then there exists a Lagrangian L of form $(\star)$ with the prescribed superlinearity ω such that the universal singular set of L contains S.

The remainder of the paper gives the construction of such a Lagrangian.

3. The construction: General discussion

Suppose $S = \bigcup_{n=1}^{\infty} S_n$, where each S_n is compact and purely unrectifiable. We construct by induction a sequence of Lagrangians L_n such that for each $n \geq 1$ we have $\mathrm{uss}(L_n) \supseteq \bigcup_{m=1}^{n} S_m$. We discuss how to do this so that the L_n converge to a function L with $\mathrm{uss}(L) \supseteq S$.

Fix a point $(x_0, y_0) \in S_n \setminus \bigcup_{m=1}^{n-1} S_m$. We construct Lagrangian L_n and function $\Phi_n \in C(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus S)$ such that there is a rectangular neighbourhood $Q(a_0, A_0; b_0, B_0)$ of (x_0, y_0) such that for any $u \in \mathrm{AC}(a_0, b_0)$ with graph lying in $Q(a_0, A_0; b_0, B_0)$, we have

$$\int_{a_0}^{b_0} L_n(x, u, u') \geq \Phi_n(U(b_0)) - \Phi_n(U(a_0)), \tag{2}$$

with equality if and only if $u'(x) = \psi_n(x, u(x))$ almost everywhere, where we ensure $\psi_n := -2(\Phi_n)_x/(\Phi_n)_y$ is well-defined on $\mathbb{R}^2 \setminus S$. We then solve the ordinary differential equation $u_0'(x) = \psi_n(x, u_0(x))$ for a locally absolutely continuous $u_0 \colon \mathbb{R} \to \mathbb{R}$ with $u_0(x_0) = y_0$. If Φ_n was constructed so that $\psi_n(x,y) \to \infty$ as $\operatorname{dist}((x,y), S_n) \to 0$, we then have $u_0'(x) \to \infty$ as $x \to x_0$. Moreover, the construction in [4] guarantees that non-monotone functions cannot be minimizers. Given this, a calibration argument implies that inequality (2) suffices to prove that u_0 is a minimizer with respect to its own boundary conditions on $[a_0, b_0]$. This shows that $(x_0, y_0) \in \operatorname{uss}(L_n)$.

Let $m \geq n$. If we have constructed our Lagrangians so that firstly $L_m \geq L_n$, we have

$$\int_{a_0}^{b_0} L_m(x, u, u') \geq \Phi_n(U(b_0)) - \Phi_n(U(a_0))$$

for all $u \in \mathrm{AC}(a_0, b_0)$. If secondly we can guarantee that $L_m(x, u_0, u_0') = L_n(x, u_0, u_0')$ for almost every $x \in (a_0, b_0)$, where u_0 is the solution of the ODE mentioned above, then we have that

$$\int_{a_0}^{b_0} L_m(x, u_0, u_0') = \Phi_n(U_0(b_0)) - \Phi_n(U_0(a_0)).$$

Thus u_0 is a minimizer of the functional given via Lagrangian L_m over $\mathrm{AC}(a_0, b_0)$ with respect to its own boundary conditions. Assuming the Lagrangians L_n converge pointwise to a Lagrangian L, we let $m \to \infty$ in these two relations to see that u_0 is a minimizer of the functional given via Lagrangian L over $\mathrm{AC}(a_0, b_0)$ with respect to its own boundary conditions.

This outline of our strategy gives us two requirements at the inductive step of constructing L_n. The details of this inductive step mimic those of the original proof in [4]. For a given superlinearity ω, they build a function $F \colon \mathbb{R}^3 \to \mathbb{R}$ of form $(\star_F)$ and define $L(x, y, p) = F(x, y, p) + \omega(p)$. The key observation to make about this proof when seeking to generalize it for our purposes is that ω may be regarded just as a Lagrangian depending only on p. Or rather, the role of ω may be taken by any Lagrangian strictly convex and superlinear in p, with partial derivatives with respect to p replacing any ω' terms. In particular, the argument may be applied to a previously constructed L_{n-1}. The arguments in [4] then tell us how to construct an F_n of form $(\star_F)$ to add to this L_{n-1}, via the construction of a potential $\Phi_n \in C(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \backslash S_n)$. The considerations of the preceding paragraphs mean the argument is rather more intricate, but the general strategy is the same.

Ensuring that $L_m \geq L_n$ for all $m \geq n$ is easy; this just requires the stipulation that each F_n is non-negative, which is already given by the methods of [4]. Harder is ensuring that for each point $(x_0, y_0) \in \bigcup_{m=1}^{n-1} S_m$, we have $L_n = L_{n-1}$ along the trajectory u_0 (except perhaps on a null set) on some fixed neighbourhood of x_0. The key fact here is that precisely by construction we know that $u_0' = \psi_m(x, u_0)$ for some $1 \leq m < n$, where $\psi_m \in C^\infty(\mathbb{R}^2 \backslash S_m)$, and therefore ψ_m is bounded on sets positively separated from S_m.

At this point it is easiest to first suppose that the $\{S_n\}_{n=1}^{\infty}$ are pairwise disjoint. Thus S_n is positively separated from $\bigcup_{m=1}^{n-1} S_m$, so we can choose a neighbourhood H_n of S_n on which ψ_m is bounded above for all $1 \leq m < n$, by M_n say. So to construct an appropriate L_n, the condition is now just that F_n is only non-zero on $H_n \times (M_n, \infty)$. A straightforward use of a cut-off function on $\mathbb{R}^2$ ensures $F_n(x, y, p) = 0$ for $(x, y) \notin H_n$. The demand that $F_n(x, y, p) = 0$ if $(x, y, p) \in H_n \times (M_n, \infty)$ reduces to certain inequalities involving the derivatives of the potential Φ_n on the set H_n. These can be satisfied using a construction of the potential similar to the construction from [4].

The existence of a pointwise limit $L(x, y, p) := \lim_{n\to\infty} L_n(x, y, p)$ is trivial if the lower bounds M_n tend to infinity: Then for each fixed $(x, y, p) \in \mathbb{R}^3$, for large enough n, L_n does not change on a neighbourhood of (x, y, p), so the limit L exists and is smooth. The arguments sketched above show that $\mathrm{uss}(L) \supseteq \bigcup_{m=1}^{\infty} S_m$.

This discussion applies directly only to the disjoint case, but the spirit of the proof is retained in the full version. The issue in the general case is that we of course no longer have positive separation of our compact sets, and hence cannot in general find an upper bound on S_n for the derivative of a minimizer u_0 witnessing $\bigcup_{m=1}^{n-1} S_m \subseteq \mathrm{uss}(L_{n-1})$. This in turn implies that the region in $\mathbb{R}^3$ on which we may modify L_{n-1} is not in general bounded below, rather as we approach $\bigcup_{m=1}^{n-1} S_m$, the base "slopes up to infinity".

Throughout the paper we adhere to the indexing suggested above. Subscripts such as m, n refer to the inductive step. Superscripts such as i, j, k, l are used to index sequences of objects discussed within the argument at a fixed inductive step. This superscript notation is retained even in arguments presented independently of the induction (e.g. in Lemma 4.2) to avoid confusion.

4. The construction: Details

Our first result is a modified version of Lemma 11 from [4]. This tells us how to modify a given Lagrangian so as to include new points in its universal singular set, but without changing it on certain "cylinders" in $\mathbb{R}^3$. We try to motivate the exact assumptions made in the next lemma by sketching its role in the inductive construction of L_n. First we note that the set G does not appear in the conclusions, only in the assumptions regarding Φ. G_n will be chosen to be a bounded open cover of S_n, but there is no loss of understanding in assuming $G = \mathbb{R}^2$ for this first lemma. We choose a sequence $\{V_n^i\}_{i=1}^{\infty}$ covering $S_n \setminus \bigcup_{m=1}^{n-1} S_m$, but such that each V_n^i is positively separated from $\bigcup_{m=1}^{n-1} S_m$, and also a sequence of upper bounds $\{M_n^i\}_{i=1}^{\infty}$ of ψ_m $(1 \leq m < n)$ on V_n^i, and thus a sequence of "cylinders" $\{V_n^i \times (M_n^i, \infty)\}_{i=1}^{\infty}$ in $\mathbb{R}^3$. Our goal, as discussed above, is to construct a function F_n of form $(\star_F)$ which is zero off all these sets. We show, just as in [4], that such an F_n is given by a potential $\Phi_n \in C(\mathbb{R}) \cap C^{\infty}(\mathbb{R}^2 \setminus S_n)$, where the derivatives of Φ_n satisfy certain inequalities. The inequalities we require are similar to but more complicated than those from [4], since we demand also some information about our resulting function F_n on the sets $V_n^i \times (M_n^i, \infty)$. We also need to fix a neighbourhood W_n of $S_n \setminus \bigcup_{m=1}^{n-1} S_m$ which will contain the graphs of $u_0 \in \mathrm{AC}(a_0, b_0)$ which witness that $(x_0, y_0) \in \mathrm{uss}(L_n)$ for each $(x_0, y_0) \in S_n \setminus \bigcup_{m=1}^{n-1} S_m$. Since L_n is already determined on $\bigcup_{m=1}^{n-1} S_m$, we keep this neighbourhood W_n in some sense as far from $\bigcup_{m=1}^{n-1} S_m$ as possible. Ideally (viz in the disjoint case) we would have that W_n is compactly contained in $V_n := \bigcup_{i=1}^{\infty} V_n^i$, but since W_n must cover $S_n \setminus \bigcup_{m=1}^{n-1} S_m$ and V_n must not intersect $\bigcup_{m=1}^{n-1} S_m$, this is not in general possible. The best we can ask for is that W_n does not approach the boundary of V_n unless it is required to do so to cover all the points of S_n, hence the condition on $\overline{W}$ below.

Before stating our result, we recall [4, Lemma 10], which we shall use just as it is used in this original paper. We do not repeat the (simple) proof.

Lemma 4.1. *There exists a C^∞ function $\gamma\colon \{(p, a, b) \in \mathbb{R}^3 : b > 0\} \to \mathbb{R}$ with the following properties:*

(4.1.a) $p \mapsto \gamma(p, a, b)$ *is convex;*
(4.1.b) $\gamma(p, a, b) = 0$ *for* $p \leq a - 1$;
(4.1.c) $\gamma(p, a, b) = b(p - a)$ *for* $p \geq a + 1$; *and*
(4.1.d) $\gamma(p, a, b) \geq \max\{0, b(p - a)\}$.

Lemma 4.2. *Let $F\colon \mathbb{R}^3 \to \mathbb{R}$ be of form $(\star_F)$, $S \subseteq \mathbb{R}^2$ be compact, and $G \supseteq S$ be open. Let $L(x, y, p) = \omega(p) + F(x, y, p)$, where ω is a given superlinearity. Suppose further that $\Phi \in C^\infty(\mathbb{R}^2 \setminus S) \cap C(\mathbb{R}^2)$, sequence $\{V^i\}_{i=1}^\infty$ of sets $V^i \subseteq \mathbb{R}^2$, and sequence of non-negative constants $\{M^i\}_{i=1}^\infty$ are such that the set $V := \bigcup_{i=1}^\infty V^i$ is open and bounded, $\overline{V} \subseteq G$, and the following conditions hold:*

(4.2.1) Φ *is decreasing in x and increasing in y on* $\mathbb{R}^2$;
(4.2.2) $-\Phi_x(x, y) \geq (2M^i + 4)\Phi_y(x, y)$ *for* $(x, y) \in V^i \setminus S$ *for all* $i \geq 1$, *and* $-\Phi_x(x, y) \geq 4\Phi_y(x, y) > 0$ *for* $(x, y) \in \mathbb{R}^2 \setminus S$;
(4.2.3) $\Phi_y(x, y) \geq 4L_p\big(x, y, -2\frac{\Phi_x}{\Phi_y}(x, y)\big)$ *for* $(x, y) \in G \setminus S$;
(4.2.4) $\lim_{0 < \mathrm{dist}((x,y),S) \to 0} \frac{\Phi_x}{\Phi_y}(x, y) = -\infty$;
(4.2.5) *for all $a < b$ and non-decreasing functions $u \in \mathrm{AC}(a, b)$, the sets $\{x : U(x) \in S\}$ and $\{(\Phi \circ U)(x) : U(x) \in S\}$ are Lebesgue null.*

Then for any $W \subseteq V$ such that $\overline{W} \setminus V \subseteq S$, there exists $\hat{F}\colon \mathbb{R}^3 \to \mathbb{R}$ of form $(\star_F)$ with the following properties:

(4.2.6) $\hat{F} \geq F$ *on* $\mathbb{R}^3$;
(4.2.7) $\hat{F} = F$ *on* $\mathbb{R}^3 \setminus \bigcup_{i=1}^\infty (V^i \times (M^i, \infty))$; *and*
(4.2.8) $\hat{L}\colon \mathbb{R}^3 \to \mathbb{R}$ *defined by*

$$\hat{L}(x, y, p) = \omega(p) + \hat{F}(x, y, p)$$

has the property that for all $a < b$ and all $u \in \mathrm{AC}(a,b)$ such that $Q(a,u(a);b,u(b)) \subseteq W$, we have

$$\int_a^b \hat{L}(x,u(x),u'(x))\,dx \geq \Phi(U(b)) - \Phi(U(a));$$

with equality if and only if $u'(x) = -2\frac{\Phi_x}{\Phi_y}(x,u(x))$ for almost every $x \in [a,b]$.

Proof. We mimic the proof of [4, Lemma 11], but now working with $L = F + \omega$, instead of just ω. The main difference in our assumptions from those in the original lemma from [4] is the dependence of the inequality in (4.2.2) on the sets V_i. This is exactly the stronger information we need to guarantee the conclusion (4.2.7) which we now require.

Define $\psi \in C^\infty(\mathbb{R}^2 \backslash S)$ and $\theta, \xi \in C^\infty(G \backslash S)$ by

$$\psi = -2\frac{\Phi_x}{\Phi_y}, \quad \theta = \Phi_y - L_p(x,y,\psi), \quad \xi = \frac{-\Phi_x + L(x,y,\psi) - \psi L_p(x,y,\psi)}{\theta}.$$

Condition (4.2.2) ensures ψ is well-defined and strictly positive everywhere on $\mathbb{R}^2 \backslash S$. By properties of ω and F, we have for all $(x,y) \in \mathbb{R}^2 \backslash S$ that $L(x,y,0) = 0$ and L is strictly convex in p. Since $\psi > 0$, we know also by properties of ω that $\omega(\psi) > 0$. So using the mean value theorem, and property $(\star_2)$ of F, we have for all $(x,y) \in \mathbb{R}^2 \backslash S$ that

$$L_p(x,y,\psi) > \frac{L(x,y,\psi) - L(x,y,0)}{\psi - 0} \geq \frac{\omega(\psi)}{\psi} > 0. \tag{3}$$

So for $(x,y) \in G \backslash S$, we have by (4.2.3) that $\theta \geq 3L_p(x,y,\psi) > 0$ and hence that ξ is well-defined.

Fix $(x,y) \in G \backslash S$. Note that by our definitions of θ and ξ,

$$L(x,y,\psi) + (p - \psi)L_p(x,y,\psi) + \theta(p - \xi) = \Phi_x + p\Phi_y. \tag{4}$$

Also note that the strict convexity of L in p and the mean value theorem give us the relation

$$L(x,y,p) \geq L(x,y,\psi) + (p - \psi)L_p(x,y,\psi) \tag{5}$$

with equality if and only if $p = \psi$. By (3) and (4.2.3) we have

$$\Phi_y > \theta \geq \Phi_y - \frac{\Phi_y}{4} = \frac{3\Phi_y}{4}. \tag{6}$$

By (5) for case $p = 0$ and the fact that $L(x,y,0) = 0$, we have that

$$\xi\theta < -\Phi_x. \tag{7}$$

Further, by (4.2.3), the definition of ψ, and the facts that $L \geq 0$ and $-\Phi_x > 0$, we also have

$$\xi\theta = -\Phi_x + L(x, y, \psi) - \frac{-2\Phi_x}{\Phi_y} L_p(x, y, \psi) \geq -\Phi_x + L(x, y, \psi) - \frac{-2\Phi_x}{4} > -\frac{\Phi_x}{2}.$$

Hence, using (6) and the fact that $\Phi_y > 0$,

$$\xi \geq -\frac{\Phi_x}{2\Phi_y}. \tag{8}$$

This implies, using (4.2.2), that

$$\xi \geq M^i + 2 > M^i + 1 \text{ if } (x, y) \in V^i \backslash S; \quad \text{and } \xi \geq 2 > 1 \text{ on } G \backslash S. \tag{9}$$

The latter gives, using the definition of ψ, (7), (6), and that $\xi\theta > 0$, that

$$\psi > \frac{2\xi\theta}{\Phi_y} \geq \frac{3\xi\Phi_y}{2\Phi_y} = \xi + \frac{\xi}{2} \geq \xi + 1 \tag{10}$$

on $G \backslash S$. Since G is open, for (x, y) sufficiently close to S we have $(x, y) \in G$, so by (8) and (4.2.4) we have that

$$\lim_{0<\operatorname{dist}((x,y),S)\to 0} \xi \geq -\frac{1}{2}\left(\lim_{0<\operatorname{dist}((x,y),S)\to 0} \frac{\Phi_x}{\Phi_y}\right) = \infty. \tag{11}$$

We now use the corner-smoothing γ constructed in Lemma 4.1 to define $f\colon G \times \mathbb{R} \to \mathbb{R}$ by

$$f(x, y, p) = \begin{cases} \gamma(p, \xi(x, y), \theta(x, y)), & (x, y) \in G \backslash S \\ 0, & (x, y) \in S. \end{cases}$$

Evidently $f \geq 0$ on $G \times \mathbb{R}$ by (4.1.a). Since $\xi \geq 1$ on $G \backslash S$ from (9), property (4.1.b) of γ implies that $f(x, y, 0) = 0$ for all $(x, y) \in G$. For fixed $(x, y) \in G$, that $p \mapsto f(x, y, p)$ is convex follows from (4.1.a). Clearly $f \in C^\infty((G \backslash S) \times \mathbb{R})$. But for given $p \in \mathbb{R}$, by (11) there is an open set $S \subseteq \Omega \subseteq G$ such that $\xi \geq p + 2$ on $\Omega \backslash S$. Hence $f = 0$ on $\Omega \times (-\infty, p + 1)$ by (4.1.b). That is, for given $(x, y, p) \in S \times \mathbb{R}$, there is an open set $\Omega \times (-\infty, p + 1)$ containing (x, y, p) on which $f = 0$. Hence $f \in C^\infty(G \times \mathbb{R})$.

Let $i \geq 1$ and suppose $(x, y, p) \in (V^i \backslash S) \times (-\infty, M^i]$. Then by (9) we see that $\xi > M^i + 1 \geq p + 1$, so $f(x, y, p) = 0$ by (4.1.b). Hence

$$f(x, y, p) = 0 \quad \text{for all } (x, y, p) \in \bigcup_{i=1}^{\infty} (V^i \times (-\infty, M^i]). \tag{12}$$

Let $W \subseteq V$ be such that $\overline{W}\backslash V \subseteq S$. Find a non-negative function $\phi \in C^\infty(\mathbb{R}^2\backslash(\overline{W}\backslash V))$ such that $\phi = 1$ on W and $\phi = 0$ off V; i.e., a smooth cut-off function which necessarily fails to be defined on $\overline{W}\backslash V$. Extend ϕ to a function defined everywhere on $\mathbb{R}^2$ by defining it as 0 on $\overline{W}\backslash V$.

Then $\phi \in C^\infty(\mathbb{R}^2\backslash((\partial V)\cap S))$ since $\overline{W}\backslash V \subseteq (\overline{V}\backslash V)\cap S = (\partial V)\cap S$. Choose an open set $G' \supseteq S \cup \overline{V}$ such that $\overline{G'} \subseteq G$, and define $\tilde{F}\colon \mathbb{R}^3 \to \mathbb{R}$ by

$$\tilde{F}(x,y,p) = \begin{cases} \phi(x,y)f(x,y,p), & (x,y) \in G' \\ 0, & (x,y) \notin G'. \end{cases}$$

We claim $\tilde{F} \in C^\infty(\mathbb{R}^3)$. Clearly $\tilde{F} \in C^\infty(((\mathbb{R}^2\backslash\overline{G'})\times\mathbb{R})\cup((G'\backslash((\partial V)\cap S))\times\mathbb{R}))$.

So consider first $(x,y,p) \in \partial G' \times \mathbb{R}$. Since $(x,y) \notin G' \supseteq \overline{V}$, we can find an open set B such that $(x,y) \in B \subseteq \mathbb{R}^2\backslash V$. So $\phi = 0$ on B, hence $\tilde{F} = 0$ on $B \times \mathbb{R}$, hence $\tilde{F} \in C^\infty(B \times \mathbb{R})$.

Consider now the case $(x,y,p) \in (G' \cap ((\partial V) \cap S)) \times \mathbb{R}$. By (11), there exists an open set Ω with $S \subseteq \Omega \subseteq G'$ such that $\xi \geq p+2$ on $\Omega\backslash S$. Since $(x,y) \in S$, we have $(x,y,p) \in \Omega \times (-\infty, p+1)$. By (4.1.b), $f = 0$ and hence $\tilde{F} = 0$ on $\Omega \times (-\infty, p+1)$. So $\tilde{F} \in C^\infty(\Omega \times (-\infty, p+1))$.

So indeed $\tilde{F} \in C^\infty(\mathbb{R}^3)$. That $\tilde{F}$ satisfies the remaining properties of $(\star_F)$ follows by the analogous properties proved above of f. Now define $\hat{F} = F + \tilde{F}$. Thus $\hat{F}$ is also of form $(\star_F)$. Property (4.2.6) follows since $\tilde{F}$ satisfies $(\star_2)$.

Let $(x,y,p) \in \mathbb{R}^3\backslash \bigcup_{i=1}^\infty (V^i \times (M^i, \infty))$. If $(x,y) \notin V$, then $\tilde{F}(x,y,p) = 0$, by choice of ϕ. If $(x,y) \in V^i \subseteq G'$ for some $i \geq 1$, then $p \leq M^i$, and so $\tilde{F}(x,y,p) = 0$ by (12). Thus $\hat{F}$ satisfies (4.2.7).

We define $\hat{L}(x,y,p) = \omega(p) + \hat{F}(x,y,p)$ and are just required to check (4.2.8). So let $(x,y) \in W\backslash S$. Since $W \subseteq V \subseteq G'$ and $\phi = 1$ on W, we have by definition and (4.1.d) that $\tilde{F}(x,y,p) = f(x,y,p) = \gamma(p,\xi,\theta) \geq \theta(p-\xi)$. Hence by (5) and (4)

$$\hat{L}(x,y,p) \geq L(x,y,p) + \theta(p-\xi) \geq L(x,y,\psi) + (p-\psi)L_p(x,y,\psi) + \theta(p-\xi) = \Phi_x + p\Phi_y.$$

For the case $p = \psi$, (10) and (4.1.c) imply that the first inequality above is an equality, thus $\hat{L}(x,y,\psi) = \Phi_x + \psi\Phi_y$. Moreover, should the equality $\hat{L}(x,y,p) = \Phi_x + p\Phi_y$ hold, then in particular the second inequality in the above calculation must be an equality, which by strict convexity of L forces $p = \psi$. That is, we have equality in this inequality if and only if $p = \psi(x,y)$.

The remainder of the proof is just as in [4]; details to supplement the following can be found there.

First suppose $a < b$ and $u \in \mathrm{AC}(a,b)$ is non-decreasing and such that $U([a,b]) \subseteq W$. Then (4.2.5) states that $U(x) \notin S$ for almost every $x \in [a,b]$, thus $(\Phi \circ U)\colon [a,b] \to \mathbb{R}$ is differentiable almost everywhere with $(\Phi \circ U)'(x) = \Phi_x(U(x)) + u'(x)(\Phi_y(U(x))$. Combining this with the above observations about $\hat{L}$

we see that for almost every $x \in [a, b]$,

$$\hat{L}(x, u(x), u'(x)) \geq (\Phi \circ U)'(x), \tag{13}$$

with equality if and only if $u'(x) = \psi(x, u(x))$. We also note that $(\Phi \circ U)$ has the Lusin property, i.e., maps null sets to null sets: (4.2.5) implies any subset of $U^{-1}(S)$ is mapped to a null set, and on $[a, b]\backslash U^{-1}(S)$ the function $(\Phi \circ U)$ is locally absolutely continuous.

Given these observations, we now check (4.2.8). Let $a < b$ and $u \in \mathrm{AC}(a, b)$ be such that $Q(a, u(a); b, u(b)) \subseteq W$. We argue that it suffices to check (4.2.8) for non-decreasing u such that $\Phi(U(a)) \leq \Phi(U(b))$. The result is trivial if $\Phi(U(a)) \geq \Phi(U(b))$ since $\hat{L} \geq 0$. By (4.2.1), we have $\Phi(U(a)) \leq \Phi(U(b))$ only if $u(a) < u(b)$. Thus if such a u is not non-decreasing, we can construct non-decreasing $v \in \mathrm{AC}(a, b)$ such that $v(a) = u(a)$, $v(b) = u(b)$, and for almost every $x \in [a, b]$ either $v(x) = u(x)$ and $v'(x) = u'(x)$, or $v'(x) = 0$. Therefore, since $\hat{L}(x, y, p) \geq \hat{L}(x, y, 0) = 0$ for all $(x, y, p) \in \mathbb{R}^3$, and since $\{x \in [a, b] : v(x) = 0\}$ must have positive measure, we see that

$$\int_a^b \hat{L}(x, u(x), u'(x))\, dx > \int_a^b \hat{L}(x, v(x), v'(x))\, dx.$$

So we can indeed assume u is non-decreasing and such that $\Phi(U(a)) \leq \Phi(U(b))$. That u is non-decreasing implies, since $Q(a, u(a); b, u(b)) \subseteq W$, that in fact $U([a, b]) \subseteq W$. So the relation (13) holds for almost every $x \in [a, b]$. We let $\{(a_j, b_j)\}_{j\in J}$ be the (at most countable) sequence of components of $(a, b)\backslash U^{-1}(S)$ such that $\Phi(U(a_j)) < \Phi(U(b_j))$. Then using that $(\Phi \circ U)$ is locally absolutely continuous on $(a, b)\backslash U^{-1}(S)$ and the fact from (4.2.5) that $(\Phi \circ U)(U^{-1}(S))$ is null, we see that

$$\begin{aligned}
\int_a^b \hat{L}(x, u(x), u'(x))\, dx &\geq \sum_{j\in J} \int_{a_j}^{b_j} \hat{L}(x, u(x), u'(x))\, dx \\
&\geq \sum_{j\in J} \int_{a_j}^{b_j} \max\{0, (\Phi \circ U)'\}\, dx \\
&\geq \sum_{j\in J} \Phi(U(b_j)) - \Phi(U(a_j)) \\
&\geq (\Phi \circ U)(b) - (\Phi \circ U)(a).
\end{aligned}$$

Equality in this relation implies that $\hat{L}(x, u(x), u'(x)) = (\Phi \circ U)'(x)$ for almost every $x \in \bigcup_{j\in J}(a_j, b_j)$, but also that $\bigcup_{j\in J}(a_j, b_j) = (a, b)\backslash U^{-1}(S)$, and also therefore that in fact $\hat{L}(x, u(x), u'(x)) = (\Phi \circ U)'(x)$ for almost every $x \in (a, b)\backslash U^{-1}(S)$. By (13) and (4.2.5) this implies that $u'(x) = \psi(x, u(x))$ for almost every $x \in [a, b]$.

Conversely, $u'(x) = \psi(x, u(x))$ almost everywhere implies

$$(\Phi \circ U)'(x) = (\Phi_x \circ U)(x) + \psi(x, u(x))(\Phi_y \circ U)(x) = (-\Phi_x \circ U)(x) \geq 0$$

almost everywhere. This, combined with the fact that $(\Phi \circ U)$ has the Lusin property (since $u' = \psi(x, u)$ implies that $u \in \mathrm{AC}(a, b)$ is non-decreasing), implies that $(\Phi \circ U)$ is absolutely continuous. Moreover, (13) implies that $\hat{L}(x, u(x), u'(x)) = (\Phi \circ U)'(x)$ almost everywhere, hence

$$\int_a^b \hat{L}(x, u(x), u'(x))\, dx = \int_a^b (\Phi \circ U)'(x)\, dx = (\Phi \circ U)(b) - (\Phi \circ U)(a)$$

as required. □

We now give the construction of the potential required for an application of this lemma. This is a version of the proof of [4, Theorem 10], i.e., the construction of a potential satisfying the conditions of their Lemma 11. This is done entirely independently of the sequence of constants $\{M^i\}_{i=1}^{\infty}$, which are therefore taken to be arbitrary. We then simply define subsets $\{V^i\}_{i=1}^{\infty}$ of $\mathbb{R}^2$ so that the required inequalities hold. The final statement (4.5.3) falls naturally out of the proof from [4]; it is only now in our case that it is relevant to emphasize it.

As part of the proof of Lemma 4.5 we recall Lemmas 12 and 13 stated and proved in [4], which are used to prove our statement in exactly the same way as they are used in [4]. We do not give the proofs. The second lemma follows easily from the first. The first relies on using the pure unrectifiability of S to find, given $\epsilon > 0$ and $C > 0$, an open set Ω around S such that the graph of any Lipschitz function from $\mathbb{R}$ to $\mathbb{R}$ with Lipschitz constant less than C intersects Ω in a set of length at most ϵ.

For two vectors $x, y \in \mathbb{R}^2$, we let $[x, y]$ denote the line segment in $\mathbb{R}^2$ with these points as endpoints.

Lemma 4.3. *Let $S \subseteq \mathbb{R}^2$ be a compact purely unrectifiable set, $e \in \mathbb{R}^2$, and $\tau > 0$. Then there is $g \in C^\infty(\mathbb{R}^2)$ such that*

- $0 \leq g(x) \leq \tau$ *for all* $x \in \mathbb{R}^2$*;*
- $\mathrm{dist}(\nabla g(x), [0, e]) < \tau$ *for all* $x \in \mathbb{R}^2$*; and*
- $\sup_{x \in S} \|\nabla g(x) - e\|_\infty < \tau$*.*

Lemma 4.4. *Let $S \subseteq \mathbb{R}^2$ be a compact purely unrectifiable set, $\Omega \supseteq S$ be open, $h^0 \in C^\infty(\mathbb{R}^2)$, $e^0, e^1 \in \mathbb{R}^2$, and $\epsilon > 0$. Then there is $h^1 \in C^\infty(\mathbb{R}^2)$ such that*

- $\|h^1 - h^0\| < \epsilon$*;*
- $h^1 = h^0$ *outside* Ω*;*
- $\mathrm{dist}(\nabla h^1(x), [e^0, e^1]) < \epsilon + \|\nabla h^0(x) - e^0\|_\infty$ *for* $x \in \mathbb{R}^2$*; and*
- $\|\nabla h^1(x) - e^1\|_\infty < \epsilon + \|\nabla h^0(x) - e^0\|_\infty$ *for* $x \in S$*.*

Lemma 4.5. *Let $S \subseteq \mathbb{R}^2$ be compact and purely unrectifiable, $G, H \subseteq \mathbb{R}^2$ be bounded such that H is open and $\overline{H} \subseteq G$, and $\{M^i\}_{i=1}^{\infty}$ be a sequence of constants. Let $F\colon \mathbb{R}^3 \to \mathbb{R}$ be such that F_p exists and is bounded above on $G \times [8, n]$ for all $n \geq 9$. Let $L(x, y, p) = \omega(p) + F(x, y, p)$, where ω is a given superlinearity.*
Then there is $\Phi \in C^{\infty}(\mathbb{R}^2 \backslash S) \cap C(\mathbb{R}^2)$ and a sequence $\{V^i\}_{i=1}^{\infty}$ of open sets $V^i \subseteq \mathbb{R}^2$ such that the conditions (4.2.1)–(4.2.5) of Lemma 4.2 hold, and

(4.5.1) *$H \cap S \subseteq V := \bigcup_{i=1}^{\infty} V^i \subseteq \overline{V} \subseteq G$;*
(4.5.2) *$V^i \subseteq \{(x, y) \in H : \operatorname{dist}((x, y), \mathbb{R}^2 \backslash H) > \frac{1}{i}\}$ for all $i \geq 1$; and*
(4.5.3) *$\psi \in C^{\infty}(\mathbb{R}^2 \backslash S)$ defined by $\psi := -2\frac{\Phi_x}{\Phi_y}$ is bounded above on any subset of $\mathbb{R}^2$ positively separated from S.*

Proof. We use a slight variant of the construction which comprises the proof of [4, Theorem 10].

We define an increasing sequence $\{c^k\}_{k=0}^{\infty}$ by, for each $k \geq 0$, choosing $c^k \geq 0$ such that $L_p(x, y, p) \leq c^k$ for all $(x, y, p) \in G \times [8, 5 \cdot 2^{k+4}]$. We now define

$$B^k = 4 + 4c^k \quad \text{and} \quad A^k = 3 \cdot 2^{k+2} B^k.$$

The construction of Φ is then similar to that in [4], with these new definitions of A^k and B^k. We sketch the proof; more details can be found in [4]. The construction relies on the exhibiting of a sequence, for $k \geq 0$, of functions $\Phi^k \in C^{\infty}(\mathbb{R}^2)$, open sets Ω^k, and $\epsilon^k > 0$ such that, where $\eta^k = 1 - 2^{k-1}$,

$$\Phi^0(x) = -A^0 x + B^0 y, \quad \Omega^0 = \mathbb{R}^2, \quad \epsilon^0 = \frac{1}{4}; \tag{14}$$

$$\|\nabla \Phi^k(x) - e^k\|_{\infty} < \eta^k \quad \text{for } x \in \overline{\Omega^k}; \tag{15}$$

if $a < b$, $u \in C([a, b])$ is non-decreasing and $\Phi \in C(\mathbb{R}^2)$ satisfies $\|\Phi - \Phi^k\|_{\infty} < 2\epsilon^k$, then

$$\lambda(\{(\Phi \circ U)(x) : U(x) \in \Omega^k\}) \leq \frac{1}{k}; \tag{16}$$

and for $k \geq 1$

$$\|\Phi^k - \Phi^{k-1}\| < \epsilon^{k-1}; \tag{17}$$

$$\Phi^1 = \Phi^0 \text{ off } B_1(S), \quad \text{and for } k \geq 2, \quad \Phi^k = \Phi^{k-1} \text{ outside } \overline{\Omega^{k-1}}; \tag{18}$$

$$\operatorname{dist}(\nabla \Phi^k(x), [e^{k-1}, e^k]) < \eta^k \quad \text{for } x \in \overline{\Omega^{k-1}}; \tag{19}$$

$$S \subseteq \Omega^k, \quad \overline{\Omega^k} \subseteq B_{2^{-k}}(S) \cap \Omega^{k-1}, \quad \epsilon^k < \frac{\epsilon^{k-1}}{2}. \tag{20}$$

(Interpret $\frac{1}{0}$ as ∞ in (16).)

This can be done inductively, using (14) to define Φ^0, Ω^0, and ϵ^0, and for $k \geq 1$ applying Lemma 4.4 with $\Omega = \Omega^{k-1}$ (except for $k = 1$ when we put $\Omega = B_1(S)$), $h^0 = \Phi^{k-1}$, $e^0 = e^{k-1}$, $e^1 = e^k$, $\epsilon = \epsilon^{k-1}$ and defining $\Phi^k = h^1$. Properties (17), (18) are immediate from the definition of Φ^k, and (19) follows by induction. Defining $\Omega^k = B_\delta(S)$ for sufficiently small $\delta > 0$, and defining $\epsilon^k = \min\left\{\frac{\epsilon^{k-1}}{2}, \delta\right\}$ gives the remaining properties (15), (16), and (20).

By (17), the sequence Φ^k converges uniformly to some $\Phi \in C(\mathbb{R}^2)$. By (18) and the nesting of $\{\Omega^k\}_{k=1}^\infty$, $\Phi^l = \Phi^k$ on $\mathbb{R}^2 \backslash \overline{\Omega^k}$ for all $l \geq k$. Hence, by (20), $\Phi \in C^\infty(\mathbb{R}^2 \backslash S)$ and $\nabla\Phi = \nabla\Phi^k$ on $\mathbb{R}^2 \backslash \overline{\Omega^k}$. For $(x, y) \in \mathbb{R}^2 \backslash S$, by (20) there is a smallest $k \geq 1$ such that $(x, y) \in \overline{\Omega^{k-1}} \backslash \overline{\Omega^k}$, and so $\nabla\Phi(x, y) = \nabla\Phi^k(x, y)$. Hence by (19)

$$\Phi_y \geq B^{k-1} - 1 \geq B^0 - 1 = 3 + 4c^0 \geq 3$$

and

$$\Phi_x \leq -A^{k-1} + 1 = -3 \cdot 2^{k+1}(4 + 4c^{k-1}) + 1 \leq -3 \cdot 4 \cdot 4 + 1 = -47.$$

Thus we have (4.2.1) and the very last inequality of (4.2.2). More precisely, by (19) there is $s \in [0, 1]$ such that

$$\begin{aligned} -\Phi_x &\geq sA^{k-1} + (1-s)A^k - 1 \\ &= s3 \cdot 2^{k+1}B^{k-1} + (1-s)3 \cdot 2^{k+2}B^k - 1 \\ &\geq 3 \cdot 2^{k+1}(sB^{k-1} + (1-s)B^k) - 1 \\ &\geq 3 \cdot 2^{k+1}(\Phi_y - 1) - 1 \\ &\geq 2^{k+1}\Phi_y. \end{aligned} \tag{21}$$

This gives the penultimate inequality of (4.2.2), since $k \geq 1$, and also (4.2.4), since as $\operatorname{dist}((x, y), S) \to 0$, we have $k \to \infty$, by (20). We now check (4.2.3). Again, there is $s \in [0, 1]$ such that

$$\begin{aligned} -\Phi_x &< sA^{k-1} + (1-s)A^k + 1 \\ &= s3 \cdot 2^{k+1}B^{k-1} + (1-s)3 \cdot 2^{k+2}B^k + 1 \\ &\leq 3 \cdot 2^{k+2}(sB^{k-1} + (1-s)B^k) + 1 \\ &\leq 3 \cdot 2^{k+2}(\Phi_y + 1) + 1 \\ &\leq 5 \cdot 2^{k+2}\Phi_y \end{aligned}$$

whence

$$\frac{-2\Phi_x}{\Phi_y} \leq 5 \cdot 2^{k+3} \quad \text{on } \mathbb{R}^2 \backslash \overline{\Omega^k}. \tag{22}$$

In particular for given $(x, y) \in G \backslash S$ there is a $k \geq 1$ such that $(x, y) \in G \backslash \overline{\Omega^k}$; thus $\Phi_y \geq B^{k-1} - 1 \geq 4c^{k-1} \geq 4L_p(x, y, \psi)$ since $(x, y, \psi) \in G \times [8, 5 \cdot 2^{k+3}]$ from (21) and (22).

Condition (22) also gives (4.5.3), since by (20) for any set $X \subseteq \mathbb{R}^2$ positively separated from S there is $k \geq 1$ such that $X \subseteq \mathbb{R}^2 \backslash \overline{\Omega^k}$, and hence $5 \cdot 2^{k+3}$ is an upper bound for ψ on X.

We are now obliged to construct $\{V^i\}_{i=1}^{\infty}$. For each $i \geq 1$, choose $k_i \geq 1$ such that $2^{k_i+2} \geq 2M^i + 4$ and define open $V^i \subseteq H$ by

$$V^i = \Omega^{k_i} \cap \left\{(x, y) \in H : \operatorname{dist}((x, y), \mathbb{R}^2 \backslash H) > \frac{1}{i}\right\}.$$

Evidently the V^i satisfy (4.5.2). For $(x, y) \in H \cap S$, we see $(x, y) \in V^i$ for $i \geq 2$ such that $B_{\frac{1}{i-1}}((x, y)) \subseteq H$. Hence $V := \bigcup_{i=1}^{\infty} V^i$ is an open set such that $H \cap S \subseteq V \subseteq H$. Since $\overline{H} \subseteq G$, we then have that $\overline{V} \subseteq G$, as required for (4.5.1).

All that remains to check of $\{V^i\}_{i=1}^{\infty}$ is (4.2.2). Let $(x, y) \in V^i \backslash S$. Then $(x, y) \in \overline{\Omega^{k-1}} \backslash \overline{\Omega^k}$ for some $k > k_i$. So by (21), and recalling $\Phi_y > 0$, we see that

$$-\Phi_x \geq 2^{k+1}\Phi_y \geq 2^{k_i+2}\Phi_y \geq (2M^i + 4)\Phi_y$$

as required.

We easily check condition (4.2.5). Let $a < b$ and $u \in \mathrm{AC}(a, b)$ be non-decreasing. The $\{x \in (a, b) : U(x) \in S\}$ is null since S is purely unrectifiable. For all $k \geq 0$, properties (17) and (20) imply that $\|\Phi - \Phi^k\| < 2\epsilon^k$ for all $k \geq 0$, and hence by property (16) that $\lambda(\{(\Phi \circ U)(x) : U(x) \in S\}) \leq \frac{1}{k}$. Hence this set is also null. $\square$

We now give the exact details of the inductive construction of our Lagrangians L_n. Let $S \subseteq \mathbb{R}^2$ be a purely unrectifiable set such that $S = \bigcup_{n=1}^{\infty} S_n$ for some compact S_n, and let ω be a fixed superlinearity. For each $n \geq 1$ define

$$G_n = B_1(S_n) \quad \text{and} \quad H_n = B_{\frac{1}{2}}(S_n) \backslash \bigcup_{m=1}^{n-1} S_m.$$

The set H_n is a neighbourhood of the set $S_n \backslash \bigcup_{m=1}^{n-1} S_m$ which we want to cover by $\operatorname{uss}(L_n)$, but contains no points of $\bigcup_{m=1}^{n-1} S_m$, which we assume to be covered by $\operatorname{uss}(L_{n-1})$. Thus $H_n \times \mathbb{R} \subseteq \mathbb{R}^3$ is the domain on which we modify a given L_{n-1}, building L_n to deal with the points in S_n, without interfering with the structure of L_{n-1} on $\bigcup_{m=1}^{n-1} S_m$. In the case that the S_n are pairwise disjoint, H_n could be chosen to be any open neighbourhood of S_n positively separated from $\bigcup_{m=1}^{n-1} S_m$.

Lemma 4.6. *For each $n \geq 1$ there exist $F_n \colon \mathbb{R}^3 \to \mathbb{R}$ of form ($\star_F$), $\Phi_n \in C^{\infty}(\mathbb{R}^2 \backslash S_n) \cap C(\mathbb{R}^2)$, sequence $\{V_n^i\}_{i=1}^{\infty}$ of open sets $V_n^i \subseteq H_n$, sequence of constants $\{M_n^i\}_{i=1}^{\infty}$, and an open set $W_n \subseteq \mathbb{R}^2$ such that the following relations hold:*

(4.6.1) $H_n \cap S_n \subseteq W_n \subseteq V_n := \bigcup_{i=1}^{\infty} V_n^i \subseteq \overline{V_n} \subseteq G_n$;
(4.6.2) $\overline{W_n} \backslash V_n \subseteq S_n$;
(4.6.3) $\{M_n^i\}_{i=1}^{\infty}$ *is a non-decreasing sequence and* $M_n^1 \geq n$;
(4.6.4) $\lim_{0<\mathrm{dist}((x,y),S_n)\to 0} \frac{(\Phi_n)_x}{(\Phi_n)_y}(x,y) = -\infty$;
(4.6.5) $L_n \colon \mathbb{R}^3 \to \mathbb{R}$ *defined by*

$$L_n(x,y,p) = \omega(p) + F_n(x,y,p)$$

has the property that for all $a < b$ *and all* $u \in \mathrm{AC}(a,b)$ *such that* $Q(a,u(a);b,u(b)) \subseteq W_n$, *we have*

$$\int_a^b L_n(x,u(x),u'(x))\,dx \geq \Phi_n(U(b)) - \Phi_n(U(a))$$

with equality if and only if $u'(x) = -2\frac{(\Phi_n)_x}{(\Phi_n)_y}(x,u(x))$ *for almost every* $x \in [a,b]$;

and for $n \geq 2$,

(4.6.6) $F_n \geq F_{n-1}$ *on* $\mathbb{R}^3$;
(4.6.7) $F_n = F_{n-1}$ *on* $\mathbb{R}^3 \backslash \bigcup_{i=1}^{\infty}(V_n^i \times (M_n^i,\infty))$; *and*
(4.6.8) $\psi_m \in C^\infty(\mathbb{R}^2 \backslash S_m)$ *defined by* $\psi_m := -2\frac{(\Phi_m)_x}{(\Phi_m)_y}$ *satisfies* $\psi_m \leq M_n^i$ *on* V_n^i *for all* $i \geq 1$, *for each* $1 \leq m < n$.

Proof. For each $n \geq 1$, we want to apply Lemma 4.5 to get a potential with which we can apply Lemma 4.2. To begin, we define $M_1^i = 1$ for all $i \geq 1$, and $F_0 \colon \mathbb{R}^3 \to \mathbb{R}$ to be the zero function.

For $n \geq 2$ we suppose $\Phi_m \in C^\infty(\mathbb{R}^2 \backslash S_m) \cap C(\mathbb{R}^2)$ to have been constructed as claimed, and moreover such that ψ_m satisfies (4.5.3) for each $1 \leq m < n$. For each $i \geq 1$ define

$$\tilde{V}_n^i = \left\{(x,y) \in H_n : \mathrm{dist}((x,y),\mathbb{R}^2 \backslash H_n) > \frac{1}{i}\right\}.$$

So for all $i \geq 1$ we have $\mathrm{dist}(\tilde{V}_n^i, \mathbb{R}^2 \backslash H_n) > 0$, and also therefore $\mathrm{dist}(\tilde{V}_n^i, S_m) > 0$ for each $1 \leq m < n$, since $\bigcup_{m=1}^{n-1} S_m \subseteq \mathbb{R}^2 \backslash H_n$. So by the assumption (4.5.3) on each ψ_m, we can choose $M_n^1 \geq n$ such that $\psi^m \leq M_n^1$ on $\tilde{V}_n^1$ for all $1 \leq m < n$, and inductively $M_n^i \geq M_n^{i-1}$ such that $\psi^m \leq M_n^i$ on $\tilde{V}_n^i$ for all $1 \leq m < n$. This gives us a sequence $\{M_n^i\}_{i=1}^{\infty}$ satisfying (4.6.3).

We can now apply Lemma 4.5 inductively for each $n \geq 1$, using data $S = S_n$, $G = G_n$, $H = H_n$, $\{M^i\}_{i=1}^{\infty} = \{M_n^i\}_{i=1}^{\infty}$, $F = F_{n-1}$. This gives us a function $\Phi = \Phi_n$ as required, and a sequence of open sets $\{V^i\}_{i=1}^{\infty} = \{V_n^i\}_{i=1}^{\infty}$ such that by (4.5.1),

$$H_n \cap S_n \subseteq V_n := \bigcup_{i=1}^{\infty} V_n^i \subseteq \overline{V_n} \subseteq G_n. \tag{23}$$

For $n \geq 2$, we have by (4.5.2) that $V_n^i \subseteq \tilde{V}_n^i$ for each $i \geq 1$, so (4.6.8) holds.

Lemma 4.5 also asserts that all the conditions of Lemma 4.2 hold, using this data, which gives us in particular (4.6.4). To apply Lemma 4.2, we need a suitable W_n.

Since Lemma 4.5 tells us V_n is open, for all $x \in H_n \cap S_n$, there is $\delta_x > 0$ such that $B_{\delta_x}(x) \subseteq V_n$. Then defining

$$W_n = \bigcup_{x \in H_n \cap S_n} B_{\frac{1}{2}\delta_x}(x)$$

gives an open set W_n which, in conjunction with (23), gives (4.6.1). We easily check that (4.6.2) holds: If $x \in \overline{W_n}$ and $B_\epsilon(x) \cap S_n = \emptyset$ for some $\epsilon > 0$, then choosing $w \in W_n \cap B_{\frac{\epsilon}{2}}(x)$ gives us a point $y \in S_n$ such that $w \in B_{\frac{\delta_y}{2}}(y)$, where $\delta_y > \epsilon$, and hence $x \in B_{\delta_y}(y) \subseteq V$. Thus $\overline{W_n} \backslash S_n \subseteq V_n$, which gives the result. Then set $F_n = \widehat{F_{n-1}}$ as given in Lemma 4.2 for this W_n. The remaining conclusions then follow directly from those of the lemma. Since Lemma 4.5 asserts that ψ_n also satisfies (4.5.3), we are able to iterate the construction to produce the required sequence. □

Proof of Theorem 2.4. By Lemma 4.6 we have a sequence $\{F_n\}_{n=1}^\infty$ of functions $F_n \colon \mathbb{R}^3 \to \mathbb{R}$ of form $(\star_F)$. Note that for $n_0 \geq 1$, we have by (4.6.3) that $p \notin (M_n^i, \infty)$ for all $i \geq 1$ and all $n \geq n_0$ whenever $p \in (-\infty, n_0)$. Hence by (4.6.7), $F_n = F_{n_0}$ on $\mathbb{R}^2 \times (-\infty, n_0)$ for all $n \geq n_0$. Then for $(x, y, p) \in \mathbb{R}^3$, choosing $n_0 > p$, we have that $F_n = F_{n_0}$ for all $n \geq n_0$ on an open set around (x, y, p). We then define $F \colon \mathbb{R}^3 \to \mathbb{R}$ by $F(x, y, p) = \lim_{n \to \infty} F_n(x, y, p)$, and it is clear that F satisfies $(\star_F)$.

So we can define Lagrangian $L \colon \mathbb{R}^3 \to \mathbb{R}$ of form $(\star)$ by defining

$$L(x, y, p) = \omega(p) + F(x, y, p).$$

We claim S lies in the universal singular set of L.

Let $(x_0, y_0) \in S$. Choose $n_0 \geq 1$ such that $(x_0, y_0) \in S_{n_0} \backslash \bigcup_{m=1}^{n_0 - 1} S_m$. As in [4], we proceed to construct a locally absolutely continuous $u_0 \colon \mathbb{R} \to \mathbb{R}$ such that $u_0'(x) = \psi_{n_0}(x, u_0(x))$ for almost every $x \in \mathbb{R}$ and $u(x_0) = y_0$, as follows. For each $k \geq 0$ we find $u^k \in C^1(\mathbb{R})$ such that $(u^k)' = \psi_{n_0}^k(x, u^k)$ for all $k \geq 0$, and show that $\{u^k\}_{k=1}^\infty$ is an equicontinuous family. Some subsequence therefore converges locally uniformly to a non-decreasing function $u_0 \in C(\mathbb{R})$ which solves $u_0'(x) = \psi_{n_0}(x, u_0(x))$ whenever $(x, u_0(x)) \notin S_{n_0}$, i.e., almost everywhere. Thus u_0 is locally absolutely continuous. We observe that $(x_0, y_0) \in S_{n_0} \cap H_{n_0} \subseteq W_{n_0}$, using (4.6.1). Since W_{n_0} is open we can choose real numbers $a_0 < b_0$ such that $(x_0, y_0) \in Q(a_0, u(a_0); b_0, u(b_0)) \subseteq W_{n_0}$.

We claim we have constructed $\{F_n\}_{n=1}^\infty$ in such a way that

$$L(x, u_0(x), u_0'(x)) = L_{n_0}(x, u_0(x), u_0'(x)) \quad \text{for almost every } x \in [a_0, b_0].$$

We show in fact that for all $n \geq n_0$,

$$L_n(x, u_0(x), u_0'(x)) = L_{n_0}(x, u_0(x), u_0'(x)) \quad \text{for almost every } x \in [a_0, b_0].$$

This suffices since a countable union of null sets is null.

We proceed by induction. The claim is obvious for $n = n_0$, so let $n > n_0$ and assume that the statement is true for $n-1$. At points where the graph of the trajectory lies outside V_n, we see the result immediately since we know L_{n-1} was not changed there: For $x \in [a_0, b_0] \backslash U_0^{-1}(V_n)$, by (4.6.7) we have

$$L_n(x, u_0(x), u_0'(x)) = L_{n-1}(x, u_0(x), u_0'(x)).$$

When the graph of the trajectory lies inside V_n, we have to use some information about the derivative. Let $i \geq 1$. By choice of u_0, (4.6.8), and (4.6.3), for almost every $x \in [a_0, b_0] \cap U_0^{-1}(V_n^i)$ we have that

$$u_0'(x) = \psi_{n_0}(x, u_0(x)) \leq M_n^i \leq M_n^j \quad \text{for all } j \geq i.$$

So for almost every $x \in [a_0, b_0] \cap U_0^{-1}(V_n^i)$, we have $u_0'(x) \notin (M_n^j, \infty)$ for all $j \geq i$. For each $x \in [a_0, b_0] \cap U_0^{-1}(V_n)$, there is a least $i \geq 1$ such that $(x, u_0(x)) \in V_n^i$; so $(x, u_0(x)) \notin V_n^j$ for all $1 \leq j < i$. Then, since $U_0^{-1}(V_n) = \bigcup_{i=1}^{\infty} U_0^{-1}(V_n^i)$ and a countable union of null sets is null, for almost every $x \in [a_0, b_0] \cap U_0^{-1}(V_n)$ we have that

$$(x, u_0(x), u_0'(x)) \in \mathbb{R}^3 \backslash \bigcup_{j=1}^{\infty} (V_n^j \times (M_n^j, \infty)).$$

But then, by (4.6.7), we see that indeed

$$L_n(x, u_0(x), u_0'(x)) = L_{n-1}(x, u_0(x), u_0'(x))$$

for almost every $x \in [a_0, b_0] \cap U_0^{-1}(V_n)$. The result then follows by the inductive hypothesis.

So applying (4.6.5) to L_{n_0}, we see, since $u_0'(x) = \psi_{n_0}(x, u_0(x))$ for almost every $x \in \mathbb{R}$,

$$\int_{a_0}^{b_0} L(x, u_0(x), u_0'(x))\, dx = \int_{a_0}^{b_0} L_{n_0}(x, u_0(x), u_0'(x))\, dx = \Phi_{n_0}(U(b_0)) - \Phi_{n_0}(U(a_0)).$$

By (4.6.6) and (4.6.5), we see

$$\int_{a_0}^{b_0} L(x, u(x), u'(x))\, dx \geq \int_{a_0}^{b_0} L_{n_0}(x, u(x), u'(x))\, dx \geq \Phi_{n_0}(U(b_0)) - \Phi_{n_0}(U(a_0))$$

for any $u \in \mathrm{AC}(a_0, b_0)$ such that $Q(a_0, u(a_0); b, u(b_0)) \subseteq W_{n_0}$. Thus u_0 is a minimizer for (1) over those functions $u \in \mathrm{AC}(a_0, b_0)$ such that $u(a_0) = u_0(a_0)$ and $u(b_0) = u_0(b_0)$. Tonelli's partial regularity result and (4.6.4) then imply that $u_0'(x_0) = \infty$. Hence (x_0, y_0) lies in the universal singular set of L, as required. $\square$

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