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Hardy Averaging Operator on Generalized Banach Function Spaces and Duality

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Abstract. Let $Af(x) := \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} f(t)\ dt$ be the n-dimensional Hardy averaging operator. It is well known that A is bounded on $L^p(\Omega)$ with an open set $\Omega \subset \mathbb{R}^n$ whenever $1 < p \leq \infty$. We improve this result within the framework of generalized Banach function spaces. We in fact find the "source" space S_X, which is strictly larger than X, and the "target" space T_X, which is strictly smaller than X, under the assumption that the Hardy-Littlewood maximal operator M is bounded from X into X, and prove that A is bounded from S_X into T_X. We prove optimality results for the action of A and its associate operator A' on such spaces and present applications of our results to variable Lebesgue spaces $L^{p(\cdot)}(\Omega)$, as an extension of A. Nekvinda and L. Pick [Math. Nachr. 283 (2010), 262–271; Z. Anal. Anwend. 30 (2011), 435–456] in the case when $n = 1$ and Ω is a bounded interval.

Keywords. Hardy averaging operator, Lebesgue spaces of variable exponent, Banach function space, optimal domain

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1. Introduction

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and Ω be an open subset of $\mathbb{R}^n$. For an integrable function u on a measurable set $E \subset \mathbb{R}^n$ of positive measure, we define the integral mean over E by

$$⨍_E u(x)\ dx = \frac{1}{|E|} \int_E u(x)\ dx,$$

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where $|E|$ denotes the Lebesgue measure of E. We denote by $B(x,r)$ the open ball with center x and of radius $r > 0$, and by $|B(x,r)|$ its Lebesgue measure. For a locally integrable function f on Ω, we consider the Hardy averaging operator A, defined by

$$Af(x) = \fint_{B(0,|x|)} f(t)\ dt$$

and the centered Hardy-Littlewood maximal operator M, defined by

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\ dy,$$

by setting $f = 0$ outside Ω (for the fundamental properties of maximal functions, see Stein [12]).

It is well known that both the operators M and A are bounded on $L^p(\Omega)$ whenever $1 < p \leq \infty$. But there is a sufficiently large family of other spaces X for which M and, consequently A are bounded on X.

In this paper we improve the result of the second author and Pick [10] in the case when $n = 1$ and Ω is a bounded interval within the framework of generalized Banach function spaces. Under the assumption $M : X \to X$, we find the 'source' space S_X and the "target" space T_X such that

(i) the Hardy averaging operator A satisfies

$$A : S_X \to T_X;$$

(ii) this result improves the classical estimate

$$A : X \to X$$

in the sense that

$$T_X \hookrightarrow X \hookrightarrow S_X;$$

(iii) this result cannot be improved any further, at least not within the environment of generalized Banach function spaces in the sense that whenever Y is a generalized Banach function space strictly larger than S_X, then

$$A : Y \not\to T_X$$

and, likewise, when Z is a generalized Banach function space strictly smaller than T_X, then

$$A : S_X \not\to Z.$$

As in [10], we treat analogous questions for variable Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and obtain several results of independent interest. The key ingredient here is a certain logarithmic control of the variation of the generating function $p(x)$, a notion which we call a weak-Lipschitz property or a log-Hölder continuity.

The paper is structured as follows. In Section 2, we introduce generalized Banach function spaces (shortly GBFS), and collect some properties on GBFS. In Section 3, we introduce the spaces T_X and S_X, and show that $A : S_X \to T_X$. Optimality of S_X and T_X is proved in Section 4. In Section 5, we present a key equivalence between two variable Lebesgue spaces whose generating functions are "close" in a certain sense. In Section 6, we introduce weak Banach function spaces. In Section 7, we present applications of our results to variable Lebesgue spaces $L^{p(\cdot)}(\Omega)$, as an extension of [10] in the case when $n = 1$ and Ω is a bounded interval. In the final section, we also prove optimality results for the action of the associate operator A' to the operator A, as an extension of [11].

2. Preliminaries

Throughout this paper, let C denote various constants independent of the variables in question, and $C(a, b, \ldots)$ a constant that depends on $a, b, \ldots$.

Let Ω be an open subset of $\mathbb{R}^n$. Let $\mathcal{M}(\Omega)$ denote the space of measurable functions on Ω with values in $[-\infty, \infty]$. Denote by χ_E the characteristic function of E. Let the symbol $|f|$ stand for the modulus of a function $f, f \in \mathcal{M}(\Omega)$. Recall the frequently used definition of Banach function spaces which can be found for instance in [1].

Definition 2.1. We say that a normed linear space $(X, \|.\|_X)$ is a Banach function space (BFS for short) if the following conditions are satisfied:

the norm $\|f\|_X$ is defined for all $f \in \mathcal{M}(\Omega)$, and $f \in X$ if and only if $\|f\|_X < \infty$; (1)

$\|f\|_X = \|\ |f|\ \|_X$ for every $f \in \mathcal{M}(\Omega)$; (2)

if $0 \le f_n \nearrow f$ a.e. in Ω, then $\|f_n\|_X \nearrow \|f\|_X$; (3)

if $E \subset \Omega$ is a measurable set of finite measure, then $\chi_E \in X$; (4)

for every measurable set $E \subset \Omega$ of finite measure, there exists a positive constant C_E such that $\int_E |f(x)|dx \le C_E \|f\|_X$. (5)

We will work with more general spaces where conditions (4) and (5) are omitted .

Definition 2.2. We say that a normed linear space $(X, \|.\|_X)$ is a generalized Banach function space (shortly GBFS) if the following conditions are satisfied:

the norm $\|f\|_X$ is defined for all $f \in \mathcal{M}(\Omega)$, and $f \in X$ if and only if $\|f\|_X < \infty$; (6)

$\|f\|_X = \|\ |f|\ \|_X$ for every $f \in \mathcal{M}(\Omega)$; (7)

if $0 \le f_n \nearrow f$ a.e. in Ω then $\|f_n\|_X \nearrow \|f\|_X$. (8)

Recall that condition (8) immediately yields the following property:

$$\text{if } 0 \le f \le g \quad \text{then } \|f\|_X \le \|g\|_X. \tag{9}$$

To see this it suffices to set $f_1 = f$, $f_n = g$ for $n \ge 2$ in (8). It is well-known that each BFS is complete and so, it is a Banach space (see [1, Theorem 1.6]). We prove now by an analogous method that each GBFS is complete.

Lemma 2.3 (Fatou's property of GBFSs). *Let $(X, \|.\|_X)$ be a GBFS. Assume that $f_n \to f$ a.e in Ω and $\liminf_{n\to\infty} \|f_n\|_X < \infty$. Then $f \in X$ and $\|f\|_X \le \liminf_{n\to\infty} \|f_n\|_X$.*

Proof. Set $h_n(x) = \inf_{m\ge n} |f_m(x)|$. Then $h_n \nearrow |f|$ a.e. and by (7) with (8) we have $\|f\|_X = \| \ |f| \ \|_X = \lim_{n\to\infty} \|h_n\|_X = \lim_{n\to\infty} \| \inf_{m\ge n} |f_m(x)| \ \|_X \le \lim_{n\to\infty} \inf_{m\ge n} \|f_m(x)\|_X = \liminf_{n\to\infty} \|f_n\|_X$. □

Lemma 2.4. *Let $(X, \|.\|_X)$ be a GBFS and $0 \le f \in X$. Denote $A = \{x \in \Omega : f(x) = \infty\}$. Then $|A| = 0$.*

Proof. Assume $|A| > 0$. Set $g = f\chi_A$. Since $g \le f$, we have by (9) an inequality $\|g\|_X \le \|f\|_X < \infty$. But $g = \infty$ in A and so, $\alpha\chi_A \le g$ for each $\alpha > 0$ which yields

$$\|\chi_A\|_X \le \frac{\|g\|_X}{\alpha} \le \frac{\|f\|_X}{\alpha} \quad \text{for each } \alpha > 0.$$

Thus, $\|\chi_A\|_X = 0$. This implies $\chi_A = 0$ a.e., which is a contradiction with $|A| > 0$. □

Lemma 2.5. *Let $(X, \|\cdot\|_X)$ be a GBFS. Assume that $f_n \in X$ and $\sum_{k=1}^{\infty} \|f_k\|_X$ is finite. Then $\sum_{k=1}^{\infty} f_k$ converges to a function f in X and $\|f\|_X \le \sum_{k=1}^{\infty} \|f_k\|_X$. Consequently, X is complete and so, a Banach space.*

Proof. Let

$$g(x) = \sum_{k=1}^{\infty} |f_k(x)|, \qquad g_n(x) = \sum_{k=1}^{n} |f_k(x)|.$$

Thus, $0 \le g_n \nearrow g$ a.e.. Since

$$\|g_n\|_X \le \sum_{k=1}^{n} \|f_k\|_X \le \sum_{k=1}^{\infty} \|f_k\|_X < \infty,$$

we have $g \in X$ by (8). The series $\sum_{k=1}^{\infty} |f_k(x)|$ is finite for almost every $x \in \Omega$ by Lemma 2.4 and so, the series $\sum_{k=1}^{\infty} f_k(x) \to f(x)$ converges for almost every $x \in \Omega$. Denote $s_n = \sum_{k=1}^{n} f_k(x)$. Then $s_n \to f$ a.e. and $s_n - s_m \to f - s_m$ a.e. as $n \to \infty$. Clearly

$$\liminf_{n\to\infty} \|s_n - s_m\|_X = \liminf_{n\to\infty} \Big\| \sum_{k=m+1}^{n} f_k \Big\|_X \le \liminf_{n\to\infty} \sum_{k=m+1}^{n} \|f_k\|_X \le \sum_{k=m+1}^{\infty} \|f_k\|_X,$$

which gives $\liminf_{n\to\infty} \|s_n - s_m\|_X \to 0$ for $m \to \infty$. Using Lemma 2.3 we obtain for each m

$$\|f - s_m\|_X \le \liminf_{n\to\infty} \|s_n - s_m\|_X,$$

which implies $\|f - s_m\|_X \to 0$ for $m \to \infty$. Moreover,

$$\|f\|_X \le \|f - s_m\|_X + \|s_m\|_X \le \|f - s_m\|_X + \sum_{k=1}^{m} \|f_k\|_X$$

and so, $\|f\|_X \le \sum_{k=1}^{\infty} \|f_k\|_X$. $\square$

Let X, Y be Banach spaces (not necessarily generalized Banach function spaces). Say that $X \hookrightarrow Y$ if $X \subset Y$ and there is $C > 0$ such that $\|f\|_Y \le C\|f\|_X$ for all $f \in X$. Recall well-known theorems on Banach function spaces (see [1, Theorem 1.8]) which assert the implication

$$(\|f\|_X < \infty \Rightarrow \|f\|_Y < \infty) \Longrightarrow X \hookrightarrow Y.$$

In what follows we will need a generalization of this remark. Remark that proof uses the same idea as in [1].

Definition 2.6. Let $(X, \|.\|_X)$ be GBFSs. Say that a mapping $T : (X, \|.\|_X) \to \mathcal{M}(\Omega)$ is a sublinear nondecreasing operator if the following conditions are satisfied for all $\alpha \in \mathbb{R}, f, g \in (X, \|.\|_X)$:

(i) $T(\alpha f) = \alpha T(f)$, $T(f + g) \le T(f) + T(g)$ a.e.
(ii) if $0 \le f \le g$ a.e., then $0 \le Tf \le Tg$ a.e.

Lemma 2.7. *Let $(X, \|.\|_X), (Y, \|.\|_Y)$ be GBFSs and T a sublinear nondecreasing operator on $\mathcal{M}(\Omega)$. Then the following two conditions are equivalent:*

(i) $\|f\|_X < \infty \Rightarrow \|Tf\|_Y < \infty$
(ii) *there is $C > 0$ such that $\|Tf\|_Y \le C\|f\|_X$ for all $f \in X$.*

Proof. We prove only the implication (i) $\Rightarrow$ (ii), since the opposite one is trivial. Assume that $T : X \to Y$ is unbounded. Then there is a sequence $0 \le f_n \in X$ such that $\|f_n\|_X \le 1$ and $\|Tf_n\|_Y \ge n^3$. Setting $f = \sum_{n=1}^{\infty} n^{-2} f_n$, we have $\|f\|_X < \infty$. Moreover, by the monotonicity of T we have $\|Tf\|_Y \ge n^{-2}\|Tf_n\|_Y \ge n$ for each n and so, $\|Tf\|_Y = \infty$ which is a contradiction with (i). $\square$

In Sections 3 and 4 we assume

$$M : X \to X. \tag{10}$$

This assumption is satisfied for a wide family of generalized Banach function spaces, for instance for Lebesgue spaces L^p, Lorenz spaces $L^{p,q}$, some Orlicz spaces L^Φ and so on.

3. Spaces T_X, S_X and boundedness of A from S_X

We will now introduce two new function spaces. Given a measurable function f on $\mathbb{R}^n$, set

$$\widetilde{f}(x) = \operatorname*{ess\,sup}_{|t|\geq|x|} |f(t)|.$$

If x is a Lebesgue point of f, then $|f(x)| \leq \widetilde{f}(x)$, so that

$$|f(x)| \leq \widetilde{f}(x) \text{ a.e.} \tag{11}$$

Definition 3.1. Let X be a GBFS and let f be a measurable function on $\mathbb{R}^n$. Set

$$\|f\|_{T_X} = \|\widetilde{f}\|_X$$

and define the corresponding space $T_X = \{f : \widetilde{f} \in X\}$.

Lemma 3.2. *Let X be a GBFS. Then T_X is a GBFS.*

Proof. We verify only (8). Assume $0 \leq f_n \nearrow f$. Since the space L^∞ is a BFS, it satisfies (3). Hence, we have for each $x \in \mathbb{R}^n$

$$\widetilde{f_n}(x) = \operatorname*{ess\,sup}_{|t|\geq|x|} |f_n(t)| \nearrow \operatorname*{ess\,sup}_{|t|\geq|x|} |f(t)| = \widetilde{f}(x)$$

and by (8) we obtain $\|f_n\|_{T_X} = \|\widetilde{f_n}\|_X \nearrow \|\widetilde{f}\|_X = \|f\|_{T_X}$, which finishes the proof. □

Theorem 3.3. *Let X be a GBFS. Then the embedding $T_X \hookrightarrow X$ holds.*

Proof. By (11), we have by the definition of T_X and (9),

$$\|f\|_X \leq \|\widetilde{f}\|_X = \|f\|_{T_X}.$$ □

Lemma 3.4. *Let X be a GBFS. Then $A : X \to T_X$.*

Proof. Fix $x \in \mathbb{R}^n$. If $|x| \leq |y| \leq 2|x|$, then

$$A|f|(y) = \fint_{B(0,|y|)} |f(w)|dw \geq C \fint_{B(0,|x|)} |f(w)|dw = CA|f|(x).$$

For $|y| \geq |x|$ we have an inclusion $B(0, 2|y|) \subset B(x, 3|y|)$ and therefore,

$$M(A|f|)(x) \geq \fint_{B(x,3|y|)} A|f|(w)dw \geq C|y|^{-n} \int_{B(0,2|y|)} A|f|(w)dw$$

and consequently

$$M(A|f|)(x) \geq C|y|^{-n} \int_{\{w:|y|\leq|w|\leq 2|y|\}} A|f|(w)dw \geq CA|f|(y).$$

Hence, setting $g(x) = \widetilde{A|f|}(x)$, we have $g(x) \leq CM(A|f|)(x)$, $x \in \mathbb{R}^n$. Moreover, $|Af(x)| \leq A|f|(x) \leq CMf(x)$. By (10), we have

$$\|Af\|_{T_X} \leq \| \widetilde{A|f|} \|_X = \|g\|_X \leq C\|M(A|f|)\|_X \leq C\|A|f|\|_X \leq C\|Mf\|_X \leq C\|f\|_X,$$

as desired. □

Definition 3.5. Let X be a GBFS. For a measurable function f, we define the norm

$$\|f\|_{S_X} = \|A|f|\ \|_{T_X},$$

and the corresponding space $S_X = \{f : \widetilde{A|f|} \in X\}$.

Lemma 3.6. *Let X be a GBFS. Then the space S_X is a GBFS.*

Proof. We verify only (8). Assume $0 \leq f_n \nearrow f$. Since the space L^1 is a BFS, it satisfies (3). Hence, we have for each $x \in \mathbb{R}^n$

$$Af_n(x) = \fint_{B(0,|x|)} f_n(t)dt \nearrow \fint_{B(0,|x|)} f(t)dt = Af(x).$$

Since T_X is a GBFS by Lemma 3.2, it satisfies (8), which gives $\|f_n\|_{S_X} = \|Af_n\|_{T_X} \nearrow \|Af\|_{T_X} = \|f\|_{S_X}$ and finishes the proof. □

Theorem 3.7. *Let X be a GBFS. Then the embedding $X \hookrightarrow S_X$ holds.*

Proof. Let $f \in X$. By the definition of S_X and Lemma 3.4, we have

$$\|f\|_{S_X} = \|A|f|\|_{T_X} \leq C\|f\|_X. \qquad \square$$

Theorem 3.8. *Let X be a GBFS. Then $A : S_X \to T_X$.*

Proof. Assume $f \in S_X$. By the definitions of T_X and S_X, we have

$$\|Af\|_{T_X} \leq \|A|f|\|_{T_X} = \|f\|_{S_X}. \qquad \square$$

4. Optimality of S_X and T_X

In this section, we shall prove optimality of S_X and T_X.

Theorem 4.1. *Assume that $Z \subsetneqq T_X$ is a GBFS. Then $A\colon T_X \not\to Z$.*

Proof. Take $g \in T_X \setminus Z$ and set $h(x) = \widetilde{g}(x)$. Then h is radially non-increasing, $h \geq g$ and $h \in T_X$. Since Z is a GBFS we have $h \notin Z$. So, $h \in T_X \setminus Z$. Since h is non-increasing, we have $Ah \geq h$ and so, $Ah \notin Z$. □

Theorem 4.2. *Assume that Z is a GBFS such that $S_X \subsetneqq Z$. Then $A\colon Z \not\to S_X$.*

Proof. Take $0 \le f \in Z \setminus S_X$. We estimate

$$\begin{aligned}
\fint_{B(0,|y|)} Af(t)dt &= \fint_{B(0,|y|)} \fint_{B(0,|t|)} f(s)ds\ dt \\
&= C \fint_{B(0,|y|)} f(s) \int_{|s|\le|t|\le|y|} |t|^{-n} dt\ ds \\
&= C \fint_{B(0,|y|)} f(s) \log \frac{|y|}{|s|}\ ds \\
&\ge Ce^{-n} \fint_{B(0,\frac{|y|}{e})} f(s)\ ds,
\end{aligned}$$

so that

$$\begin{aligned}
\|Af\|_{S_X} &\ge Ce^{-n} \Big\| \operatorname*{ess\,sup}_{|y|\ge|x|} \fint_{B(0,\frac{|y|}{e})} f(s)\ ds \Big\|_X \\
&= Ce^{-n} \Big\| \operatorname*{ess\,sup}_{|z|\ge\frac{|x|}{e}} \fint_{B(0,|z|)} f(s)\ ds \Big\|_X \\
&\ge Ce^{-n} \Big\| \operatorname*{ess\,sup}_{|z|\ge|x|} \fint_{B(0,|z|)} f(s)\ ds \Big\|_X .
\end{aligned}$$

Since $f \notin S_X$, we see that $Af \notin S_X$, which completes the proof. □

Remark 4.3. By Theorems 3.3 and 3.7, $T_X \hookrightarrow S_X$. It follows from Theorems 4.1 and 4.2 that the action of the operator $A : S_X \to T_X$ is optimal in the sense that neither the source space nor the target one can be improved.

5. Variable Lebesgue spaces $L^{p(\cdot)}$

We will frequently use the notation $\mathbf{B}$ for the unit ball $B(0,1)$ in $\mathbb{R}^n$.

Let $1 \le p < \infty$ and $1 \le p_\infty < \infty$. In this section, we consider continuous exponents $p(\cdot)$ on $\mathbb{R}^n$ such that

(P1) $1 \le p^- \equiv \inf_{x\in\mathbb{R}^n} p(x) \le \sup_{x\in\mathbb{R}^n} p(x) \equiv p^+ < \infty$

(P2) $|p(x) - p| \le \dfrac{C}{\log(e + \frac{1}{|x|})}$ whenever $x \in \mathbb{R}^n$

(P3) $|p(x) - p_\infty| \le \dfrac{C}{\log(e + |x|)}$ whenever $x \in \mathbb{R}^n$.

If p satisfies (P2), then p is said to satisfy the weak-Lipschitz condition at zero with respect to p. Moreover, we say that $p(\cdot)$ is weak-Lipschitz or log-Hölder if

(P4) $|p(x) - p(y)| \le \dfrac{C}{\log(e + \frac{1}{|x-y|})}$ whenever $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$.

Definition 5.1. Let Ω be an open subset of $\mathbb{R}^n$. Let us consider the family $L^{p(\cdot)}(\Omega)$ of all measurable functions f on Ω satisfying

$$\int_\Omega \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy < \infty$$

for some $\lambda > 0$. We define the norm on this space by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \le 1\right\}.$$

The following remark is proved in [3].

Remark 5.2. $L^{p(\cdot)}(\Omega)$ is a BFS.

Lemma 5.3 (cf. [10, Lemma 3.1])**.** *Suppose that $p(\cdot)$ and $q(\cdot)$ satisfy* (P1)*. Assume that*

$$|p(x) - q(x)| \le \frac{C_1}{\log(e + \frac{1}{|x|})} \quad \textit{whenever } x \in \mathbb{R}^n.$$

Let $a > 0$. Let f be a nonnegative measurable functions on $\mathbb{R}^n$ satisfying

$$f(x) \le C_2|x|^{-a}, \quad |x| \le 1. \tag{12}$$

Then

$$\int_{\mathbf{B}} f(x)^{p(x)}dx < \infty \quad \textit{if and only if} \quad \int_{\mathbf{B}} f(x)^{q(x)}dx < \infty.$$

Moreover, there is $C > 1$ such that

$$C^{-1}\|f\|_{L^{q(\cdot)}(\mathbf{B})} \le \|f\|_{L^{p(\cdot)}(\mathbf{B})} \le C\|f\|_{L^{q(\cdot)}(\mathbf{B})}.$$

Proof. By symmetry, it suffices to prove just the "only if" part. To this end, suppose that $\int_{\mathbf{B}} f(x)^{p(x)}dx \le 1$. Write

$$f = f\chi_{\{y:f(y)\ge 1\}} + f\chi_{\{y:0\le f(y)<1\}} = f_1 + f_2,$$

where χ_E denotes the characteristic function of E. Let $a > 0$. Since $f(x) \le C_2|x|^{-a}$, $|x| \le 1$, we have

$$\int_{\mathbf{B}} f_1(x)^{q(x)}dx \le \int_{\mathbf{B}} f_1(x)^{p(x)+|q(x)-p(x)|}dx \le \int_{\mathbf{B}} f_1(x)^{p(x)} f_1(x)^{\frac{C_1}{\log\left(e+\frac{1}{|x|}\right)}} dx$$

and hence

$$\int_{\mathbf{B}} f_1(x)^{q(x)}dx \le \int_{\mathbf{B}} f_1(x)^{p(x)}(C_2|x|^{-a})^{\frac{C_1}{\log\left(e+\frac{1}{|x|}\right)}} dx \le C\int_{\mathbf{B}} f_1(x)^{p(x)}dx \le C.$$

Since $\int_{\mathbf{B}} f_2(x)^{q(x)}dx \le C$, we obtain $\int_{\mathbf{B}} f(x)^{q(x)}dx \le C$. This shows the desired norm inequality. $\square$

Lemma 5.4 (cf. [9, Lemmas 3.12 and 3.15])**.** *Suppose that* $p(\cdot)$ *and* $q(\cdot)$ *satisfy* (P1)*. Assume that there is* $C_3 > 0$ *such that*

$$|p(x) - q(x)| \le \frac{C_3}{\log(e + |x|)} \quad \textit{whenever } |x| > 1.$$

Let $b > 0$*. Let* f *be a nonnegative measurable functions on* $\mathbb{R}^n$ *satisfying*

$$f(x) \le C_4 |x|^b, \quad |x| > 1. \tag{13}$$

Then

$$\int_{\mathbb{R}^n \setminus \mathbf{B}} f(x)^{p(x)} dx < \infty \quad \textit{if and only if} \quad \int_{\mathbb{R}^n \setminus \mathbf{B}} f(x)^{q(x)} dx < \infty.$$

Moreover, there is $C > 1$ *such that*

$$C^{-1} \|f\|_{L^{q(\cdot)}(\mathbb{R}^n \setminus \mathbf{B})} \le \|f\|_{L^{p(\cdot)}(\mathbb{R}^n \setminus \mathbf{B})} \le C \|f\|_{L^{q(\cdot)}(\mathbb{R}^n \setminus \mathbf{B})}.$$

Proof. By symmetry, it suffices to prove just the "only if" part. To this end, suppose that $\int_{\mathbb{R}^n \setminus \mathbf{B}} f(x)^{p(x)} dx \le 1$. Write

$$f = f\chi_{\{y: f(y) \ge 1\}} + f\chi_{\{y: 0 \le f(y) < 1\}} = f_1 + f_2.$$

Let $b > 0$. Since $f(x) \le C_4 |x|^b$, $|x| > 1$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus \mathbf{B}} f_1(x)^{q(x)} dx &\le \int_{\mathbb{R}^n \setminus \mathbf{B}} f_1(x)^{p(x) + |q(x) - p(x)|} dx \\
&\le \int_{\mathbb{R}^n \setminus \mathbf{B}} f_1(x)^{p(x)} f_1(x)^{\frac{C_3}{\log(e + |x|)}} dx \\
&\le \int_{R^n \setminus \mathbf{B}} f_1(x)^{p(x)} (C_4 |x|^b)^{\frac{C_3}{\log(e + |x|)}} dx \\
&\le C \int_{R^n \setminus \mathbf{B}} f_1(x)^{p(x)} dx \\
&\le C.
\end{aligned}$$

Let $m > n$. Since $(1 + |x|)^{m|p(x) - q(x)|} \le C$ and $q(x) + |p(x) - q(x)| > p(x)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus \mathbf{B}} f_2(x)^{q(x)} dx \\
&= \int_{\{x \in \mathbb{R}^n \setminus \mathbf{B}: 0 \le f_2 \le (1 + |x|)^{-m}\}} f_2(x)^{q(x)} dx + \int_{\{x \in \mathbb{R}^n \setminus \mathbf{B}: f_2 > (1 + |x|)^{-m}\}} f_2(x)^{q(x)} dx \\
&\le \int_{\mathbb{R}^n \setminus \mathbf{B}} (1 + |x|)^{-m} dx + \int_{\mathbb{R}^n \setminus \mathbf{B}} f_2(x)^{q(x)} \left(\frac{f_2(x)}{(1 + |x|)^{-m}} \right)^{|p(x) - q(x)|} dx
\end{aligned}$$

Hence

$$\int_{\mathbb{R}^n\setminus\mathbf{B}} f(x)^{q(x)}dx \le C + C\int_{\mathbb{R}^n\setminus\mathbf{B}} f_2(x)^{q(x)+|p(x)-q(x)|}dx \le C + C\int_{\mathbb{R}^n\setminus\mathbf{B}} f_2(x)^{p(x)}dx \le C.$$

This shows the desired norm inequality. □

6. Weak Banach function spaces

In the next section we will use a slightly more general concept of Banach function spaces than in Definition 2.1. The last two axioms are weakened and so, we will call these spaces weak Banach function spaces.

Definition 6.1. We say that a normed linear space $(X, \|.\|_X)$ is a weak Banach function space (WBFS for short) if the following conditions are satisfied:

$$\text{the norm } \|f\|_X \text{ is defined for all } f \in \mathcal{M}(\Omega) \text{ and } f \in X \text{ if and only if } \|f\|_X < \infty; \tag{14}$$

$$\|f\|_X = \|\, |f|\, \|_X \text{ for every } f \in \mathcal{M}(\Omega); \tag{15}$$

$$\text{if } 0 \le f_n \nearrow f \text{ a.e. in } \Omega, \text{ then } \|f_n\|_X \nearrow \|f\|_X; \tag{16}$$

$$\text{if } E \subset \Omega \text{ is a bounded measurable set, then } \chi_E \in X; \tag{17}$$

$$\text{for every bounded measurable set } E, E \subset \Omega, \text{ there exists a positive constant } C_E \text{ such that } \int_E |f(x)|dx \le C_E\|f\|_X. \tag{18}$$

Theorem 6.2. *Each WBFS is complete and consequently, it is a Banach space.*

Proof. The assertion immediately follows from the evident fact that each WBFS is a GBFS and from Lemma 2.5. □

Definition 6.3. Let $(X, \|.\|_X)$ be a WBFS. Define the associate space X' as the collection of all functions in $\mathcal{M}(\Omega)$ with finite norm

$$\|f\|_{X'} = \sup\left\{\int_\Omega f(x)g(x)dx : \|g\|_X \le 1\right\}.$$

The following theorems are proved in [1] for BFSs, but the proofs can be copied for WBFSs without changes. It suffices to only consider bounded subsets instead of subsets with finite measures.

Theorem 6.4. *Let X be a WBFS. Then X' is a WBFS.*

Theorem 6.5. *Let X, Y be WBFSs and $X \subsetneqq Y$. Then $Y' \subsetneqq X'$.*

Theorem 6.6. *Let X be a WBFS. Then $X'' = X$ and $\|f\|_{X''} = \|f\|_X$ for each $f \in X$.*

Definition 6.7. Let X, Y be WBFSs and $T : X \to Y$ be a bounded linear operator. Define an associate operator T' by

$$\int_\Omega Tf(x)g(x)dx = \int_\Omega f(x)T'g(x)dx$$

for all $f \in X$ and $g \in Y'$.

Theorem 6.8. *Let X, Y be WBFSs and $T : X \to Y$ be a bounded linear operator. Then $T' : Y' \to X'$ is bounded, too.*

7. Spaces T_p and S_p and boundedness of A

In this section, we will give applications of our results to variable Lebesgue spaces $L^{p(\cdot)}(\Omega)$, as an extension of [10] in the case when $n = 1$ and Ω is a bounded interval.

Definition 7.1. Let Ω be an open subset of $\mathbb{R}^n$. Let us consider the family $T_{p(\cdot)}(\Omega)$ of all measurable functions f on Ω satisfying

$$\int_\Omega \left(\operatorname*{ess\,sup}_{|t|\ge|x|} \left| \frac{f(t)}{\lambda} \right| \right)^{p(x)} dx < \infty$$

for some $\lambda > 0$. We define the norm on this space by

$$\|f\|_{T_{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\operatorname*{ess\,sup}_{|t|\ge|x|} \left| \frac{f(t)}{\lambda} \right| \right)^{p(x)} dx \le 1 \right\}.$$

If $p(\cdot)$ is a constant p, then we write $T_p(\Omega)$ and $\|f\|_{T_p(\Omega)}$.

Remark that $T_{p(\cdot)}(\Omega) = T_X$ for $X = L^{p(\cdot)}(\Omega)$.

The following Hölder's inequality is well-known (see [7, Theorem 2.1]). If $p'(x) = \frac{p(x)}{p(x)-1}$, then

$$\int_\Omega |f(x)g(x)|dx \le C\|f\|_{L^{p'(\cdot)}(\Omega)}\|g\|_{L^{p(\cdot)}(\Omega)}$$

for some constant C.

Lemma 7.2. *Any $T_{p(\cdot)}(\Omega)$ is a WBFS.*

Proof. We know from Lemma 3.2 that $T_{p(\cdot)}(\Omega)(= T_{L^{p(\cdot)}(\Omega)})$ is a GBFS. To complete the proof, it suffices to verify conditions (17) and (18). Let $E \subset \Omega$ be a bounded measurable set. Then there is $R > 0$ with $E \subset B(0, R)$.

Verification of (17). Take $\lambda \geq 1$. Then

$$\int_{\Omega} \left| \frac{\operatorname{ess\,sup}_{|t|\geq|x|} \chi_E(t)}{\lambda} \right|^{p(x)} dx \leq \int_{\Omega\cap B(0,R)} \left|\frac{1}{\lambda}\right|^{p(x)} dx + \int_{\Omega\setminus B(0,R)} \left|\frac{0}{\lambda}\right|^{p(x)} dx < \infty.$$

Proof of (18). By Hölder's inequality, we obtain

$$\int_E |f(x)|dx \leq C\|1\|_{L^{p'(\cdot)}(E)}\|f\|_{L^{p(\cdot)}(E)} \leq C\|\widetilde{f}\|_{L^{p(\cdot)}(\Omega)} = C\|f\|_{T_{p(\cdot)}(\Omega)}. \qquad \square$$

The following theorem is proved in [10] for the one-dimensional case (see Theorem 4.3).

Theorem 7.3. *Suppose that* $p(\cdot)$ *satisfies* (P1) *and* (P2). *Then the norms in* $T_{p(\cdot)}(\mathbf{B})$ *and* $T_p(\mathbf{B})$ *are equivalent.*

Proof. First suppose $f \in T_p(\mathbf{B})$. Since $\widetilde{f}$ is radially non-increasing, we have

$$\widetilde{f}(x) \leq \fint_{B(0,|x|)} \widetilde{f}(y)dy \leq \left(\fint_{B(0,|x|)} \widetilde{f}(y)^p dy\right)^{\frac{1}{p}} \leq C|x|^{-\frac{n}{p}}.$$

Hence (12) holds. Thus, in view of Lemma 5.3, we see that $\|f\|_{T_{p(\cdot)}(\mathbf{B})} \leq C\|f\|_{T_p(\mathbf{B})}$.

Next suppose $f \in T_{p(\cdot)}(\mathbf{B})$ with $\|f\|_{T_{p(\cdot)}(\mathbf{B})} \leq 1$. Since $\widetilde{f}$ is radially non-increasing and $\int_{\mathbf{B}} \widetilde{f}(y)^{p(y)}dy \leq 1$, we have

$$\widetilde{f}(x) \leq \fint_{B(0,|x|)} \widetilde{f}(y)dy \leq C \fint_{B(0,|x|)} \widetilde{f}(y)^{p(y)}dy + 1 \leq C|x|^{-n} + 1 \leq C|x|^{-n}.$$

Hence (12) holds. Thus, in view of Lemma 5.3, we see that $\|f\|_{T_p(\mathbf{B})} \leq C$. This implies that $\|f\|_{T_p(\mathbf{B})} \leq C\|f\|_{T_{p(\cdot)}(\mathbf{B})}$. $\square$

In view of Theorem 7.3 and the definition of $T_{p(\cdot)}(\mathbf{B})$, we have the following.

Corollary 7.4 (cf. [10, Theorem 6.1])**.** *Suppose* $p(\cdot)$ *satisfies* (P1) *and* (P2). *Then* $T_p(\mathbf{B}) \hookrightarrow L^{p(\cdot)}(\mathbf{B})$.

Theorem 7.5. *Suppose that* $p(\cdot)$ *satisfies* (P1) *and* (P3). *Then the norms in* $T_{p(\cdot)}(\mathbb{R}^n \setminus \mathbf{B})$ *and* $T_{p_\infty}(\mathbb{R}^n \setminus \mathbf{B})$ *are equivalent.*

Proof. Since $\widetilde{f}$ is radially nonincreasing, (13) with f replaced by $\widetilde{f}$ holds as in the proof of Theorem 7.3. Hence, this theorem follows from Lemma 5.4. $\square$

By Theorem 7.5 instead of Theorem 7.3, we can prove the following.

Corollary 7.6. *Suppose that* $p(\cdot)$ *satisfies* (P1) *and* (P3). *Then* $T_{p_\infty}(\mathbb{R}^n \setminus \mathbf{B}) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n \setminus \mathbf{B})$.

Here we consider the following condition:

(P1′) $1 < p^- \le p^+ < \infty$.

Remark that (P1′) and (P3) imply $1 < p_\infty < \infty$. We know the boundedness of maximal functions in $L^{p(\cdot)}(\mathbb{R}^n)$, due to [2].

Lemma 7.7. *Suppose that* $p(\cdot)$ *satisfies* (P1′), (P3) *and* (P4). *Then there exists a positive constant* C *such that*

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for all measurable functions $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

Since $|Af(x)| \le CMf(x)$, we obtain the following by Lemma 7.7.

Lemma 7.8. *Suppose that* $p(\cdot)$ *satisfies* (P1′), (P3) *and* (P4). *Then* $A : L^{p(\cdot)}(\mathbb{R}^n) \to L^{p(\cdot)}(\mathbb{R}^n)$.

An analogy of the following lemma can be found in [10, Lemma 5.2].

Lemma 7.9. *Suppose that* $p(\cdot)$ *satisfies* (P1′), (P3) *and* (P4). *Then* $A : L^{p(\cdot)}(\mathbb{R}^n) \to T_{p(\cdot)}(\mathbb{R}^n)$.

Proof. This lemma follows immediately from Lemmas 7.7 and 3.4. □

The following two lemmas were borrowed from [10] where the 1-dimensional case is investigated. But it is easy to see the same assertion in the n-dimensional case.

Lemma 7.10. *Let* $C > 0$. *Then the function* $q(x) = p - \frac{C}{\log(e+\frac{1}{|x|})}$ *satisfies* (P4).

Lemma 7.11. *Let* $p(\cdot)$ *and* $q(\cdot)$ *satisfy* (P4) *with constants* C_1 *and* C_2, *respectively. Then the function* $h(x) = \max\{p(x), q(x)\}$ *satisfies* (P4) *with the constant* $\max\{C_1, C_2\}$.

Theorem 7.12 (cf. [10, Theorem 5.5])**.** *Let* $1 < p < \infty$. *Suppose that* $p(\cdot)$ *satisfies* (P1′) *and* (P2). *Then* $A : L^{p(\cdot)}(\mathbf{B}) \to T_p(\mathbf{B})$.

Proof. By our assumption, $|p(x) - p| \le \frac{C}{\log(e+\frac{1}{|x|})}$ whenever $x \in \mathbf{B}$. We set $d = \inf_{x\in\mathbf{B}} p(x)$ and

$$q(x) = \max\left\{ d, p - \frac{C}{\log(e + \frac{1}{|x|})} \right\}.$$

Then $q(x) \le p(x)$ for $x \in \mathbf{B}$ and $q(\cdot)$ satisfies (P1′). Hence $L^{p(\cdot)}(\mathbf{B}) \hookrightarrow L^{q(\cdot)}(\mathbf{B})$ (see e.g. [7]). Next, by Lemmas 7.10 and 7.11, q satisfies (P4). Thus, by Lemma 7.9, $A : L^{q(\cdot)}(\mathbf{B}) \to T_{q(\cdot)}(\mathbf{B})$ holds. Finally, in view of Theorem 7.3, $T_{q(\cdot)}(\mathbf{B}) \hookrightarrow T_p(\mathbf{B})$. □

Definition 7.13. Let us consider the family $S_p(\Omega)$ of all measurable functions f on Ω satisfying $\|f\|_{S_p(\Omega)} < \infty$ with the norm

$$\|f\|_{S_p(\Omega)} = \left(\int_\Omega \operatorname*{ess\,sup}_{|t|\geq|x|} \left(\fint_{B(0,|t|)} |f(s)|ds\right)^p dx\right)^{\frac{1}{p}} = \|A|f|\|_{T_p(\Omega)}.$$

Lemma 7.14. *Let $1 < p < \infty$. Then $S_p(\Omega)$ is a WBFS.*

Proof. We know from Lemma 3.6 that $S_p(\Omega)$ is a GBFS. To complete the proof, it suffices to verify conditions (17) and (18). Let $E \subset \Omega$ be a bounded measurable set with $|E| > 0$. Take $R > 0$ such that

$$|E \cap \{x : |x| \geq R\}| = 0 \quad \text{and} \quad |E \cap \{x : |x| \geq R - \varepsilon\}| > 0 \tag{19}$$

for each $\varepsilon > 0$. Consequently,

$$|\Omega \cap \{x : |x| \geq R/2\}| > 0. \tag{20}$$

$$\text{Verification of (17). Clearly,} \quad A|\chi_E|(x) \leq \begin{cases} 1 & \text{if } |x| \leq R \\ \frac{R^n}{|x|^n} & \text{if } |x| > R \end{cases}$$

$$\text{and consequently,} \quad \widetilde{A|\chi_E|}(x) \leq \begin{cases} 1 & \text{if } |x| \leq R \\ \frac{R^n}{|x|^n} & \text{if } |x| > R. \end{cases}$$

Then $\|\chi_E\|^p_{S_p(\Omega)} = \left\|\widetilde{A|\chi_E|}\right\|^p_{L^p(\Omega)} \leq \int_{B(0,R)} dx + \int_{\Omega\setminus B(0,R)} \frac{R^{pn}}{|x|^{pn}} dx < \infty.$

Proof of (18). By (20) we have

$$0 < L := \int_{\Omega\cap\{x:|x|\geq\frac{R}{2}\}} |x|^{-np} dx < \infty.$$

Let $f \geq 0$ be measurable. Then we can write

$$\begin{aligned}
\int_E f(s)ds &= L^{-\frac{1}{p}} \int_E f(s)ds \Big(\int_{\Omega\cap\{x:|x|\geq\frac{R}{2}\}} |x|^{-np} dx\Big)^{\frac{1}{p}} \\
&= L^{-\frac{1}{p}} \Big(\int_{\Omega\cap\{x:|x|\geq\frac{R}{2}\}} \Big(|x|^{-n} \int_E f(s)ds\Big)^p dx\Big)^{\frac{1}{p}} \\
&\leq L^{-\frac{1}{p}} \Big(\int_{\Omega\cap\{x:|x|\geq\frac{R}{2}\}} \Big(\operatorname*{ess\,sup}_{|t|\geq|x|} |t|^{-n} \int_E f(s)ds\Big)^p dx\Big)^{\frac{1}{p}} \\
&\overset{(19)}{\leq} L^{-\frac{1}{p}} \Big(\int_{\Omega\cap\{x:|x|\geq\frac{R}{2}\}} \Big(\operatorname*{ess\,sup}_{|t|\geq|x|} |t|^{-n} \int_{B(0,R)} f(s)ds\Big)^p dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

Since $|t| \geq |x| \geq \frac{R}{2}$, we have $2|t| \geq R$ and so,

$$\begin{aligned}\int_E f(s)ds &\leq L^{-\frac{1}{p}} \Big(\int_{\Omega\cap\{x:|x|\geq\frac{R}{2}\}} \Big(\operatorname*{ess\,sup}_{|t|\geq|x|} |t|^{-n} \int_{B(0,2|t|)} f(s)ds \Big)^p dx \Big)^{\frac{1}{p}} \\ &= L^{-\frac{1}{p}}\ 2^n \Big(\int_{\Omega\cap\{x:|x|\geq\frac{R}{2}\}} \Big(\operatorname*{ess\,sup}_{2|t|\geq 2|x|} (2|t|)^{-n} \int_{B(0,2|t|)} f(s)ds \Big)^p dx \Big)^{\frac{1}{p}}.\end{aligned}$$

Write now $r = 2t$ and increase the integration domain from $\Omega \cap \{x : |x| \geq R/2\}$ to Ω. We obtain an estimate

$$\begin{aligned}\int_E f(s)ds &\leq L^{-\frac{1}{p}}\ 2^n \Big(\int_\Omega \Big(\operatorname*{ess\,sup}_{|r|\geq 2|x|} |r|^{-n} \int_{B(0,|r|)} f(s)ds \Big)^p dx \Big)^{\frac{1}{p}} \\ &\leq L^{-\frac{1}{p}}\ 2^n \Big(\int_\Omega \Big(\operatorname*{ess\,sup}_{|r|\geq |x|} |r|^{-n} \int_{B(0,|r|)} f(s)ds \Big)^p dx \Big)^{\frac{1}{p}} \\ &= C\|f\|_{S_p(\Omega)},\end{aligned}$$

which finishes the proof. $\square$

By Theorems 7.12 and 3.3, we obtain the following.

Corollary 7.15 (cf. [10, Theorem 6.2])**.** *Let* $1 < p < \infty$. *Suppose that* $p(\cdot)$ *satisfies* (P1′) *and* (P2). *Then* $L^{p(\cdot)}(\mathbf{B}) \hookrightarrow S_p(\mathbf{B}) \hookrightarrow L^1(\mathbf{B})$.

For the second embedding, note that

$$\|f\|_{S_p(\mathbf{B})} \geq \Big(\int_{\mathbf{B}} \Big(\fint_{\mathbf{B}} |f(s)|ds \Big)^p dx \Big)^{\frac{1}{p}} = C \int_{\mathbf{B}} |f(s)|ds = C\|f\|_{L^1(\mathbf{B})}.$$

Theorem 7.16. *Suppose that* $p(\cdot)$ *satisfies* (P1′) *and* (P3). *Then* $A : L^{p(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \to T_{p_\infty}(\mathbb{R}^n \setminus \mathbf{B})$.

Proof. To show $\|Af\|_{T_{p_\infty}(\mathbb{R}^n\setminus\mathbf{B})} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n\setminus\mathbf{B})}$, suppose that

$$\int_{\mathbb{R}^n\setminus\mathbf{B}} |f(x)|^{p(x)}dx \leq 1. \tag{21}$$

By our assumption, $|p(x) - p_\infty| \leq \frac{C}{\log(e+|x|)}$ whenever $x \in \mathbb{R}^n \setminus \mathbf{B}$.

Set $d = \inf_{x\in\mathbb{R}^n\setminus\mathbf{B}} p(x)$. Then by (P3)

$$q(x) := \max\left\{ d, p_\infty - \frac{C}{\log(e+|x|)} \right\} \leq p(x) \leq p_\infty + \frac{C}{\log(e+|x|)} := \tilde{q}(x).$$

Then $q(x) \leq p(x) \leq \tilde{q}(x)$ for $x \in \mathbb{R}^n \setminus \mathbf{B}$ and $q(\cdot)$ and $\tilde{q}(\cdot)$ satisfy (P1′) and (P3).

Since $\left|\nabla\left(\frac{1}{\log(e+|x|)}\right)\right| \le \frac{1}{e}$, the functions q and $\widetilde{q}$ are Lipschitz and so, both satisfy (P4). Thus, by Lemma 7.9, $A : L^{q(\cdot)}(\mathbb{R}^n) \to T_{q(\cdot)}(\mathbb{R}^n)$ and $A : L^{\widetilde{q}(\cdot)}(\mathbb{R}^n) \to T_{\widetilde{q}(\cdot)}(\mathbb{R}^n)$. If we consider functions vanishing on $\mathbf{B}$, then we obtain

$$A : L^{q(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \to T_{q(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}), \quad A : L^{\tilde{q}(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \to T_{\tilde{q}(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}). \tag{22}$$

Moreover, in view of Theorem 7.5 we have

$$T_{q(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \hookrightarrow T_{p_\infty}(\mathbb{R}^n \setminus \mathbf{B}), \quad T_{\tilde{q}(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \hookrightarrow T_{p_\infty}(\mathbb{R}^n \setminus \mathbf{B}). \tag{23}$$

Write

$$f = f\chi_{\{y:f(y)\ge 1\}} + f\chi_{\{y:0\le f(y)<1\}} = f_1 + f_2. \tag{24}$$

By (21) and (24) we obtain

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\mathbf{B}} |f_1(x)|^{q(x)}dx + \int_{\mathbb{R}^n\setminus\mathbf{B}} |f_2(x)|^{\tilde{q}(x)}dx &\le \int_{\mathbb{R}^n\setminus\mathbf{B}} |f_1(x)|^{p(x)}dx + \int_{\mathbb{R}^n\setminus\mathbf{B}} |f_2(x)|^{p(x)}dx \\ &= \int_{\mathbb{R}^n\setminus\mathbf{B}} |f(x)|^{p(x)}dx \\ &\le 1.\end{aligned}$$

By (22) we have

$$\int_{\mathbb{R}^n\setminus\mathbf{B}} (\operatorname*{ess\,sup}_{|t|\ge|x|} |Af_1(t)|)^{q(x)}dx \le C, \quad \int_{\mathbb{R}^n\setminus\mathbf{B}} (\operatorname*{ess\,sup}_{|t|\ge|x|} |Af_2(t)|)^{\tilde{q}(x)}dx \le C.$$

Finally, (23) yields

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus\mathbf{B}} (\operatorname*{ess\,sup}_{|t|\ge|x|} |Af(t)|)^{p_\infty}dx \\ &\le C\int_{\mathbb{R}^n\setminus\mathbf{B}} (\operatorname*{ess\,sup}_{|t|\ge|x|} |Af_1(t)|)^{p_\infty}dx + C\int_{\mathbb{R}^n\setminus\mathbf{B}} (\operatorname*{ess\,sup}_{|t|\ge|x|} |Af_2(t)|)^{p_\infty}dx \\ &\le C,\end{aligned}$$

which finishes the proof with Lemma 2.7. □

Definition 7.17. Let us consider the Herz type space $L^{1,\beta}(\Omega)$ of all functions f on Ω satisfying

$$\sup_{r>1} r^\beta \fint_{B(0,r)\cap\Omega} |f(y)|dy < \infty.$$

Corollary 7.18. *Let* $1 < p_\infty < \infty$. *Suppose that* $p(\cdot)$ *satisfies* (P1′) *and* (P3). *Then* $L^{p(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \hookrightarrow S_{p_\infty}(\mathbb{R}^n \setminus \mathbf{B}) \hookrightarrow L^{1,\frac{n}{p_\infty}}(\mathbb{R}^n \setminus \mathbf{B})$.

Proof. By Theorem 7.16, we have $L^{p(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \hookrightarrow S_{p_\infty}(\mathbb{R}^n \setminus \mathbf{B})$. Next we will show that $S_{p_\infty}(\mathbb{R}^n \setminus \mathbf{B}) \hookrightarrow L^{1,\frac{n}{p_\infty}}(\mathbb{R}^n \setminus \mathbf{B})$. Let $f \in S_{p_\infty}(\mathbb{R}^n \setminus \mathbf{B})$. For $r > 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\mathbf{B}} \operatorname*{ess\,sup}_{|t|\geq|x|} \left(\fint_{B(0,|t|)} |f(s)|ds \right)^{p_\infty} dx &\geq \int_{\mathbb{R}^n\setminus\mathbf{B}} \left(\fint_{B(0,|x|)} |f(s)|ds \right)^{p_\infty} dx \\ &\geq C \int_{\{x:|x|>r\}} |x|^{-np_\infty} \left(\int_{B(0,r)} |f(s)|ds \right)^{p_\infty} dx \\ &\geq C \left(r^{\frac{n}{p_\infty}} \fint_{B(0,r)} |f(s)|ds \right)^{p_\infty}.\end{aligned}$$

Hence $f \in L^{1,\frac{n}{p_\infty}}(\mathbb{R}^n \setminus \mathbf{B})$. □

Remark 7.19. For every $p \in (1,\infty)$, we have

$$T_p(\mathbf{B}) \hookrightarrow S_p(\mathbf{B}) \quad \text{and} \quad T_p(\mathbb{R}^n \setminus \mathbf{B}) \hookrightarrow S_p(\mathbb{R}^n \setminus \mathbf{B}).$$

In fact, these follow from Corollaries 7.4, 7.6, 7.15 and 7.18.

All the assertions of the following corollary follow immediately from Theorem 3.8, 7.12, 7.16, Corollaries 7.4, 7.6, and Remark 7.19.

Corollary 7.20 (cf. [10, Corollary 7.2])**.** *Let* $1 < p < \infty$. *Then*

$$\begin{array}{ll} A : S_p(\mathbb{R}^n) \to T_p(\mathbb{R}^n), & \text{(by Theorem 3.8)} \\ A : S_p(\mathbf{B}) \to S_p(\mathbf{B}) & \text{(by Theorem 3.8 and Remark 7.19),} \\ A : S_p(\mathbb{R}^n \setminus \mathbf{B}) \to S_p(\mathbb{R}^n \setminus \mathbf{B}) & \text{(by Theorem 3.8 and Remark 7.19),} \\ A : T_p(\mathbf{B}) \to T_p(\mathbf{B}) & \text{(by Corollary 7.4 and Theorem 7.12)} \end{array}$$

and

$$A : T_p(\mathbb{R}^n \setminus \mathbf{B}) \to T_p(\mathbb{R}^n \setminus \mathbf{B}) \qquad \text{(by Corollary 7.6 and Theorem 7.16).}$$

Moreover suppose that $r(\cdot), s(\cdot)$ *satisfy* (P1′) *and* (P2) *with a same* p. *Then*

$$A : L^{r(\cdot)}(\mathbf{B}) \to L^{s(\cdot)}(\mathbf{B}) \qquad \text{(by Corollary 7.4 and Theorem 7.12).}$$

Suppose that $r(\cdot), s(\cdot)$ *satisfy* (P1′) *and* (P3) *with the same* p. *Then*

$$A : L^{r(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \to L^{s(\cdot)}(\mathbb{R}^n \setminus \mathbf{B}) \qquad \text{(by Corollary 7.6 and Theorem 7.16).}$$

In view of Theorems 4.1 and 4.2, we can prove the following corollaries.

Corollary 7.21 (cf. [10, Theorem 8.2]). *Let* $1<p<\infty$. *Assume that* $Z \subsetneqq T_p(\mathbf{B})$ *is a GBFS. Then* $A : T_p(\mathbf{B}) \not\to Z$.

Corollary 7.22 (cf. [10, Theorem 8.3]). *Let* $1 < p < \infty$. *Assume that* Z *is a GBFS such that* $S_p(\mathbf{B}) \subsetneqq Z$. *Then* $A : Z \not\to S_p(\mathbf{B})$.

Corollary 7.23. *Let* $1 < p < \infty$. *Assume that* $Z \subsetneqq T_p(\mathbb{R}^n \setminus \mathbf{B})$ *is a GBFS. Then* $A : T_p(\mathbb{R}^n \setminus \mathbf{B}) \not\to Z$.

Corollary 7.24. *Let* $1 < p < \infty$. *Assume that* Z *is a GBFS such that* $S_p(\mathbb{R}^n \setminus \mathbf{B}) \subsetneqq Z$. *Then* $A : Z \not\to S_p(\mathbb{R}^n \setminus \mathbf{B})$.

8. Associate operator A' and associativity between T_p and S_p

Consider $\Omega = \mathbb{R}^n$ in what follows. Note that the associate operator A' to the operator A is given by

$$A'f(x) = \sigma_n^{-1} \int_{\{y:|x|\le|y|\}} |y|^{-n} f(y)\, dy$$

for a locally integrable function f on $\mathbb{R}^n$, where σ_n is the volume of the unit ball in $\mathbb{R}^n$. In fact,

$$\int f(x)Ag(x)\, dx = \int g(y)A'f(y)\, dy \tag{25}$$

for nonnegative measurable functions f and g on $\mathbb{R}^n$.

By Theorem 6.8, we have the following lemma.

Lemma 8.1. *Let* $p > 1$. *Then* $A' : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.

Lemma 8.2. *If* $f \in \mathcal{M}(\mathbb{R}^n)$ *is nonnegative, then* $\widetilde{Af}(z) \le CM(A'f)(z)$.

Proof. Let $|t| \ge |z|$. If we set $g = \chi_{B(0,r)}$ in (25), we find

$$\int_{B(0,r)} f(x)\, dx \le \int_{B(0,r)} A'f(y)\, dy,$$

so that

$$⨍_{B(0,|t|)} f(x)dx \le ⨍_{B(0,|t|)} A'f(y)dy \le C ⨍_{B(z,2|t|)} A'f(y)dy \le CM(A'f)(z).$$

Hence $\widetilde{Af}(z) \le CM(A'f)(z)$. ∎

In view of Hardy's inequality (see [8]), we see that if $p > 1$ and $\alpha < \frac{n}{p'}$, then

$$\int \left(|y|^{\alpha-n} \int_{\{x:|x|\le|y|\}} f(x)|x|^{-\alpha}dx \right)^p dy \le C \int f(x)^p dx$$

and if $p > 1$ and $\alpha > \frac{n}{p'}$, then

$$\int \left(|y|^{\alpha-n} \int_{\{x:|y|\le|x|\}} f(x)|x|^{-\alpha}dx \right)^p dy \le C \int f(x)^p dx \tag{26}$$

for nonnegative measurable functions f on $\mathbb{R}^n$.

Let $U_p(\mathbb{R}^n) = \{f \in \mathcal{M}(\mathbb{R}^n) : A'|f| \in L^p(\mathbb{R}^n)\}$.

Lemma 8.3. *If $p > 1$, then $U_p(\mathbb{R}^n) = S_p(\mathbb{R}^n)$.*

Proof. First we show $U_p(\mathbb{R}^n) \subset S_p(\mathbb{R}^n)$. Let $f \in U_p(\mathbb{R}^n)$ be nonnegative. By Lemma 8.2 and the boundedness of M, we have

$$\int \widetilde{A}f(x)^p dx \le C \int M(A'f)(x)^p dx \le C \int A'f(x)^p dx. \tag{27}$$

Therefore $\widetilde{A}f \in L^p(\mathbb{R}^n)$, which implies $U_p(\mathbb{R}^n) \subset S_p(\mathbb{R}^n)$.

Next we show $S_p(\mathbb{R}^n) \subset U_p(\mathbb{R}^n)$. Let $f \in S_p(\mathbb{R}^n)$ be nonnegative. Note that

$$\begin{aligned}
\int_{\{y:|x|\le|y|\}} |y|^{-n} f(y)\, dy &= \sum_{j=1}^{\infty} \int_{\{y:2^{j-1}|x|\le|y|\le 2^j|x|\}} |y|^{-n} f(y)\, dy \\
&\le \sum_{j=1}^{\infty} (2^{j-1}|x|)^{-n} \int_{\{y:|y|\le 2^j|x|\}} f(y)\, dy \\
&\le C \sum_{j=1}^{\infty} \int_{2^j|x|}^{2^{j+1}|x|} \left(\int_{B(0,t)} f(y)\, dy \right) t^{-n-1} dt \\
&= C \int_{2|x|}^{\infty} \left(\int_{B(0,t)} f(y)\, dy \right) t^{-n-1} dt.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\int A'f(x)^p dx &= \sigma_n^{-p} \int \left(\int_{\{y;|x|\le|y|\}} |y|^{-n} f(y)\, dy \right)^p dx \\
&\le C \int \left(\int_{|x|}^{\infty} \left(\int_{B(0,t)} f(y)\, dy \right) t^{-n-1} dt \right)^p dx \\
&\le C \int \left(\int_{\{z;|x|\le|z|\}} Af(z)|z|^{-n} dz \right)^p dx.
\end{aligned}$$

Now, using (26) with $\alpha = n$, we have

$$\int \Big(\int_{\{z;|x|\le|z|\}} Af(z)|z|^{-n}dz \Big)^p dx \le C \int Af(x)^p dx,$$

which gives $\int A'f(x)^p dx \le C \int Af(x)^p dx \le C \int \widetilde{Af}(x)^p dx$ and therefore $A'f \in L^p(\mathbb{R}^n)$. This implies that $S_p(\mathbb{R}^n) \subset U_p(\mathbb{R}^n)$. □

Since $\widetilde{A'f}(x) = A'f(x)$, we have the following corollary in view of the proof of Lemma 8.3.

Corollary 8.4. *Let $p > 1$. Then $A' : S_p(\mathbb{R}^n) \to T_p(\mathbb{R}^n)$.*

By Theorems 3.3 and 3.7, we have immediately the following.

Corollary 8.5. *Let $p > 1$. Then $A' : S_p(\mathbb{R}^n) \to S_p(\mathbb{R}^n)$, $A' : T_p(\mathbb{R}^n) \to T_p(\mathbb{R}^n)$.*

The following lemma is proved easily.

Lemma 8.6. *For nonnegative measurable functions f and g on $\mathbb{R}^n$, there holds*

$$\int f(x)g(x)\ dx \le \int \left(\operatorname*{ess\,sup}_{|x|\ge|y|} f(x) \right) A'g(y)\ dy.$$

Proof.

$$\begin{aligned}
\int f(x)g(x)\ dx &= \int f(x)g(x) \left(\fint_{B(0,|x|)} dy \right) dx \\
&= \sigma_n^{-1} \int \left(\int_{\{x;|y|\le|x|\}} |x|^{-n} f(x)g(x)dx \right) dy \\
&\le \sigma_n^{-1} \int \left(\operatorname*{ess\,sup}_{|x|\ge|y|} f(x) \right) \left(\int_{\{x;|y|\le|x|\}} |x|^{-n} g(x)dx \right) dy \\
&= \int \left(\operatorname*{ess\,sup}_{|x|\ge|y|} f(x) \right) A'g(y)\ dy. \qquad \square
\end{aligned}$$

Theorem 8.7. *Let $p > 1$. Then $(T_p(\mathbb{R}^n))' = S_{p'}(\mathbb{R}^n)$ and their norms are equivalent.*

Proof. Take $g \in S_{p'}(\mathbb{R}^n) \ge 0$. By Hölder's inequality, Lemmas 8.6 and 8.3, we get

$$\int |f(x)g(x)|dx \le \|\tilde{f}\|_{L^p(\mathbb{R}^n)}\ \|A'g\|_{L^{p'}(\mathbb{R}^n)} \le C\|f\|_{T_p(\mathbb{R}^n)}\ \|g\|_{S_{p'}(\mathbb{R}^n)},$$

from which it follows that $\|g\|_{(T_p(\mathbb{R}^n))'} \le C\|g\|_{S_{p'}(\mathbb{R}^n)}$ and $S_{p'}(\mathbb{R}^n) \subset (T_p(\mathbb{R}^n))'$.

We show the converse norm inequality. First take g with $\|g\|_{S_{p'}(\mathbb{R}^n)} = 1$. We set

$$f_0 = A((A'g)^{p'-1}).$$

Then, by Lemmas 8.1–8.3 and by the boundedness of maximal operator on L^p

$$\|f_0\|^p_{T_p(\mathbb{R}^n)} = \int \tilde{f}_0(x)^p dx \le C\int \big(M(A((A'g)^{p'-1}))(x)\big)^p dx \le C\int \big(A((A'g)^{p'-1})(x)\big)^p dx,$$

hence

$$\|f_0\|^p_{T_p(\mathbb{R}^n)} \le C\int A'g(x)^{(p'-1)p} dx = C\int A'g(x)^{p'} dx \le C\|g\|^{p'}_{S_{p'}(\mathbb{R}^n)} = C.$$

Moreover, by relation (25) we get $\int |f_0(x)g(x)|dx = \int A((A'g)^{p'-1})(x)g(x)dx = \int A'g(x)^{p'-1}A'g(x)dx = \int A'g(x)^{p'} dx = \int \widetilde{A'g}(x)^{p'} dx = \|g\|^{p'}_{S_{p'}(\mathbb{R}^n)} = 1$. Thus

$$\|g\|_{(T_p(\mathbb{R}^n))'} = \sup_{\|f\|_{T_p(\mathbb{R}^n)} \le 1} \int |f(x)g(x)|\, dx \ge C \int |f_0(x)g(x)|\, dx = C.$$

Hence

$$C^{-1}\|g\|_{S_{p'}(\mathbb{R}^n)} \le \|g\|_{(T_p(\mathbb{R}^n))'} \le C\|g\|_{S_{p'}(\mathbb{R}^n)} \tag{28}$$

for $g \in S_{p'}(\mathbb{R}^n)$.

Next take $g \ge 0$ with $g \notin S_{p'}(\mathbb{R}^n)$, that is, $\|g\|_{S_{p'}(\mathbb{R}^n)} = \infty$. Set $g_n(x) = \min(g(x), n)\ \chi_{B(0,n)}(x)$. Since g_n are bounded with bounded support and $S_{p'}$ is a WBFS by Lemma 7.14, we have $\|g_n\|_{S_{p'}(\mathbb{R}^n)} < \infty$ for all n. Moreover, $g_n \nearrow g$ a.e. and so, $\|g_n\|_{S_{p'}(\mathbb{R}^n)} \nearrow \|g\|_{S_{p'}(\mathbb{R}^n)} = \infty$. In view of Theorem 6.4, we see from (28) that $\|g\|_{(T_p(\mathbb{R}^n))'} = \infty$, which implies $g \notin (T_p(\mathbb{R}^n))'$ and hence $(T_p(\mathbb{R}^n))' \subset S_{p'}(\mathbb{R}^n)$, as required. □

The following theorem shows the optimality of spaces $S_p(\mathbb{R}^n), T_p(\mathbb{R}^n)$ for the operator A'.

Theorem 8.8. *Let $p > 1$ and let Y, Z be WBFSs with $Z \supsetneqq S_p(\mathbb{R}^n)$ and $T_p(\mathbb{R}^n) \supsetneqq Y$. Then $A' : Z \not\to S_p(\mathbb{R}^n)$, $A' : T_p(\mathbb{R}^n) \not\to Y$.*

Proof. Remark first that we have

$$A'f(x) \ge \sigma_n^{-1} \int_{B(0,2|x|)\setminus B(0,|x|)} |y|^{-n} f(y)dy \ge Cf(2x) \tag{29}$$

for each radially non-increasing $f \ge 0$.

Take a WBFS Y such that $T_p(\mathbb{R}^n) \supsetneqq Y$. Let $g \in T_p(\mathbb{R}^n) \setminus Y$ and set $h(x) = \widetilde{g}(\frac{x}{2})$. Then h is radially non-increasing, $h \ge \widetilde{g} \ge g$ and $h \in T_p(\mathbb{R}^n)$.

Since Y is a WBFS, we have $h \notin Y$. So, $h \in T_p(\mathbb{R}^n) \setminus Y$. Since h is radially non-increasing, we have by (29) $A'h(x) \geq Ch(2x) \geq Cg(x)$ which implies that $A'h \notin Y$.

Next take a WBFS Z such that $Z \supsetneqq S_p(\mathbb{R}^n)$. If $h \in Z \setminus S_p(\mathbb{R}^n)$, then we see from (27) that $A'h \notin L^p(\mathbb{R}^n)$. Since $A'h$ is radially non-increasing, we find from (29) an inequality $A'(A'h)(x) \geq CA'h(2x)$ and we obtain by Lemma 8.3 that $A'(A'h) \notin L^p(\mathbb{R}^n)$, or $A'h \notin S_p(\mathbb{R}^n)$. □

As an immediate consequence, we obtain the following corollary.

Corollary 8.9. *Let $p > 1$ and let Y, Z be WBFSs with $Z \supsetneqq S_p(\mathbb{R}^n)$ and $T_p(\mathbb{R}^n) \supsetneqq Y$. Then $A' : Z \not\to T_p(\mathbb{R}^n)$, $A' : S_p(\mathbb{R}^n) \not\to Y$.*

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