Existence of Mild Solutions for Semilinear Nonlocal Cauchy Problems in Separable Banach Spaces

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Abstract. In the paper we study the existence of mild solutions of a semilinear evolution equation with nonlocal initial conditions under the assumptions of the Hausdorff measure of noncompactness in separable Banach space.

Keywords. Evolution system, measure of noncompactness, mild solution, semilinear equation of evolution

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1. Introduction

We consider the following semilinear equation of evolution with nonlocal initial conditions having the form

$$x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in J \tag{1}$$
$$x(0) = g(x), \tag{2}$$

where $J = [0, T]$, $A(t) : D_t \subset E \to E$ generates an evolution system

$$\{U(t, s)\}_{0 \leq s \leq t \leq T}$$

on a separable Banach space E, $g : C(J, E) \to E$ and $f : J \times E \to E$ are given mappings.

Recently there have appeared a lot of papers concerned with the existence of integral or mild solutions for Equation (1) with (2) or similar problem (cf. [1, 2, 5, 9–12, 14, 16–22]). In all those papers there are imposed conditions requiring the compactness of at least one of the mappings f and g or compactness or equicontinuity of the evolution system $\{U(t, s)\}$ (or semigroup $\{G(t)\}$).

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In this paper we give conditions guaranteeing the existence of mild solutions of Equation (1) with (2) without assumptions on the compactness of f, g and $\{U(t, s)\}$. Our considerations will be conducted in a separable Banach space E and we assume that the mappings g, f are condensing with respect to a measure of noncompactness and the evolution system $\{U(t, s)\}$ is strongly continuous. The proofs of results obtained in this paper are based on a new calculation method which employs the technique of measures of noncompactness.

The paper is organized as follows. In Section 2 there are given notations and auxiliary facts needed further on. In Section 3 we formulate and prove two theorems on the existence of mild solutions of Equation (1) with (2). Section 4 is devoted to discussion of some hypotheses assumed on the functions involved in Equation (1).

2. Notation and auxiliary facts

In this section, we collect some definitions and results which will be needed later. Let $(E, \|\cdot\|)$ be a real Banach space with the zero element θ. Denote by $B(x, r)$ the closed ball in E centered at x and with radius r. The collection of all linear and bounded operators from E into itself will be denoted by $B(E)$. If X is a subset of E we write $\overline{X}$, $\mathrm{Conv}X$ in order to denote the closure and the convex closure of X, respectively.

Throughout this paper, we will also accept the following definition of the concept of measure of noncompactness [6].

Definition 2.1. A function μ, defined on bounded subsets of a real Banach space E with real values, is said to be a measure of noncompactness if it satisfies the following conditions:

1° $\mu(X) = 0$ implies that X is relatively compact.
2° $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3° $\mu(\mathrm{Conv}X) = \mu(X)$.
4° $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
5° If (X_n) is a sequence of nonempty, bounded and closed subsets of E such that $X_{n+1} \subset X_n$ $(n = 1, 2, \ldots)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Remark 2.2. Let us notice that the intersection set X_∞ described in axiom 5° satisfies the equality $\mu(X_\infty) = 0$. In fact, the inequality $\mu(X_\infty) \leq \mu(X_n)$ for $n = 1, 2, \ldots$ implies that $\mu(X_\infty) = 0$. This property of the set X_∞ will be very important in our investigations.

The most frequently applied measure of noncompactness is that called the Hausdorff measure of noncompactness which is defined in the following way

$$\beta(X) = \inf\{r > 0 : X \text{ can be covered by finitely many balls of radius } r\}.$$

Other facts concerning measures of noncompactness may be found in [3,4,6,15]. In the sequel, we will work in the space $C(J, E)$ consisting of all functions defined and continuous on J with values in the Banach space E. The space $C(J, E)$ is furnished with the standard norm

$$\|x\|_C = \sup\{\|x(t)\| : t \in J\}.$$

We will use a measure of noncompactness in the space $C(J, E)$ which was investigated in [3,4,6–8,15]. In order to define this measure let us fix a nonempty bounded subset X of the space $C(J, E)$ and a positive number $t \in J$. For $x \in X$ and $\varepsilon \geq 0$ denote by $\omega^t(x, \varepsilon)$ *the modulus of continuity* of the function x on the interval $[0, t]$, i.e.

$$\omega^t(x, \varepsilon) = \sup\{\|x(t_2) - x(t_1)\| : t_1, t_2 \in [0, t], |t_2 - t_1| \leq \varepsilon\}.$$

Further, let us put:

$$\omega^t(X, \varepsilon) = \sup\{\omega^t(x, \varepsilon) : x \in X\}, \quad \omega_0^t(X) = \lim_{\varepsilon \to 0+} \omega^t(X, \varepsilon).$$

Apart from this we put

$$\overline{\beta}(X) = \sup\{\beta(X(t)) : t \in J\},$$

where β denotes Hausdorff measure of noncompactness in E. Finally, we define the function μ on the family of bounded subsets of the space $C(J, E)$ by putting

$$\mu(X) = \omega_0^T(X) + \overline{\beta}(X).$$

It may be shown that the function μ is the measure of noncompactness in the space $C(J, E)$ (see [3,4,6–8,15]). The kernel $\ker \mu = \{X \subset C(J, E) : \mu(X) = 0, X \neq \emptyset\}$ is the family of all nonempty and bounded sets X such that functions belonging to X are equicontinuous on J and the set $X(t)$ is relatively compact in E for $t \in J$. This property will be crucial in our further study.

Next, for a given nonempty and bounded subset X of the space $C(J, E)$, let us denote

$$\int_0^t X(s)ds = \left\{\int_0^t x(s)ds : x \in X\right\}, \ t \in J, \quad X([0, t]) = \{x(s) : x \in X, s \in [0, t]\}.$$

Lemma 2.3 ([13, Corollary 3.1(b), Remarks (c)]). *If the Banach space E is separable and a set $X \subset C(J, E)$ is bounded, then the function $t \mapsto \beta(X(t))$ is measurable and*

$$\beta\left(\int_0^t X(s)ds\right) \leq \int_0^t \beta(X(s))ds \quad \text{for each } t \in J.$$

Remark 2.4. Observe that in the above lemma we do not require the equicontinuity of functions from the set X.

For our further purposes we will also need the following lemma.

Lemma 2.5. *Assume that a set $X \subset C(J, E)$ is bounded. Then*

$$\beta(X([0, t])) \leq \omega_0^t(X) + \sup_{s \leq t} \beta(X(s)), \quad \text{for } t \in J. \tag{3}$$

Proof. Fix arbitrarily $\delta > 0$. Then there exists $\varepsilon > 0$ such that $\omega^t(X, \varepsilon) \leq \omega_0^t(X) + \frac{\delta}{2}$. Let us take a partition $0 = t_0 < t_1 < \cdots < t_k = t$ such that $t_i - t_{i-1} \leq \varepsilon$ for $i = 1, \ldots, k$. Then for each $t' \in [t_{i-1}, t_i]$ and $x \in X$ the following inequality is fulfilled

$$\|x(t') - x(t_i)\| \leq \omega_0^t(X) + \frac{\delta}{2}. \tag{4}$$

Let us notice that for each $i = 1, 2, \ldots, k$ there are points $z_{ij} \in E$ $(j = 1, \ldots, n_i)$ such that

$$X(t_i) \subset \bigcup_{j=1}^{n_i} B\left(z_{ij}, \sup_{s \leq t} \beta(X(s)) + \frac{\delta}{2}\right). \tag{5}$$

We show that

$$X([0, t]) = \bigcup_{i=1}^{k} \bigcup_{j=1}^{n_i} B\left(z_{ij}, \sup_{s \leq t} \beta(X(s)) + \omega_0^t(X) + \delta\right). \tag{6}$$

Let us choose an arbitrary element $v \in X([0, t])$. Then, we can find $t' \in [0, t]$ and $x \in X$, such that $v = x(t')$. Choosing i such that $t' \in [t_{i-1}, t_i]$ and j such that $x(t_i) \in B(z_{ij}, \sup_{s \leq t} \beta(X(s)) + \frac{\delta}{2})$ we obtain from (4) and (5)

$$\|v - z_{ij}\| = \|x(t') - z_{ij}\| \leq \|x(t') - x(t_i)\| + \|x(t_i) - z_{ij}\| \leq \omega_0^t(X) + \sup_{s \leq t} \beta(X(s)) + \delta$$

and this verifies (6). Condition (6) yields that

$$\beta(X([0, t])) \leq \omega_0^t(X) + \sup_{s \leq t} \beta(X(s)) + \delta.$$

Letting $\delta \to 0+$ we get (3). $\qquad \square$

Now we give an example of a set X and a Banach space E such that the sign equality in (3) is attained.

Example 2.6. Consider an infinite dimensional Banach space E and a sequence of vectors $\{e_k\} \subset E$ such that $\|e_k\| = 1$, $k = 1, 2, \ldots$ and

$$\beta(\{e_k : k \in \mathbb{N}\}) = 1, \tag{7}$$

for example $E = c_0$ and $e_k = (0, \ldots, 0, 1, 0, \ldots)$, $k = 1, 2, \ldots$.

Next, we define the sequence $\{f_n\}$ of continuous and piece-wise linear functions $f_n : [0, T] \to \mathbb{R}$, $n = 1, 2, \ldots$, given by formula

$$f_n(t) = \begin{cases} 0 & \text{for} \quad t \in [0, \frac{T}{n+1}] \cup [\frac{T}{n}, T] \\ 2n(n+1)t - 2n & \text{for} \quad t \in (\frac{T}{n+1}, \frac{(2n+1)T}{2n(n+1)}] \\ -2n(n+1)t + 2(n+1) & \text{for} \quad t \in (\frac{(2n+1)T}{2n(n+1)}, \frac{T}{n}). \end{cases}$$

Further, let us put

$$x_n(t) = (1 + f_n(t))e_n, \quad t \in [0, T], \ n = 1, 2, \ldots, \quad \text{and} \quad X = \{x_n : n \in \mathbb{N}\}.$$

Observe that

$$X(s) = \begin{cases} \{e_k : k \in \mathbb{N}\} & \text{for} \quad s = 0 \\ \{e_k : k \neq n\} \cup \{(1 + f_n(s))e_n\} & \text{for} \quad s \in (\frac{T}{n+1}, \frac{T}{n}]. \end{cases}$$

Hence

$$\beta(X(s)) = 1 \quad \text{for } s \in [0, T], \tag{8}$$
$$X([0, t]) = \{e_k : k \in \mathbb{N}\} \quad \text{for } t = 0,$$

and

$$\bigcup_{c \in [1,2]} c \cdot \{e_k : k > n\} \subset X([0, t]) \subset \bigcup_{c \in [1,2]} c \cdot \{e_k : k \in \mathbb{N}\} \quad \text{for } t \in \left(\frac{T}{n+1}, \frac{T}{n}\right].$$

Then, in virtue of (7) we get

$$\beta(X([0, t])) = \begin{cases} 1 & \text{for} \quad t = 0 \\ 2 & \text{for} \quad t \in (0, T]. \end{cases} \tag{9}$$

Moreover, we have

$$\omega_0^t(X) = \begin{cases} 0 & \text{for} \quad t = 0 \\ 1 & \text{for} \quad t \in (0, T]. \end{cases} \tag{10}$$

Linking (8)–(10), we conclude that the sign equality in (3) is attained for $t \in J$.

3. Main result

In this section we prove two existence results for the semilinear equation of evolution (1) and (2). First, we will assume that the functions involved in Equations (1) and (2) satisfy the following conditions:

(HA) $A(t)$ is a linear operator acting from $D_t \subset E$ to E for each $t \in J$ and $A(t)$ generates a strongly continuous *evolution system* $\{U(t,s)\}_{0 \le s \le t \le T}$ such that

    (i) $U(t,s)$ is a function, defined on $0 \le s \le t \le T$, which takes values in $B(E)$,

    (ii) $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for $0 \le s \le r \le t \le T$,

    (iii) the map $(t,s) \to U(t,s)x$ is continuous for every $x \in E$.

Further, let us denote:

$$N(t) = \sup\{\|U(s,0)\| : 0 \le s \le t\}, \quad \overline{N}(t) = \sup\{\|U(s,\tau)\| : 0 \le \tau \le s \le t\}.$$

(Hf)   (i) The mapping $f : J \times E \to E$ satisfies Carathéodory conditions, i.e. $f(\cdot, x)$ is measurable for $x \in E$ and $f(t, \cdot)$ is continuous for a.e. $t \in J$,

    (ii) the mapping f is bounded on bounded subsets of $C(J, E)$,

    (iii) there exists a constant $k_f \ge 0$ such that for any bounded set $X \subset C(J, E)$, the inequality

$$\beta(f([0,t] \times X)) \le k_f \beta(X([0,t]))$$

holds for $t \in J$, where $f([0,t] \times X) = \{f(s, x(s)) : 0 \le s \le t, x \in X\}$.

(Hg)   (i) The function $g : C(J, E) \to E$ is continuous,

    (ii) there exists a constant $k_g \ge 0$ such that

$$\beta(g(X)) \le k_g \beta(X(J)),$$

for each bounded set $X \subset C(J, E)$.

(H1) There exists a constant $r > 0$ such that for any $t \in J$

$$N(t) \sup_{x \in B(\theta, r)} \|g(x)\| + \overline{N}(t) \sup_{x \in B(\theta, r)} \int_0^t \|f(s, x(s))\| ds \le r,$$

where $B(\theta, r)$ is closed ball in $C(J, E)$ centered at θ and with radius r.

(H2)

$$3k_g N(T) + 3k_f T \overline{N}(T) < 1.$$

Definition 3.1. A continuous function $x : J \to E$ such that

$$x(t) = U(t,0)g(x) + \int_0^t U(t,s)f(s,x(s))ds \quad \text{for } t \in J$$

will be called *the mild solution* of Equation (1) with initial condition (2).

Next, consider the operators $H, G : C(J,E) \to C(J,E)$ defined by the formulas

$$(Hx)(t) = U(t,0)g(x)$$

$$(Gx)(t) = \int_0^t U(t,s)f(s,x(s))ds.$$

Lemma 3.2. *Assume that assumptions* (HA) *and* (Hf) *are satisfied and a set* $X \subset C(J,E)$ *is bounded. Then*

$$\omega_0^t(GX) \le 2t\overline{N}(t)\beta(f([0,t] \times X)) \quad \text{for } t \in J.$$

Proof. Fix $t \in J$ and denote $W = f([0,t] \times X)$,

$$\nu^t(\varepsilon) = \sup\{\|(U(t_2,s) - U(t_1,s))w\| : 0 \le s \le t_1 \le t_2 \le t, t_2 - t_1 \le \varepsilon, w \in W\}.$$

At the beginning we show that

$$\lim_{\varepsilon \to 0+} \nu^t(\varepsilon) \le 2\overline{N}(t)\beta(W). \tag{11}$$

Suppose contrary. Then there exists a number d such that

$$\lim_{\varepsilon \to 0+} \nu^t(\varepsilon) > d > 2\overline{N}(t)\beta(W). \tag{12}$$

Fix $\delta > 0$ such that
$$d > 2\overline{N}(t)(\beta(W) + \delta). \tag{13}$$

Condition (12) yields that there exist sequences $(t_{2,n}), (t_{1,n}), (s_n) \subset J$ and $(w_n) \subset W$, such that $t_{2,n} \to t', t_{1,n} \to t', s_n \to s$ and

$$\|(U(t_{2,n}, s_n) - U(t_{1,n}, s_n))w_n\| > d.$$

Let the points $y_1, y_2, \ldots, y_k \in E$ be such that $W \subset \bigcup_{i=1}^k B(y_i, \beta(W) + \delta)$. Then there exists a point y_j and a subsequence of (w_n), (which is further denoted by (w_n)) such that $w_n \in B(y_j, \beta(W) + \delta)$, i.e.

$$\|y_j - w_n\| \le \beta(W) + \delta \quad \text{for } n = 1, 2, \ldots.$$

Further, we obtain

$$\|U(t_{2,n}, s_n)w_n - U(t_{1,n}, s_n)w_n\|$$
$$\leq \|U(t_{2,n}, s_n)w_n - U(t_{2,n}, s_n)y_j\| + \|U(t_{2,n}, s_n)y_j - U(t_{1,n}, s_n)y_j\|$$
$$+ \|U(t_{1,n}, s_n)y_j - U(t_{1,n}, s_n)w_n\|$$
$$\leq 2\overline{N}(t)\|y_j - w_n\| + \|U(t_{2,n}, s_n)y_j - U(t_{1,n}, s_n)y_j\|$$
$$\leq 2\overline{N}(t)(\beta(W) + \delta) + \|U(t_{2,n}, s_n)y_j - U(t_{1,n}, s_n)y_j\|.$$

Letting $n \to \infty$ and using the properties of the evolution system $\{U(t, s)\}$, from the above estimate we get

$$\limsup_{n \to \infty} \|U(t_{2,n}, s_n)w_n - U(t_{1,n}, s_n)w_n\| \leq 2\overline{N}(t)(\beta(W) + \delta).$$

This contradicts (12) and (13).

Now, fix $\varepsilon > 0$ and $t_1, t_2 \in [0, t], 0 \leq t_2 - t_1 \leq \varepsilon$. Applying (Hf)(ii) we obtain

$$\|(Gx)(t_2) - (Gx)(t_1)\|$$
$$\leq \int_0^{t_1} \|(U(t_2, s) - U(t_1, s))f(s, x(s))\| ds + \int_{t_1}^{t_2} \|U(t_2, s)f(s, x(s))\| ds$$
$$\leq \int_0^t \|(U(t_2, s) - U(t_1, s))f(s, x(s))\| ds$$
$$+ \varepsilon \overline{N}(t) \sup\{\|f(s, x(s))\| : s \in [0, t], x \in X\}.$$

Hence, we derive the following inequality

$$\omega^t(GX, \varepsilon)$$
$$\leq \sup\left\{\int_0^t \|(U(t_2, s) - U(t_1, s))f(s, x(s))\| ds : t_1, t_2 \in [0, t], 0 \leq t_2 - t_1 \leq \varepsilon, x \in X\right\}$$
$$+ \varepsilon \overline{N}(t) \sup\{\|f(s, x(s))\| : s \in [0, t], x \in X\}.$$

Letting $\varepsilon \to 0+$ and keeping in mind (11) we complete the proof.   □

Lemma 3.3. *Assume that assumptions* (HA), (Hg) *are satisfied and a set* $X \subset C(J, E)$ *is bounded. Then*

$$\omega_0^t(HX) \leq 2N(t)\beta(g(X)) \quad for \ t \in J.$$

The simple proof is omitted. Then we can formulate our first result.

Theorem 3.4. *If the Banach space* E *is separable then under assumptions* (HA), (Hg), (Hf), (H1), *and* (H2), *Equation* (1) *with initial condition* (2) *has at least one mild solution* $x = x(t)$.

Proof. Consider the operator F defined by formula

$$(Fx)(t) = U(t,0)g(x) + \int_0^t U(t,s)f(s,x(s))ds.$$

For an arbitrarily fixed $x \in C(J,E)$ and $t \in J$ we get:

$$\|(Fx)(t)\| \le N(t)\|g(x)\| + \overline{N}(t)\int_0^t \|f(s,x(s))\|ds.$$

From the above estimate and assumption (H1) we infer that there exists a constant $r > 0$ such that the operator F transforms the closed ball $B(\theta, r)$ into itself.

Now, we prove that operator F is continuous in $B(\theta, r)$. To do this, let us fix $x \in B(\theta, r)$ and take arbitrary sequence $(x_n) \in B(\theta, r)$ such that $x_n \to x$ in $C(J,E)$. Next, we have

$$\|Fx - Fx_n\|_C \le N(T)\|g(x) - g(x_n)\| + \overline{N}(T)\int_0^T \|f(s,x(s)) - f(s,x_n(s))\|ds.$$

Applying Lebesgue dominated convergence theorem and (Hg)(i) and (Hf)(i) we derive that F is continuous on $B(\theta, r)$.

Now, we consider the sequence of sets (Ω_n) defined by induction as follows:

$$\Omega_0 = B(\theta, r), \Omega_{n+1} = \text{Conv}(F\Omega_n) \quad \text{for } n = 0, 1, \ldots.$$

This sequence is decreasing, i.e. $\Omega_n \supset \Omega_{n+1}$ for $n = 0, 1, 2, \ldots$.

Further, let us put

$$u_n(t) = \beta(\Omega_n([0,t])), \quad w_n(t) = \omega_0^t(\Omega_n).$$

Observe that each of functions $u_n(t)$ and $w_n(t)$ is nondecreasing, while sequences $(u_n(t))$ and $(w_n(t))$ are nonincreasing at any fixed $t \in J$. Put

$$u_\infty(t) = \lim_{n\to\infty} u_n(t), \quad w_\infty(t) = \lim_{n\to\infty} w_n(t)$$

for $t \in J$. Using Lemmas 2.5, 3.3 and (Hg)(ii) we obtain

$$
\begin{aligned}
\beta(H\Omega_n([0,t])) &\le \omega_0^t(H\Omega_n) + \sup_{s\le t}\beta(H\Omega_n(s)) \\
&\le 2N(t)\beta(g(\Omega_n)) + \sup_{s\le t}N(s)\beta(g(\Omega_n)) \\
&\le 3N(t)\beta(g(\Omega_n)) \\
&\le 3k_gN(t)\beta(\Omega_n([0,T])) \\
&= 3k_gN(t)u_n(T).
\end{aligned}
\tag{14}
$$

Moreover, taking into account Lemmas 2.3, 2.5, 3.2 and (Hf)(iii) we infer the following estimate

$$\beta(G\Omega_n([0,t])) \leq \omega_0^t(G\Omega_n) + \sup_{s\leq t}\beta(G\Omega_n(s))$$

$$\leq 2t\overline{N}(t)\beta(f([0,t]\times\Omega_n)) + \sup_{s\leq t}\beta\left(\int_0^s U(s,\tau)f(\tau,\Omega_n(\tau))d\tau\right)$$

$$\leq 2k_f t\overline{N}(t)\beta(\Omega_n([0,t])) + \sup_{s\leq t}\overline{N}(s)\int_0^s \beta(f(\tau,\Omega_n(\tau)))d\tau$$

$$\leq 2k_f t\overline{N}(t)u_n(t) + \overline{N}(t)\int_0^t \beta(f([0,\tau]\times\Omega_n))d\tau$$

$$\leq 2k_f t\overline{N}(t)u_n(t) + k_f\overline{N}(t)\int_0^t u_n(\tau)d\tau.$$

Linking this estimate with (14) we obtain

$$u_{n+1}(t) = \beta(\Omega_{n+1}([0,t])) = \beta(F\Omega_n([0,t])) \leq \beta(H\Omega_n([0,t])) + \beta(G\Omega_n([0,t]))$$

and consequently $u_{n+1}(t) \leq 3k_g N(t)u_n(T) + 2k_f t\overline{N}(t)u_n(t) + k_f\overline{N}(t)\int_0^t u_n(\tau)d\tau$. Letting $n\to\infty$ we get

$$u_\infty(t) \leq 3k_g N(t)u_\infty(T) + 2k_f t\overline{N}(t)u_\infty(t) + k_f\overline{N}(t)\int_0^t u_\infty(\tau)d\tau.$$

Hence, putting $t = T$, we get in view of (H2)

$$u_\infty(T) = 0. \tag{15}$$

Moreover, applying Lemmas 3.3, 3.2, (Hg)(ii) and (Hf)(iii) we derive $w_{n+1}(t) = \omega_0^t(\Omega_{n+1}) = \omega_0^t(F\Omega_n) \leq \omega_0^t(H\Omega_n) + \omega_0^t(G\Omega_n) \leq 2k_g N(t)u_n(T) + 2k_f t\overline{N}(t)u_n(t)$. Letting $n\to\infty$ we get

$$w_\infty(t) \leq 2k_g N(t)u_\infty(T) + 2k_f t\overline{N}(t)u_\infty(t).$$

Putting $t = T$ and applying (15) we conclude that $w_\infty(T) = 0$. This fact together with (15) implies that $\lim_{n\to\infty}\mu(\Omega_n) = 0$. Hence, in view of Remark 2.2, we deduce that the set $\Omega_\infty = \bigcap_{n=0}^\infty \Omega_n$ is nonempty, compact and convex. Finally, linking all above obtained facts concerning the set Ω_∞ and the operator $F : \Omega_\infty \to \Omega_\infty$ and using the classical Schauder fixed point principle we infer that the operator F has at least one fixed point x in the set Ω_∞. Obviously the function $x = x(t)$ is a mild solution of Equations (1) and (2). $\square$

Now we will investigate Equations (1) and (2) under the following hypotheses:

(Hg') (i) This condition is identical with (Hg)(i),

   (ii) there exists a nonnegative constant $k_g \geq 0$ such that

$$\beta(g(X)) \leq k_g \sup_{t \leq T} \beta(X(t)),$$

   for each bounded set $X \subset C(J, E)$.

(Hf') (i) The mapping $f : J \times E \to E$ is uniformly continuous on bounded subsets of $J \times E$,

   (ii) there is a constant $k_f \geq 0$ such that

$$\beta(f(t, Y)) \leq k_f \beta(Y),$$

   for $t \in J$ and for each bounded $Y \subset E$.

(H3)

$$\min\{k_g N(T) + k_f \overline{N}(T), 2k_f T \overline{N}(T)\} < 1.$$

For the proof of next theorem we need the following lemma.

Lemma 3.5. *Assume that assumptions* (HA) *and* (Hf') *are satisfied and a set $X \subset C(J, E)$ is bounded. Then*

$$\omega_0^t(GX) \leq 2k_f t \overline{N}(t) \beta(X([0, t])) \quad for\ t \in J.$$

Proof. Fix $t \in J$ and denote

$$\overline{\nu}^t(\varepsilon) = \sup\{\|(U(t_2, s) - U(t_1, s))w\| : 0 \leq s \leq t_1 \leq t_2 \leq t, t_2 - t_1 \leq \varepsilon, x \in X\}.$$

At the beginning we show that

$$\lim_{\varepsilon \to 0+} \overline{\nu}^t(\varepsilon) \leq 2k_f \overline{N}(t) \beta(X([0, t])).$$

Suppose contrary. Then there exists a number d such that

$$\lim_{\varepsilon \to 0+} \overline{\nu}^t(\varepsilon) > d > 2k_f \overline{N}(t) \beta(X([0, t])).$$

Fix $\delta > 0$ such that

$$\lim_{\varepsilon \to 0+} \overline{\nu}^t(\varepsilon) > d + \delta > d > 2\overline{N}(t)(k_f \beta(X([0, t])) + \delta). \qquad (16)$$

Condition (16) yields that there exist sequences $(t_{2,n}), (t_{1,n}), (s_n) \subset J$ and $(x_n) \subset X$, such that $t_{2,n} \to t', t_{1,n} \to t', s_n \to s'$ and

$$\|(U(t_{2,n}, s_n) - U(t_{1,n}, s_n))f(s_n, x_n(s_n))\| > d + \delta.$$

Hence, in view of the uniform continuity of the mapping f we obtain

$$\|(U(t_{2,n}, s_n) - U(t_{1,n}, s_n))f(s', x_n(s_n))\| > d,$$

for n sufficiently large. Further, we choose points $y_1, y_2, \ldots, y_k \in E$ such that $f(s', X([0, t])) \subset \bigcup_{i=1}^{k} B(y_i, \beta(f(s', X([0, t]))) + \delta)$. Then there is a point y_j and a subsequence of $(f(s', x_n(s_n)))$, (which is further denoted by $(f(s', x_n(s_n)))$) such that

$$\|f(s', x_n(s_n)) - y_j\| \leq \beta(f(s', X([0, t]))) + \delta \leq k_f \beta(X([0, t])) + \delta.$$

Next, arguing analogously as in Lemma 3.2 we derive

$$\|U(t_{2,n}, s_n)f(s', x_n(s_n)) - U(t_{1,n}, s_n)f(s', x_n(s_n))\|$$
$$\leq 2\overline{N}(t)\|f(s', x_n(s_n)) - y_j\| + \|U(t_{2,n}, s_n)y_j - U(t_{1,n}, s_n)y_j\|$$
$$\leq 2\overline{N}(t)(k_f \beta(X([0, t])) + \delta) + \|U(t_{2,n}, s_n)y_j - U(t_{1,n}, s_n)y_j\|.$$

Letting $n \to \infty$ and using the properties of the evolution system $\{U(t, s)\}$, this estimate implies $\limsup_{n\to\infty}\|U(t_{2,n}, s_n)f(s', x_n(s_n)) - U(t_{1,n}, s_n)f(s', x_n(s_n))\| \leq 2\overline{N}(t)(k_f \beta(X([0, t])) + \delta)$. This contradicts (16). The remainder of the proof proceeds analogously as the proof of Lemma 3.2 and is therefore omitted. $\square$

Now, we can formulate the second existence result.

Theorem 3.6. *If the Banach space E is separable then under assumptions* (HA), (Hg'), (Hf'), (H1) *and* (H3), *Equation* (1) *with initial condition* (2) *has at least one mild solution $x = x(t) \in C(J, E)$.*

Proof. Similarly as in proof of Theorem 3.4 we can show that the mapping $F : B(\theta, r) \to B(\theta, r)$ is continuous and we define analogously the sequence $(\Omega_n)_{n\in\mathbb{N}}$. Let us put

$$v_n(t) = \sup_{s \leq t} \beta(\Omega_n(s)), \quad v_\infty(t) = \lim_{n\to\infty} v_n(t).$$

Using (Hg')(ii) and (Hf')(ii) we get

$$\beta(\Omega_{n+1}(s)) = \beta(F\Omega_n(s))$$
$$\leq \beta(H\Omega_n(s)) + \beta(G\Omega_n(s))$$
$$\leq N(s)\beta(g(\Omega_n)) + \beta\left(\int_0^s U(s, \tau)f(\tau, \Omega_n(\tau))d\tau\right)$$
$$\leq k_g N(s) \sup_{s \leq T} \beta(\Omega_n(s)) + \int_0^s \beta(U(s, \tau)f(\tau, \Omega_n(\tau)))d\tau$$
$$\leq k_g N(s)v_n(T) + \overline{N}(s) \int_0^s \beta(f(\tau, \Omega_n(\tau)))d\tau$$
$$\leq k_g N(s)v_n(T) + k_f \overline{N}(s) \int_0^s \beta(\Omega_n(\tau))d\tau$$
$$\leq k_g N(s)v_n(T) + k_f \overline{N}(s) \int_0^s v_n(\tau)d\tau.$$

Hence $v_{n+1}(t) = \sup_{s \leq t} \beta(\Omega_{n+1}(s)) \leq k_g N(t) v_n(T) + k_f \overline{N}(t) \int_0^t v_n(\tau) d\tau$. Letting $n \to \infty$ we obtain

$$v_\infty(t) \leq k_g N(t) v_\infty(T) + k_f \overline{N}(t) \int_0^t v_\infty(\tau) d\tau.$$

Now, put $t = T$. Using (H3) we get

$$v_\infty(T) = 0. \tag{17}$$

Then, keeping in mind Lemmas 2.5 and 3.5 we have $w_{n+1}(t) = \omega_0^t(\Omega_{n+1}) = \omega_0^t(F\Omega_n)$ and

$$\begin{aligned}
w_{n+1}(t) &\leq \omega_0^t(H\Omega_n) + \omega_0^t(G\Omega_n) \\
&\leq 2N(t)\beta(g(\Omega_n)) + 2k_f t\overline{N}(t)\beta(\Omega_n([0,t])) \\
&\leq 2k_g N(t) v_n(T) + 2k_f t\overline{N}(t)(\omega_0^t(\Omega_n) + \sup_{\tau \leq t} \beta(\Omega_n(\tau))) \\
&\leq 2k_g N(t) v_n(T) + 2k_f t\overline{N}(t)(w_n(t) + v_n(t)).
\end{aligned}$$

Letting $n \to \infty$ and putting $t = T$ we obtain

$$w_\infty(T) \leq 2k_f T\overline{N}(T) w_\infty(T) + (2k_g N(T) + 2k_f T\overline{N}(T)) v_\infty(T),$$

$$w_\infty(T)(1 - 2k_f T\overline{N}(T)) \leq (2k_g N(T) + 2k_f T\overline{N}(T)) v_\infty(T).$$

Linking above established fact, (17) and (H3) we get $w_\infty(T) = 0$. This implies that $\lim_{n \to \infty} \mu(\Omega_\infty) = 0$ and therefore the set $\Omega_\infty = \bigcap_{n=0}^\infty \Omega_n$ is nonempty, compact and convex. Using the classical Schauder fixed point principle for the operator $F : \Omega_\infty \to \Omega_\infty$ we infer that the operator F has at least one fixed point in $x \in \Omega_\infty$. This completes the proof. $\qquad \square$

4. Final remarks

In this section we are going to discuss hypotheses (Hf)(iii), (Hf')(ii), (H1), (Hg)(ii) and (Hg')(ii). We provide more convenient sufficient conditions allowing to replace these hypotheses.

First, we list some assumptions:

(F1) The mappings $f_1, f_2 : J \times E \to E$ satisfy Carathéodory conditions.

(F2) The mapping $f_1(\cdot, x)$ is continuous on J for each $x \in E$.

(F3) The mapping f_2 is compact on $C(J, E)$.

(F4) There exists a constant $k_f \geq 0$ such that

$$\|f_1(t, x) - f_1(t, y)\| \leq k_f \|x - y\|$$

for any $t \in J$ and for all $x, y \in E$.

(F5) There are two integrable functions $a_f, b_f : J \to \mathbb{R}_+$ such that

$$\|f(t, x)\| \leq a_f(t) + b_f(t)\|x\|$$

for a.e. $t \in J$ and for each $x \in E$.

Moreover, we define the mapping $f : J \times E \to E$ by the formula

$$f(t, x) = f_1(t, x) + f_2(t, x).$$

Proposition 4.1. *Under the assumptions* (F1)–(F4) *the mapping f satisfies the hypotheses* (Hf)(iii).

Proof. Let us take a nonempty and bounded subset X of $C(J, E)$ and fix arbitrarily $t \in J$ and $\varepsilon > 0$. Moreover, we put $r = \beta(X([0, t]))$. Then there exist points $a_1, a_2, \ldots, a_n \in E$ such that

$$X([0, t]) \subset \bigcup_{i=1}^{n} B(a_i, r + \varepsilon).$$

Keeping in mind the continuity of the function $f_1(\cdot, a_i)$ on J for $i = 1, 2, \ldots, n$ we deduce that there is a partition $0 = s_0 < s_1 < \cdots < s_m = t$ of the interval $[0, t]$ such that for each $s \in [s_{j-1}, s_j]$ we have

$$\|f_1(s, a_i) - f_1(s_j, a_i)\| \leq \varepsilon \tag{18}$$

for $i = 1, 2, \ldots, n$. Let us fix $s \in J$ and $x \in X$. Choosing j and i such that $s \in [s_{j-1}, s_j]$ and

$$x(s) \in B(a_i, r + \varepsilon) \tag{19}$$

we obtain from (F4), (18) and (19)

$$\begin{aligned}
\|f_1(s, x(s)) - f_1(s_j, a_i)\| &\leq \|f_1(s, x(s)) - f_1(s, a_i)\| + \|f_1(s, a_i) - f_1(s_j, a_i)\| \\
&\leq k_f \|x(s) - a_i\| + \varepsilon \\
&\leq k_f(r + \varepsilon) + \varepsilon.
\end{aligned}$$

This implies that $f_1([0, t] \times X) \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} B(f_1(s_j, a_i), k_f(r + \varepsilon) + \varepsilon)$ and therefore

$$\beta(f_1([0, t] \times X)) \leq k_f(r + \varepsilon) + \varepsilon, \quad \beta(f_1([0, t] \times X)) \leq k_f \beta(X([0, t])).$$

Linking this inequality and (F3) we get

$$\beta(f([0, t] \times X)) \leq \beta(f_1([0, t] \times X)) + \beta(f_2([0, t] \times X)) \leq k_f \beta(X([0, t])). \quad \square$$

Proposition 4.2. *Under the assumptions* (F1), (F3), (F4) *the mapping f satisfies hypothesis* (Hf')(ii).

The similar proof is omitted.

Now we discuss hypothesis (H1). The mapping g is usually defined in two ways:

$$g(x) = \sum_{i=1}^{n} c_i x(t_i), \tag{20}$$

where $0 \le t_1 < t_2 < \cdots < t_n \le T, c_1, c_2, \ldots, c_n$ are given constants, or

$$g(x) = \int_0^T h(t, x(t)) dt. \tag{21}$$

The function h involved in formula (21) will be studied under the following assumptions:

(G1) The mapping $h : J \times E \to E$ satisfies Carathéodory conditions.

(G2) There are two integrable functions $a_h, b_h : J \to \mathbb{R}_+$ such that

$$\|h(t, x)\| \le a_h(t) + b_h(t)\|x\|$$

for a.e. $t \in J$ and each $x \in E$.

(G3) There exists a constant $k_h \ge 0$ such that

$$\|h(t, x) - h(t, y)\| \le k_h \|x - y\|$$

for any $t \in J$ and for all $x, y \in E$. Moreover, the function $h(\cdot, x)$ is continuous on J for each $x \in E$.

(G4) There exists a integrable function $k : J \to \mathbb{R}_+$ such that

$$\beta(h(t, Y)) \le k(t)\beta(Y)$$

for a.e. $t \in J$ and every a bounded subset $Y \subset E$.

Now we can formulate the next proposition.

Proposition 4.3. *If the mapping g is given by formula* (20), *f satisfies* (F5) *and moreover*

$$N(T) \sum_{i=1}^{n} |c_i| + \overline{N}(T) \int_0^T b_f(t) dt < 1,$$

then the hypothesis (H1) *is satisfied.*

Proof. It is easy to check that r can be given by the formula

$$r = \overline{N}(T) \int_0^T a_f(t) dt \cdot \left(1 - N(T) \sum_{i=1}^{n} |c_i| - \overline{N}(T) \int_0^T b_f(t) dt \right)^{-1}. \qquad \square$$

Proposition 4.4. *Assume that g is given by formula* (21), *the conditions* (F5), (G1), (G2) *are satisfied and moreover*

$$N(T) \int_0^T b_h(t)dt + \overline{N}(T) \int_0^T b_f(t)dt < 1.$$

Then the hypothesis (H1) *is satisfied.*

The similar proof is omitted.

Now we discuss hypotheses (Hg)(ii) and (Hg')(ii).

Proposition 4.5. *If g is given by formula* (20), *then the hypotheses* (Hg)(ii) *and* (Hg')(ii) *are satisfied with the constant $k_g = \sum_{i=1}^n |c_i|$.*

Proof. We will prove (Hg)(ii). Let X be a nonempty and bounded subset of $C(J, E)$. Then

$$\beta(g(X)) \leq \beta\Big(\sum_{i=1}^n c_i X(t_i)\Big) \leq \sum_{i=1}^n |c_i|\beta(X(t_i)) \leq \sum_{i=1}^n |c_i| \cdot \beta(X(J)).$$

The proof for (Hg')(ii) is omitted.   □

Proposition 4.6. *Let g be described by* (21) *and suppose that conditions* (G1) *and* (G4) *are satisfied. Then the hypothesis* (Hg')(ii) *is fulfilled with the constant $k_g = \int_0^T k(t)dt$.*

Proof. Let X be a nonempty and bounded subset of $C(J, E)$. We have

$$\beta(g(X)) \leq \beta\left(\int_0^T h(t, X(t))dt\right) \leq \int_0^T k(t)\beta(X(t))dt \leq \int_0^T k(t)dt \cdot \sup_{t \in J} \beta(X(t))$$

and therefore $k_g = \int_0^T k(t)dt$.   □

Proposition 4.7. *If g is defined by* (21) *and the conditions* (G1) *and* (G3) *are fulfilled, then the hypothesis* (Hg)(ii) *is satisfied with the constant $k_g \leq Tk_h$.*

The proof is similar to the proof of Proposition 4.1 and will be omitted.

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