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Implicit Difference Methods for Differential Functional Parabolic Equations with Dirichlet's Condition

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Abstract. Classical solutions of nonlinear second-order partial differential functional equations of parabolic type with Dirichlet's condition are approximated in the paper by solutions of associated implicit difference functional equations. The functional dependence is of the Volterra type. Nonlinear estimates of the generalized Perron type for given functions are assumed. The convergence and stability results are proved with the use of discrete functional inequalities and the comparison technique. In particular, these theorems cover quasi-linear equations. However, such equations are also treated separately. The known results on similar difference methods can be obtained as particular cases of our simple result.

Keywords. Parabolic differential functional equations, difference methods, stability and convergence, nonlinear estimates of the generalized Perron type

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1. Introduction

Let functions $f : \Delta \to \mathbf{R}$ and $\varphi : E_0 \cup \partial_0 E \to \mathbf{R}$ be given (the relevant sets are defined in Section 2.1). Consider a nonlinear second-order partial differential functional equation of parabolic type of the form

$$\partial_t z\,(t,x) = f\,(t,x,z,\partial_x z\,(t,x)\,,\partial_{xx} z\,(t,x)) \tag{1}$$

with the *initial condition* and the *boundary condition of the Dirichlet type*

$$z\,(t,x) = \varphi\,(t,x) \quad \text{on } E_0 \cup \partial_0 E, \tag{2}$$

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where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$, $\partial_{xx} z = \left[\partial_{x_i x_j} z\right]_{i,j=1}^{n}$. The aim of this paper is to give a consistent, convergent and stable implicit finite difference method for finding an approximate solution of problem (1), (2). The equation may be nonlinear with respect to second derivatives. Such an equation is called strongly nonlinear. The functional dependence is of the Volterra type (e.g., delays or Volterra type integrals).

Partial differential equations of parabolic type give mathematical models of nonstationary processes of heat exchange or mass transport. Some complicated kinds of these phenomena involve equations with a functional term. Differential difference equations (e.g., with time or spatial delays) describe fast heat changes in nuclear reactors, while differential integral equations are used for integral heat sources in an anisotropic medium. Both can be connected with our equation. Such equations also describe nuclear reactor dynamics.

We prove a theorem on error estimates between an exact and approximate solutions of implicit discrete functional equations of the Volterra type. The error is estimated by a solution of the initial comparison problem for a recurrent discrete inequality. We also give a theorem on the existence of the exact solution. We apply this general idea in the investigation of the convergence and stability of implicit difference functional schemes generated by problem (1), (2). A similar technique for explicit problems was studied by Z. Kamont, H. Leszczyński [6,8] and by L. Sapa, K. Kropielnicka [13,26]. Moreover, such a technique for implicit quasi-linear problems was considered by Z. Kamont [10].

Let $a_{ij} : \Delta^A \to \mathbf{R}$ and $F : \Delta^F \to \mathbf{R}$, $i, j = 1, \ldots, n$, be given functions (see Section 2.1). If we assume that each a_{ij} is non-positive or non-negative in Δ^A, then these results in particular cover a quasi-linear differential functional equation of the form

$$\partial_t z(t,x) = \sum_{i,j=1}^{n} a_{ij}(t,x,z)\, \partial_{x_i x_j} z(t,x) + F(t,x,z,\partial_x z(t,x)). \tag{3}$$

To omit this condition, another scheme is also studied.

We assume the existence of a classical solution of problems (1), (2) and (3), (2). Theorems on the existence and uniqueness of such solutions for some special parabolic differential functional equations with different boundary conditions can be found in [3–5, 21, 31] and the references therein.

Our results can be extended to weakly coupled systems.

Explicit or implicit difference methods for general strongly nonlinear parabolic differential functional equations with Dirichlet's or a nonlinear boundary condition have been considered by Z. Kamont, K. Kropielnicka, H. Leszczyński, M. Malec, C. Mączka, W. Voigt, M. Rosati, M. Netka [6, 8, 9, 15–17, 20] and others. In those papers, the Lipschitz or Perron conditions with respect to z are assumed. In our paper, we generalize the Perron estimate, multiplying a function σ by some nondecreasing function ρ (see assumption (F_4) in Section 4). The similar generalized Perron type estimate was introduced by L. Sapa in [26] and was considered in [13] also, where the explicit methods were studied. If f is differentiable with respect to z, then our generalization admits $\partial_z f$ unbounded with respect to p, q. This considerably extends the class of problems which are solvable with the method described. Under the assumptions adopted, our nonlinear equation includes as special cases the quasi-linear equation (3) and a strongly nonlinear equation with a quasi-linear term (see Examples 6.1, 6.2). Neither of these cases appears in the cited papers. This result is new, even for equations without a functional term or another type (see [11, 14, 18, 19]). Moreover, unlike [9], we do not assume the strong monotonicity condition with respect to z. Add that, unlike [26], the Courant-Friedrichs-Levy condition on the steps of a mesh is omitted (see Remark 5.6). These are the main results of our paper.

An implicit finite difference method for quasi-linear parabolic differential functional equations similar to (3) with Dirichlet's condition has been considered by K. Kropielnicka [12].

The results concerning numerical methods, differential functional and difference functional inequalities or the uniqueness theory, appearing in the papers of P. Besala and G. Paszek [1,2], C. V. Pao [22–24], R. Redheffer and W. Walter [25, 30], J. Szarski [27–29] and numerous others, do not apply to nonlinear equations and quasi-linear equations with such a general functional dependence as in our paper.

The paper is organized in the following way. In Section 2 notation is introduced and some definitions are formulated. Section 3 deals with the theorems on the existence and uniqueness of the exact solution and on error estimates of approximate solutions for discrete functional equations of the Volterra type. The assumptions for the differential functional problem (1), (2), the definition of the implicit finite difference functional scheme and the assumptions on the steps of a mesh are given in Section 4. In Section 5 the convergence of the implicit difference methods for (1), (2) and (3), (2) is proved. Finally, in Section 6 the numerical examples are presented.

2. Notation and definitions

2.1. Sets and function spaces. Let $T > 0$, $X = (X_1, \ldots, X_n)$, $\tau_0 \geq 0$, $\tau = (\tau_1, \ldots, \tau_n)$, where $X_i > 0$, $\tau_i \geq 0$ for $i = 1, \ldots, n$, be given. Define

$$\begin{aligned} E &= [0,T] \times (-X, X) \subset \mathbf{R}^{1+n}, \\ E_0 &= [-\tau_0, 0] \times [-X-\tau, X+\tau] \subset \mathbf{R}^{1+n}, \\ \partial_0 E &= [0,T] \times ([-X-\tau, X+\tau] \setminus (-X, X)) \subset \mathbf{R}^{1+n}. \end{aligned} \tag{4}$$

Let, moreover,

$$\begin{aligned} \Omega &= E \cup E_0 \cup \partial_0 E, \\ \Omega_t &= \Omega \cap \left([-\tau_0, t] \times \mathbf{R}^n\right), \quad t \in [0, T]. \end{aligned} \tag{5}$$

Denote by $M_{n\times n}$ the class of all $n \times n$ symmetric real matrices. Define the sets

$$\begin{aligned} \Delta &= E \times C(\Omega, \mathbf{R}) \times \mathbf{R}^n \times M_{n\times n}, \\ \Delta^A &= E \times C(\Omega, \mathbf{R}), \\ \Delta^F &= E \times C(\Omega, \mathbf{R}) \times \mathbf{R}^n. \end{aligned} \tag{6}$$

The *maximum norms* in $\mathbf{R}^n$ and $M_{n\times n}$ are denoted by $\|\cdot\|$, while in the *space of continuous functions* $C(\Omega, \mathbf{R})$ by $\|\cdot\|_\Omega$.

For a fixed $t \in [0, T]$,

$$\|z\|_{\Omega_t} = \max\left\{\left|z\left(\widetilde{t}, x\right)\right| : \left(\widetilde{t}, x\right) \in \Omega_t\right\} \tag{7}$$

is a seminorm in $C(\Omega, \mathbf{R})$, where $z \in C(\Omega, \mathbf{R})$.

For a fixed $t \in [0, T]$, the symbol $\| \ \|_{C(\Omega,\mathbf{R})_t}$ stands for a semi-norm in the space of linear and continuous functionals $L(C(\Omega, \mathbf{R}), \mathbf{R})$, generated by the semi-norm $\| \ \|_{\Omega_t}$ in the space $C(\Omega, \mathbf{R})$, i.e.

$$\|A\|_{C(\Omega,\mathbf{R})_t} = \inf\{\mu \geq 0 : \ \forall h \in C(\Omega, \mathbf{R}) \ |Ah| \leq \mu \|h\|_{\Omega_t}\}, \tag{8}$$

where $A \in L(C(\Omega, \mathbf{R}), \mathbf{R})$.

2.2. Discretization, difference and interpolating operators. We use vectorial inequalities to mean that the same inequalities hold between the corresponding components. We write $x \diamond y = (x_1 y_1, \ldots, x_n y_n)$ for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$. Define a mesh on the set Ω in the following way. Let $(h_0, h') = h$, $h' = (h_1, \ldots, h_n)$, stand for the steps of the mesh. Denote by H the set of all h such that there exist $N_0 \in \mathbf{Z}$ and $N = (N_1, \ldots, N_n) \in \mathbf{N}^n$ with the properties: $N_0 h_0 = \tau_0$, $N \diamond h' = X + \tau$. Obviously, $H \neq \emptyset$ and there are $K_0 \in \mathbf{N}$ and $K = (K_1, \ldots, K_n) \in \mathbf{Z}^n$ such that $K_0 h_0 \leq T < (K_0 + 1) h_0$, $K \diamond h' < X \leq (K + 1) \diamond h'$. For $h \in H$ and $(\mu, m) \in \mathbf{Z}^{1+n}$, $m = (m_1, \ldots, m_n)$, we define nodal points $\left(t^{(\mu)}, x^{(m)}\right)$, $x^{(m)} = \left(x_1^{(m_1)}, \ldots, x_n^{(m_n)}\right)$, in the following way

$$t^{(\mu)} = \mu h_0, \quad x^{(m)} = m \diamond h'.$$

For $h \in H$, we put

$$R_h^{1+n} = \left\{\left(t^{(\mu)}, x^{(m)}\right) : \ (\mu, m) \in \mathbf{Z}^{1+n}\right\}. \tag{9}$$

Define the discrete sets

$$\begin{aligned} E_h &= E \cap R_h^{1+n}, \\ E_{0.h} &= E_0 \cap R_h^{1+n}, \\ \partial_0 E_h &= \partial_0 E \cap R_h^{1+n}, \\ \Omega_h &= E_h \cup E_{0.h} \cup \partial_0 E_h, \\ \Omega_{h.\mu} &= \Omega_h \cap \left(\left[-\tau_0, t^{(\mu)} \right] \times \mathbf{R}^n \right), \quad \mu = 0, \ldots, K_0. \end{aligned} \tag{10}$$

Let, moreover,

$$E_h^+ = \left\{ \left(t^{(\mu)}, x^{(m)} \right) \in E_h : \ 0 \le \mu \le K_0 - 1 \right\}, \tag{11}$$

$$I_h = \left\{ t^{(\mu)} : \ 0 \le \mu \le K_0 \right\}, \quad I_h^+ = \left\{ t^{(\mu)} : \ 0 \le \mu \le K_0 - 1 \right\}. \tag{12}$$

For a *mesh function* $z : \Omega_h \supset A_h \to \mathbf{R}$ and a point $\left(t^{(\mu)}, x^{(m)} \right) \in A_h$, we put $z^{(\mu,m)} = z\left(t^{(\mu)}, x^{(m)} \right)$, $|z|^{(\mu,m)} = |z^{(\mu,m)}|$. We denote the space of all such functions by $\mathfrak{F}(A_h, \mathbf{R})$ and call it the *space of mesh functions.* In $\mathfrak{F}(A_h, \mathbf{R})$, we introduce the *maximum norm*

$$\|z\|_{A_h} = \max \left\{ \left| z^{(\mu,m)} \right| : \ \left(t^{(\mu)}, x^{(m)} \right) \in A_h \right\}, \tag{13}$$

where $z \in \mathfrak{F}(A_h, \mathbf{R})$.

For a fixed $\mu \in \{0, 1, \ldots, K_0\}$,

$$\|z\|_{\Omega_{h.\mu}} = \max \left\{ \left| z^{(\tilde{\mu},m)} \right| : \ \left(t^{(\tilde{\mu})}, x^{(m)} \right) \in \Omega_{h.\mu} \right\} \tag{14}$$

is a seminorm in the space $\mathfrak{F}(\Omega_h, \mathbf{R})$, where $z \in \mathfrak{F}(\Omega_h, \mathbf{R})$. For a function $z : \ I_h \supset A_h \to \mathbf{R}_+$, we put $z^{(\mu)} = z\left(t^{(\mu)} \right)$, $t^{(\mu)} \in A_h$, where $\mathbf{R}_+ = [0, +\infty)$.

Put $\chi = 1 + 2n^2$ and

$$\Lambda = \{ \lambda = (\lambda_1, \ldots, \lambda_n) : \ \lambda_i \in \{-1, 0, 1\}, \ i = 1, \ldots, n, \ |\lambda| \le 2 \}, \tag{15}$$

where $|\lambda| = |\lambda_1| + \cdots + |\lambda_n|$. Note that χ is the number of elements of Λ. Let $\psi : \Lambda \to \{1, \ldots, \chi\}$ be a function such that $\psi(\lambda) \ne \psi\left(\overline{\lambda} \right)$ for $\lambda \ne \overline{\lambda}$. Put

$$i_0 = \psi(0). \tag{16}$$

We assume that $\prec$ is an order in Λ defined in the following way: $\lambda \prec \overline{\lambda}$ if $\psi(\lambda) \le \psi\left(\overline{\lambda} \right)$. Elements of the space $\mathbf{R}^\chi$ we denote by $\xi = (\xi_1, \ldots, \xi_\chi)$. For a function $z \in \mathfrak{F}(\Omega_h, \mathbf{R})$ and a point $\left(t^{(\mu)}, x^{(m)} \right) \in E_h$ we define the vector $z_{<\mu,m>} = (z_1, \ldots, z_\chi) \in \mathbf{R}^\chi$, $z_i = z^{\left(\mu, m + \psi^{-1}(i) \right)}$, $i = 1, \ldots, \chi$, where ψ^{-1} is the inverse function of ψ.

Write $\Gamma = \{(i,j) : \ 1 \le i, j \le n, \ i \ne j\}$ and suppose that $\Gamma_+, \Gamma_- \subset \Gamma$ are such that $\Gamma_+ \cup \Gamma_- = \Gamma$, $\Gamma_+ \cap \Gamma_- = \emptyset$ (in particular, it may happen that $\Gamma_+ = \emptyset$

or $\Gamma_- = \emptyset$). We assume that $(i,j) \in \Gamma_+$ when $(j,i) \in \Gamma_+$ and $(i,j) \in \Gamma_-$ when $(j,i) \in \Gamma_-$.

Let $z \in \mathfrak{F}(\Omega_h, \mathbf{R})$ and $\left(t^{(\mu)}, x^{(m)}\right) \in E_h$. Set

$$\delta_i^+ z^{(\mu,m)} = \frac{1}{h_i}\left[z^{(\mu,m+e_i)} - z^{(\mu,m)}\right], \quad \delta_i^- z^{(\mu,m)} = \frac{1}{h_i}\left[z^{(\mu,m)} - z^{(\mu,m-e_i)}\right], \tag{17}$$

where $e_i = (0,\ldots,0,1,0,\ldots,0)$ with 1 in the ith entry, $i = 1,\ldots,n$. We apply the *difference quotients* δ_0, $\delta = (\delta_1,\ldots,\delta_n)$, $\delta^{(2)} = [\delta_{ij}]_{i,j=1}^n$ given by

$$\begin{aligned}
\delta_0 z^{(\mu,m)} &= \frac{1}{h_0}\left[z^{(\mu+1,m)} - z^{(\mu,m)}\right], \\
\delta_i z^{(\mu,m)} &= \frac{1}{2}\left[\delta_i^+ z^{(\mu,m)} + \delta_i^- z^{(\mu,m)}\right] && \text{for } i = 1,\ldots,n, \\
\delta_{ii} z^{(\mu,m)} &= \delta_i^+ \delta_i^- z^{(\mu,m)} && \text{for } i = 1,\ldots,n, \\
\delta_{ij} z^{(\mu,m)} &= \frac{1}{2}\left[\delta_i^+ \delta_j^- z^{(\mu,m)} + \delta_i^- \delta_j^+ z^{(\mu,m)}\right] && \text{for } (i,j) \in \Gamma_-, \\
\delta_{ij} z^{(\mu,m)} &= \frac{1}{2}\left[\delta_i^+ \delta_j^+ z^{(\mu,m)} + \delta_i^- \delta_j^- z^{(\mu,m)}\right] && \text{for } (i,j) \in \Gamma_+.
\end{aligned} \tag{18}$$

We use these operators to approximate derivatives in equations (1) and (3).

We say that an operator $G_h : \mathfrak{F}(\Omega_h, \mathbf{R}) \to C(\Omega, \mathbf{R})$ is an *interpolating operator* if it has the properties:

(i) for all $z \in C^{1,2}(\Omega, \mathbf{R})$

$$\lim_{h\to 0} \|G_h[Z] - z\|_\Omega = 0,$$

where $Z := z|_{\Omega_h}$ is the restriction of z to Ω_h,

(ii) there is $D > 0$ such that for all $z, \overline{z} \in \mathfrak{F}(\Omega_h, \mathbf{R})$

$$\|G_h[z] - G_h[\overline{z}]\|_{\Omega_{t^{(\mu)}}} \le D\,\|z - \overline{z}\|_{\Omega_{h.\mu}}, \quad \mu = 0,\ldots,K_0.$$

We apply these operators to approximate the functional term in equations (1) and (3). An example of G_h is the well-known linear operator T_h introduced in [7]. For T_h we may put $D = 1$.

3. Discrete functional equations and inequalities

We consider an implicit discrete functional equation with the initial boundary condition. Next, we give two theorems respectively on the existence and uniqueness of a solution of this problem and on the estimate of the difference between the exact and approximate solutions. They will be applied in the proofs of the theorems on a convergence of the difference methods in Section 5.

Suppose that a functional $\mathcal{F}_h : E_h^+ \times \mathfrak{F}(\Omega_h, \mathbf{R}) \times \mathbf{R}^\chi \to \mathbf{R}$ is given. For $\left(t^{(\mu)}, x^{(m)}, z, \xi\right) \in E_h^+ \times \mathfrak{F}(\Omega_h, \mathbf{R}) \times \mathbf{R}^\chi$, we write $\mathcal{F}_h[z,\xi]^{(\mu,m)} = \mathcal{F}_h\left(t^{(\mu)}, x^{(m)}, z, \xi\right)$. Given $\varphi_h \in \mathfrak{F}(E_{0.h} \cup \partial_0 E_h, \mathbf{R})$, we consider the discrete functional equation

$$z^{(\mu+1,m)} = \mathcal{F}_h\left[z, z_{<\mu+1,m>}\right]^{(\mu,m)} \tag{19}$$

with the initial boundary condition

$$z^{(\mu,m)} = \varphi_h^{(\mu,m)} \quad \text{on } E_{0.h} \cup \partial_0 E_h. \tag{20}$$

Note that the numbers $z^{\left(\mu+1,m+\psi^{-1}(i)\right)}$, $i = 1, \ldots, \chi$, appear in $z_{<\mu+1,m>}$ so (19), (20) is an implicit problem.

We say that the functional $\mathcal{F}_h$ satisfies the Volterra condition when for all $\left(t^{(\mu)}, x^{(m)}\right) \in E_h^+$, $z, \overline{z} \in \mathfrak{F}(\Omega_h, \mathbf{R})$ and $\xi \in \mathbf{R}^\chi$, if $z|_{\Omega_{h.\mu}} = \overline{z}|_{\Omega_{h.\mu}}$, then $\mathcal{F}_h[z,\xi]^{(\mu,m)} = \mathcal{F}_h[\overline{z},\xi]^{(\mu,m)}$. Observe that the Volterra condition states that the value of $\mathcal{F}_h$ at $\left(t^{(\mu)}, x^{(m)}, z, \xi\right)$ depends on $\left(t^{(\mu)}, x^{(m)}, \xi\right)$ and the restriction of the function z to the set $\Omega_{h.\mu}$ only. However, this well-known condition does not imply the existence of a solution for (19), (20) so we give a suitable theorem.

The following assumptions on $\mathcal{F}_h$ will be needed.

Assumption $H[\mathcal{F}_h]$**.**

(H_1) $\mathcal{F}_h$, $h \in H$, is of the Volterra type.

(H_2) There exist the partial derivatives $\partial_{\xi_i}\mathcal{F}_h$ on $E_h^+ \times \mathfrak{F}(\Omega_h, \mathbf{R}) \times \mathbf{R}^\chi$, $i = 1, \ldots, \chi$, and $\partial_{\xi_{i_0}}\mathcal{F}_h[z,\cdot]^{(\mu,m)}$ is bounded for each $\left(t^{(\mu)}, x^{(m)}, z\right) \in E_h^+ \times \mathfrak{F}(\Omega_h, \mathbf{R})$, where i_0 is defined in (16).

(H_3) The conditions

$$\partial_{\xi_i}\mathcal{F}_h[z,\xi]^{(\mu,m)} \geq 0, \quad i = 1, \ldots, \chi,\ i \neq i_0, \tag{21}$$

$$\sum_{i=1}^{\chi} \partial_{\xi_i}\mathcal{F}_h[z,\xi]^{(\mu,m)} = 0 \tag{22}$$

are satisfied at each $\left(t^{(\mu)}, x^{(m)}, z, \xi\right) \in E_h^+ \times \mathfrak{F}(\Omega_h, \mathbf{R}) \times \mathbf{R}^\chi$.

Theorem 3.1. *If assumption $H[\mathcal{F}_h]$ is satisfied, then there exists exactly one solution $v \in \mathfrak{F}(\Omega_h, \mathbf{R})$ of problem* (19), (20).

Proof. We use induction on μ and the Banach fixed-point theorem. By (20), the vectors $v^{(\mu,\cdot)}$, $\mu = -N_0, \ldots, 0$, are known. Suppose that $0 \leq \mu \leq K_0 - 1$ is fixed and that the solution v of problem (19), (20) is given on $\Omega_{h.\mu}$. We prove that the vector $v^{(\mu+1,\cdot)}$ exists and that it is unique. Define $\widetilde{v} \in \mathfrak{F}(\Omega_h, \mathbf{R})$ as follows: $\widetilde{v}^{(\widetilde{\mu},m)} = v^{(\widetilde{\mu},m)}$ for $\left(t^{(\widetilde{\mu})}, x^{(m)}\right) \in \Omega_{h.\mu}$, $\widetilde{v}^{(\widetilde{\mu},m)} = 0$ for $\left(t^{(\widetilde{\mu})}, x^{(m)}\right) \in \Omega_h \backslash \Omega_{h.\mu}$. It is sufficient to show that there exists exactly one solution of the system of equations

$$z^{(\mu+1,m)} = \mathcal{F}_h\left[\widetilde{v}, z_{<\mu+1,m>}\right]^{(\mu,m)}, \quad -K \leq m \leq K, \tag{23}$$

with the boundary condition

$$z^{(\mu+1,m)} = \varphi_h^{(\mu+1,m)} \quad \text{on } \partial_0 E_h. \tag{24}$$

From (H_2) there is $Q_h > 0$ such that $Q_h \geq -\partial_{\xi_{i_0}} \mathcal{F}_h [\widetilde{v}, \xi]^{(\mu,m)}$, $\xi \in \mathbf{R}^\chi$, $-K \leq m \leq K$. It is clear that system (23) is equivalent to the following one

$$z^{(\mu+1,m)} = \frac{1}{Q_h+1}\left[Q_h z^{(\mu+1,m)} + \mathcal{F}_h \left[\widetilde{v}, z_{<\mu+1,m>}\right]^{(\mu,m)}\right], \tag{25}$$

$-K \leq m \leq K$. Define

$$S_h = \left\{x^{(m)} :\ x^{(m)} \in [-X-\tau, X+\tau]\right\},$$
$$\partial_0 S_h = \left\{x^{(m)} :\ x^{(m)} \in [-X-\tau, X+\tau] \setminus (-X, X)\right\}.$$

We consider the space of mesh functions $\mathfrak{F}(S_h, \mathbf{R})$. For $\zeta \in \mathfrak{F}(S_h, \mathbf{R})$, we write $\zeta^{(m)} = \zeta\left(x^{(m)}\right)$ and for a point $x^{(m)} \in (-X, X)$, we put $\zeta_{<m>} = (\zeta_1, \ldots, \zeta_\chi) \in \mathbf{R}^\chi$, $\zeta_i = \zeta^{\left(m+\psi^{-1}(i)\right)}$, $i = 1, \ldots, \chi$. The norm in the space $\mathfrak{F}(S_h, \mathbf{R})$ is defined as $\|\zeta\|_* = \max\left\{\left|\zeta^{(m)}\right| :\ x^{(m)} \in S_h\right\}$. Consider the complete metric space

$$X_h = \left\{\zeta \in \mathfrak{F}(S_h, \mathbf{R}) :\ \zeta^{(m)} = \varphi_h^{(\mu+1,m)},\ x^{(m)} \in \partial_0 S_h\right\}$$

with a metric generated by the norm $\| \ \|_*$. We apply the operator $W_h : X_h \to X_h$ defined by

$$W_h[\zeta]^{(m)} = \frac{1}{Q_h+1}\left[Q_h \zeta^{(m)} + \mathcal{F}_h \left[\widetilde{v}, \zeta_{<m>}\right]^{(\mu,m)}\right] \quad \text{for } x^{(m)} \in S_h \setminus \partial_0 S_h \tag{26}$$

and

$$W_h[\zeta]^{(m)} = \varphi_h^{(\mu+1,m)} \quad \text{for } x^{(m)} \in \partial_0 S_h. \tag{27}$$

We prove that

$$\left\|W_h[\zeta] - W_h\left[\overline{\zeta}\right]\right\|_* \leq \frac{Q_h}{Q_h+1}\left\|\zeta - \overline{\zeta}\right\|_* \quad \text{on } X_h. \tag{28}$$

It follows from (26) and the mean value theorem that

$$\begin{aligned}
&W_h[\zeta]^{(m)} - W_h\left[\overline{\zeta}\right]^{(m)} \\
&= \frac{1}{Q_h+1}\left[Q_h\left(\zeta-\overline{\zeta}\right)^{(m)} + \sum_{i=1}^{\chi} \partial_{\xi_i}\mathcal{F}_h\left[\widetilde{v}, P^{(m)}\right]^{(\mu,m)}\left(\zeta-\overline{\zeta}\right)^{\left(m+\psi^{-1}(i)\right)}\right] \\
&= \frac{1}{Q_h+1}\Bigg[\left(Q_h + \partial_{\xi_{i_0}}\mathcal{F}_h\left[\widetilde{v}, P^{(m)}\right]^{(\mu,m)}\right)\left(\zeta-\overline{\zeta}\right)^{(m)} \\
&\quad + \sum_{i\neq i_0, i=1}^{\chi} \partial_{\xi_i}\mathcal{F}_h\left[\widetilde{v}, P^{(m)}\right]^{(\mu,m)}\left(\zeta-\overline{\zeta}\right)^{\left(m+\psi^{-1}(i)\right)}\Bigg]
\end{aligned} \tag{29}$$

for $x^{(m)} \in S_h \setminus \partial_0 S_h$, where $P^{(m)} \in \mathbf{R}^{\chi}$ are intermediate points. The above relation and (H_3) give the estimate

$$\left| W_h \left[\zeta\right]^{(m)} - W_h \left[\overline{\zeta}\right]^{(m)} \right| \leq \frac{Q_h}{Q_h + 1} \left\| \zeta - \overline{\zeta} \right\|_* \quad \text{for } x^{(m)} \in S_h \setminus \partial_0 S_h. \tag{30}$$

According to (27) we have $W_h \left[\zeta\right]^{(m)} - W_h \left[\overline{\zeta}\right]^{(m)} = 0$ for $x^{(m)} \in \partial_0 S_h$. This completes the proof of (28). It follows from the Banach fixed-point theorem that the operator W_h has exactly one fixed point $\zeta^* \in X_h$ and consequently, $v^{(\mu+1,\cdot)} := \zeta^*$ is the unique solution of (23), (24). Hence the proof is complete by induction. □

Let $Y_h \subset \mathfrak{F}(\Omega_h, \mathbf{R})$ be a fixed subset. Suppose that a function $w \in Y_h$, a function $\overline{\gamma} : I_h^+ \to \mathbf{R}_+$ and $\overline{\gamma}_0 \in \mathbf{R}_+$ satisfy the conditions

$$\left| z^{(\mu+1,m)} - \mathcal{F}_h \left[z, z_{<\mu+1,m>}\right]^{(\mu,m)} \right| \leq \overline{\gamma}^{(\mu)} \quad \text{on } E_h^+, \tag{31}$$

$$\left| z^{(\mu,m)} - \varphi_h^{(\mu,m)} \right| \leq \overline{\gamma}_0 \quad \text{on } E_{0.h} \cup \partial_0 E_h. \tag{32}$$

The function w satisfying the above relations is considered an *approximate solution of* (19), (20). We give a theorem on the estimate of the difference between the exact and approximate solutions of (19), (20).

Theorem 3.2. *Suppose that Assumption* $H[\mathcal{F}_h]$ *is satisfied and*

(i) $\sigma_h : I_h^+ \times \mathbf{R}_+ \to \mathbf{R}_+$ *is nondecreasing with respect to the second variable and*

$$\left| \mathcal{F}_h \left[z, \overline{z}_{<\mu+1,m>}\right]^{(\mu,m)} - \mathcal{F}_h \left[\overline{z}, \overline{z}_{<\mu+1,m>}\right]^{(\mu,m)} \right| \leq \sigma_h \left(t^{(\mu)}, \| z - \overline{z} \|_{\Omega_{h.\mu}} \right) \tag{33}$$

for $\left(t^{(\mu)}, x^{(m)}\right) \in E_h^+$, $z \in \mathfrak{F}(\Omega_h, \mathbf{R})$, $\overline{z} \in Y_h$,

(ii) $w \in Y_h$ *and there are* $\overline{\gamma} : I_h^+ \to \mathbf{R}_+$, $\overline{\gamma}_0 \in \mathbf{R}_+$ *such that relations* (31), (32) *hold,*

(iii) $\beta : I_h \to \mathbf{R}_+$ *is nondecreasing and satisfies the recurrent inequality*

$$\beta^{(\mu+1)} \geq \sigma_h \left(t^{(\mu)}, \beta^{(\mu)} \right) + \overline{\gamma}^{(\mu)}, \quad \mu = 0, \ldots, K_0 - 1, \tag{34}$$

and $\beta^{(0)} \geq \overline{\gamma}_0$.

Then

$$\| w - v \|_{\Omega_{h.\mu}} \leq \beta^{(\mu)}, \quad \mu = 0, \ldots, K_0, \tag{35}$$

where $v \in \mathfrak{F}(\Omega_h, \mathbf{R})$ *is the unique solution of problem* (19), (20).

Proof. The existence of the unique solution $v \in \mathfrak{F}(\Omega_h, \mathbf{R})$ of (19), (20) follows immediately from Theorem 3.1. We prove assertion (35) by induction on μ.

It follows from (20), (32) and assumptions (ii), (iii) that inequality (35) is satisfied for $\mu = 0$.

Assuming (35) to hold for a fixed μ, $0 \le \mu \le K_0 - 1$, we prove it for $\mu + 1$. Define $\widetilde{m} \in \mathbf{Z}^n$ as follows

$$|w - v|^{(\mu+1,\widetilde{m})} = \max\left\{|w - v|^{(\mu+1,m)} : \left(t^{(\mu+1)}, x^{(m)}\right) \in \Omega_h\right\}. \tag{36}$$

We show that

$$|w - v|^{(\mu+1,\widetilde{m})} \le \beta^{(\mu+1)}. \tag{37}$$

If $\left(t^{(\mu+1)}, x^{(\widetilde{m})}\right) \in \partial_0 E_h$, then (37) follows from assumption (iii). Consider the case when $\left(t^{(\mu+1)}, x^{(\widetilde{m})}\right) \in E_h$. Equation (19) gives

$$\begin{aligned}
&(w - v)^{(\mu+1,\widetilde{m})} \\
&= \mathcal{F}_h\left[w, w_{<\mu+1,\widetilde{m}>}\right]^{(\mu,\widetilde{m})} - \mathcal{F}_h\left[v, w_{<\mu+1,\widetilde{m}>}\right]^{(\mu,\widetilde{m})} + w^{(\mu+1,\widetilde{m})} \\
&\quad - \mathcal{F}_h\left[w, w_{<\mu+1,\widetilde{m}>}\right]^{(\mu,\widetilde{m})} + \mathcal{F}_h\left[v, w_{<\mu+1,\widetilde{m}>}\right]^{(\mu,\widetilde{m})} - \mathcal{F}_h\left[v, v_{<\mu+1,\widetilde{m}>}\right]^{(\mu,\widetilde{m})}.
\end{aligned} \tag{38}$$

From (38), assumption (H_2) and the mean value theorem, we obtain

$$\begin{aligned}
&(w - v)^{(\mu+1,\widetilde{m})}\left[1 - \partial_{\xi_{i_0}}\mathcal{F}_h\left[v, P^{(\mu+1,\widetilde{m})}\right]^{(\mu,\widetilde{m})}\right] \\
&= \mathcal{F}_h\left[w, w_{<\mu+1,\widetilde{m}>}\right]^{(\mu,\widetilde{m})} - \mathcal{F}_h\left[v, w_{<\mu+1,\widetilde{m}>}\right]^{(\mu,\widetilde{m})} - \mathcal{F}_h\left[w, w_{<\mu+1,\widetilde{m}>}\right]^{(\mu,\widetilde{m})} \\
&\quad + w^{(\mu+1,\widetilde{m})} + \sum_{i \ne i_0, i=1}^{\chi} \partial_{\xi_i}\mathcal{F}_h\left[v, P^{(\mu+1,\widetilde{m})}\right]^{(\mu,\widetilde{m})} (w - v)^{\left(\mu+1,\widetilde{m}+\psi^{-1}(i)\right)}
\end{aligned} \tag{39}$$

where $P^{(\mu+1,\widetilde{m})} \in \mathbf{R}^{\chi}$ is an intermediate point. Relations (31), (39), assumptions (H_3), (i)–(iii) and the induction assumption lead to the estimate

$$\begin{aligned}
&|w - v|^{(\mu+1,\widetilde{m})}\left[1 - \partial_{\xi_{i_0}}\mathcal{F}_h\left[v, P^{(\mu+1,\widetilde{m})}\right]^{(\mu,\widetilde{m})}\right] \\
&\le \sigma_h\left(t^{(\mu)}, \beta^{(\mu)}\right) + \overline{\gamma}^{(\mu)} + |w - v|^{(\mu+1,\widetilde{m})} \sum_{i \ne i_0, i=1}^{\chi} \partial_{\xi_i}\mathcal{F}_h\left[v, P^{(\mu+1,\widetilde{m})}\right]^{(\mu,\widetilde{m})}.
\end{aligned} \tag{40}$$

Inequality (40) and assumptions (H_3), (iii) imply (37). Hence, by the induction assumption and the monotonicity of β, the proof is complete by induction. □

Remark 3.3. Let the assumptions of Theorem 3.2 be satisfied with

$$\sigma_h(t, y) := (1 + Lh_0)\, y, \ (t, y) \in I_h^+ \times \mathbf{R}_+,$$

where $L \ge 0$ and there is $\widetilde{\gamma} \in \mathbf{R}_+$ such that $\overline{\gamma}^{(\mu)} \le h_0\widetilde{\gamma}$, $\mu = 0, \dots, K_0 - 1$. Then

(i) if $L > 0$, then

$$\|w-v\|_{\Omega_{h.\mu}} \leq (1+Lh_0)^{\mu}\,\overline{\gamma}_0 + \widetilde{\gamma}\frac{(1+Lh_0)^{\mu}-1}{L} \leq \exp{(LT)}\overline{\gamma}_0 + \widetilde{\gamma}\frac{\exp{(LT)}-1}{L}$$

for $\mu = 0, \ldots, K_0$;

(ii) if $L = 0$, then

$$\|w-v\|_{\Omega_{h.\mu}} \leq \overline{\gamma}_0 + \mu h_0 \widetilde{\gamma} \leq \overline{\gamma}_0 + T\widetilde{\gamma}$$

for $\mu = 0, \ldots, K_0$.

These estimates may be obtained by solving the initial comparison problem

$$\beta^{(\mu+1)} = (1+Lh_0)\,\beta^{(\mu)} + \overline{\gamma}^{(\mu)}, \quad \beta^{(0)} = \overline{\gamma}_0, \quad \mu = 0, \ldots, K_0 - 1 \tag{41}$$

(see assumption (iii)).

4. Differential and difference functional problems

We need the following assumptions on the functions f, φ, the interpolating operator G_h and the regularity of a solution u of (1), (2).

Assumption $F[f, u, G_h]$**.**

(F_1) f of variables $(t, x, z, p, q) \in \Delta$ is continuous on Δ.

(F_2) There exist the partial derivatives

$$\partial_p f = (\partial_{p_1} f, \ldots, \partial_{p_n} f), \quad \partial_q f = \left[\partial_{q_{ij}} f\right]_{i,j=1}^{n}$$

on Δ and $\partial_{p_i} f$, $\partial_{q_{ij}} f$, $i, j = 1, \ldots, n$, are bounded on Δ.

(F_3) The matrix $\partial_q f$ is symmetric and

$$\partial_{q_{ij}} f\,(P) \geq 0 \quad \text{and} \quad \partial_{q_{ij}} f\,(P) \not\equiv 0 \quad \text{for } (i,j) \in \Gamma_+,$$
$$\partial_{q_{ij}} f\,(P) \leq 0 \quad \text{for } (i,j) \in \Gamma_-$$

at each $P \in \Delta$.

(F_4) There are functions $\sigma : [0, T] \times \mathbf{R}_+ \to \mathbf{R}_+$, $\rho : \mathbf{R}_+^2 \to \mathbf{R}_+$ such that:

(i) σ is continuous and nondecreasing with respect to both variables, moreover, $\sigma\,(t, 0) = 0$ for $t \in [0, T]$,

(ii) ρ is nondecreasing with respect to both variables,

(iii) for each $c \geq 0$ and $\varepsilon, \varepsilon_0 \geq 0$, the maximal solution of the Cauchy problem

$$\omega^{'}(t) = c\sigma\,(t, D\omega\,(t)) + \varepsilon, \quad \omega\,(0) = \varepsilon_0 \tag{42}$$

is defined on $[0, T]$ and the function $\widetilde{\omega}\,(t) = 0$ for $t \in [0, T]$ is the maximal solution of (42) for each $c \geq 0$ and $\varepsilon, \varepsilon_0 = 0$, where D appears in the definition of the interpolating operator G_h,

(iv) the *generalized Perron type estimate*

$$\left|f(t,x,z,p,q)-f(t,x,\overline{z},p,q)\right|\leq\rho\left(\|p\|,\|q\|\right)\sigma\left(t,\|z-\overline{z}\|_{\Omega_t}\right) \quad (43)$$

holds on Δ.

(F_5) $u\in C^{1,2}(\Omega,\mathbf{R})$ is a solution of (1), (2).

(F_6) For each $z,\overline{z}\in\mathfrak{F}(\Omega_h,\mathbf{R})$ if $z|_{\Omega_{h.\mu}}=\overline{z}|_{\Omega_{h.\mu}}$, then $G_h[z]|_{\Omega_{t^{(\mu)}}}=G_h[\overline{z}]|_{\Omega_{t^{(\mu)}}}$, $\mu=0,\ldots,K_0$, $h\in H$.

Remark 4.1. Assumptions (i) and (iv) in (F_4) imply that the function f is of the Volterra type. That is, if $(t,x)\in E$ and $z,\overline{z}\in C(\Omega,\mathbf{R})$, $z|_{\Omega_t}=\overline{z}|_{\Omega_t}$, then $f(t,x,z,p,q)=f(t,x,\overline{z},p,q)$ for $p\in\mathbf{R}^n$, $q\in M_{n\times n}$.

Remark 4.2. It is required in assumption (F_3) that for each $(i,j)\in\Gamma$ the function $g_{ij}(P)=\operatorname{sign}\partial_{q_{ij}}f(P)$, $P\in\Delta$, is non-positive on Δ or non-negative on Δ. This assumption can be also considered as a definition of the sets Γ_+ and Γ_-. Moreover, simple calculations show that assumption (F_6) is true for $G_h=T_h$ (see [7]).

Remark 4.3. Let the Fréchet derivative $\partial_z f(t,x,z,p,q)\in L(C(\Omega,\mathbf{R}),\mathbf{R})$ for $(t,x,z,p,q)\in\Delta$. Assumption (F_4) holds for example if $\|\partial_z f(t,x,z,p,q)\|_{C(\Omega,\mathbf{R})_t}\leq\rho(\|p\|,\|q\|)$ on Δ, where $\rho:\mathbf{R}_+^2\to\mathbf{R}_+$ is nondecreasing with respect to both variables, e.g. $\rho(y_1,y_2)=ay_1+by_2+c$, $a,b,c=\text{const}\geq 0$, $y_1,y_2\in\mathbf{R}_+$ (see Examples 6.1–6.3). Then we may put $\sigma(t,y)=y$, $t\in[0,T]$, $y\in\mathbf{R}_+$. It is true in particular for all f satisfying the Lipschitz condition with respect to z. Examples of nonlinear σ are given in [7,9].

We now define an *implicit finite difference functional scheme* which will be applied to approximate a classical solution of the differential functional problem (1), (2). It is the system of algebraic equations

$$\begin{cases}\delta_0 z^{(\mu,m)}=f\left(t^{(\mu)},x^{(m)},G_h[z],\delta z^{(\mu+1,m)},\delta^{(2)}z^{(\mu+1,m)}\right)\\ \quad z^{(\mu,m)}=\varphi_h^{(\mu,m)} \quad \text{on } E_{0.h}\cup\partial_0 E_h,\end{cases} \quad (44)$$

where $\varphi_h\in\mathfrak{F}(E_{0.h}\cup\partial_0 E_h,\mathbf{R})$ is a given function, G_h is a given interpolating operator and $z\in\mathfrak{F}(\Omega_h,\mathbf{R})$.

We shall use the following assumptions on the steps h of the mesh Ω_h.

Assumption $S[h]$**.**

(S_1) The steps $h=\left(h_0,h'\right)\in H$ are such that

$$-\frac{h_i}{2}\left|\partial_{p_i}f(P)\right|+\partial_{q_{ii}}f(P)-h_i\sum_{j\neq i,j=1}^{n}\frac{1}{h_j}\left|\partial_{q_{ij}}f(P)\right|\geq 0 \quad (45)$$

at each $P\in\Delta$, $i=1,\ldots,n$.

(S_2) There is $c_0>0$ such that $h_ih_j^{-1}\leq c_0$ for $i,j=1,\ldots,n$.

Remark 4.4. For the mixed derivatives $\partial_{q_{ij}} f$, $(i,j) \in \Gamma$, the sign conditions are formulated in assumption (F_3) (see Remark 4.2). Moreover, inequality (45) can be fulfilled only in case $\partial_{q_{ii}} f(P) \geq 0$ at each $P \in \Delta$, $i = 1, \ldots, n$.

5. Theoretical study of the scheme

5.1. Convergence of the difference method. We now turn to the main problem of this paper, the convergence of the difference method (44). We begin with a useful lemma.

For $\xi = (\xi_1, \ldots, \xi_\chi) \in \mathbf{R}^\chi$ we put

$$\delta_i^+ \xi_{i_0} = \frac{1}{h_i}\left[\xi_{\psi(e_i)} - \xi_{i_0}\right], \quad \delta_i^- \xi_{i_0} = \frac{1}{h_i}\left[\xi_{i_0} - \xi_{\psi(-e_i)}\right], \tag{46}$$

$i = 1, \ldots, n$ (see (16)). The expressions

$$\delta \xi_{i_0} = (\delta_1 \xi_{i_0}, \ldots, \delta_n \xi_{i_0}), \quad \delta^{(2)} \xi_{i_0} = \left[\delta_{ij} \xi_{i_0}\right]_{i,j=1}^n$$

are defined in the following way

$$\begin{aligned}
\delta_i \xi_{i_0} &= \frac{1}{2}\left[\delta_i^+ \xi_{i_0} + \delta_i^- \xi_{i_0}\right] && \text{for } i = 1, \ldots, n, \\
\delta_{ii} \xi_{i_0} &= \delta_i^+ \delta_i^- \xi_{i_0} && \text{for } i = 1, \ldots, n, \\
\delta_{ij} \xi_{i_0} &= \frac{1}{2}\left[\delta_i^+ \delta_j^- \xi_{i_0} + \delta_i^- \delta_j^+ \xi_{i_0}\right] && \text{for } (i,j) \in \Gamma_-, \\
\delta_{ij} \xi_{i_0} &= \frac{1}{2}\left[\delta_i^+ \delta_j^+ \xi_{i_0} + \delta_i^- \delta_j^- \xi_{i_0}\right] && \text{for } (i,j) \in \Gamma_+.
\end{aligned} \tag{47}$$

Consider the functional $\mathcal{F}_h : E_h^+ \times \mathfrak{F}(\Omega_h, \mathbf{R}) \times \mathbf{R}^\chi \to \mathbf{R}$ defined by

$$\mathcal{F}_h[z, \xi]^{(\mu,m)} = z^{(\mu,m)} + h_0 f\left(t^{(\mu)}, x^{(m)}, G_h[z], \delta\xi_{i_0}, \delta^{(2)}\xi_{i_0}\right). \tag{48}$$

Note that

$$\mathcal{F}_h\left[z, z_{<\mu+1,m>}\right]^{(\mu,m)} = z^{(\mu,m)} + h_0 f\left(t^{(\mu)}, x^{(m)}, G_h[z], \delta z^{(\mu+1,m)}, \delta^{(2)} z^{(\mu+1,m)}\right).$$

Therefore difference scheme (44) and problem (19), (20) with $\mathcal{F}_h$ defined in (48) are the same.

Lemma 5.1. *Let Assumptions* $F[f, u, G_h]$ *and* $S[h]$ *hold. Then the functional* $\mathcal{F}_h$ *defined by* (48) *satisfies Assumption* $H[\mathcal{F}_h]$.

The proof of the above lemma is analogous to that of [9, Lemma 4.6] and it is therefore omitted (see also the proof of Lemma 5.4).

Let $U := u|_{\Omega_h} \in \mathfrak{F}(\Omega_h, \mathbf{R})$ be the restriction of a solution $u \in C^{1,2}(\Omega, \mathbf{R})$ of the differential functional problem (1), (2) to the mesh Ω_h and let $v \in \mathfrak{F}(\Omega_h, \mathbf{R})$ be the solution of the finite difference functional scheme (44). We say that the difference method (44) is *uniformly convergent* if

$$\lim_{h\to 0} \|U - v\|_{\Omega_h} = 0.$$

Theorem 5.2. *Let Assumptions $F[f, u, G_h]$ and $S[h]$ hold and suppose that there is a function $\gamma_0 : H \to \mathbf{R}_+$ such that*

$$\left|\varphi^{(\mu,m)} - \varphi_h^{(\mu,m)}\right| \le \gamma_0(h) \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h \quad \text{and} \quad \lim_{h\to 0} \gamma_0(h) = 0. \tag{49}$$

Under these assumptions:

(i) *there exists the unique solution $v \in \mathfrak{F}(\Omega_h, \mathbf{R})$ of* (44),

(ii) *there is an $\alpha : H \to \mathbf{R}_+$ such that*

$$\|U - v\|_{\Omega_{h.\mu}} \le \alpha(h) \quad \text{for } 0 \le \mu \le K_0 \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0. \tag{50}$$

Proof. Let $\mathcal{F}_h : E_h^+ \times \mathfrak{F}(\Omega_h, \mathbf{R}) \times \mathbf{R}^\chi \to \mathbf{R}$ be defined by (48). The existence of the unique solution $v \in \mathfrak{F}(\Omega_h, \mathbf{R})$ of (44) follows from Theorem 3.1 and Lemma 5.1.

To prove (ii) we apply Theorem 3.2 and Lemma 5.1. The solution v satisfies (19), (20) and there is a function $\gamma : H \to \mathbf{R}_+$ such that

$$\left|U^{(\mu+1,m)} - \mathcal{F}_h\left[U, U_{<\mu+1,m>}\right]^{(\mu,m)}\right| \le h_0 \gamma(h) \quad \text{on } E_h^+$$

and $\lim_{h\to 0} \gamma(h) = 0$. Let a constant $d \ge 0$ be such that

$$\left|\partial_{x_i} u(t,x)\right|, \ \left|\partial_{x_i x_j} u(t,x)\right| \le d \quad \text{for } (t,x) \in \Omega, \ i,j = 1,\ldots,n \tag{51}$$

(see (F_5)). We denote by Y_h the class of all functions $z \in \mathfrak{F}(\Omega_h, \mathbf{R})$ with the property:

$$\left|\delta_i z^{(\mu,m)}\right|, \ \left|\delta_{ij} z^{(\mu,m)}\right| \le d \quad \text{for } \left(t^{(\mu)}, x^{(m)}\right) \in E_h, \ i,j = 1,\ldots,n.$$

Obviously, $U \in Y_h$. Suppose that $z \in \mathfrak{F}(\Omega_h, \mathbf{R})$, $\overline{z} \in Y_h$ and $\left(t^{(\mu)}, x^{(m)}\right) \in E_h^+$. We prove that

$$\begin{aligned} &\left|\mathcal{F}_h\left[z, \overline{z}_{<\mu+1,m>}\right]^{(\mu,m)} - \mathcal{F}_h\left[\overline{z}, \overline{z}_{<\mu+1,m>}\right]^{(\mu,m)}\right| \\ &\le \|z - \overline{z}\|_{\Omega_{h.\mu}} + h_0 \rho(d,d)\, \sigma\left(t^{(\mu)}, D\,\|z - \overline{z}\|_{\Omega_{h.\mu}}\right). \end{aligned} \tag{52}$$

It follows from Assumption $F[f, u, G_h]$ that

$$\begin{aligned} &\left|\mathcal{F}_h\left[z, \overline{z}_{<\mu+1,m>}\right]^{(\mu,m)} - \mathcal{F}_h\left[\overline{z}, \overline{z}_{<\mu+1,m>}\right]^{(\mu,m)}\right| \\ &\le \|z - \overline{z}\|_{\Omega_{h.\mu}} + h_0 \rho\left(\left\|\delta \overline{z}^{(\mu+1,m)}\right\|, \left\|\delta^{(2)} \overline{z}^{(\mu+1,m)}\right\|\right) \sigma\left(t^{(\mu)}, \|G_h[z] - G_h[\overline{z}]\|_{\Omega_{t^{(\mu)}}}\right). \end{aligned} \tag{53}$$

The use of the monotonicity of ρ, σ and the properties of the interpolating operator G_h in (53) implies (52).

Denote by $\eta : I_h \to \mathbf{R}_+$ the solution of the initial comparison difference problem

$$\begin{cases} \eta^{(\mu+1)} = \eta^{(\mu)} + h_0 \rho(d,d)\, \sigma\left(t^{(\mu)}, D\eta^{(\mu)}\right) + h_0 \gamma(h), & \mu = 0, \ldots, K_0 - 1, \\ \quad \eta^{(0)} = \gamma_0(h). \end{cases} \tag{54}$$

It follows from Theorem 3.2 and Lemma 5.1 that

$$\|U - v\|_{\Omega_{h.\mu}} \le \eta^{(\mu)}, \quad \mu = 0, \ldots, K_0. \tag{55}$$

Consider the Cauchy problem

$$\omega^{'}(t) = \rho(d,d)\, \sigma(t, D\omega(t)) + \gamma(h), \quad \omega(0) = \gamma_0(h) \tag{56}$$

and its maximal solution $\omega(\cdot; h) : [0,T] \to \mathbf{R}_+$ (see (F_4)). It easily follows that

$$\eta^{(\mu)} \le \omega\left(t^{(\mu)}; h\right) \le \omega(T; h) \quad \text{for } \mu = 0, \ldots, K_0 \tag{57}$$

and $\lim_{h\to 0} \omega(t; h) = 0$ uniformly on $[0,T]$. Put $\alpha(h) = \omega(T; h)$. The proof is complete. □

5.2. Quasi-linear equation. We are interested in the mumerical approximation of a classical solution of problem (3), (2).

We need the following assumptions on the functions F, φ, coefficients a_{ij}, the interpolating operator G_h and the regularity of a solution u of (3), (2), as well as on the steps h of the mesh Ω_h.

Assumption $QF[F, A, u, G_h]$**.**

(QF_1) F of variables $(t,x,z,p) \in \Delta^F$ and a_{ij}, $i,j = 1, \ldots, n$, of variables $(t,x,z) \in \Delta^A$, are continuous on Δ^F and Δ^A, respectively.

(QF_2) There exists the partial derivative $\partial_p F = (\partial_{p_1} F, \ldots, \partial_{p_n} F)$ on Δ^F and $\partial_{p_i} F$, $i = 1, \ldots, n$, are bounded on Δ^F; a_{ij}, $i,j = 1, \ldots, n$, are bounded on Δ^A.

(QF_3) $A = [a_{ij}]_{i,j=1}^n$ is symmetric.

(QF_4) There are functions $\sigma : [0,T] \times \mathbf{R}_+ \to \mathbf{R}_+$, $\rho_1 : \mathbf{R}_+ \to \mathbf{R}_+$ such that:

(i) σ is continuous and nondecreasing with respect to both variables, moreover, $\sigma(t, 0) = 0$ for $t \in [0,T]$,

(ii) ρ_1 is nondecreasing,

(iii) for each $c \ge 0$ and $\varepsilon, \varepsilon_0 \ge 0$, the maximal solution of the Cauchy problem

$$\omega^{'}(t) = c\sigma(t, D\omega(t)) + \varepsilon, \quad \omega(0) = \varepsilon_0 \tag{58}$$

is defined on $[0,T]$ and the function $\widetilde{\omega}(t) = 0$ for $t \in [0,T]$ is the maximal solution of (58) for each $c \ge 0$ and $\varepsilon, \varepsilon_0 = 0$, where D appears in the definition of the interpolating operator G_h,

(iv) the *generalized Perron type estimate* and *Perron type estimate*

$$|F(t,x,z,p) - F(t,x,\overline{z},p)| \le \rho_1(\|p\|)\,\sigma\left(t, \|z-\overline{z}\|_{\Omega_t}\right), \tag{59}$$
$$|a_{ij}(t,x,z) - a_{ij}(t,x,\overline{z})| \le \sigma\left(t, \|z-\overline{z}\|_{\Omega_t}\right), \tag{60}$$

where $i,j = 1,\ldots,n$, hold on Δ^F and Δ^A, respectively.

(QF_5) $u \in C^{1,2}(\Omega, \mathbf{R})$ is a solution of (3), (2).

(QF_6) For each $z, \overline{z} \in \mathfrak{F}(\Omega_h, \mathbf{R})$ if $z|_{\Omega_{h.\mu}} = \overline{z}|_{\Omega_{h.\mu}}$, then $G_h[z]\,|_{\Omega_{t(\mu)}} = G_h[\overline{z}]\,|_{\Omega_{t(\mu)}}$, $\mu = 0,\ldots,K_0$, $h \in H$.

Assumption $QS[h]$**.**

(QS_1) The steps $h = \left(h_0, h'\right) \in H$ are such that

$$-\frac{h_i}{2}\left|\partial_{p_i}F(P)\right| + a_{ii}(t,x,z) - h_i \sum_{j\neq i, j=1}^{n} \frac{1}{h_j}\left|a_{ij}(t,x,z)\right| \ge 0 \tag{61}$$

for all $(t,x,z) \in \Delta^A$ and $P \in \Delta^F$, $i = 1,\ldots,n$.

(QS_2) There is $c_0 > 0$ such that $h_i h_j^{-1} \le c_0$ for $i,j = 1,\ldots,n$.

Remark 5.3. Assumptions (i) and (iv) in (QF_4) imply that the function F and coefficients a_{ij} are of the Volterra type; see Remark 4.1.

We now put

$$f(t,x,z,p,q) = \sum_{i,j=1}^{n} a_{ij}(t,x,z)\,q_{ij} + F(t,x,z,p) \tag{62}$$

for $(t,x,z,p,q) \in \Delta$, and consider difference method (44) with this f for (2), (3). If we apply Theorem 5.2, then we need Assumptions QF$[F,A,u,G_h]$, QS$[h]$ and the following assumption on the matrix A: for each $(i,j) \in \Gamma$, the function

$$\widetilde{a}_{ij}(t,x,z) = \operatorname{sign} a_{ij}(t,x,z) \quad \text{for } (t,x,z) \in \Delta^A$$

is non-positive on Δ^A or non-negative on Δ^A (see (F_3)). It is easily seen that $\rho(y_1,y_2) = n^2 y_2 + \rho_1(y_1)$ for $y_1, y_2 \in \mathbf{R}_+$ satisfies (F_4).

We prove that the condition of the coefficients a_{ij} being of the same sign in Δ^A can be omitted if we modify the difference operator $\delta^{(2)}$. More precisely, we consider problem (44) with δ_0, δ, δ_{ii}, $i = 1,\ldots,n$, given in Section 2, and we define δ_{ij}, $i,j = 1,\ldots,n$, $i \neq j$, by

$$\begin{aligned}
\delta_{ij} z^{(\mu+1,m)} &= \frac{1}{2}\left[\delta_i^+\delta_j^- z^{(\mu+1,m)} + \delta_i^-\delta_j^+ z^{(\mu+1,m)}\right] \quad \text{if } a_{ij}\left(t^{(\mu)}, x^{(m)}, G_h[z]\right) < 0,\\
\delta_{ij} z^{(\mu+1,m)} &= \frac{1}{2}\left[\delta_i^+\delta_j^+ z^{(\mu+1,m)} + \delta_i^-\delta_j^- z^{(\mu+1,m)}\right] \quad \text{if } a_{ij}\left(t^{(\mu)}, x^{(m)}, G_h[z]\right) \ge 0,
\end{aligned} \tag{63}$$

where $z \in \mathfrak{F}(\Omega_h, \mathbf{R})$, $\left(t^{(\mu)}, x^{(m)}\right) \in E_h$. Observe that the finite difference functional scheme (44) with f given by (62) and δ_{ij} by (63) depends on the sign of a_{ij} at $\left(t^{(\mu)}, x^{(m)}, G_h[z]\right)$ and this sign does not have to be the same in Δ^A.

Consider the functional $\mathcal{F}_h : E_h^+ \times \mathfrak{F}(\Omega_h, \mathbf{R}) \times \mathbf{R}^\chi \to \mathbf{R}$ defined by

$$\mathcal{F}_h[z, \xi]^{(\mu,m)} = z^{(\mu,m)} + h_0 \sum_{i,j=1}^{n} a_{ij}\left(t^{(\mu)}, x^{(m)}, G_h[z]\right) \delta_{ij}\xi_{i_0} \\ + h_0 F\left(t^{(\mu)}, x^{(m)}, G_h[z], \delta\xi_{i_0}\right) \tag{64}$$

(see (48)). The expressions $\delta\xi_{i_0}$, $\delta_{ii}\xi_{i_0}$, $i = 1, \ldots, n$, are defined by (47), (46) and

$$\begin{aligned} \delta_{ij}\xi_{i_0} &= \frac{1}{2}\left[\delta_i^+\delta_j^-\xi_{i_0} + \delta_i^-\delta_j^+\xi_{i_0}\right] \quad \text{if } a_{ij}\left(t^{(\mu)}, x^{(m)}, G_h[z]\right) < 0, \\ \delta_{ij}\xi_{i_0} &= \frac{1}{2}\left[\delta_i^+\delta_j^+\xi_{i_0} + \delta_i^-\delta_j^-\xi_{i_0}\right] \quad \text{if } a_{ij}\left(t^{(\mu)}, x^{(m)}, G_h[z]\right) \geq 0, \end{aligned} \tag{65}$$

where $\xi \in \mathbf{R}^\chi$, $z \in \mathfrak{F}(\Omega_h, \mathbf{R})$, $\left(t^{(\mu)}, x^{(m)}\right) \in E_h^+$, $i, j = 1, \ldots, n$, $i \neq j$. Note that

$$\mathcal{F}_h\left[z, z_{<\mu+1,m>}\right]^{(\mu,m)} = z^{(\mu,m)} + h_0 \sum_{i,j=1}^{n} a_{ij}\left(t^{(\mu)}, x^{(m)}, G_h[z]\right) \delta_{ij} z^{(\mu+1,m)} \\ + h_0 F\left(t^{(\mu)}, x^{(m)}, G_h[z], \delta z^{(\mu+1,m)}\right).$$

It is clear that difference scheme (44), with f defined in (62) and δ_{ij} modified in (63), and problem (19), (20) with $\mathcal{F}_h$ defined in (64) are the same.

Lemma 5.4. *Let Assumptions* $QF[F, A, u, G_h]$ *and* $QS[h]$ *hold. Then the functional* $\mathcal{F}_h$ *defined by* (64) *satisfies Assumption* $H[\mathcal{F}_h]$.

Proof. It follows from Remark 5.3 and the properties of G_h that assumption (H_1) is satisfied.

Write $Q^{(\mu,m)}[z] = \left(t^{(\mu)}, x^{(m)}, G_h[z]\right)$, $P^{(\mu,m)}[z, \xi] = \left(t^{(\mu)}, x^{(m)}, G_h[z],\ \delta\xi_{i_0}\right)$. Define the sets

$$\Gamma_+^{(\mu,m)} = \left\{(i,j) \in \Gamma :\ a_{ij}\left(Q^{(\mu,m)}[z]\right) \geq 0\right\}, \quad \Gamma_-^{(\mu,m)} = \Gamma \setminus \Gamma_+^{(\mu,m)}.$$

Let, moreover,

$$\begin{aligned} \overline{\Lambda} &= \{\lambda \in \Lambda :\ \exists\, i \in \{1, \ldots, n\}\ \lambda = e_i \text{ or } \lambda = -e_i\}, \\ \Lambda_+^{(\mu,m)} &= \left\{\lambda \in \Lambda :\ \exists\,(i,j) \in \Gamma_+^{(\mu,m)}\ \lambda = e_i + e_j \text{ or } \lambda = -e_i - e_j\right\}, \\ \Lambda_-^{(\mu,m)} &= \left\{\lambda \in \Lambda :\ \exists\,(i,j) \in \Gamma_-^{(\mu,m)}\ \lambda = e_i - e_j \text{ or } \lambda = -e_i + e_j\right\}, \\ \Lambda_*^{(\mu,m)} &= \Lambda \setminus \left(\{0\} \cup \overline{\Lambda} \cup \Lambda_+^{(\mu,m)} \cup \Lambda_-^{(\mu,m)}\right). \end{aligned}$$

By (QF_2), (QF_3), we have

$$\begin{aligned}
\partial_{\xi_{i_0}}\mathcal{F}_h\left[z,\xi\right]^{(\mu,m)} &= -2h_0\sum_{i=1}^{n}\frac{1}{h_i^2}a_{ii}\left(Q^{(\mu,m)}\left[z\right]\right) \\
&\quad + h_0\sum_{(i,j)\in\Gamma}\frac{1}{h_ih_j}\left|a_{ij}\left(Q^{(\mu,m)}\left[z\right]\right)\right|, \\
\partial_{\xi_{\psi(e_i)}}\mathcal{F}_h\left[z,\xi\right]^{(\mu,m)} &= \frac{h_0}{2h_i}\partial_{p_i}F\left(P^{(\mu,m)}\left[z,\xi\right]\right) + \frac{h_0}{h_i^2}a_{ii}\left(Q^{(\mu,m)}\left[z\right]\right) \\
&\quad - h_0\sum_{j\neq i,j=1}^{n}\frac{1}{h_ih_j}\left|a_{ij}\left(Q^{(\mu,m)}\left[z\right]\right)\right|, \\
\partial_{\xi_{\psi(-e_i)}}\mathcal{F}_h\left[z,\xi\right]^{(\mu,m)} &= -\frac{h_0}{2h_i}\partial_{p_i}F\left(P^{(\mu,m)}\left[z,\xi\right]\right) + \frac{h_0}{h_i^2}a_{ii}\left(Q^{(\mu,m)}\left[z\right]\right) \\
&\quad - h_0\sum_{j\neq i,j=1}^{n}\frac{1}{h_ih_j}\left|a_{ij}\left(Q^{(\mu,m)}\left[z\right]\right)\right|, \\
\partial_{\xi_{\psi(e_i+e_j)}}\mathcal{F}_h\left[z,\xi\right]^{(\mu,m)} &= \partial_{\xi_{\psi(-e_i-e_j)}}\mathcal{F}_h\left[z,\xi\right]^{(\mu,m)} \\
&= \frac{h_0}{2h_ih_j}a_{ij}\left(Q^{(\mu,m)}\left[z\right]\right), \quad (i,j)\in\Gamma_+^{(\mu,m)}, \\
\partial_{\xi_{\psi(e_i-e_j)}}\mathcal{F}_h\left[z,\xi\right]^{(\mu,m)} &= \partial_{\xi_{\psi(-e_i+e_j)}}\mathcal{F}_h\left[z,\xi\right]^{(\mu,m)} \\
&= -\frac{h_0}{2h_ih_j}a_{ij}\left(Q^{(\mu,m)}\left[z\right]\right), \quad (i,j)\in\Gamma_-^{(\mu,m)}, \\
\partial_{\xi_{\psi(\lambda)}}\mathcal{F}_h\left[z,\xi\right]^{(\mu,m)} &= 0, \quad \lambda\in\Lambda_*^{(\mu,m)},
\end{aligned}$$

$i=1,\ldots,n$. The above relations and assumption (QS_1) imply (H_2), (H_3). This completes the proof. □

Theorem 5.5. *Let Assumptions* $QF[F,A,u,G_h]$ *and* $QS[h]$ *hold and suppose that there is a function* $\gamma_0 : H\to\mathbf{R}_+$ *such that*

$$\left|\varphi^{(\mu,m)}-\varphi_h^{(\mu,m)}\right|\le\gamma_0\left(h\right)\quad\text{on } E_{0.h}\cup\partial_0E_h\quad\text{and}\quad\lim_{h\to0}\gamma_0\left(h\right)=0. \tag{66}$$

Under these assumptions:

(i) *there exists the unique solution* $v\in\mathfrak{F}\left(\Omega_h,\mathbf{R}\right)$ *of* (44),

(ii) *there is an* $\alpha : H\to\mathbf{R}_+$ *such that*

$$\left\|U-v\right\|_{\Omega_{h.\mu}}\le\alpha\left(h\right)\quad\text{for } 0\le\mu\le K_0\quad\text{and}\quad\lim_{h\to0}\alpha\left(h\right)=0. \tag{67}$$

Proof. The proof of this theorem is similar to that of Theorem 5.2. We apply Theorems 3.1, 3.2 and Lemma 5.4. Let $\mathcal{F}_h : E_h^+\times\mathfrak{F}\left(\Omega_h,\mathbf{R}\right)\times\mathbf{R}^\chi\to\mathbf{R}$ be defined by (64).

The existence of the unique solution $v \in \mathfrak{F}(\Omega_h, \mathbf{R})$ of (44) follows from Theorem 3.1 and Lemma 5.4.

To prove (ii) we apply Theorem 3.2 and Lemma 5.4. The solution v satisfies (19), (20) and there is a function $\gamma : H \to \mathbf{R}_+$ such that

$$\left|U^{(\mu+1,m)} - \mathcal{F}_h\left[U, U_{<\mu+1,m>}\right]^{(\mu,m)}\right| \leq h_0 \gamma(h) \quad \text{on } E_h^+$$

and $\lim_{h\to 0}\gamma(h) = 0$. Let a constant $d \geq 0$ be such that

$$\left|\partial_{x_i} u(t,x)\right|,\ \left|\partial_{x_i x_j} u(t,x)\right| \leq d \quad \text{for } (t,x) \in \Omega,\ i,j = 1,\dots,n \tag{68}$$

(see (QF_5)). We denote by Y_h the class of all functions $z \in \mathfrak{F}(\Omega_h, \mathbf{R})$ with the property:

$$\left|\delta_i z^{(\mu,m)}\right|,\ \left|\delta_{ij} z^{(\mu,m)}\right| \leq d \quad \text{for } \left(t^{(\mu)}, x^{(m)}\right) \in E_h, \quad i,j = 1,\dots,n.$$

Obviously, $U \in Y_h$. Suppose that $z \in \mathfrak{F}(\Omega_h, \mathbf{R})$, $\overline{z} \in Y_h$ and $\left(t^{(\mu)}, x^{(m)}\right) \in E_h^+$. We prove that

$$\begin{aligned}&\left|\mathcal{F}_h\left[z, \overline{z}_{<\mu+1,m>}\right]^{(\mu,m)} - \mathcal{F}_h\left[\overline{z}, \overline{z}_{<\mu+1,m>}\right]^{(\mu,m)}\right| \\ &\leq \|z - \overline{z}\|_{\Omega_{h.\mu}} + h_0\left(n^2 d + \rho_1(d)\right)\sigma\left(t^{(\mu)}, D\,\|z - \overline{z}\|_{\Omega_{h.\mu}}\right).\end{aligned} \tag{69}$$

It follows from Assumption $QF[F, A, u, G_h]$ that

$$\begin{aligned}&\left|\mathcal{F}_h\left[z, \overline{z}_{<\mu+1,m>}\right]^{(\mu,m)} - \mathcal{F}_h\left[\overline{z}, \overline{z}_{<\mu+1,m>}\right]^{(\mu,m)}\right| \\ &\leq \|z - \overline{z}\|_{\Omega_{h.\mu}} \\ &\quad + h_0\left[\sum_{i,j=1}^{n}\left|\delta_{ij}\overline{z}^{(\mu+1,m)}\right| + \rho_1\left(\left\|\delta\overline{z}^{(\mu+1,m)}\right\|\right)\right]\sigma\left(t^{(\mu)}, \|G_h[z] - G_h[\overline{z}]\|_{\Omega_{t^{(\mu)}}}\right).\end{aligned} \tag{70}$$

The use of the monotonicity of ρ_1, σ and the properties of the interpolating operator G_h in (70) implies (69).

An analysis similar to that in the proof of Theorem 5.2 shows that assertion (67) is satisfied with $\alpha(h) = \omega(T; h)$, where $\omega(\cdot; h) : [0, T] \to \mathbf{R}_+$ is the maximal solution of the Cauchy problem (56) with $\rho(y_1, y_2) = n^2 y_2 + \rho_1(y_1)$, $y_1, y_2 \in \mathbf{R}_+$. This concludes the proof. □

Remark 5.6. Observe that we do not assume in Theorems 5.2 and 5.5 the Courant-Friedrichs-Levy conditions

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2}\partial_{q_{ii}} f(P) + h_0 \sum_{(i,j)\in\Gamma} \frac{1}{h_i h_j}\left|\partial_{q_{ij}} f(P)\right| \geq 0, \tag{71}$$

$$1-2h_0\sum_{i=1}^{n}\frac{1}{h_i^2}a_{ii}\left(t,x,z\right)+h_0\sum_{(i,j)\in\Gamma}\frac{1}{h_ih_j}\left|a_{ij}\left(t,x,z\right)\right|\geq 0, \tag{72}$$

$P\in\Delta$, $(t,x,z)\in\Delta^A$ respectively, which are typical in explicit methods (see [26]).

Remark 5.7. Let Assumptions of Theorems 5.2 or 5.5 hold and let f be Lipschitz continuous with respect to z,p,q. Put $G_h=T_h$, where T_h is well-known interpolating operator introduced in [7]. It follows from the properties of the difference quotients and T_h that if $u\in C^{2,3}(\Omega,\mathbf{R})$ and $\gamma_0=O(\|h\|)$, then $U-v=O(\|h\|)$, and if $u\in C^{2,4}(\Omega,\mathbf{R})$ and $\gamma_0=O(h_0+\|h'\|^2)$, then $U-v=O\left(h_0+\|h'\|^2\right)$.

Remark 5.8. Suppose that the assumptions of Theorems 5.2 or 5.5 are satisfied and, moreover, there is a constant $\overline{c}>0$ such that

$$\left\|\delta w^{(\mu,m)}\right\|,\ \left\|\delta^{(2)}w^{(\mu,m)}\right\|\leq\overline{c}\quad\text{on } E_h \tag{73}$$

for all solutions $w\in\mathfrak{F}(\Omega_h,\mathbf{R})$ of perturbed finite difference functional schemes of (44). It follows from an analysis of the proofs of these theorems that the difference methods presented are stable. It is enough to replace U by w. If $\rho,\rho_1=\text{const}$, then condition (73) can be omitted.

Remark 5.9. All the results of this paper can be extended to weakly coupled differential functional systems. One part of each system may be strongly nonlinear and the other quasi-linear. This is a new result even in the case of systems without functional terms. For simplicity we consider one equation only.

6. Numerical results

To illustrate the class of problems which can be treated with our methods, we consider a strongly nonlinear differential equation with a quasi-linear term and two quasi-linear differential integral equations with deviated variables. Dirichlet's problems below cannot be solved with the numerical methods known to date.

Put $n=2$. Let $E=[0,0.01]\times(-0.01,0.01)^2$, $E_0=\{0\}\times[-0.01,0.01]^2$ and $\partial E_0=[0,0.01]\times\left([-0.01,0.01]^2\setminus(-0.01,0.01)^2\right)$.

Example 6.1. Consider the strongly nonlinear differential equation

$$\begin{aligned}\partial_t z\left(t,x,y\right)=&\arctan\left[\partial_{xx}z\left(t,x,y\right)+\partial_{xy}z\left(t,x,y\right)+\partial_{yy}z\left(t,x,y\right)\right]\\&+\left[2+\cos z\left(t,x,y\right)\right]\left[\partial_{xx}z\left(t,x,y\right)+\partial_{xy}z\left(t,x,y\right)\right.\\&\left.+\partial_{yy}z\left(t,x,y\right)\right]+\left[\sin z\left(t,x,y\right)\right]\partial_x z\left(t,x,y\right)+g\left(t,x,y\right)\end{aligned} \tag{74}$$

for $(t, x, y) \in E$, with the initial-boundary condition

$$z(t, x, y) = \sin t \cos(x + y) \quad \text{for } (t, x, y) \in E_0 \cup \partial E_0, \tag{75}$$

where $g(t, x, y) = \arctan[3 \sin t \cos(x + y)] + (6 \sin t + \cos t) \cos(x + y) + 3 \sin t$ $\cos(x + y) \cos[\sin t \cos(x + y)] + \sin t \sin(x + y) \sin[\sin t \cos(x + y)]$.

Observe that the right-hand side of (74) has a strongly nonlinear term and a quasi-linear term. Note that $f(t, x, y, z, p, q) = \operatorname{arctg}\left(q_{11} + \frac{1}{2}q_{12} + \frac{1}{2}q_{21} + q_{22}\right) + [2 + \cos z(t, x, y)]\left(q_{11} + \frac{1}{2}q_{12} + \frac{1}{2}q_{21} + q_{22}\right) + [\sin z(t, x, y)]\, p_1 + g(t, x, y)$ does not fulfill neither the Lipschitz nor the classical Perron conditions, but the generalized Perron condition (43) is true with $\rho(y_1, y_2) = y_1 + 3y_2$ and $\sigma(t, y) = y$. The function $u(t, x, y) = \sin t \cos(x + y)$ is an analytic solution of (74), (75). Put $h_0 = h_1 = h_2 = 10^{-3}$. For each $t^{(\mu)}$ we use one hundred iterations of the Newton method to solve the implicit difference scheme. Let ε_{max}, ε_{mean} be the largest and mean values, respectively, of the errors $|U - v|$ at time $t^{(\mu)}$.

$t^{(\mu)}$	ε_{max}	ε_{mean}
0.001	1.31×10^{-6}	4.61×10^{-7}
0.002	1.31×10^{-6}	4.64×10^{-7}
0.003	1.31×10^{-6}	4.64×10^{-7}
0.004	1.31×10^{-6}	4.64×10^{-7}
0.005	1.31×10^{-6}	4.64×10^{-7}
0.006	1.31×10^{-6}	4.64×10^{-7}
0.007	1.31×10^{-6}	4.64×10^{-7}
0.008	1.31×10^{-6}	4.64×10^{-7}
0.009	1.31×10^{-6}	4.64×10^{-7}
0.010	1.31×10^{-6}	4.64×10^{-7}

Table 1: Errors of the difference method with T_h

Note that the Courant-Friedrichs-Levy condition (71) for such steps is not satisfied and the explicit method given in [26] is not convergent. In fact, the errors ε_{max}, ε_{mean} of that method exceeded 10^{46} and 10^{44}, respectively.

Example 6.2. Consider the quasi-linear differential integral equation with deviated variables

$$\begin{aligned} \partial_t z(t, x, y) = & \left[2 + \cos\left(\int_{-x}^{x}\int_{-y}^{y} z(t, \xi, \zeta)\, d\zeta d\xi\right)\right] \\ & \times [\partial_{xx} z(t, x, y) + \partial_{xy} z(t, x, y) + \partial_{yy} z(t, x, y)] \\ & + [\sin z(t, x, y)]\, \partial_x z(t, x, y) + z(0.5t, 0, 0) + g(t, x, y) \end{aligned} \tag{76}$$

for $(t, x, y) \in E$, with the initial-boundary condition

$$z(t, x, y) = \sin t \cos(x + y) \quad \text{for } (t, x, y) \in E_0 \cup \partial E_0, \tag{77}$$

where $g(t,x,y) = (6\sin t + \cos t)\cos(x+y) + 3\sin t\cos(x+y)\cos(4\sin t\sin x\sin y) + \sin t\sin(x+y)\sin[\sin t\cos(x+y)] - \sin(0.5t)$.

Note that $f(t,x,y,z,p,q) = \left[2 + \cos\left(\int_{-x}^{x}\int_{-y}^{y} z(t,\xi,\zeta)\, d\zeta d\xi\right)\right]\left(q_{11} + \frac{1}{2}q_{12} + \frac{1}{2}q_{21} + q_{22}\right) + [\sin z(t,x,y)]\, p_1 + z(0.5t,0,0) + g(t,x,y)$ does not fulfill neither the Lipschitz nor the classical Perron conditions, but the generalized Perron condition (43) is true with $\rho(y_1,y_2) = y_1 + 12y_2 + 1$ and $\sigma(t,y) = y$. The function $u(t,x,y) = \sin t\cos(x+y)$ is an analytic solution of (76), (77). Put $h_0 = h_1 = h_2 = 10^{-3}$. For each $t^{(\mu)}$ we use the method of an inverse matrix to solve the implicit difference scheme. Let ε_{max}, ε_{mean} be the largest and mean values, respectively, of the errors $|U - v|$ at time $t^{(\mu)}$.

$t^{(\mu)}$	ε_{max}	ε_{mean}
0.001	9.99×10^{-4}	7.03×10^{-4}
0.002	1.99×10^{-3}	1.40×10^{-3}
0.003	2.99×10^{-3}	2.11×10^{-3}
0.004	3.99×10^{-3}	2.81×10^{-3}
0.005	4.99×10^{-3}	3.51×10^{-3}
0.006	5.99×10^{-3}	4.22×10^{-3}
0.007	6.99×10^{-3}	4.92×10^{-3}
0.008	7.99×10^{-3}	5.63×10^{-3}
0.009	8.99×10^{-3}	6.33×10^{-3}
0.010	9.99×10^{-3}	7.03×10^{-3}

Table 2: Errors of the difference method with T_h

Note that the Courant-Friedrichs-Levy condition (72) for such steps is not satisfied and the explicit method given in [26] is not convergent. In fact, the errors ε_{max}, ε_{mean} of that method exceeded 10^{42} and 10^{40}, respectively.

Example 6.3. Consider the quasi-linear differential integral equation with deviated variables

$$\begin{aligned} \partial_t z(t,x,y) = {} & \partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) \\ & + \left[\cos\left(\int_{-x}^{x}\int_{-y}^{y} z(t,\xi,\zeta)\, d\zeta d\xi\right)\right] \partial_{xy} z(t,x,y) \\ & + [\sin z(t,x,y)]\, \partial_x z(t,x,y) + z(0.5t,0,0) + g(t,x,y) \end{aligned} \tag{78}$$

for $(t,x,y) \in E$, with the initial-boundary condition

$$z(t,x,y) = \sin t\cos(x+y) \quad \text{for } (t,x,y) \in E_0 \cup \partial E_0, \tag{79}$$

where $g(t,x,y) = (2\sin t + \cos t)\cos(x+y) + \sin t\cos(x+y)\cos(4\sin t\sin x\sin y) + \sin t\sin(x+y)\sin[\sin t\cos(x+y)] - \sin(0.5t)$.

Note that $f(t,x,y,z,p,q) = q_{11} + q_{22} + \frac{1}{2}\left[\cos\left(\int_{-x}^{x}\int_{-y}^{y} z(t,\xi,\zeta)\,d\zeta d\xi\right)\right] q_{12}$ $+\frac{1}{2}\left[\cos\left(\int_{-x}^{x}\int_{-y}^{y} z(t,\xi,\zeta)\,d\zeta d\xi\right)\right] q_{21} + [\sin z(t,x,y)]\, p_1 + z(0.5t,0,0) + g(t,x,y)$ does not fulfill neither the Lipschitz nor the classical Perron conditions, but the generalized Perron condition (43) is true with $\rho\,(y_1,y_2) = y_1 + 6y_2 + 1$ and $\sigma\,(t,y) = y$. The function $u\,(t,x,y) = \sin t \cos\,(x+y)$ is an analytic solution of (78), (79). Put $h_0 = h_1 = h_2 = 10^{-3}$. For each $t^{(\mu)}$ we use the method of an inverse matrix to solve the implicit difference scheme. Let ε_{max}, ε_{mean} be the largest and mean values, respectively, of the errors $|U - v|$ at time $t^{(\mu)}$.

$t^{(\mu)}$	ε_{max}	ε_{mean}
0.001	9.98×10^{-4}	7.03×10^{-4}
0.002	1.99×10^{-3}	1.40×10^{-3}
0.003	2.99×10^{-3}	2.11×10^{-3}
0.004	3.99×10^{-3}	2.81×10^{-3}
0.005	4.99×10^{-3}	3.51×10^{-3}
0.006	5.99×10^{-3}	4.22×10^{-3}
0.007	6.99×10^{-3}	4.92×10^{-3}
0.008	7.99×10^{-3}	5.63×10^{-3}
0.009	8.99×10^{-3}	6.33×10^{-3}
0.010	9.99×10^{-3}	7.03×10^{-3}

Table 3: Errors of the difference method with T_h

Note that the Courant-Friedrichs-Levy condition (72) for such steps is not satisfied and the explicit method given in [26] is not convergent. In fact, the errors ε_{max}, ε_{mean} of that method exceeded 10^{21} and 10^{20}, respectively.

The results shown in the tables are consistent with our mathematical analysis. The tables of errors are typical of difference methods.

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