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Asymptotic Formulae for Linear Combinations of Generalized Sampling Operators

Carlo Bardaro and Ilaria Mantellini

Abstract. We give some Voronovskaja formula for linear combinations of generalized sampling operators and we furnish also a quantitative version in terms of the classical Peetre K-functional. This provides a better order of approximation in the asymptotic formula. We apply the general theory to various kernels: Bochner-Riesz kernel, translates of B-splines and Jackson type kernel.

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1. Introduction

In the study of the order of approximation to a function f by sequences of linear operators T_n, it is well known that if T_n is positive the rate of (pointwise) convergence is at most $\mathcal{O}(n^{-2})$ for functions which are not smoother than C^2. The existence of derivatives of higher order of $f(x)$ cannot improve this order of approximation. This is due to the Korovkin theorem which states that the optimal rate of convergence cannot be faster than C^2-functions (see [1]). It is therefore interesting to construct (non-positive) linear operators to improve the order of approximation. A classical approach is based on the construction of suitable linear combinations of positive linear operators. This idea comes from the classical work of P. L. Butzer for Bernstein polynomials [13] and has then been developed by several authors (see e.g. [3, 18, 23, 28–30]).

In recent years we have developed a theory for the study of linear combinations of the Mellin integral operators which includes the Mellin-Gauss-Weierstrass operator, the Mellin-Picard operator, moment operator and others,

C. Bardaro, I. Mantellini: Department of Mathematics and Informatics, University of Perugia, Via Vanvitelli 1, 06123 Perugia, Italy;
bardaro@unipg.it; mantell@dmi.unipg.it

see [8,9,11,12,22]. This approach for integral operators is very suitable because from the original kernel, we obtain a new kernel, not necessarily positive, which generate a Mellin operator of the same kind.

Now we apply the same approach in the study of linear combinations of certain discrete operators not necessarily positive. An important class of such operators is given by the generalized sampling series introduced by P. L. Butzer and his school in Aachen (see [15,16,25]), which have fundamental applications in signal processing, in particular in linear prediction by samples from the past of the signal to reconstruct. These operators are defined by

$$(G_n f)(x) = \sum_{k=-\infty}^{+\infty} \varphi\left(n\left(x - \frac{k}{n}\right)\right) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N},\ x \in \mathbb{R},$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function and the signal f belongs to suitable function spaces.

Thus, it is certainly of interest to investigate the order of pointwise or uniform convergence of $G_n f$ to f. Under certain moment conditions on the kernel function φ, in [16] it is proved that the rate of uniform convergence, in the space of all the uniformly continuous and bounded functions on $\mathbb{R}$, may be of order $\mathcal{O}(n^{-r})$, for any integer $r \in \mathbb{N}$. However, largest is the constant r, the harder it is to build examples of kernels which satisfy the above mentioned moment conditions. These examples are mainly costructed by means of the solutions of certain linear systems based on the Poisson summation formula (see [14,16]). In the present paper we consider a simple approach for the construction of linear combinations of generalized sampling operators, which give a better order of pointwise or uniform convergence. In particular we are interested in the asymptotic behaviour, which gives Voronovskaja type formulae for these combinations. Given $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{R} \setminus \{0\}$ such that $\alpha_1 + \cdots + \alpha_s = 1$, we define the operator:

$$(G_n^s f)(x) = \sum_{i=1}^{s} \alpha_i (G_{in} f)(x),$$

and we look for coefficients α_i such that certain moments of higher order are null. In this way we obtain a linear system whose solution gives an operator with a high order of approximation. We apply this method to suitable particular cases: we examine the Bochner-Riesz kernel, a Jackson type kernel and a kernel defined by combinations of translates of central B-splines. The latter operators are of interest in the linear prediction theory, since they have a compact support. We furnish also certain quantitative versions of the asymptotic formulae, in terms of the Peetre K-functional (see [2, 17, 21, 24]), using an approach introduced in [19].

Note that the new operators $G_n^s f$ are not of the same type of the original ones, because they are not generated by a single kernel function φ, unlike what happens for Mellin operators. Indeed, in Mellin frame, a linear combination produces a new kernel which generates another (non-positive) Mellin operator.

2. Convergence properties of generalized sampling series

Let $L^\infty(\mathbb{R})$ be the space of all the essentially bounded real functions defined on $\mathbb{R}$ and by $C^0 = C^0(\mathbb{R})$ the subspace of all uniformly continuous and bounded functions $f : \mathbb{R} \to \mathbb{R}$, provided with the usual supnorm $\|f\|_\infty$. For $k \geq 1$ by $C^k = C^k(\mathbb{R})$ we denote the space whose elements f are k-times continuosly differentiable and $f^{(k)} \in C^0$.

Let $\varphi \in C^0$ be fixed. For any $\nu \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $u \in \mathbb{R}$ let us define

$$m_\nu(\varphi, u) := \sum_{k=-\infty}^{+\infty} \varphi(u-k)(k-u)^\nu,$$

$$M_\nu(\varphi) := \sup_{u\in\mathbb{R}} \sum_{k=-\infty}^{+\infty} |\varphi(u-k)||k-u|^\nu.$$

Let the function φ satisfy the following assumptions:

i) for every $u \in \mathbb{R}$ we have

$$m_0(\varphi, u) = \sum_{k=-\infty}^{+\infty} \varphi(u-k) = 1,$$

ii) for every $u \in \mathbb{R}$ we have for $\nu = 1, 2, \ldots, r-1$,

$$m_\nu(\varphi, u) = 0, \quad m_r(\varphi, u) = A_r$$

for a given constant $A_r \in \mathbb{R} \setminus \{0\}$,

iii) $M_r(\varphi) < +\infty$ and

$$\lim_{w\to+\infty} \sum_{|k-u|>w} |\varphi(u-k)||k-u|^r = 0$$

uniformly with respect to $u \in \mathbb{R}$.

From now on we will write $m_\nu(\varphi) = m_\nu(\varphi, u)$ for $\nu \in \mathbb{N}_0$ and $u \in \mathbb{R}$. Note that (see [6]) for $\mu,\ \nu \in \mathbb{N}_0$ with $\mu < \nu$, $M_\nu(\varphi) < +\infty$ implies $M_\mu(\varphi) < +\infty$. When φ has compact support, we immediately have that $M_\nu(\varphi) < +\infty$ for every $\nu \in \mathbb{N}_0$ and iii) holds.

For $n \in \mathbb{N}$, the generalized sampling operator generated by φ is defined as (see e.g. [15, 16, 25])

$$(G_n f)(x) = \sum_{k=-\infty}^{+\infty} \varphi\left(n\left(x - \frac{k}{n}\right)\right) f\left(\frac{k}{n}\right), \quad x \in \mathbb{R}.$$

Under the above assumptions we have $L^\infty(\mathbb{R}) \subset \bigcap_{n\in\mathbb{N}} \operatorname{Dom} G_n$ where $\operatorname{Dom} G_n$ is the space of all functions $f : \mathbb{R} \to \mathbb{R}$ for which the series defining $G_n f(x)$ is absolutely convergent for every $x \in \mathbb{R}$.

At first we consider some quantitative estimates of the uniform convergence in the space C^0 in terms of the classical modulus of continuity. For $\varepsilon > 0$, we define the (usual) modulus of continuity of C^0 by

$$\omega(f,\varepsilon) := \sup_{|x-y|<\varepsilon} |f(x) - f(y)|.$$

We have the following theorem (similar estimates can be found in [10]).

Theorem 2.1. *Let $f \in C^0$, $\delta > 0$ and $j \in \mathbb{N}$ be fixed. If Assumption* i) *holds and $M_j(\varphi) < +\infty$, we have*

$$\|G_n f - f\|_\infty \leq M_0(\varphi)\omega(f,\delta) + \frac{2\|f\|_\infty}{n^j\delta^j} M_j(\varphi).$$

Proof. For every $x \in \mathbb{R}$, we obtain

$$\begin{aligned}
|(G_n f)(x) - f(x)| &\leq \sum_{k=-\infty}^{+\infty} |\varphi(nx-k)| \left| f\left(\frac{k}{n}\right) - f(x)\right| \\
&\leq \sum_{|\frac{k}{n}-x|<\delta} |\varphi(nx-k)|\omega(f,\delta) + \sum_{|\frac{k}{n}-x|\geq\delta} |\varphi(nx-k)| \left| f\left(\frac{k}{n}\right) - f(x)\right| \\
&\leq M_0(\varphi)\omega(f,\delta) + 2\|f\|_\infty \sum_{|k-nx|\geq n\delta} |\varphi(nx-k)| \frac{|nx-k|^j}{|nx-k|^j} \\
&\leq M_0(\varphi)\omega(f,\delta) + \frac{2\|f\|_\infty}{n^j\delta^j} M_j(\varphi).
\end{aligned}$$

Thus the assertion follows from the arbitrariness of x. □

Another result is the following.

Theorem 2.2. *Let $f \in C^0$, $\delta > 0$. If Assumption* i) *holds and $M_1(\varphi) < +\infty$, we have*

$$\|G_n f - f\|_\infty \leq \omega(f,\delta)\left(M_0(\varphi) + \frac{M_1(\varphi)}{n\delta}\right).$$

Proof. Since $\omega(f,\lambda\delta) \leq (\lambda+1)\omega(f,\delta)$, for $\lambda > 0$, we have

$$\begin{aligned}
|(G_n f)(x) - f(x)| &\leq \sum_{k=-\infty}^{+\infty} |\varphi(nx-k)|\, \omega\left(f, \left|\frac{k}{n} - x\right|\right) \\
&\leq \sum_{k=-\infty}^{+\infty} |\varphi(nx-k)| \left(1 + \frac{|\frac{k}{n} - x|}{\delta}\right) \omega(f,\delta) \\
&\leq \omega(f,\delta)\left(M_0(\varphi) + \frac{M_1(\varphi)}{n\delta}\right).
\end{aligned}$$

Thus the assertion follows from the arbitrariness of x. □

Remark 2.3. a) If we take, $\delta = \frac{1}{n}$ in Theorem 2.2, we get

$$\|G_n f - f\|_\infty \leq C\,\omega\left(f, \frac{1}{n}\right),$$

where $C = M_0(\varphi) + M_1(\varphi)$.

b) If $f \in C^r \cap C^0$, under the assumptions i), ii), iii), using the Taylor formula with integral remainder, we can obtain the following uniform estimate (see also [16]):

$$\|G_n f - f\|_\infty \leq \frac{M_r(\varphi)}{n^r\ r!}\|f^{(r)}\|_\infty.$$

Now we state an asymptotic formula for $G_n f$, for a particular case see [5,6], in which only the case $r = 2$ is studied.

Theorem 2.4. *Let $f \in L^\infty(\mathbb{R})$ be a function such that $f^{(r)}(x)$ exists at a point $x \in \mathbb{R}$. Under the Assumptions* i)–iii) *we obtain*

$$\lim_{n\to+\infty} n^r[(G_n f)(x) - f(x)] = A_r \frac{f^{(r)}(x)}{r!}.$$

Proof. Using the local Taylor formula for the function f, there exists a bounded function h such that $\lim_{y\to 0} h(y) = 0$ and

$$f\left(\frac{k}{n}\right) = f(x)+f'(x)\left(\frac{k}{n} - x\right)+\cdots+\frac{f^{(r)}(x)}{r!}\left(\frac{k}{n} - x\right)^r+h\left(\frac{k}{n} - x\right)\left(\frac{k}{n} - x\right)^r.$$

Thus we have

$$n^r[(G_n f)(x) - f(x)] = A_r\frac{f^{(r)}(x)}{r!} + \sum_{k=-\infty}^{+\infty} \varphi\left(n\left(x - \frac{k}{n}\right)\right) h\left(\frac{k}{n} - x\right)(k - nx)^r.$$

Now we estimate the term

$$I := \sum_{k=-\infty}^{+\infty} \varphi(nx - k) h\left(\frac{k}{n} - x\right)(k - nx)^r.$$

Let $\varepsilon > 0$ be fixed. There exists $\delta > 0$ such that $|h(y)| \leq \varepsilon$ for every $|y| \leq \delta$. Moreover there exists $\overline{w} > 0$ such that for every $w > \overline{w}$

$$\sum_{|k-u|>w} |\varphi(u - k)||k - u|^r < \varepsilon$$

uniformly with respect to $u \in \mathbb{R}$. Let $\overline{n}$ be such that $\delta n > \overline{w}$, for every $n > \overline{n}$, so we have

$$\begin{aligned}
|I| &\leq \sum_{|\frac{k}{n}-x|<\delta} \left|\varphi(nx-k)h\left(\frac{k}{n}-x\right)\right| |k-nx|^r \\
&\quad + \sum_{|\frac{k}{n}-x|\geq\delta} \left|\varphi(nx-k)h\left(\frac{k}{n}-x\right)\right| |k-nx|^r \\
&\leq \varepsilon M_r(\varphi) + \|h\|_\infty \sum_{|k-nx|\geq n\delta} |\varphi(nx-k)||k-nx|^r \\
&\leq \varepsilon(M_r(\varphi) + \|h\|_\infty).
\end{aligned}$$

Thus, passing to the limit, we have $\lim_{n\to+\infty} n^r[(G_nf)(x) - f(x)] = A_r \frac{f^{(r)}(x)}{r!}$, that is the assertion. □

Remark 2.5. Following the same reasoning as in [6], we can relax the boundedness assumption on f assuming that there are $r+1$ positive constants a_0, $a_1, \ldots, a_r$ such that

$$|f(x)| \leq a_0 + a_1x + \cdots + a_rx^r, \quad \text{for every } x \in \mathbb{R}.$$

Our next aim is to determine the order of approximation in Theorem 2.4, using the classical Peetre K-functional defined by ([24]):

$$K(\varepsilon, f) \equiv K(\varepsilon, f, C^0, C^1) := \inf\{\|f-g\|_\infty + \varepsilon\|g'\|_\infty : g \in C^1\}$$

for $f \in C^0$ and $\varepsilon \geq 0$. The K-functional is related to the modulus of continuity ω, by means of the following lemma (see [2, 17, 24]).

Lemma 2.6. *For every $f \in C^0$ we have*

$$K\left(\frac{\varepsilon}{2}, f, C^0, C^1\right) = \frac{1}{2}\widetilde{\omega}(f,\varepsilon), \quad \varepsilon \geq 0.$$

Here $\widetilde{\omega}(f,\cdot)$ denotes the least concave majorant of $\omega(f,\cdot)$, (see [2, 17]).

Our next result is based on the following version of the Taylor formula (see [19]):

$$f(x) = \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_m(f;x_0,x),$$

for $x_0, x \in \mathbb{R},\ m \geq 1$ in which the remainder $R_m(f;x_0,x)$ is estimated by

$$|R_m(f;x_0,x)| \leq \frac{|x-x_0|^m}{m!}\,\omega(f^{(m)};|x-x_0|).$$

In [19] the following estimate of the remainder $R_m(f;x_0,x)$ in terms of $\widetilde{\omega}$, is also established:

Lemma 2.7. *For $m \in \mathbb{N}_0$ let $f \in C^m$ and $x, x_0 \in \mathbb{R}$. Then we have*

$$|R_m(f; x_0, x)| \le \frac{|x - x_0|^m}{m!}\, \widetilde{\omega}\bigg(f^{(m)}, \frac{|x - x_0|}{m+1}\bigg).$$

We now give a quantitative version of Theorem 2.4.

Theorem 2.8. *Let $f \in C^r$ and $x \in \mathbb{R}$ be fixed. Under the assumptions of Theorem 2.4, if moreover $M_{r+1}(\varphi) < +\infty$ we have*

$$\left| n^r[(G_n f)(x) - f(x)] - A_r \frac{f^{(r)}}{r!} \right| \le \frac{M_r(\varphi)}{r!}\, \widetilde{\omega}\bigg(f^{(r)}, \frac{1}{r+1} \frac{1}{n} \frac{M_{r+1}(\varphi)}{|A_r|}\bigg).$$

Proof. The proof follows by the same arguments as in [6, Theorem 2]. □

3. Linear combinations of generalized sampling series

In this section, using a direct method, we will construct suitable linear combinations of generalized sampling operators in order to improve the order of approximation in Voronovskaja formula. In what follows, we will assume that the moments $m_\nu(\varphi)$ of the function φ are independent of $u \in \mathbb{R}$ for every $\nu \in \mathbb{N}$, when they exist.

3.1. General theory. Let $\alpha_i,\ i = 1, \dots, s$ be non-zero real numbers such that $\alpha_1 + \dots + \alpha_s = 1$ and for $n \in \mathbb{N}$ let us consider the following linear combination

$$(G_n^s f)(x) = \sum_{i=1}^{s} \alpha_i (G_{in} f)(x), \quad x \in \mathbb{R}.$$

We define for $\nu \in \mathbb{N}_0$ and $x \in \mathbb{R}$

$$\mathcal{M}_\nu^s(\varphi, n) := \sum_{i=1}^{s} \alpha_i \sum_{k=-\infty}^{+\infty} \varphi(inx - k) \left(\frac{k}{in} - x\right)^\nu.$$

Note that

$$\mathcal{M}_\nu^s(\varphi, n) = m_\nu(\varphi) \sum_{i=1}^{s} \frac{\alpha_i}{(in)^\nu}.$$

By the assumption ii) we have that $\mathcal{M}_\nu^s(\varphi, n) = 0$, for $\nu = 1, \dots, r-1$.

Concerning the uniform estimate with the modulus of continuity ω, we have the following:

Corollary 3.1. *Under the assumptions of Theorem 2.2, there is an absolute constant $D > 0$ such that*

$$\|G_n^s f - f\|_\infty \le D\, \omega\bigg(f, \frac{1}{n}\bigg).$$

Proof. By Theorem 2.2 with $\delta = \frac{1}{in}$, there is a constant $C > 0$ independent of i such that

$$\|G_n^s f - f\|_\infty \leq \sum_{i=1}^{s} |\alpha_i| \|G_{in} f - f\|_\infty \leq C \sum_{i=1}^{s} |\alpha_i| \, \omega\left(f, \frac{1}{in}\right) \leq D \, \omega\left(f, \frac{1}{n}\right),$$

where $D := C \sum_{i=1}^{s} |\alpha_i|$. □

Note that also in this case, we can obtain an analogous uniform estimate for functions $f \in C^r$ (see Remark 2.3b)).

Now we show that the above linear combinations provide a better order in the Voronovskaja formula. In particular we will obtain an order strictly greater than r in Theorem 2.4 for functions belonging to C^j for $j > r$.

Theorem 3.2. *Let $f \in L^\infty(\mathbb{R})$ be a function such that $f^{(j)}(x)$ exists at a point $x \in \mathbb{R}$ with $j \geq r$. Under the assumptions of Theorem 2.4, if moreover $M_j(\varphi)$ is finite for every $\nu = r+1, \ldots, j$ then*

$$(G_n^s f)(x) - f(x) = \sum_{\nu=r}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \sum_{i=1}^{s} \frac{\alpha_i}{i^\nu} + o(n^{-j}), \quad n \to +\infty.$$

Proof. Following similar arguments as in Theorem 2.4, using the local Taylor formula for the function f, there exists a bounded function h such that $\lim_{y \to 0} h(y) = 0$ and

$$f\left(\frac{k}{in}\right) = f(x) + f'(x)\left(\frac{k}{in} - x\right) + \cdots + \frac{f^{(j)}(x)}{j!}\left(\frac{k}{in} - x\right)^j + h\left(\frac{k}{in} - x\right)\left(\frac{k}{in} - x\right)^j.$$

We have

$$\begin{aligned}
&(G_n^s f)(x) - f(x) \\
&= \sum_{i=1}^{s} \alpha_i \sum_{k=-\infty}^{+\infty} \varphi(inx - k) \left(f\left(\frac{k}{in}\right) - f(x) \right) \\
&= \sum_{\nu=1}^{j} \frac{f^{(\nu)}(x)}{\nu!} \mathcal{M}_\nu^s(\varphi, n) + \sum_{i=1}^{s} \alpha_i \sum_{k=-\infty}^{+\infty} \varphi(inx - k) \, h\left(\frac{k}{in} - x\right) \left(\frac{k}{in} - x\right)^j \\
&= \sum_{\nu=r}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \sum_{i=1}^{s} \frac{\alpha_i}{i^\nu} + I
\end{aligned}$$

where

$$I = \sum_{i=1}^{s} \alpha_i \sum_{k=-\infty}^{+\infty} \varphi(inx - k) \, h\left(\frac{k}{in} - x\right) \left(\frac{k}{in} - x\right)^j.$$

Now we evaluate I. Let $\varepsilon > 0$ be fixed. There exists $\delta > 0$ such that $|h(y)| \leq \varepsilon$ for every $|y| \leq \delta$. So, using the same method as in Theorem 2.4, we get

$$|I| \leq \sum_{i=1}^{s} \frac{|\alpha_i|}{(in)^j} (\varepsilon M_j(\varphi) + \|h\|_\infty o(1)) = o(n^{-j}). \qquad \square$$

Now, we assume that $\mathcal{M}_\nu^s(\varphi, n) = 0$, for $\nu = r, \ldots, j-1$, for an integer $j > r$ and $\mathcal{M}_j^s(\varphi, n) \neq 0$. This implies that $m_j(\varphi) := A_j \neq 0$. In this instance, from Theorem 3.2 we can obtain the following Voronovskaja formula, for functions f such that $f^{(j)}$ exists at the point x:

$$\lim_{n \to +\infty} n^j [(G_n^s f)(x) - f(x)] = \frac{f^{(j)}(x)}{j!} A_j \sum_{i=1}^{s} \frac{\alpha_i}{i^j}.$$

As in Section 2 we can obtain the following quantitative version of the above Voronovskaja formula.

Theorem 3.3. *Let $f \in C^j$, $j > r$ and $x \in \mathbb{R}$ be fixed. Under the assumptions of Theorem 3.2, if moreover $M_{j+1}(\varphi) < +\infty$ we have*

$$\left| n^j [(G_n^s f)(x) - f(x)] - \frac{f^{(j)}(x)}{j!} A_j \sum_{i=1}^{s} \frac{\alpha_i}{i^j} \right| \leq \frac{2E}{j!} M_j(\varphi) K\left(\frac{M_{j+1}(\varphi)}{2|A_j|(j+1)n}, f^{(j)} \right)$$

where $E := \sum_{i=1}^{s} |\alpha_i|$.

Proof. We have for $x \in \mathbb{R}$

$$\begin{aligned} &\left| (G_n^s f)(x) - f(x) - \frac{f^{(j)}(x)}{j! n^j} A_j \sum_{i=1}^{s} \frac{\alpha_i}{i^j} \right| \\ &= \left| \sum_{i=1}^{s} \alpha_i \sum_{k=-\infty}^{+\infty} \varphi(inx - k)\, h\left(\frac{k}{in} - x \right) \left(\frac{k}{in} - x \right)^j \right|. \end{aligned}$$

Now, putting $R_j(f, x, \frac{k}{in}) = h(\frac{k}{in} - x)(\frac{k}{in} - x)^j$ and using Lemma 2.7, we get

$$\begin{aligned} I &:= \left| \sum_{i=1}^{s} \alpha_i \sum_{k=-\infty}^{+\infty} \varphi(inx - k)\, h\left(\frac{k}{in} - x \right) \left(\frac{k}{in} - x \right)^j \right| \\ &\leq \sum_{i=1}^{s} |\alpha_i| \sum_{k=-\infty}^{+\infty} |\varphi(inx - k)| \frac{|\frac{k}{in} - x|^j}{j!}\, \widetilde{\omega}\left(f^{(j)}, \frac{|\frac{k}{in} - x|}{j+1} \right) \\ &= 2 \sum_{i=1}^{s} |\alpha_i| \sum_{k=-\infty}^{+\infty} |\varphi(inx - k)| \frac{|\frac{k}{in} - x|^j}{j!}\, K\left(\frac{|\frac{k}{in} - x|}{2(j+1)}, f^{(j)} \right). \end{aligned}$$

For any $g \in C^{j+1}$ we have

$$\begin{aligned} I &\leq 2\sum_{i=1}^{s}|\alpha_i| \sum_{k=-\infty}^{+\infty} |\varphi(inx-k)| \frac{|\frac{k}{in}-x|^j}{j!}\left(\|(f-g)^{(j)}\|_\infty + \frac{|\frac{k}{in}-x|}{2(j+1)}\|g^{(j+1)}\|_\infty\right) \\ &= \frac{2}{j!}\sum_{i=1}^{s}|\alpha_i| \sum_{k=-\infty}^{+\infty} |\varphi(inx-k)|\left|\frac{k}{in}-x\right|^j \\ &\times \left(\|(f-g)^{(j)}\|_\infty + \frac{\|g^{(j+1)}\|_\infty}{2(j+1)} \frac{\sum_{k=-\infty}^{+\infty}|\varphi(inx-k)||\frac{k}{in}-x|^{j+1}}{\sum_{k=-\infty}^{+\infty}|\varphi(inx-k)||\frac{k}{in}-x|^{j}}\right). \end{aligned}$$

Taking the infimum over all the functions $g \in C^{j+1}$ we finally get

$$I \leq \frac{2E}{j!\, n^j} M_r(\varphi) K\left(\frac{M_{j+1}(\varphi)}{2|A_j|(j+1)n}, f^{(j)}\right). \qquad \square$$

3.2. The construction of specific linear combinations. Let $\nu_1,\dots,\nu_{s-1} \in \mathbb{N}$, with $\nu_1 = r < \nu_2 < \cdots < \nu_{s-1}$, be integers such that $m_{\nu_j}(\varphi) \neq 0$, $j = 1,\dots,s-1$, and assume that $m_\nu(\varphi) = 0$, for $r < \nu < \nu_{s-1}$, $\nu \neq \nu_i$. We look for constants α_i in such a way that $\mathcal{M}^s_{\nu_j}(\varphi, n) = 0$ for $j = 1, \dots s-1$. We obtain the following linear system

$$\begin{cases} \displaystyle\sum_{i=1}^{s} \alpha_i = 1 \\ \displaystyle\sum_{i=1}^{s} \frac{\alpha_i}{i^r} = 0 \\ \qquad \vdots \\ \displaystyle\sum_{i=1}^{s} \frac{\alpha_i}{i^{\nu_{s-1}}} = 0. \end{cases}$$

The solution gives a linear combination with order at least $\nu_{s-1}+1$ for functions f for which $f^{(\nu_{s-1}+1)}(x)$ exists.

For example, we examine the case $s = 2$. In this instance the linear system reduces to

$$\begin{cases} \alpha_1 + \alpha_2 = 1 \\ \alpha_1 + \dfrac{\alpha_2}{2^r} = 0 \end{cases}$$

with solution $\overline{\alpha}_1 = -\frac{1}{2^r-1}$, $\overline{\alpha}_2 = \frac{2^r}{2^r-1}$. So

$$(G_n^2 f)(x) = -\frac{1}{2^r-1}(G_n f)(x) + \frac{2^r}{2^r-1}(G_{2n} f)(x)$$

and for every $j \geq r+1$ we obtain, for functions f for which $f^{(j)}(x)$ exists,

$$(G_n^2 f)(x) - f(x) = \sum_{\nu=r+1}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \left(-\frac{1}{2^r - 1} + \frac{2^{r-\nu}}{2^r - 1} \right) + o(n^{-j})$$
$$= \sum_{\nu=r+1}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \frac{2^{r-\nu} - 1}{2^r - 1} + o(n^{-j}).$$

In particular, for $j = r+1$, we get

$$\lim_{n \to +\infty} n^{r+1}((G_n^2 f)(x) - f(x)) = -\frac{f^{(r+1)}(x)}{(r+1)!} \frac{1}{2(2^r - 1)} m_{r+1}(\varphi).$$

Concerning the case $s = 3$, putting $\nu_1 = r$ and denoting $\nu_2 = \mu > r$, the linear system reduces to

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 1 \\ \alpha_1 + \dfrac{\alpha_2}{2^r} + \dfrac{\alpha_3}{3^r} = 0 \\ \alpha_1 + \dfrac{\alpha_2}{2^\mu} + \dfrac{\alpha_3}{3^\mu} = 0 \end{cases}$$

with solution $\overline{\alpha}_1 = \frac{2^{\mu-r} - 3^{\mu-r}}{H}$, $\overline{\alpha}_2 = \frac{2^\mu(3^{\mu-r}-1)}{H}$, $\overline{\alpha}_3 = \frac{3^\mu(1-2^{\mu-r})}{H}$, where we have put $H := 2^{\mu-r} - 3^{\mu-r} + 2^\mu \cdot 3^{\mu-r} - 2^\mu + 3^\mu - 3^\mu 2^{\mu-r}$. So we obtain the linear combination

$$(G_n^3 f)(x) = \overline{\alpha}_1 (G_n f)(x) + \overline{\alpha}_2 (G_{2n} f)(x) + \overline{\alpha}_3 (G_{3n} f)(x)$$

and for every $j \geq \mu + 1$ we get, for functions f for which $f^{(j)}(x)$ exists,

$$(G_n^3 f)(x) - f(x)$$
$$= \sum_{\nu=\mu+1}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \left(\overline{\alpha}_1 + \frac{\overline{\alpha}_2}{2^\nu} + \frac{\overline{\alpha}_3}{3^\nu} \right) + o(n^{-j})$$
$$= \sum_{\nu=\mu+1}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \frac{1}{H} \left(2^{\mu-r} - 3^{\mu-r} + \frac{3^{\mu-r} - 1}{2^{\nu-\mu}} + \frac{1 - 2^{\mu-r}}{3^{\nu-\mu}} \right) + o(n^{-j}).$$

In particular if $j = \mu + 1$ we obtain

$$\lim_{n \to +\infty} n^{\mu+1}((G_n^3 f)(x) - f(x)) = \frac{f^{(\mu+1)}(x)}{(\mu+1)!} \left(\frac{2^{\mu-r+2} - 3^{\mu-r+1} - 1}{6H} \right) m_{\mu+1}(\varphi).$$

4. Examples

In this section we will apply the previous theory to various specific examples of interest in sampling and prediction theory. We limit ourselves to the Voronovskaja formulae. Their quantitative versions are obtained in a similar way. In the following, given a function $g \in L^1(\mathbb{R})$, we define the Fourier transform of g as

$$\widehat{g}(v) = \int_{-\infty}^{+\infty} g(u)e^{-iuv}du, \quad v \in \mathbb{R}.$$

I) *Bochner-Riesz kernel.* Let us consider the Bochner-Riesz kernel defined by (see e.g. [14, 27])

$$\varphi(x) \equiv b^\gamma(x) = \frac{2^\gamma}{\sqrt{2\pi}}\Gamma(\gamma+1)(|x|)^{-\frac{1}{2}-\gamma}J_{\frac{1}{2}+\gamma}(|x|)$$

for $\gamma > 0$, where J_λ is the Bessel function of order λ. It is well known that

$$\widehat{b^\gamma}(v) = \begin{cases} (1-v^2)^\gamma, & |v| \leq 1 \\ 0, & |v| > 1. \end{cases}$$

Using the Poisson summation formula

$$(-i)^j \sum_{k=-\infty}^{+\infty} \varphi(u-k)(u-k)^j \sim \sum_{k=-\infty}^{+\infty} \widehat{\varphi}^{(j)}(2\pi k)e^{i2\pi ku},$$

we have for $u \in \mathbb{R}$

$$m_0(b^\gamma) = \sum_{k=-\infty}^{+\infty} b^\gamma(u-k) = \widehat{b^\gamma}(0) = 1$$

and

$$m_1(b^\gamma) = -\sum_{k=-\infty}^{+\infty} b^\gamma(u-k)(u-k) = 0, \quad m_2(b^\gamma) = \sum_{k=-\infty}^{+\infty} b^\gamma(u-k)(u-k)^2 = 2\gamma.$$

Moreover, in [6] it is shown that for $\gamma > 3$ we have $M_3(b^\gamma) < +\infty$ and

$$\lim_{r\to+\infty} \sum_{k\notin U_r(u)} |b^\gamma(u-k)|(u-k)^2 = 0,$$

uniformly with respect to $u \in \mathbb{R}$, so condition iii) holds for $r = 2$ and as a consequence the following Voronovskaja formula for the Bochner-Riesz sampling operators, denoted by B_n is given $\lim_{n\to+\infty} n^2((B_nf)(x) - f(x)) = \gamma f''(x)$, provided that f'' exists at the point x.

Concerning the linear combinations, we begin with the case $s = 2$. Using the Poisson formula, we easily get

$$m_3(b^\gamma) = -\sum_{k=-\infty}^{+\infty} b^\gamma(u-k)(u-k)^3 = 0, \quad m_4(b^\gamma) = \sum_{k=-\infty}^{+\infty} b^\gamma(u-k)(u-k)^4 = 12\gamma(\gamma-1).$$

Then, from the general theory, we have, for functions f for which $f^{(j)}(x)$ exists, $j > 3$,

$$\begin{aligned}(B_n^2 f)(x) - f(x) &= \sum_{\nu=3}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \left(-\frac{1}{3} + \frac{2^{2-\nu}}{3} \right) + o(n^{-j}) \\ &= \sum_{\nu=4}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \frac{2^{2-\nu} - 1}{3} + o(n^{-j}).\end{aligned}$$

In particular, for $j = 4$, $\lim_{n\to+\infty} n^4((B_n^2 f)(x) - f(x)) = -\frac{f^{(4)}(x)}{8}\gamma(\gamma-1)$ follows.

Now we consider the case $s = 3$. Since we have $m_3(b^\gamma) = 0$, in this case we look for coefficients $\alpha_1, \alpha_2, \alpha_3$ such that $\mathcal{M}_2^3(b^\gamma, n) = \mathcal{M}_4^3(b^\gamma, n) = 0$, i.e., we put $\nu_1 = r = 2, \nu_2 = \mu = 4$. We obtain the system

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 1 \\ \alpha_1 + \dfrac{\alpha_2}{4} + \dfrac{\alpha_3}{9} = 0 \\ \alpha_1 + \dfrac{\alpha_2}{16} + \dfrac{\alpha_3}{81} = 0. \end{cases}$$

Since in this instance $H = -120$, its solution is given by $\overline{\alpha}_1 = \frac{1}{24}$, $\overline{\alpha}_2 = -\frac{16}{15}$, $\overline{\alpha}_3 = \frac{81}{40}$. We obtain the linear combination

$$(B_n^3 f)(x) = \frac{1}{24}(B_n f)(x) - \frac{16}{15}(B_{2n} f)(x) + \frac{81}{40}(B_{3n} f)(x).$$

Again, by the Poisson formula, we have

$$m_5(b^\gamma) = -\sum_{k=-\infty}^{+\infty} b^\gamma(u-k)(u-k)^5 = 0$$

and

$$m_6(b^\gamma) = \sum_{k=-\infty}^{+\infty} b^\gamma(u-k)(u-k)^6 = 120\gamma(\gamma-1)(\gamma-2).$$

Then we have, for $j \geq 6$,

$$(B_n^3 f)(x) - f(x) = \sum_{\nu=6}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \left(\frac{1}{24} - \frac{1}{15 \cdot 2^{\nu-4}} + \frac{1}{40 \cdot 3^{\nu-4}} \right) + o(n^{-j}).$$

For $j = 6$ we get $\lim_{n\to+\infty} n^6((B_n^3 f)(x) - f(x)) = \frac{f^{(6)}(x)}{216}\gamma(\gamma-1)(\gamma-2)$.

II) *Translates of central B-splines.* Let us consider the central B-splines of order $h \in \mathbb{N}$ defined by

$$B_h(x) := \frac{1}{(h-1)!}\sum_{\ell=0}^{h}(-1)^\ell \begin{pmatrix} h \\ \ell \end{pmatrix} \left(\frac{h}{2} + x - \ell\right)_+^{h-1}$$

where $x_+^r := \max\{x^r, 0\}$. It is well known that the Fourier transform of the functions B_h is given by

$$\widehat{B}_h(v) = \left(\frac{\sin\frac{v}{2}}{\frac{v}{2}}\right)^h, \quad v \in \mathbb{R},\ h \in \mathbb{N}$$

(see [16, 26]). We apply the previous theory to the kernel function defined by

$$\varphi(x) = \left(3 - \frac{h}{24}\right) B_h(x-1) + \left(\frac{h}{12} - 3\right) B_h(x-2) + \left(1 - \frac{h}{24}\right) B_h(x-3)$$

for a fixed $h \geq 6$. The Fourier transform of φ is given by

$$\widehat{\varphi}(v) = \widehat{B}_h(v)\left(\left(3 - \frac{h}{24}\right) e^{-iv} + \left(\frac{h}{12} - 3\right) e^{-2iv} + \left(1 - \frac{h}{24}\right) e^{-3iv}\right).$$

By elementary calculations we have

$$\widehat{\varphi}(0) = 1,\ \widehat{\varphi}'(0) = 0,\ \widehat{\varphi}''(0) = 0,\ \widehat{\varphi}'''(0) = \left(6 - \frac{h}{2}\right) i,\ \widehat{\varphi}^{(4)}(0) = -\frac{h^2}{48} - \frac{251h}{120} + 36,$$

$$\widehat{\varphi}^{(5)}(0) = \left(\frac{5}{12}h^2 + \frac{5}{2}h - 150\right) i, \quad \widehat{\varphi}^{(6)}(0) = -540 - \frac{1255}{63}h + \frac{125}{48}h^2 + \frac{5}{288}h^3.$$

Then, since $\widehat{\varphi}^{(j)}(2k\pi) = 0$ for every $k \neq 0$, we have by the Poisson summation formula

$$m_1(\varphi) = m_2(\varphi) = 0, \quad m_3(\varphi) = -6 + \frac{h}{2}, \quad m_4(\varphi) = -\frac{h^2}{48} - \frac{251h}{120} + 36$$

and

$$m_5(\varphi) = \frac{5h^2}{12} + \frac{5h}{2} - 150, \quad m_6(\varphi) = 540 + \frac{1255}{63}h - \frac{125}{48}h^2 - \frac{5}{288}h^3.$$

Since φ has compact support contained in the interval $[1 - \frac{h}{2}, 3 + \frac{h}{2}]$, we have that $M_\nu(\varphi) < +\infty$ for every $\nu \in \mathbb{N}$ and conditions ii) and iii) are satisfied with $r = 3$, if $h \neq 12$. In this case we obtain the following Voronovskaja formula for the sampling operators generated by φ, denoted by S_n :

$$\lim_{n\to+\infty} n^3((S_n f)(x) - f(x)) = \frac{h-12}{12} f'''(x),$$

provided that f''' exists at the point x.

If $h = 12$ then we obtain a formula of fourth order taking into account that in this instance $m_3(\varphi) = 0$. So, we have $r = 4$ and

$$\lim_{n\to+\infty} n^4((S_n f)(x) - f(x)) = \frac{79}{240} f^{(4)}(x),$$

provided that $f^{(4)}$ exists at the point x.

Concerning the linear combinations, we consider the case $s = 2$. If $h \neq 12$, we have $\nu_1 = r = 3$ and we look for coefficients α_1, α_2 such that $\mathcal{M}_3^2(\varphi, n) = 0$. The solutions are $\overline{\alpha}_1 = -\frac{1}{7}, \overline{\alpha}_2 = \frac{8}{7}$ and the operator takes the form

$$(S_n^2 f)(x) = -\frac{1}{7}(S_n f)(x) + \frac{8}{7}(S_{2n} f)(x).$$

Then, from the general theory, we have, for functions f for which $f^{(j)}(x)$ exists,

$$(S_n^2 f)(x) - f(x) = \sum_{\nu=4}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \frac{2^{3-\nu} - 1}{7} + o(n^{-j}).$$

In particular, for $j = 4$, $\lim_{n\to+\infty} n^4((S_n^2 f)(x) - f(x)) = -\frac{f^{(4)}(x)}{4!}\frac{1}{14}\left(36 - \frac{h^2}{48} - \frac{251h}{120}\right)$ follows.

If $h = 12$ we take $r = 4$ and so we look for coefficients α_1, α_2 such that $\mathcal{M}_4^2(\varphi, n) = 0$. In this case the solution is $\overline{\alpha}_1 = -\frac{1}{15}, \overline{\alpha}_2 = \frac{16}{15}$ and the operator takes the form

$$(S_n^2 f)(x) = -\frac{1}{15}(S_n f)(x) + \frac{16}{15}(S_{2n} f)(x).$$

Then, from the general theory if $j \geq 5$ we have, for functions f for which $f^{(j)}(x)$ exists,

$$(S_n^2 f)(x) - f(x) = \sum_{\nu=5}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \frac{2^{4-\nu} - 1}{15} + o(n^{-j}).$$

In particular, we obtain, for $j = 5$, $\lim_{n\to+\infty} n^5((S_n^2 f)(x) - f(x)) = \frac{f^{(5)}(x)}{60}$.

Now we consider the case $s = 3$. At first we consider $h \neq 12$. In this case we look for coefficients $\alpha_1, \alpha_2, \alpha_3$ such that $\mathcal{M}_3^3(\varphi, n) = \mathcal{M}_4^3(\varphi, n) = 0$, i.e., we put $\nu_1 = r = 3, \nu_2 = \mu = 4$. We obtain the system

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 1 \\ \alpha_1 + \dfrac{\alpha_2}{8} + \dfrac{\alpha_3}{27} = 0 \\ \alpha_1 + \dfrac{\alpha_2}{16} + \dfrac{\alpha_3}{81} = 0. \end{cases}$$

Since in this instance $H = -50$, its solution is given by $\overline{\alpha}_1 = \frac{1}{50}$, $\overline{\alpha}_2 = -\frac{16}{25}$, $\overline{\alpha}_3 = \frac{81}{50}$. We obtain the linear combination

$$(S_n^3 f)(x) = \frac{1}{50}(S_n f)(x) - \frac{16}{25}(S_{2n} f)(x) + \frac{81}{50}(S_{3n} f)(x).$$

Then we have, for $j \geq 5$,

$$(S_n^3 f)(x) - f(x) = \sum_{\nu=5}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \left(-\frac{1}{50}\right)\left(-1 + \frac{1}{2^{\nu-5}} - \frac{1}{3^{\nu-4}}\right) + o(n^{-j}).$$

For $j = 5$ we get $\lim_{n\to+\infty} n^5((S_n^3 f)(x) - f(x)) = \frac{f^{(5)}(x)}{18000}\left(\frac{5h^2}{12} + \frac{5h}{2} - 150\right)$.

Moreover if $h = 12$ we look for coefficients $\alpha_1, \alpha_2, \alpha_3$ such that $\mathcal{M}_4^3(\varphi, n) = \mathcal{M}_5^3(\varphi, n) = 0$, i.e., we put $\nu_1 = r = 4, \nu_2 = \mu = 5$. We obtain the system

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 1 \\ \alpha_1 + \dfrac{\alpha_2}{16} + \dfrac{\alpha_3}{81} = 0 \\ \alpha_1 + \dfrac{\alpha_2}{32} + \dfrac{\alpha_3}{243} = 0. \end{cases}$$

Since in this instance $H = -180$, its solution is given by $\overline{\alpha}_1 = \frac{1}{180}$, $\overline{\alpha}_2 = -\frac{16}{45}$, $\overline{\alpha}_3 = \frac{27}{20}$. We obtain the linear combination

$$(S_n^3 f)(x) = \frac{1}{180}(S_n f)(x) - \frac{16}{45}(S_{2n} f)(x) + \frac{27}{20}(S_{3n} f)(x).$$

Then we have, for $j \geq 6$,

$$(S_n^3 f)(x) - f(x) = \sum_{\nu=6}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(\varphi)}{n^\nu} \left(-\frac{1}{180}\right)\left(-1 + \frac{1}{2^{\nu-6}} - \frac{1}{3^{\nu-5}}\right) + o(n^{-j}).$$

For $h = 12$ we get $m_6(\varphi) = \frac{7855}{21}$ and so in particular, for $j = 6$, we obtain $\lim_{n\to+\infty} n^6((S_n^3 f)(x) - f(x)) = \frac{1571}{1632960} f^{(6)}(x)$.

The method can be applied to many other examples of this type, starting from combinations of translates of type

$$\varphi(x) = a_0 B_h(x - b_0) + a_1 B_h(x - b_1) + a_2 B_h(x - b_2)$$

for suitable real numbers a_0, a_1, a_2 and b_0, b_1, b_2 such that $a_0 + a_1 + a_2 = 1$ and $b_0 < b_1 < b_2$ (see [6]).

III) *Generalized Jackson kernel.* Let us consider the generalized Jackson operators with kernel

$$\varphi(x) = J_{\gamma,\beta}(x) = c_{\gamma,\beta}\mathrm{sinc}^{2\beta}\left(\frac{x}{2\gamma\beta\pi}\right),$$

with $x \in \mathbb{R}$, $\beta \in \mathbb{N}, \gamma \geq 1$, $c_{\gamma,\beta}$ is a normalization constant and sinc $u := \frac{\sin(\pi u)}{\pi u}$. It is well known (see [4, 14]) that $J_{\gamma,\beta}$ is bandlimited to the interval $[-\frac{1}{\gamma}, \frac{1}{\gamma}]$. In this instance, our operator takes the form

$$(J_{n,\gamma,\beta}f)(x) = \sum_{k=-\infty}^{+\infty} J_{\gamma,\beta}(nx-k)f\left(\frac{k}{n}\right), \quad n \in \mathbb{N},\ x \in \mathbb{R}.$$

Since $J_{\gamma,\beta}$ is bandlimited to $[-\frac{1}{\gamma}, \frac{1}{\gamma}]$ and using the Poisson summation formula we have that

$$(-i)^j \sum_{k=-\infty}^{+\infty} J_{\gamma,\beta}(u-k)(u-k)^j = \widehat{J}^{(j)}_{\gamma,\beta}(0), \quad u \in \mathbb{R}.$$

Therefore, see also [7], we easily get

$$m_1(J_{\gamma,\beta}) = -i\widehat{J}'_{\gamma,\beta}(0) = 0, \quad m_2(J_{\gamma,\beta}) = -\widehat{J}''_{\gamma,\beta}(0) = \int_{-\infty}^{+\infty} x^2 J_{\gamma,\beta}(x)dx =: A_{\gamma,\beta}$$

and

$$m_3(J_{\gamma,\beta}) = 0, \quad m_4(J_{\gamma,\beta}) = -\int_{-\infty}^{+\infty} x^4 J_{\gamma,\beta}(x)dx =: -B_{\gamma,\beta}.$$

Conditions i) and ii) are satisfied with $r = 2$. Moreover if $\beta \geq 3$ then (see [4, Remark 3.2(d)]), also condition iii) is satisfied and the following Voronovskaja formula for the generalized Jackson operator holds (see [7])

$$\lim_{n\to+\infty} n^2[(J_{n,\gamma,\beta}f)(x) - f(x)] = A_{\gamma,\beta}\frac{f''(x)}{2},$$

at every point $x \in \mathbb{R}$ in which $f''(x)$ exists.

As to the calculation of the constant $A_{\gamma,\beta}$ we have

$$A_{\gamma,\beta} = 16c_{\gamma,\beta}\beta^3\gamma^3 \int_0^{+\infty} \frac{\sin^{2\beta} x}{x^{2\beta-2}}dx.$$

For example, for $\beta = 3$, using [20, Formula 12, p. 454], we get $c_{\gamma,3} = \frac{10}{33\pi\gamma}$, $A_{\gamma,3} = \frac{180}{11}\gamma^2$. Concerning the linear combinations, we begin with the case $s = 2$. From the general theory we get $\overline{\alpha}_1 = -\frac{1}{3}$ and $\overline{\alpha}_2 = \frac{4}{3}$. So we have

$$(J^2_{n,\gamma,\beta}f)(x) = -\frac{1}{3}(J_{n,\gamma,\beta}f)(x) + \frac{4}{3}(J_{2n,\gamma,\beta}f)(x).$$

Then, from the general theory, for $j \geq 4$, we have, for functions f for which $f^{(j)}(x)$ exists,

$$(J^2_{n,\gamma,\beta}f)(x) - f(x) = \sum_{\nu=3}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(J_{\gamma,\beta})}{n^\nu} \left(-\frac{1}{3} + \frac{2^{2-\nu}}{3}\right) + o(n^{-j})$$
$$= \sum_{\nu=4}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(J_{\gamma,\beta})}{n^\nu} \frac{2^{2-\nu}-1}{3} + o(n^{-j}).$$

In particular, for $j = 4$, we get $\lim_{n\to+\infty} n^4((J^2_{n,\gamma,\beta}f)(x) - f(x)) = B_{\gamma,\beta}\frac{1}{96}f^{(4)}(x)$ at every point $x \in \mathbb{R}$ in which $f^{(4)}(x)$ exists.

As to the calculation of the constant $B_{\gamma,\beta}$ we have

$$B_{\gamma,\beta} = 64c_{\gamma,\beta}\beta^5\gamma^5 \int_0^{+\infty} \frac{\sin^{2\beta} x}{x^{2\beta-4}} dx.$$

For example, for $\beta = 3$, we get $c_{\gamma,3} = \frac{10}{33\pi\gamma}$, $B_{\gamma,3} = \frac{9720}{11}\gamma^4$.

Now we consider the case $s = 3$. Since we have $m_3(J_{\gamma,\beta}) = 0$, in this case we look for coefficients $\alpha_1, \alpha_2, \alpha_3$ such that $\mathcal{M}^3_2(J_{\gamma,\beta}, n) = \mathcal{M}^3_4(J_{\gamma,\beta}, n) = 0$, i.e., we put $\nu_1 = r = 2, \nu_2 = \mu = 4$. We obtain the same system as before with $H = -120$ and $\overline{\alpha}_1 = \frac{1}{24}$, $\overline{\alpha}_2 = -\frac{16}{15}$, $\overline{\alpha}_3 = \frac{81}{40}$. We obtain the linear combination

$$(J^3_{n,\gamma,\beta}f)(x) = \frac{1}{24}(J_{n,\gamma,\beta}f)(x) - \frac{16}{15}(J_{2n,\gamma,\beta}f)(x) + \frac{81}{40}(J_{3n,\gamma,\beta}f)(x).$$

Again, by the Poisson formula, we have for $\beta \geq 4$

$$m_5(J_{\gamma,\beta}) = 0, \quad m_6(J_{\gamma,\beta}) = \int_{-\infty}^{+\infty} x^6 J_{\gamma,\beta}(x)dx =: C_{\gamma,\beta}.$$

Then, for $j \geq 6$ and for every function f for which $f^{(j)}(x)$ exists

$$(J^3_{n,\gamma,\beta}f)(x) - f(x) = \sum_{\nu=6}^{j} \frac{f^{(\nu)}(x)}{\nu!} \frac{m_\nu(J_{\gamma,\beta})}{n^\nu} \left(\frac{1}{24} - \frac{1}{15\cdot 2^{\nu-4}} + \frac{1}{40\cdot 3^{\nu-4}}\right) + o(n^{-j}).$$

In particular, for $j = 6$, $\lim_{n\to+\infty} n^6((J^3_{n,\gamma,\beta}f)(x) - f(x)) = \frac{f^{(6)}(x)}{25920}C_{\gamma,\beta}$ follows.

As to the calculation of the constant $C_{\gamma,\beta}$ we have

$$C_{\gamma,\beta} = 128c_{\gamma,\beta}\beta^6\gamma^6 \int_0^{+\infty} \frac{\sin^{2\beta} x}{x^{2\beta-6}} dx$$

and for $\beta = 4$ we get $C_{\gamma,4} = \frac{3225600}{151}\gamma^5$.

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