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Characterisation by Local Means of Anisotropic Lizorkin-Triebel Spaces with Mixed Norms

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Abstract. This is a contribution to the theory of Lizorkin-Triebel spaces having mixed Lebesgue norms and quasi-homogeneous smoothness. We discuss their characterisation in terms of general quasi-norms based on convolutions. In particular, this covers the case of local means, in Triebel's terminology. The main step is an extension of some crucial inequalities due to Rychkov to the case with mixed norms.

Keywords. Local means, mixed norms, moment conditions, Tauberian conditions

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1. Introduction

This paper is devoted to a study of anisotropic Lizorkin-Triebel spaces $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ with mixed norms, which has grown out of work of the first and third author, cf. [12, 13].

First Sobolev embeddings and completeness of the scale $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ were covered in [12]. As the foundation for this, the Nikol'skiĭ-Plancherel-Polya inequality for sequences of functions in the mixed-norm space $L_{\vec{p}}(\mathbb{R}^n)$ was established in [12] with fairly elementary proofs. Then a detailed trace theory for hyperplanes in $\mathbb{R}^n$ was worked out in [13], e.g. with the novelty that the well-known borderline $s = \frac{1}{p}$ has to be shifted upwards in some cases, because of the mixed norms.

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In the present paper we obtain some general characterisations of the space $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, that may be specialised to kernels of local means. We have at least two motivations for this. One is that local means have emerged in the last decade as the natural foundation for a discussion of wavelet bases for Sobolev spaces and their generalisations to the Besov and Lizorkin-Triebel scales; cf. works of Triebel [22, Theorem 1.20] and e.g. Vybiral [23, Theorem 2.12], Hansen [8, Theorem 4.3.1].

Secondly, local means will be crucial for the entire strategy in our forthcoming paper [9], in which we establish invariance of $F^{s,\vec{a}}_{\vec{p},q}$ under diffeomorphisms in order to carry over trace results from [13] to spaces over smooth domains. More precisely, because of the anisotropic structure of the $F^{s,\vec{a}}_{\vec{p},q}$-spaces, we consider them over smooth cylindrical sets in Euclidean space in [9] and develop results for traces on the flat and curved parts of the boundary of the cylinder in [10].

To elucidate the importance of the results here and in [9, 10], we recall that $F^{s,\vec{a}}_{\vec{p},q}$-spaces have applications to parabolic differential equations with initial and boundary value conditions: when solutions are sought in a *mixed-norm* Lebesgue space $L_{\vec{p}}$ (e.g. to allow for different properties in the space and time directions), then $F^{s,\vec{a}}_{\vec{p},q}$-spaces are in general *inevitable* for a correct description of non-trivial data on the *curved* boundary.

This conclusion was obtained in works of Weidemaier [24–26], who treated several special cases; the reader may consult the introduction of [13] for details.

To give a brief review of the present results, we recall that the norm $\|\cdot|F^{s,\vec{a}}_{\vec{p},q}\|$ of $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ is defined in a well-known Fourier-analytic way by splitting the frequency space by means of a Littlewood-Paley partition of unity. But to have "complete" freedom, it is natural first of all to work with convolutions $\psi_j * f$ defined from more arbitrary sequences $(\psi_j)_{j\in\mathbb{N}_0}$ of Schwartz functions with dilations $\psi_j = 2^{j|\vec{a}|}\psi(2^{j\vec{a}}\cdot)$ for $j \geq 1$. This requires both the Tauberian conditions that $\widehat{\psi}_0(\xi)$, $\widehat{\psi}(\xi)$ have no zeroes for $|\xi|_{\vec{a}} < 2\varepsilon$ and $\frac{\varepsilon}{2} < |\xi|_{\vec{a}} < \varepsilon$, respectively; and the moment condition that $D^\alpha\widehat{\psi}(0) = 0$ for $|\alpha| \leq M_\psi$.

Secondly, one may work with anisotropic Peetre-Fefferman-Stein maximal functions $\psi^*_{j,\vec{a}}f$, and with these our main result can be formulated as follows:

Theorem. *If $s < (M_\psi + 1)\min(a_1, \dots, a_n)$ and $0 < p_j < \infty$, $0 < q \leq \infty$, the following quasi-norms are equivalent on the space of temperate distributions:*

$$\|f|F^{s,\vec{a}}_{\vec{p},q}\|, \quad \|\{2^{sj}\psi_j * f\}_{j=0}^\infty|L_{\vec{p}}(\ell_q)\|, \quad \|\{2^{sj}\psi^*_{j,\vec{a}}f\}_{j=0}^\infty|L_{\vec{p}}(\ell_q)\|. \tag{1}$$

Thus $f \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ if and only if one (hence all) of these expressions are finite.

In the isotropic case, i.e., when $\vec{a} = (1, \dots, 1)$ and unmixed L_p-norms are used, the theorem has been known since the important work of Rychkov [15], albeit in another formulation. In our generalisation we follow Rychkov's proof strategy closely, but with some corrections; cf. Remark 1.1 below.

Another particular case is when the functions ψ_0 and ψ have compact support, in which case the convolutions may be interpreted as local means, as observed by Triebel [21]. Thus we develop the mentioned characterisations by local means for the anisotropic $F^{s,\vec{a}}_{\vec{p},q}$-spaces in Theorem 5.2 below, and as far as we know, already this part of their theory is a novelty. As indicated above, it will enter directly into the proofs of our paper [9].

However, it deserves to be mentioned that the arguments in [9] also rely on a stronger estimate than the inequalities underlying the above theorem. In fact we need to consider parameter dependent functions ψ_θ, $\theta \in \Theta$ (an index set), that satisfy the moment conditions in a uniform way. Theorem 4.4 below gives the precise details and our estimate of

$$\| \{2^{sj} \sup_{\theta\in\Theta} \psi^*_{\theta,j,\vec{a}} f\}_{j=0}^\infty \,| L_{\vec{p}}(\ell_q) \|. \tag{2}$$

Similar quasi-norms were introduced by Triebel in the proof of [21, Proposition 4.3.2] for the purpose of showing diffeomorphism invariance of the isotropic scale $F^s_{p,q}(\mathbb{R}^n)$. However, he only claimed the equivalence of the quasi-norms for f belonging a priori to $F^s_{p,q}$ and details of proof were not given. Since our estimate of (2) is valid for arbitrary distributions $f \in \mathcal{S}'$, it should be well motivated that we develop this important tool with a full explanation here.

Remark 1.1. The fact that the arguments in [15] are incomplete was observed in the Ph.D. thesis of M. Hansen [8, Remark 3.2.4], where it was exemplified that in general a certain O-condition is unfulfilled; cf. Remark 4.7 below. Another flaw is pointed out here in Remark 2.9. However, to obtain the full generality with arbitrary temperate distributions in Proposition 4.6 below, we have preferred to reinforce the original proofs of Rychkov. Hence we have found it best to aim at a self-contained exposition in this paper.

Contents. The paper is organized as follows. Section 2 reviews our notation and gives a discussion of the anisotropic spaces of Lizorkin-Triebel type with a mixed norm. Section 3 presents some maximal inequalities for mixed Lebesgue norms. Quasi-norms defined from general systems of Schwartz functions subjected to moment and Tauberian conditions are estimated in Section 4, following works of Rychkov. In Section 5 these spaces are characterised by such general norms, and by local means.

2. Preliminaries

2.1. Notation. Vectors $\vec{p} = (p_1, \dots, p_n)$ with $p_i \in]0, \infty]$ for $i = 1, \dots, n$ are written $0 < \vec{p} \le \infty$, as throughout inequalities for vectors are understood componentwise; as are functions, e.g. $\vec{p}! = p_1! \cdots p_n!$.

By $L_{\vec{p}}(\mathbb{R}^n)$ we denote the set of all functions $u : \mathbb{R}^n \to \mathbb{C}$ that are Lebesgue measurable and such that

$$\| u \,|\, L_{\vec{p}}(\mathbb{R}^n)\| := \Big(\int_{\mathbb{R}}\Big(\dots\Big(\int_{\mathbb{R}}|u(x_1,\dots,x_n)|^{p_1}\,dx_1\Big)^{\frac{p_2}{p_1}}\dots\Big)^{\frac{p_n}{p_{n-1}}}dx_n\Big)^{\frac{1}{p_n}} < \infty, \quad (3)$$

with the modification of using the essential supremum over x_j in case $p_j = \infty$. Equipped with this quasi-norm, $L_{\vec{p}}(\mathbb{R}^n)$ is a quasi-Banach space; it is normed if $\min(p_1, \dots, p_n) \ge 1$.

Furthermore, for $0 < q \le \infty$ we shall use the notation $L_{\vec{p}}(\ell_q)(\mathbb{R}^n)$ for the space of sequences $(u_k)_{k\in\mathbb{N}_0} = \{u_k\}_{k=0}^\infty$ of Lebesgue measurable functions fulfilling

$$\| \{u_k\}_{k=0}^\infty \,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n)\| := \Big\| \Big(\sum_{k=0}^\infty |u_k|^q\Big)^{\frac{1}{q}} \Big| L_{\vec{p}}(\mathbb{R}^n)\Big\| < \infty, \quad (4)$$

with supremum over k in case $q = \infty$. For brevity, we write $\| u_k \,|L_{\vec{p}}(\ell_q)\|$ instead of $\| \{u_k\}_{k=0}^\infty \,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n)\|$; as customary for $\vec{p} = (p, \dots, p)$, we simplify $L_{\vec{p}}$ to L_p etc. If $\max(p_1, \dots, p_n, q) < \infty$, sequences of C_0^∞-functions are dense in $L_{\vec{p}}(\ell_q)$.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of all smooth, rapidly decreasing functions; it is equipped with the family of seminorms, using $\langle x\rangle^2 := 1 + |x|^2$,

$$p_M(\varphi) := \sup\big\{\langle x\rangle^M |D^\alpha\varphi(x)| \,\big|\, x \in \mathbb{R}^n, |\alpha| \le M\big\}, \quad M \in \mathbb{N}_0, \quad (5)$$

whereby $D^\alpha := (-\mathrm{i}\partial_{x_1})^{\alpha_1} \cdots (-\mathrm{i}\partial_{x_n})^{\alpha_n}$ for each multi-index $\alpha \in \mathbb{N}_0^n$; or with

$$p_{\alpha,\beta}(\varphi) := \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \varphi(x)|, \quad \alpha, \beta \in \mathbb{N}_0^n. \quad (6)$$

The Fourier transformation $\mathcal{F}g(\xi) = \widehat{g}(\xi) = \int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\xi} g(x)\,dx$ for $g \in \mathcal{S}(\mathbb{R}^n)$ extends by duality to the dual space $\mathcal{S}'(\mathbb{R}^n)$ of temperate distributions.

Throughout generic constants will mainly be denoted by c or C, and in case their dependence on certain parameters is relevant this will be explicitly stated.

2.2. Lizorkin-Triebel spaces with a mixed norm. As a motivation for the general mixed-norm Lizorkin-Triebel spaces $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, we first mention that for $1 < \vec{p} < \infty$ a temperate distribution u belongs to a class $F^{s,\vec{a}}_{\vec{p},2}(\mathbb{R}^n)$ having natural numbers $m_j := \frac{s}{a_j}$ for each $j = 1, \dots, n$ if and only if u belongs to the mixed-norm Sobolev space $W^{\vec{m},\vec{a}}_{\vec{p}}(\mathbb{R}^n)$, $\vec{m} = (m_1, \dots, m_n)$, defined by

$$\| u \,|L_{\vec{p}}(\mathbb{R}^n)\| + \sum_{i=1}^n \Big\| \frac{\partial^{m_i} u}{\partial x_i^{m_i}} \Big| L_{\vec{p}}(\mathbb{R}^n)\Big\| < \infty. \quad (7)$$

This expression defines the norm on $W^{\vec{m},\vec{a}}_{\vec{p}}$, which is equivalent to that on $F^{s,\vec{a}}_{\vec{p},2}$.

More generally, mixed-norm Lizorkin-Triebel spaces generalise the fractional Sobolev (Bessel potential) spaces $H^{s,\vec{a}}_{\vec{p}}(\mathbb{R}^n)$, since for $1 < \vec{p} < \infty$, $s \in \mathbb{R}$,

$$u \in H^{s,\vec{a}}_{\vec{p}}(\mathbb{R}^n) \iff u \in F^{s,\vec{a}}_{\vec{p},2}(\mathbb{R}^n). \tag{8}$$

Here the norms are also equivalent; the former is given by $\| \mathcal{F}^{-1}(\langle\xi\rangle^{-s}_{\vec{a}}\widehat{u}(\xi)) \,|L_{\vec{p}}\|$, whereby $\langle\xi\rangle_{\vec{a}}$ is an anisotropic version of $\langle\xi\rangle$ compatible with $\vec{a}$; cf. the following.

To account for the Fourier-analytic definition of $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, we first recall the anisotropic structure used for derivatives. Each coordinate x_j in $\mathbb{R}^n$ is given a weight $a_j \ge 1$, collected in $\vec{a} = (a_1, \dots, a_n)$. Based on the quasi-homogeneous dilation $t^{\vec{a}}x := (t^{a_1}x_1, \dots, t^{a_n}x_n)$ for $t \ge 0$, and $t^{s\vec{a}}x := (t^s)^{\vec{a}}x$ for $s \in \mathbb{R}$, in particular $t^{-\vec{a}}x = (t^{-1})^{\vec{a}}x$, the anisotropic distance function $|x|_{\vec{a}}$ is introduced for $x \neq 0$ as the unique $t > 0$ such that $t^{-\vec{a}}x \in S^{n-1}$ (with $|0|_{\vec{a}} = 0$); i.e.,

$$\frac{x_1^2}{t^{2a_1}} + \dots + \frac{x_n^2}{t^{2a_n}} = 1. \tag{9}$$

For the reader's convenience we recall that $|\cdot|_{\vec{a}}$ is C^∞ on $\mathbb{R}^n \setminus \{0\}$ by the Implicit Function Theorem. The formula $|t^{\vec{a}}x|_{\vec{a}} = t|x|_{\vec{a}}$ is seen directly, and this implies the triangle inequality,

$$|x + y|_{\vec{a}} \le |x|_{\vec{a}} + |y|_{\vec{a}}. \tag{10}$$

The relation to e.g. the Euclidean norm $|x|$ can be deduced from

$$\max\Big(|x_1|^{\frac{1}{a_1}}, \dots, |x_n|^{\frac{1}{a_n}}\Big) \le |x|_{\vec{a}} \le |x_1|^{\frac{1}{a_1}} + \dots + |x_n|^{\frac{1}{a_n}}. \tag{11}$$

For the above-mentioned weight function, one can e.g. let $\langle\xi\rangle_{\vec{a}} = |(\xi, 1)|_{(\vec{a},1)}$, using the anisotropic distance given by $(\vec{a}, 1)$ on $\mathbb{R}^{n+1}$; analogously to $\langle\xi\rangle$ in the isotropic case.

We pick for convenience a fixed Littlewood-Paley decomposition, written $1 = \sum_{j=0}^\infty \Phi_j(\xi)$, in the anisotropic setting as follows: Let $\psi \in C_0^\infty$ be a function such that $0 \le \psi(\xi) \le 1$ for all ξ, $\psi(\xi) = 1$ if $|\xi|_a \le 1$, and $\psi(\xi) = 0$ if $|\xi|_a \ge \frac{3}{2}$. Then we set $\Phi = \psi - \psi(2^{\vec{a}}\cdot)$ and define

$$\Phi_0(\xi) = \psi(\xi), \quad \Phi_j(\xi) = \Phi(2^{-j\vec{a}}\xi),\ j = 1, 2, \dots \tag{12}$$

Definition 2.1. The Lizorkin-Triebel space $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, where $0 < \vec{p} < \infty$ is a vector of integral exponents, $s \in \mathbb{R}$ a smoothness index, and $0 < q \le \infty$ a sum exponent, is the space of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\| u \,|\, F^{s,\vec{a}}_{\vec{p},q}\| := \Big\| \Big(\sum_{j=0}^\infty 2^{jsq} \left|\mathcal{F}^{-1}\left(\Phi_j(\xi)\mathcal{F}u(\xi)\right)(\cdot)\right|^q \Big)^{\frac{1}{q}} \Big| L_{\vec{p}}(\mathbb{R}^n) \Big\| < \infty. \tag{13}$$

For simplicity, we omit $\vec{a}$ when $\vec{a} = (1, \dots, 1)$ and shall often set

$$u_j(x) = \mathcal{F}^{-1}\left(\Phi_j(\xi)\, \mathcal{F}u(\xi)\right)(x), \quad x \in \mathbb{R}^n,\ j \in \mathbb{N}_0. \tag{14}$$

Occasionally, we need to consider Besov spaces, which are defined similarly:

Definition 2.2. For $0 < \vec{p} \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$ the Besov space $B^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ consists of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\| u \,|\, B^{s,\vec{a}}_{\vec{p},q} \| := \Big(\sum_{j=0}^{\infty} 2^{jsq} \| u_j \,| L_{\vec{p}}(\mathbb{R}^n) \|^q \Big)^{\frac{1}{q}} < \infty. \tag{15}$$

Remark 2.3. The Lizorkin-Triebel spaces $F^{s,\vec{a}}_{\vec{p},q}$ have a long history, as they give back e.g. the mixed-norm Sobolev spaces $W^{\vec{m}}_{\vec{p}}$, cf. (7). Anisotropic Sobolev (Bessel potential) spaces $H^{s,\vec{a}}_p$ with $1 < p < \infty$ (partly for $s > 0$) have been investigated in the monographs of Nikol'skiĭ [14] and Besov, Il'in and Nikol'skiĭ [2]; here the point of departure was a definition based on derivatives and differences. In the second edition [3] also Lizorkin-Triebel spaces with mixed norms were treated in Ch. 6.29–30. For characterisation of $F^{s,\vec{a}}_{p,q}$ by differences we refer also to Yamazaki [28, Theorem 4.1] and Seeger [17].

The $F^{s,\vec{a}}_{\vec{p},q}$-spaces were considered for $n = 2$ by Schmeisser and Triebel [18], who used the Fourier-analytic characterisation, which we prefer for its efficacy what concerns application of powerful tools from Fourier analysis and distribution theory. (The definition of the anisotropy in terms of $|\cdot|_{\vec{a}}$ is a well-known procedure going back to the 1960's; historical remarks and some basic properties of $|\cdot|_{\vec{a}}$ can be found in e.g. [27].)

For later use we recall some properties of these classes. First standard arguments, cf. [12, 13], yield the following:

Lemma 2.4. *Each $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ is a quasi-Banach space, which is normed if both $\vec{p} \ge 1$ and $q \ge 1$. More precisely, for $u, v \in F^{s,\vec{a}}_{\vec{p},q}$ and $d := \min(1, p_1, \dots, p_n, q)$,*

$$\| u + v \,| F^{s,\vec{a}}_{\vec{p},q} \|^d \le \| u \,| F^{s,\vec{a}}_{\vec{p},q} \|^d + \| v \,| F^{s,\vec{a}}_{\vec{p},q} \|^d. \tag{16}$$

Furthermore, there are continuous embeddings

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n), \tag{17}$$

where $\mathcal{S}$ is dense in $F^{s,\vec{a}}_{\vec{p},q}$ for $q < \infty$. Also, the classes $F^{s,\vec{a}}_{\vec{p},q}$ do not depend on the chosen anisotropic decomposition of unity (up to equivalent quasi-norms).

Lemma 2.5 ([13]). *For $\lambda > 0$ so large that $\lambda\vec{a} \ge 1$, the space $F^{s,\vec{a}}_{\vec{p},q}$ coincides with $F^{\lambda s,\lambda\vec{a}}_{\vec{p},q}$ and the corresponding quasi-norms are equivalent.*

The lemma suggests to introduce a normalisation for the vector $\vec{a}$, and often one has fixed the value of $|\vec{a}|$ in the literature. In this paper we just adopt the flexible framework with $\vec{a} \geq 1$, though.

Remark 2.6. In Lemma 2.5 the inequalities $\vec{a} \geq 1$ and $\lambda\vec{a} \geq 1$ are redundant. In fact one can define $F^{s,\vec{a}}_{\vec{p},q}$ for arbitrary $\vec{a} > 0$, as in [13]. This gives another set-up on $\mathbb{R}^n$, where (10), and hence (11), has to be changed, for then

$$|x+y|^d_{\vec{a}} \leq |x|^d_{\vec{a}} + |y|^d_{\vec{a}}, \quad d = \min(1, a_1, \dots, a_n). \tag{18}$$

The basic results on the $F^{s,\vec{a}}_{\vec{p},q}$-scale can then be derived similarly for $\vec{a} > 0$; only a few constants need to be slightly changed because of (18). Thus one finds e.g. Lemma 2.5 for all $\lambda > 0$, cf. the end of Section 3 in [13] (the details in [13, Section 3] only cover $\vec{a} \geq 1$, but are extended to all $\vec{a} > 0$ as just indicated; in fact $\rho(x,y) = |x-y|_{\vec{a}}$ is then a quasi-distance, a framework widely used by e.g. Stein [20]). However, in view of this lemma, it is simplest henceforth just to assume that $F^{s,\vec{a}}_{\vec{p},q}$ is defined in terms of an anisotropy $\vec{a} \geq 1$; which has been done throughout in the present paper.

2.3. Summation lemmas. For later reference we give two minor results.

Lemma 2.7. *When $(g_j)_{j\in\mathbb{N}_0}$ is a sequence of nonnegative measurable functions on $\mathbb{R}^n$ and $\delta > 0$, then $G_j(x) := \sum_{k=0}^{\infty} 2^{-\delta|j-k|} g_j(x)$ fulfils for $0 < \vec{p} < \infty$, $0 < q \leq \infty$ that*

$$\| G_j \,|L_{\vec{p}}(\ell_q)\| \leq C_{\delta,q} \| g_j \,|L_{\vec{p}}(\ell_q)\|, \tag{19}$$

whereby the constant is $C_{\delta,q} = (\sum_{k\in\mathbb{Z}} 2^{-\delta|k|\tilde{q}})^{\frac{1}{\tilde{q}}}$ for $\tilde{q} = \min(1,q)$.

Like for the unmixed case in [15, Lemma 2], the above lemma is obtained by pointwise application of Minkowski's inequality to a convolution in $\ell_q(\mathbb{Z})$.

Lemma 2.8. *Let $(b_j)_{j\in\mathbb{N}_0}$ and $(d_j)_{j\in\mathbb{N}_0}$ be two sequences in $[0,\infty]$ and $0 < r \leq 1$. If for some $j_0 \geq 0$ there exists real numbers $C, N_0 > 0$ such that*

$$d_j \leq C 2^{jN_0} \quad \text{for } j \geq j_0, \tag{20}$$

and if for every $N > 0$ there exists a real number C_N such that

$$d_j \leq C_N \sum_{k=j}^{\infty} 2^{(j-k)N} b_k d_k^{1-r}, \quad \text{for } j \geq j_0, \tag{21}$$

then the same constants C_N, $N > 0$, fulfil that

$$d_j^r \leq C_N \sum_{k=j}^{\infty} 2^{(j-k)Nr} b_k, \quad \text{for } j \geq j_0. \tag{22}$$

Proof. With $D_{j,N} = \sup_{k\ge j} 2^{(j-k)N} d_k$ it follows from (21) that for $j \ge j_0$, $N > 0$,

$$D_{j,N} \le \sup_{k\ge j} C_N \sum_{l\ge k} 2^{(j-l)N} b_l d_l^{1-r} \le C_N(\sum_{l\ge j} 2^{(j-l)Nr} b_l) D_{j,N}^{1-r}. \qquad (23)$$

Clearly $D_{j_1,N} = 0$ implies $d_j = 0$ for $j \ge j_1$, so (22) is trivial for such j. We thus only need to consider the $D_{j,N} > 0$. Now (20) yields that $D_{j,N} < \infty$ for all $j \ge j_0$ when $N \ge N_0$, so then (22) follows from (23) by division by $D_{j,N}^{1-r}$.

Given any $N \in]0, N_0[$, we may in the just proved cases of (22) decrease N_0 to N, which gives a version of (22) with N in the exponent and the constant C_{N_0}. Analogously to (23), one therefore finds from the definition of $D_{j,N}$ that $D_{j,N} \le C_{N_0}^{\frac{1}{r}} (\sum_{l\ge j} 2^{(j-l)Nr} b_l)^{\frac{1}{r}}$ for $j \ge j_0$. Here the right-hand side may be assumed finite (as else (22) is trivial for this N), whence we may proceed as before by division in (23). □

Remark 2.9. Lemma 2.8 was essentially crystallised by Rychkov [15, Lemma 3], albeit with three unnecessary assumptions: $d_j < \infty$ (a consequence of (20)), that $b_j, d_j > 0$ and that $j_0 = 0$. For our proof of Proposition 4.6 below, it is essential to consider $j_0 > 0$, and it would be cumbersome there to reduce to strict positivity of b_j, d_j. In [15] no justification was given for this strictness in the application of [15, Lemma 3], but this is remedied by Lemma 2.8 above.

3. Some maximal inequalities

In this section we obtain some maximal inequalities in the mixed-norm set-up. This part of the theory of the $F^{s,\vec{a}}_{\vec{p},q}$-spaces is interesting in its own right, and also important for the authors' work [9]. Moreover, the methods are similar to those adopted in the set-up in Section 4 below, but are rather cleaner here.

For distributions u that for some $R > 0$ and $j \in \mathbb{N}$ satisfy

$$\operatorname{supp} \widehat{u} \subset \big\{ \xi \in \mathbb{R}^n \bigm| |\xi_k| \le R\, 2^{ja_k}, k = 1, \dots, n \big\} \qquad (24)$$

the Peetre-Fefferman-Stein maximal function $u^*(x)$ is given by

$$u^*(x) = \sup_{y\in\mathbb{R}^n} \frac{|u(y)|}{\prod_{l=1}^n (1 + R\, 2^{ja_l}|x_l - y_l|)^{r_l}}, \quad \vec{r} > 0. \qquad (25)$$

It obviously fulfils

$$|u(x)| \le u^*(x) \le \| u \,|L_\infty\|, \quad x \in \mathbb{R}^n. \qquad (26)$$

When u in addition is in $L_{\vec{p}}$, the Nikol'skiĭ-Plancherel-Polya inequality for mixed norms, cf. [12, Proposition 4], gives the finiteness of the right-hand side, hence u^*

is finite everywhere. Thus, analogously to [11, Section 2], the maximal function is continuous.

To prepare for the theorem below, we first show the following pointwise estimate of $u^*(x)$ by combining the proof ingredients from [11, Proposition 2.2], which the reader may consult for more details. Now their order is crucial:

Proposition 3.1. *When* $0 < \vec{q}, \vec{r} \le \infty$ *then there is a constant* $c_{\vec{q},\vec{r}}$ *such that every* $u \in \mathcal{S}'$ *fulfilling* (24) *also satisfies*

$$u^*(x) \le c_{\vec{q},\vec{r}} \left\| \frac{u(x - R^{-1}\, 2^{-j\vec{a}} z)}{\prod_{l=1}^n (1+|z_l|)^{r_l}} \,\middle|\, L_{\vec{q}}\,(\mathbb{R}^n_z) \right\| \quad \text{for } x \in \mathbb{R}^n. \tag{27}$$

Proof. Taking $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\widehat{\psi} \equiv 1$ on $[-1,1] \times \cdots \times [-1,1]$ and such that $\operatorname{supp} \widehat{\psi} \subset [-2,2] \times \cdots \times [-2,2]$, we have $u = \mathcal{F}^{-1}(\widehat{\psi}(R^{-1} 2^{-j\vec{a}} \cdot)) * u$, which may be written with an integral since u is C^∞ with polynomial growth,

$$u(y) = \int \cdots \int R^n\, 2^{j|\vec{a}|}\, \psi(R\, 2^{j\vec{a}}(y-z))\, u(z)\, dz_1 \cdots dz_n. \tag{28}$$

Now $\vec{q} = (q_<, q_\ge)$ is split into two groups $q_<$ and $q_\ge$ according to whether $q_k < 1$ or $q_k \ge 1$ holds. The groups may be interlaced, but for simplicity this is ignored in the notation; the important thing is to treat the two groups separately.

First (28) is estimated by the norm of $L_1(\mathbb{R}^n)$, which then is controlled in terms of the norm of $L_{(q_<, 1_\ge)}$, whereby interlacing of the groups $q_<$ and $1_\ge$ is unimportant: for fixed y, the spectrum of the integrand in (28) is contained in $[-3R\, 2^{ja_1}, 3R\, 2^{ja_1}] \times \cdots \times [-3R\, 2^{ja_n}, 3R\, 2^{ja_n}]$, so the Nikol'skiĭ-Plancherel-Polya inequality for mixed norms applies, cf. [12, Proposition 4], which for $q_k < 1$ gives an estimate by the norms of L_{q_k} with respect to z_k; that is,

$$|u(y)| \le c \prod_{q_k<1} (3R\, 2^{ja_k})^{\frac{1}{q_k}-1} \left\| R^n 2^{j|\vec{a}|} \psi(R 2^{j\vec{a}}(y-\cdot))\, u \,\middle|\, L_{(q_<, 1_\ge)} \right\|. \tag{29}$$

(The integration order in this norm is as stated in (28).)

Secondly, using Hölder's inequality in the variables where $q_k \ge 1$, and gathering their dual exponents q_k^* in $(q_\ge)^*$, gives for $x \in \mathbb{R}^n$

$$\begin{aligned} &\frac{|u(y)|}{\prod_l (1 + R 2^{ja_l}|x_l - y_l|)^{r_l}} \\ &\le c \prod_{q_k<1} (3R 2^{ja_k})^{\frac{1}{q_k}-1} \left\| \frac{R^n 2^{j|\vec{a}|} u(z)}{\prod_l (1 + R 2^{ja_l}|x_l - z_l|)^{r_l}} \,\middle|\, L_{\vec{q}} \right\| \\ &\quad \times \left\| \prod_l (1 + R 2^{ja_l}|y_l - z_l|)^{r_l} \psi(R 2^{j\vec{a}}(y-z)) \,\middle|\, L_{(\infty_<, (q_\ge)^*)} \right\|. \end{aligned} \tag{30}$$

Since $\psi \in \mathcal{S}$, a change of coordinates $z_k \mapsto R^{-1}\, 2^{-ja_k} z_k$ yields (27) with the constant $c_{\vec{q},\vec{r}} = c \prod_{q_k<1} 3^{\frac{1}{q_k}-1} \| \prod_{l=1}^n (1+|z_l|)^{r_l} \psi \,|\, L_{(\infty_<, (q_\ge)^*)} \| < \infty$. □

We now obtain an elementary proof of the *mixed-norm* boundedness of u^*, by adapting the proof of the isotropic L_p-result in [11, Theorem 2.1]:

Theorem 3.2. *Let* $0 < \vec{p} \leq \infty$ *and suppose*

$$r_l > \frac{1}{\min(p_1, \dots, p_l)}\,, \quad l = 1, \dots, n. \tag{31}$$

Then there exists a constant c such that

$$\| u^* \,|L_{\vec{p}}\| \leq c \,\| u \,|L_{\vec{p}}\| \tag{32}$$

holds for all $u \in L_{\vec{p}} \cap \mathcal{S}'$ *satisfying the spectral condition* (24)*.*

Proof. We use (27) with $q_k = \min(p_1, \dots, p_k)$ for $k = 1, \dots, n$ and calculate the L_{p_j}-norms successively on both sides. Since $p_j \geq q_k$ for all $k \geq j$, we may apply the generalised Minkowski inequality $n-(j-1)$ times, as well as the translation invariance of $dx_1, \dots, dx_n$, which gives

$$\| u^* \,|L_{\vec{p}}\| \leq c_{\vec{q},\vec{r}} \Big(\prod_{l=1}^{n} \|(1+|z_l|)^{-r_l} \,|L_{q_l}\| \Big) \| u \,|L_{\vec{p}}\|. \tag{33}$$

Here (31) yields the finiteness of the L_{q_l}-norms. □

The following result is convenient for certain convolution estimates. Since the embedding $B^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ holds for $s > \vec{a} \cdot \frac{1}{\vec{p}}$, or for $s = \vec{a} \cdot \frac{1}{\vec{p}}$ if $q \leq 1$, it is a result pertaining to continuous functions.

Corollary 3.3. *If* $C > 0$ *and* $\vec{r}$ *fulfils* (31)*,* $d = \min(1, p_1, \dots, p_n)$ *yields*

$$\Big\| \sup_{|x-y|<C} |u(y)| \,\Big| L_{\vec{p}}(\mathbb{R}^n_x) \Big\| \leq c \,\| u \,|B^{s,\vec{a}}_{\vec{p},d}\| \quad \textit{for } s = \vec{a} \cdot \vec{r}. \tag{34}$$

Proof. Since $\| \cdot |L_{\vec{p}}\|^d$ is subadditive, simple arguments yield

$$\begin{aligned} \Big\| \sup_{|x-y|<C} |u(y)| \,\Big| L_{\vec{p}}(\mathbb{R}^n_x) \Big\|^d &\leq \Big\| \sup_{|x-y|<C} \sum_{j=0}^{\infty} |u_j(y)| \,\Big|\, L_{\vec{p}}(\mathbb{R}^n_x) \Big\|^d \\ &\leq \sum_{j=0}^{\infty} \prod_{\ell=1}^{n} (1 + C\, 2^{ja_\ell})^{d\,r_\ell} \| u_j^* \,|L_{\vec{p}}\|^d. \end{aligned} \tag{35}$$

Since $\prod_{\ell=1}^{n} (1 + C\, 2^{ja_\ell})^{d\,r_\ell} \leq (1+C)^{d\,|\vec{r}|}\, 2^{jd\,\vec{a}\cdot\vec{r}}$, the right-hand side is seen to be less than $c \,\| u \,|\, B^{s,\vec{a}}_{\vec{p},d}\|^d$ for $s = \vec{a} \cdot \vec{r}$ by application of Theorem 3.2. □

Remark 3.4. In [9] Corollary 3.3 enters our estimates for certain $u \in F^{s,\vec{a}}_{\vec{p},q}$ with $\sum_{\ell=1}^{n} \frac{a_\ell}{\min(p_1,\dots,p_\ell)} < s$. Then one can pick $\vec{r}$ satisfying (31) and such that $\vec{a} \cdot \vec{r} < s$, hence elementary embeddings yield $\| \sup_{|x-y|<C} |u(y)| \,|L_{\vec{p}}(\mathbb{R}^n_x)\| \leq c \,\| u \,|F^{s,\vec{a}}_{\vec{p},q}\|$.

4. Rychkov's inequalities

In the systematic theory of the $F^{s,\vec{a}}_{\vec{p},q}$-spaces, it is of course important to dispense from the requirement in Definition 2.1 that the Schwartz functions Φ_j have compact support. In so doing, we shall largely follow Rychkov's treatment of the isotropic case [15].

In the following $\vec{a} = (a_1, \dots, a_n)$ is a fixed anisotropy with $\vec{a} \geq 1$; we set

$$a_0 = \min(a_1, \dots, a_n). \tag{36}$$

Throughout this section we consider $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ that fulfil Tauberian conditions in terms of some $\varepsilon > 0$ and/or a moment condition of order M_ψ,

$$|\mathcal{F}\psi_0(\xi)| > 0 \quad \text{on } \{\xi \mid |\xi|_{\vec{a}} < 2\varepsilon\}, \tag{37}$$

$$|\mathcal{F}\psi(\xi)| > 0 \quad \text{on } \Big\{\xi \;\Big|\; \frac{\varepsilon}{2} < |\xi|_{\vec{a}} < 2\varepsilon\Big\}, \tag{38}$$

$$D^\alpha(\mathcal{F}\psi)(0) = 0 \quad \text{for } |\alpha| \leq M_\psi. \tag{39}$$

Hereby $M_\psi \in \mathbb{N}_0$, or we take $M_\psi = -1$ when the condition (39) is void. Note that if (37) is verified for the Euclidean distance, it holds true also in the anisotropic case, perhaps with a different ε; cf. (11).

In this section we also change notation by setting

$$\varphi_j(x) = 2^{j|\vec{a}|}\varphi(2^{j\vec{a}}x), \quad \varphi \in \mathcal{S},\ j \in \mathbb{N}. \tag{40}$$

For ψ_0 this gives rise to the sequence $\psi_{0,j}(x) := 2^{j|\vec{a}|}\psi_0(2^{j\vec{a}}x)$, but we shall mainly deal with $(\psi_j)_{j\in\mathbb{N}_0}$ that mixes ψ_0 and ψ. Note that $\psi_0 = \psi_{0,0}$.

To elucidate the Tauberian conditions, we recall in the lemma below a well-known fact on Calderón's reproducing formula:

$$u = \sum_{j=0}^{\infty} \lambda_j * \psi_j * u, \quad \text{for } u \in \mathcal{S}'(\mathbb{R}^n). \tag{41}$$

Lemma 4.1. *When $\psi_0, \psi \in \mathcal{S}$ fulfil the Tauberian conditions* (37), (38) *there exist $\lambda_0, \lambda \in \mathcal{S}$ fulfilling* (41) *for every $u \in \mathcal{S}'$. Moreover, it can be arranged that $\widehat{\lambda}_0$ and $\widehat{\lambda}$ are supported by the sets in* (37), *respectively* (38).

Proof. By Fourier transformation (41) is carried over to

$$\mathcal{F}\lambda_0(\xi)\,\mathcal{F}\psi_0(\xi) + \sum_{j=1}^{\infty} \mathcal{F}\lambda(2^{-j\vec{a}}\xi)\,\mathcal{F}\psi(2^{-j\vec{a}}\xi) = 1, \quad \xi \in \mathbb{R}^n. \tag{42}$$

To find λ_0, λ reduces to a Littlewood-Paley construction: taking $h \in C_0^\infty$ such that $0 \leq h \leq 1$ on $\mathbb{R}^n$, $\operatorname{supp} h \subset \{\xi \mid |\xi|_{\vec{a}} < 2\varepsilon\}$ and $h(\xi) = 1$ if $|\xi|_{\vec{a}} \leq \frac{3}{2}\varepsilon$, then $\widehat{\lambda}_0 = h\,\widehat{\psi_0}^{\,-1}$ and $\widehat{\lambda} = \big(h - h(2^{\vec{a}}\cdot)\big)\widehat{\psi}^{\,-1}$ fulfil (42) and the support inclusions. □

A general reference to Calderon's formula could be [7, Chapter 6]. More refined versions have been introduced by Rychkov [16].

To comment on the moment condition, we use for $M \geq -1$ the subspace

$$\mathcal{S}_M := \Big\{ \mu \in \mathcal{S}(\mathbb{R}^n) \,\Big|\, D^\alpha(\mathcal{F}\mu)(0) = 0 \quad \text{for all } |\alpha| \leq M \Big\}. \tag{43}$$

It is recalled that in addition to the $p_{\alpha,\beta}$ in (6) also the following family of seminorms induces the topology on $\mathcal{S}$:

$$q_{N,\alpha}(\psi) := \int_{\mathbb{R}^n} \langle x \rangle^N |D^\alpha \psi(x)|\, dx, \quad N \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n. \tag{44}$$

This is convenient for the fact that moment conditions, also in case of the anisotropic dilation $t^{\vec{a}}$, induce a rate of convergence to 0 in $\mathcal{S}$:

Lemma 4.2. *For $\alpha, \beta \in \mathbb{N}_0^n$ there is an estimate for $0 < t \leq 1, \nu \in \mathcal{S}$ and $\mu \in \mathcal{S}_M$,*

$$p_{\alpha,\beta}(t^{-|\vec{a}|}\mu(t^{-\vec{a}}\,\cdot) * \nu) \leq C_\alpha\, t^{(M+1)a_0} \max p_{0,\zeta}(\widehat{\mu}) \cdot q_{M+1,\gamma}(\widehat{D^\beta \nu}), \tag{45}$$

where the maximum is over all ζ with $|\zeta| \leq M+1$ or $\zeta \leq \alpha$; and over $\gamma \leq \alpha$.

Proof. The continuity of $\mathcal{F}^{-1} = (2\pi)^{-n}\overline{\mathcal{F}} : L_1 \to L_\infty$ and Leibniz' rule give that

$$\begin{aligned} p_{\alpha,\beta}(t^{-|\vec{a}|}\mu(t^{-\vec{a}}\,\cdot) * \nu) &= \sup_{z \in \mathbb{R}^n} \left| \mathcal{F}^{-1}\left(D_\xi^\alpha (t^{-|\vec{a}|}\widehat{\mu(t^{-\vec{a}}\,\cdot)}\, \widehat{D^\beta \nu}) \right)(z) \right| \\ &\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \int t^{a\cdot(\alpha-\gamma)} |D^{\alpha-\gamma}\widehat{\mu}(t^{\vec{a}}\xi)|\, |D^\gamma \widehat{D^\beta \nu}(\xi)|\, d\xi. \end{aligned} \tag{46}$$

For $|\alpha - \gamma| \leq M$ the integral is estimated using a Taylor expansion of order $N := M - |\alpha - \gamma|$. All terms except the remainder vanish, because μ has vanishing moments up to order M. The integral is therefore bounded by

$$\begin{aligned} &\int t^{\vec{a}\cdot(\alpha-\gamma)} \Big| \sum_{|\zeta|=N+1} \frac{N+1}{\zeta!} (t^{\vec{a}}\xi)^\zeta \int_0^1 (1-\theta)^N \partial_\xi^\zeta D_\xi^{\alpha-\gamma} \widehat{\mu}(\theta t^{\vec{a}}\xi) d\theta \Big|\, |D^\gamma \widehat{D^\beta \nu}(\xi)|\, d\xi \\ &\leq t^{(M+1)a_0} \max_{|\zeta| \leq M+1} \| D^\zeta \widehat{\mu} \mid L_\infty \| \int |\xi|^{N+1} |D^\gamma \widehat{D^\beta \nu}(\xi)|\, d\xi \\ &\leq t^{(M+1)a_0} \max_{|\zeta| \leq M+1} p_{0,\zeta}(\widehat{\mu})\, q_{M+1,\gamma}(\widehat{D^\beta \nu}). \end{aligned}$$

For $|\alpha - \gamma| \geq M+1$ the integral in (46) is easily seen to be estimated by

$$t^{(M+1)a_0} \max_{\zeta \leq \alpha} p_{0,\zeta}(\widehat{\mu})\, q_{0,\gamma}(\widehat{D^\beta \nu}). \tag{47}$$

The claim is obtained by taking the largest of the bounds. □

4.1. Comparison of norms. For any $\vec{r} = (r_1, \dots, r_n) > 0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we deal in this section with the non-linear maximal operators of Peetre-Fefferman-Stein type induced by $\{\psi_j\}_{j\in\mathbb{N}_0}$,

$$\psi_j^* f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\psi_j * f(y)|}{\prod_{\ell=1}^n (1 + 2^{ja_\ell} |x_\ell - y_\ell|)^{r_\ell}}, \quad x \in \mathbb{R}^n,\ j \in \mathbb{N}_0. \tag{48}$$

For simplicity their dependence on $\vec{a}$ and $\vec{r}$ is omitted. (Compared to (25), no R is in the denominator here, as $\psi_j * f$ need not have compact spectrum.)

To give the background, we recall an important technical result of Rychkov:

Proposition 4.3 ([15, (8')]). *Let $\psi_0, \psi \in \mathcal{S}$ be given such that* (39) *holds, while $\varphi_0, \varphi \in \mathcal{S}$ fulfil the Tauberian conditions* (37), (38) *in terms of some $\varepsilon' > 0$. When $0 < p < \infty$, $0 < q \le \infty$ and $s < (M_\psi + 1)\, a_0$ there exists a constant $c > 0$ such that for $f \in \mathcal{S}'$,*

$$\| 2^{sj} \psi_j^* f \,| L_p(\ell_q) \| \le c \, \| 2^{sj} \varphi_j^* f \,| L_p(\ell_q) \|. \tag{49}$$

We shall extend this to a mixed-norm version, which even covers parameter-dependent families of the spectral cut-off functions; this will be crucial for our results in [9]. So if Θ denotes an index set and $\psi_{\theta,0}, \psi_\theta \in \mathcal{S}(\mathbb{R}^n)$, $\theta \in \Theta$, we set $\psi_{\theta,j}(x) := 2^{j|\vec{a}|} \psi_\theta(2^{j\vec{a}} x)$ for $j \in \mathbb{N}$. Not surprisingly we need to assume that the ψ_θ fulfil the same moment condition, i.e., uniformly with respect to θ:

Theorem 4.4. *Let $\psi_{\theta,0}, \psi_\theta \in \mathcal{S}(\mathbb{R}^n)$ be given such that* (39) *holds for some M_{ψ_θ} independent of $\theta \in \Theta$, while $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ fulfil* (37), (38) *in terms of an $\varepsilon' > 0$. Also let $0 < \vec{p} < \infty$, $0 < q \le \infty$ and $s < (M_{\psi_\theta} + 1)\, a_0$. For a given $\vec{r}$ in* (48) *and an integer $M \ge -1$ chosen so large that $(M+1)a_0 + s > 2\vec{a}\cdot\vec{r}$, we assume that*

$$\begin{aligned}
A &:= \sup_{\theta\in\Theta} \max \| D^\alpha \mathcal{F}\psi_\theta \,| L_\infty \| && < \infty,\\
B &:= \sup_{\theta\in\Theta} \max \| (1 + |\xi|)^{M+1} D^\gamma \mathcal{F}\psi_\theta(\xi) \,| L_1 \| && < \infty,\\
C &:= \sup_{\theta\in\Theta} \max \| D^\alpha \mathcal{F}\psi_{\theta,0} \,| L_\infty \| && < \infty,\\
D &:= \sup_{\theta\in\Theta} \max \| (1 + |\xi|)^{M+1} D^\gamma \mathcal{F}\psi_{\theta,0}(\xi) \,| L_1 \| && < \infty,
\end{aligned}$$

where the maxima are over all α with $|\alpha| \le M_{\psi_\theta} + 1$ or $\alpha \le \lceil \vec{r} + 2 \rceil$, respectively $\gamma \le \lceil \vec{r} + 2 \rceil$. Then there exists a constant $c > 0$ such that for $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\| 2^{sj} \sup_{\theta\in\Theta} \psi_{\theta,j}^* f \,| L_{\vec{p}}(\ell_q) \| \le c(A + B + C + D) \, \| 2^{sj} \varphi_j^* f \,| L_{\vec{p}}(\ell_q) \|. \tag{50}$$

Hereby $\lceil t \rceil$ denotes the smallest integer $k \geq t$, and $\lceil \vec{r} \rceil := (\lceil r_1 \rceil, \dots, \lceil r_n \rceil)$.

In the proof of the estimate (50) we choose $\lambda_0, \lambda \in \mathcal{S}(\mathbb{R}^n)$ by applying Lemma 4.1 to the given $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$. Following [15], we then consider the auxiliary integrals

$$I_{j,k} := \int |\psi_{\theta,j} * \lambda_k(z)| \prod_{\ell=1}^{n} (1 + 2^{ka_\ell} |z_\ell|)^{r_\ell} \, dz\,, \quad j,k \in \mathbb{N}_0. \tag{51}$$

The integrand may be estimated using that $\psi_{\theta,j} * \lambda_k(z) = 2^{k|\vec{a}|}\psi_{\theta,j-k} * \lambda(2^{k\vec{a}}z)$, so the Binomial Theorem and Lemma 4.2 with $\beta = 0$, $t^{-1} = 2^{j-k} \geq 1$ yield

$$\begin{aligned} &|\psi_{\theta,j} * \lambda_k(z)| \prod_{l=1}^{n} (1 + |2^{ka_l} z_l|)^{r_l} \\ &\leq 2^{k|\vec{a}|} \sum_{\alpha \leq \lceil \vec{r} \rceil} \binom{\lceil \vec{r} \rceil}{\alpha} p_{\alpha,0}(\psi_{\theta,j-k} * \lambda) \\ &\leq C_{\lceil \vec{r} \rceil} 2^{(k-j)(M_{\psi_\theta}+1)a_0 + k|\vec{a}|} \max{}' p_{0,\zeta}(\widehat{\psi_\theta}) \cdot q_{M_{\psi_\theta}+1,\gamma}(\widehat{\lambda}), \end{aligned} \tag{52}$$

where $\max'$ denotes a maximum over finitely many multi-indices, in this case over ζ fulfilling $|\zeta| \leq M_{\psi_\theta} + 1$ or $\zeta \leq \lceil \vec{r} \rceil$, respectively $\gamma \leq \vec{r}$.

Lemma 4.5. *For any integer $M \geq -1$ there exists a constant $c = c_{M,M_\psi,\vec{r},\lambda_0,\lambda}$ such that for $k, j \in \mathbb{N}_0$,*

$$I_{j,k} \leq c\,(A + B + C + D) \times \begin{cases} 2^{(k-j)(M_{\psi_\theta}+1)a_0} & \text{for } k \leq j, \\ 2^{-(k-j)((M+1)a_0 - \vec{a}\cdot\vec{r})} & \text{for } j \leq k \end{cases} \tag{53}$$

when $\psi_{\theta,0}, \psi_\theta \in \mathcal{S}$ and the ψ_θ fulfil (39) for some M_{ψ_θ} independent of $\theta \in \Theta$.

Proof. First we consider the case $j \geq k \geq 1$, where (52) yields

$$\begin{aligned} I_{j,k} &\leq \sup_{z \in \mathbb{R}^n} |\psi_{\theta,j} * \lambda_k(z)| \prod_{l=1}^{n} (1 + 2^{ka_l}|z_l|)^{r_l+2} \int \prod_{l=1}^{n} 2^{-ka_l}(1 + |x_l|)^{-2} \, dx \\ &\leq C_{\vec{r}}\, 2^{(k-j)(M_{\psi_\theta}+1)a_0} \max{}' \, \| D^\zeta \widehat{\psi_\theta} \mid L_\infty \| \cdot q_{M_{\psi_\theta}+1,\gamma}(\widehat{\lambda}) \\ &\leq C_{\vec{r},M_{\psi_\theta},\lambda}\, 2^{(k-j)(M_{\psi_\theta}+1)a_0} A. \end{aligned} \tag{54}$$

For $k \geq j \geq 1$ one can replace 2^{ka_l} in (51) by 2^{ja_l} at the cost of the factor $2^{(k-j)\vec{a}\cdot\vec{r}}$ in front of the integral. Then the roles of ψ_θ and λ can be interchanged, since the support information on $\widehat{\lambda}$ yields $\lambda \in \bigcap_M \mathcal{S}_M$. This gives, with $\rho = \lceil \vec{r} + 2 \rceil$,

$$I_{j,k} \leq c 2^{(k-j)\vec{a}\cdot\vec{r}} \sum_{\alpha \leq \rho} \binom{\rho}{\alpha} p_{\alpha,0}(\psi_\theta * \lambda_{k-j}) \leq C_{M,\vec{r},\lambda} 2^{-(k-j)((M+1)a_0 - \vec{a}\cdot\vec{r})} B. \tag{55}$$

Similar estimates are obtained for $I_{j,0}$, $I_{0,k}$ and $I_{0,0}$ with C, D as factors. □

Using Lemma 4.5, the proof given in [15] is now extended to a

Proof of Theorem 4.4. The identity (41) gives for $f \in \mathcal{S}'$ and $j \in \mathbb{N}$ that

$$\psi_{\theta,j} * f = \sum_{k=0}^{\infty} \psi_{\theta,j} * \lambda_k * \varphi_k * f. \tag{56}$$

By Lemma 4.5 with M chosen so large that $(M+1)a_0 + s > 2\vec{a}\cdot\vec{r}$, there exists a θ-independent constant $c > 0$ such that the summands can be crudely estimated,

$$\begin{aligned} &|\psi_{\theta,j} * \lambda_k * \varphi_k * f(y)| \\ &\le \varphi_k^* f(y) \int |\psi_{\theta,j} * \lambda_k(z)| \prod_{l=1}^{n} (1 + 2^{ka_l}|z_l|)^{r_l}\, dz \\ &\le c\,(A+B+C+D)\,\varphi_k^* f(y) \times \begin{cases} 2^{(k-j)(M_{\psi_\theta}+1)a_0} & \text{for } k \le j, \\ 2^{-(k-j)((M+1)a_0 - \vec{a}\cdot\vec{r})} & \text{for } j \le k. \end{cases} \end{aligned} \tag{57}$$

Here $\varphi_k^* f(y) \le \varphi_k^* f(x) \max\left(1, 2^{(k-j)\vec{a}\cdot\vec{r}}\right) \prod_{l=1}^{n} (1 + 2^{ja_l}|x_l - y_l|)^{r_l}$ is easily verified for $x, y \in \mathbb{R}^n$ and $j, k \in \mathbb{N}_0$ by elementary calculations, so therefore

$$\begin{aligned} &\sup_{y\in\mathbb{R}^n} \frac{|\psi_{\theta,j} * \lambda_k * \varphi_k * f(y)|}{\prod_{l=1}^{n}(1 + 2^{ja_l}|x_l - y_l|)^{r_l}} \\ &\le c(A+B+C+D)\,\varphi_k^* f(x) \times \begin{cases} 2^{(k-j)(M_{\psi_\theta}+1)a_0} & \text{for } k \le j, \\ 2^{-(k-j)((M+1)a_0 - 2\vec{a}\cdot\vec{r})} & \text{for } j \le k. \end{cases} \end{aligned} \tag{58}$$

Inserting into (56) and using that $\delta := \min\big((M_{\psi_\theta}+1)a_0 - s, (M+1)a_0 - 2\vec{a}\cdot\vec{r} + s\big) > 0$ by the assumptions, the above implies for $j \ge 0$,

$$2^{js} \sup_{\theta\in\Theta} \psi_{\theta,j}^* f(x) \le c(A+B+C+D) \sum_{k=0}^{\infty} 2^{ks} \varphi_k^* f(x)\, 2^{-|j-k|\delta}. \tag{59}$$

Now Lemma 2.7 yields (50). □

4.2. Control by convolutions. Since $\widehat{\psi}$ need not have compact support, Proposition 3.1 is replaced by a pointwise estimate with a sum representing the higher frequencies:

Proposition 4.6. *Let $\psi_0, \psi \in \mathcal{S}$ satisfy the Tauberian conditions* (37), (38). *For $N, \vec{r}, \tau > 0$ there exists a constant $C_{N,\vec{r},\tau}$ such that for $f \in \mathcal{S}'$ and $j \in \mathbb{N}_0$,*

$$\left(\psi_j^* f(x)\right)^\tau \le C_{N,\vec{r},\tau} \sum_{k\ge j} 2^{(j-k)N\tau} \int \frac{2^{k|\vec{a}|}|\psi_k * f(z)|^\tau}{\prod_{l=1}^{n}(1 + 2^{ka_l}|x_l - z_l|)^{r_l\tau}}\, dz. \tag{60}$$

As a proof ingredient we use the $\mathcal{S}'$-order of $f \in \mathcal{S}'(\mathbb{R}^n)$, written $\operatorname{ord}_{\mathcal{S}'}(f)$, that is the smallest $N \in \mathbb{N}_0$ for which there exists $c > 0$ such that, cf. (5),

$$|\langle f, \psi \rangle| \leq c\, p_N(\psi) \quad \text{for all } \psi \in \mathcal{S}(\mathbb{R}^n). \tag{61}$$

Remark 4.7. Our proof of Proposition 4.6 follows that of Rychkov [15], although his exposition leaves a heavy burden with the reader, since the application of Lemma 3 there is only justified when $\operatorname{ord}_{\mathcal{S}'}(f)$ is sufficiently small; cf. the O-condition (66) below. In a somewhat different context, Rychkov gave a verbal explanation after (2.17) in [16] (with similar reasoning in [8, 19]) that perhaps could be carried over to the present situation. But we have found it simplest to reinforce [15] by showing that the central O-condition *is* indeed fulfilled whenever f is such that the right-hand side of (60) is finite. In so doing, we give the full argument for the sake of completeness.

Proof. Step 1. First we choose two functions $\lambda_0, \lambda \in \mathcal{S}$ with $\widehat{\lambda} = 0$ around $\xi = 0$ by applying Lemma 4.1 to the given $\psi_0, \psi \in \mathcal{S}$. Using Calderón's reproducing formula, cf. (41), on $f(2^{-j\vec{a}}\cdot)$, dilating and convolving with ψ_j, we obtain

$$\psi_j * f = (\lambda_{0,j} * \psi_{0,j}) * (\psi_j * f) + \sum_{k=j+1}^{\infty} (\psi_j * \lambda_k) * (\psi_k * f). \tag{62}$$

To estimate $\psi_j * \lambda_k$ we use (52) for an arbitrary integer $M_\lambda \geq -1$ to get

$$|\psi_j * \lambda_k(z)| \leq C_{\vec{r}} \frac{2^{j|\vec{a}|}\, 2^{(j-k)(M_\lambda+1)a_0}}{\prod_{l=1}^{n}(1 + 2^{ja_l}|z_l|)^{r_l}} \max{}' p_{0,\varsigma}(\widehat{\lambda}) \cdot q_{M_\lambda+1,\gamma}(\widehat{\psi}). \tag{63}$$

An analogous estimate is obtained for $\lambda_{0,j} * \psi_{0,j}$, when (52) is applied with $t = 1$, $M_{\lambda_0} = -1$. Inserting these bounds into (62) yields for $C_{M_\lambda,\vec{r}} = C_{M_\lambda,\vec{r},\lambda_0,\lambda,\psi_0,\psi}$,

$$|\psi_j * f(y)| \leq C_{M_\lambda,\vec{r}} \sum_{k=j}^{\infty} 2^{(j-k)(M_\lambda+1)a_0} \int \frac{2^{j|\vec{a}|}|\psi_k * f(y-z)|}{\prod_{l=1}^{n}(1 + 2^{ja_l}|z_l|)^{r_l}}\, dz. \tag{64}$$

Since $j \mapsto 2^{j\vec{a}\cdot\vec{r}} \prod_{l=1}^{n}(1 + 2^{ja_l}|x_l - z_l|)^{-r_l}$ is monotone increasing, (64) entails that for $N = (M_\lambda + 1)a_0 - \vec{a} \cdot \vec{r}$,

$$\begin{aligned} \psi_j^* f(x) &\leq C_{M_\lambda,\vec{r}} \sum_{k \geq j} 2^{(j-k)(M_\lambda+1)a_0} \int \frac{2^{j|\vec{a}|}|\psi_k * f(z)|}{\prod_{l=1}^{n}(1 + 2^{ja_l}|x_l - z_l|)^{r_l}}\, dz \\ &\leq C_N \sum_{k \geq j} 2^{(j-k)N} \int \frac{2^{k|\vec{a}|}|\psi_k * f(z)|^{\tau}}{\prod_{l=1}^{n}(1 + 2^{ka_l}|x_l - z_l|)^{r_l\tau}}\, dz \;\; \left(\psi_k^* f(x)\right)^{1-\tau}. \end{aligned} \tag{65}$$

Here N can be lowered in the exponent, so (65) holds for all $N \geq -\vec{a} \cdot \vec{r}$, with $N \mapsto C_{N,\vec{r}}$ piecewise constant; i.e., constant on intervals having the form $](k-1)a_0, ka_0] - \vec{a} \cdot \vec{r}$ for $k \in \mathbb{N}_0$. Obviously this yields (60) in case $\tau = 1$.

Step 2. To cover a given $\tau \in]0,1[$ we apply Lemma 2.8 with b_j as the last integral in (65): because of the inequality (65), the estimate (60) with $C_{N,\vec{r},\tau} = C_N$ follows for all $N > 0$ by the lemma if we can only verify the last assumption that, for some $N_0 > 0$,

$$d_j := \psi_j^* f(x) = O\left(2^{jN_0}\right). \tag{66}$$

In case $\omega \le \vec{r}$ for $\omega = \operatorname{ord}_{\mathcal{S}'} f$, this estimate follows for all $j \ge 0$ from standard calculations by applying (61) to the numerator in $\psi_j^* f(x)$.

In the remaining cases, where $\omega > r_l$ for some $l \in \{1,\dots,n\}$, we shall show a similar estimate unless (60) is trivial. First we choose $\vec{q}$ such that $\vec{q} \ge \max(r_1,\dots,r_n,\omega)$. Then (60) holds true for $\vec{q}$ and the right-hand side gets larger by replacing each q_l with r_l in the denominator. Hence we have for $N > 0$,

$$|\psi_j * f(y)|^\tau \le C_{N,\vec{q},\tau} \sum_{k\ge j} 2^{(j-k)N\tau} \int \frac{2^{k|\vec{a}|}|\psi_k * f(z)|^\tau}{\prod_{l=1}^n (1+2^{ka_l}|y_l - z_l|)^{r_l\tau}}\, dz. \tag{67}$$

Using monotonicity as in Step 1, the above is seen to imply, say for $N > \vec{a}\cdot\vec{r}$, $j \in \mathbb{N}_0$ that

$$\left(\psi_j^* f(x)\right)^\tau \le C_{N,\vec{q},\tau} \sum_{k\ge j} 2^{(j-k)(N-\vec{a}\cdot\vec{r})\tau} \int \frac{2^{k|\vec{a}|}|\psi_k * f(z)|^\tau}{\prod_{l=1}^n (1+2^{ka_l}|x_l - z_l|)^{r_l\tau}}\, dz. \tag{68}$$

(The constant depends on $\vec{q}$, i.e., on f.) We can assume the sum on the right-hand side is finite for some $j_1 \ge 0$, $N_1 > \vec{a}\cdot\vec{r}$, for else (60) is trivial. Then

$$\begin{aligned} &\sup_{m\ge j_1} 2^{(j_1-m)(N_1-\vec{a}\cdot\vec{r})}\psi_m^* f(x) \\ &\le C_{N_1,\vec{q},\tau}^{\frac{1}{\tau}} \Big(\sum_{k\ge j_1} 2^{(j_1-k)(N_1-\vec{a}\cdot\vec{r})\tau} \int \frac{2^{k|\vec{a}|}|\psi_k * f(z)|^\tau}{\prod_l (1+2^{ka_l}|x_l - z_l|)^{r_l\tau}}\, dz\Big)^{\frac{1}{\tau}} < \infty. \end{aligned} \tag{69}$$

This implies (66) at once for $j \ge j_1$ and $N_0 := N_1 - \vec{a}\cdot\vec{r}$, so now Lemma 2.8 yields (60) for $j \ge j_1$. When considering the smallest such j_1, the right-hand side of (60) is infinite for every $j < j_1$ (any N) so that (60) is trivial.

Step 3. For $\tau > 1$ we deduce (64) with $r_l + 1$ for all l and afterwards apply Hölder's inequality with dual exponents $\tau, \tau' > 1$ with respect to Lebesgue measure and the counting measure. Simple calculations then yield (60). □

Now we can briefly modify the arguments in [15] to obtain the next result.

Theorem 4.8. *Let* $\psi_0, \psi \in \mathcal{S}$ *satisfy the Tauberian conditions* (37), (38). *When* $0 < \vec{p} < \infty$, $0 < q \le \infty$, $-\infty < s < \infty$ *and the* $\psi_j^* f$ *are defined for* $\vec{r}$ *satisfying*

$$r_l \min(q, p_1, \dots, p_n) > 1, \quad l = 1,\dots,n, \tag{70}$$

then there exists a constant $c > 0$ *such that for* $f \in \mathcal{S}'$,

$$\|2^{sj}\psi_j^* f \,|L_{\vec{p}}(\ell_q)\| \le c\, \|2^{sj}\psi_j * f \,|L_{\vec{p}}(\ell_q)\|. \tag{71}$$

Proof. The proof relies on the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{\operatorname{meas}(B(0,r))} \int_{B(0,r)} |f(x+y)|\,dy. \tag{72}$$

When applied only in one variable x_l, we denote it by M_l; i.e., using the splitting $x = (x', x_l, x'')$ we have $M_l u(x_1, \dots, x_n) := (Mu(x', \cdot, x''))(x_l)$. By assumption on $\vec{r}$, we may pick τ such that $\max_{1\le l\le n} \frac{1}{r_l} < \tau < \min(q, p_1, \dots, p_n)$. This implies that $(1 + |z_l|)^{-r_l\tau} \in L_1(\mathbb{R})$, and since it is also radially decreasing, iterated application of the majorant property of the Hardy-Littlewood maximal function, described in e.g. [20, p. 57], yields a bound of the convolution on the right-hand side of (60), hence

$$\psi_j^* f(x) \le C_{N,\vec{r}}^{\frac{1}{\tau}} \Big(\sum_{k\ge j} 2^{(j-k)N\tau} M_n(\dots M_2(M_1|\psi_k * f|^\tau)\dots)(x)\Big)^{\frac{1}{\tau}}. \tag{73}$$

Here application of Lemma 2.7 gives

$$\|2^{js}\psi_j^* f\,|L_{\vec{p}}(\ell_q)\| \le C_{N,\vec{r}}\,\|2^{js\tau} M_n(\dots(M_1|\psi_j * f|^\tau)\dots)|L_{\frac{\vec{p}}{\tau}}(\ell_{\frac{q}{\tau}})\|^{\frac{1}{\tau}}, \tag{74}$$

whence (71) follows by n-fold application of the maximal inequality of Bagby [1] on the space $L_{\frac{\vec{p}}{\tau}}(\ell_{\frac{q}{\tau}})$, since $\tau < \min(q, p_1, \dots, p_n)$; cf. also [13, Section 3.4]. □

5. General quasi-norms and local means

First of all, Theorems 4.4, 4.8 give some very general characterisations of $F^{s,\vec{a}}_{\vec{p},q}$. In fact the next result shows that in Definition 2.1 the Littlewood-Paley partition of unity is not essential: the quasi-norm can be replaced by a more general one in which the summation to 1 or the compact supports, or both, are lost:

Theorem 5.1. *Let $s \in \mathbb{R}$, $0 < \vec{p} < \infty$, $0 < q \le \infty$ and let ψ_0, ψ in $\mathcal{S}(\mathbb{R}^n)$ be given such that the Tauberian conditions* (37), (38) *are fulfilled together with a moment condition of order M_ψ so that $s < (M_\psi + 1)\min(a_1, \dots, a_n)$, cf.* (39). *When $\psi^*_{j,\vec{a}} f$ is defined with $\vec{r} > \min(q, p_1, \dots, p_n)^{-1}$, cf.* (48), *then the following properties of $f \in \mathcal{S}'(\mathbb{R}^n)$ are equivalent:*

$$\begin{aligned} &\text{(i)} && f \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n),\\ &\text{(ii)} && \|\{2^{sj}\psi_j * f\}_{j=0}^\infty|L_{\vec{p}}(\ell_q)\| < \infty,\\ &\text{(iii)} && \|\{2^{sj}\psi^*_{j,\vec{a}} f\}_{j=0}^\infty|L_{\vec{p}}(\ell_q)\| < \infty. \end{aligned}$$

Moreover, the $F^{s,\vec{a}}_{\vec{p},q}$-quasi-norm is equivalent to those in (ii) *and* (iii).

Proof. Since $\psi_j * f(x) \le \psi^*_{j,\vec{a}} f(x)$ is trivial, clearly (iii) $\Longrightarrow$ (ii); the converse holds by Theorem 4.8. To obtain (iii) $\Longrightarrow$ (i), one may in the Lizorkin-Triebel norm estimate the convolutions by $(\mathcal{F}^{-1}\Phi)^*_{j,\vec{a}} f$, and the resulting norm is estimated by the one in (iii) by means of Theorem 4.4 (with a trivial index set like $\Theta = \{1\}$). That (i) $\Longrightarrow$ (iii) follows by using Theorem 4.4 to estimate from above by the quasi-norm defined from $(\mathcal{F}^{-1}\Phi)^*_{j,\vec{a}} f$, with all r_l so large that Theorem 4.8 gives control by the $\mathcal{F}^{-1}\Phi_j * f$. □

From the above it is e.g. obvious that the space $F^{s,\vec{a}}_{\vec{p},q}$ does not depend on the Littlewood-Paley partition of unity in (12), and that different choices yield equivalent quasi-norms.

As an immediate corollary of Theorem 5.1, there is the following characterisation of $F^{s,\vec{a}}_{\vec{p},q}$ in terms of integration kernels. It has been well known in the isotropic case:

Theorem 5.2. *Let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ such that $\int k_0(x)\,dx \neq 0 \neq \int k^0(x)\,dx$ and set $k(x) = \Delta^N k^0(x)$ for some $N \in \mathbb{N}$. When $0 < \vec{p} < \infty$, $0 < q \le \infty$, and $s < 2N \min(a_1, \dots, a_n)$, then a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ if and only if*

$$\| f \,|F^{s,\vec{a}}_{\vec{p},q}\|^* := \| k_0 * f \,|L_{\vec{p}}\| + \|\{2^{sj} k_j * f\}_{j=1}^{\infty} \,|L_{\vec{p}}(\ell_q)\| < \infty. \tag{75}$$

Furthermore, $\| f \,|F^{s,\vec{a}}_{\vec{p},q}\|^*$ is an equivalent quasi-norm on $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$.

In (75), the functions k_j, $j \ge 1$ are given by $k_j(x) = 2^{j|\vec{a}|} k(2^{j\vec{a}} x)$; cf. (40).

Remark 5.3. Obviously, we may choose k_0, k^0 such that both functions have compact support. In this case Triebel termed k_0 and k kernels of *local means*, and in [21, 2.4.6] he proved that (75) is an equivalent quasi-norm on the f belonging a priori to the isotropic space $F^s_{p,q}$. This was carried over to anisotropic, but unmixed spaces by Farkas [5]. Extension to function spaces with generalised smoothness has been done by Farkas and Leopold [6]; and to spaces of dominating mixed smoothness by Vybiral [23] and Hansen [8].

Remark 5.4. Bui, Paluszinki and Taibleson [4] obtained a characterisation, i.e., equivalence for all $f \in \mathcal{S}'$, in the isotropic (but weighted) case, which Rychkov [15] simplified to the present discrete Littlewood-Paley decompositions. Our Theorem 5.2 generalises this in two ways, i.e., we prove a characterisation of $F^{s,\vec{a}}_{\vec{p},q}$ that has anisotropies both in terms of $\vec{a}$ and mixed norms.

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