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On a Singular Logistic Equation with the p-Laplacian

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Abstract. We prove the existence and nonexistence of positive solutions for the boundary value problems

$$\begin{cases} -\Delta_p u = g(x,u) - \dfrac{h(x)}{u^\alpha} & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), p > 1,\ \Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega,\ \alpha \in (0,1), g : \Omega \times (0,\infty) \to \mathbb{R}$ is possibly singular at $u = 0$. An application to a singular logistic-like equation is given.

Keywords. Sup-supersolutions, singular, positive solutions
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1. Introduction

Consider the boundary value problem

$$\begin{cases} -\Delta_p u = g(x,u) - \dfrac{h(x)}{u^\alpha} & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u),\ p > 1, \Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega,\ h : \Omega \to \mathbb{R},\ g : \Omega \times (0,\infty) \to \mathbb{R}$, and $0 < \alpha < 1$.

In [4, Theorem 5.3], Drabek and Hernandez show that the logistic equation involving the p-Laplacian

$$\begin{cases} -\Delta_p u = \lambda m(x) u^{p-1} - u^{\gamma-1} & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

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where $1 < p < \gamma, m \in L^r(\Omega), r > \frac{N(\gamma-1)}{p(\gamma-p)}$, $m(x) \geq m_0 > 0$ in Ω, has a unique positive solution u with $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for $\lambda > \lambda_1$. Here λ_1 denotes the first eigenvalue of

$$\begin{cases} -\Delta_p u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Note that the nonlinearity $g(x,u) = \lambda m(x)u^{p-1} - u^{\gamma-1}$ is continuous in u for a.e. $x \in \Omega$, and satisfies

$$\lim_{u\to\infty} \frac{g(x,u)}{m(x)u^{p-1}} = -\infty, \quad \lim_{u\to 0^+} \frac{g(x,u)}{m(x)u^{p-1}} = \lambda > \lambda_1. \tag{4}$$

uniformly for $x \in \Omega$.

When $p = 2$, Lee et al. [8] consider the singular problem

$$\begin{cases} -\Delta u = \lambda u - f(u) - \dfrac{c}{u^\alpha} & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where λ, c, α are positive constants with $\alpha < 1$, $f : [0,\infty) \to \mathbb{R}$ is continuous and satisfies

$$\lambda u - M \leq f(u) \leq Au^q$$

for all $u \geq 0$, where M, A, q are positive constants with $q > 1$. Under these assumptions, they show that (5) has a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ for $\lambda > \frac{2\lambda_1}{1+\alpha}$ and c is sufficiently small [8, Theorem 2.1]. Here λ_1 corresponds to $m(x) \equiv 1$. Note that the nonlinearity $g(u) = \lambda u - f(u)$ is continuous and satisfies

$$\limsup_{u\to\infty} \frac{g(u)}{u} \leq 0, \quad \liminf_{u\to 0^+} \frac{g(u)}{u} \geq \lambda > \frac{2\lambda_1}{1+\alpha}. \tag{6}$$

Note that for $f(u) = u^q$, (5) is a singular perturbation problem of (2) with $p = 2$ and $m(x) \equiv 1$, but the result in [8] is not as good as the corresponding one in [4] when $c = 0$. In this paper, we shall study positive solutions to the general problem (1) when h is a bounded function with small $\sup_\Omega h$ and $g(., u)$ is allowed to be singular at $u = 0$ and satisfies a weaker condition than (4) and (6). To be precise, we shall assume the following:

(A1) $m \in L^\infty(\Omega)$ and there exists a constant $m_0 > 0$ such that $m(x) \geq m_0$ for a.e. $x \in \Omega$.

(A2) $g : \Omega \times (0,\infty) \to \mathbb{R}$ is continuous and

$$\limsup_{u\to\infty} \frac{g(x,u)}{m(x)u^{p-1}} < \lambda_1, \quad \liminf_{u\to 0^+} \frac{g(x,u)}{m(x)u^{p-1}} > \lambda_1$$

uniformly for $x \in \Omega$.

(A3) There exists $\alpha \in (0,1)$ such that

$$\limsup_{u\to 0^+} u^\alpha g(x,u) < \infty$$

uniformly for $x \in \Omega$.

In particular, our result can be applied to the following singular perturbation problem of (2)

$$\begin{cases} -\Delta_p u = \lambda m(x)u^{p-1} - u^{\gamma-1} - \dfrac{h(x)}{u^\alpha} & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $1 < p < \gamma$, $\alpha \in (0,1)$, m is as above, gives the existence of a positive solution $u \in C^{1,\beta}(\bar{\Omega})$ for some $\beta \in (0,1)$ when $\lambda > \lambda_1$ and $\sup_\Omega h$ is sufficiently small. Also, if h is a constant, there exists a constant $h^* > 0$ such that (7) has a positive solution for $h < h^*$ and no positive solution for $h > h^*$.

Our main result complements the result in [4] and improves the corresponding result in [8] in many ways. Our approach is based on the method of sub- and supersolutions developed in [6] for singular problems. However, the type of nonlinearities $g(u)$ covered in [6] does not apply here as it requires

$$\lim_{u\to\infty} \frac{g(u)}{u^{p-1}} = 0 \quad \text{and} \quad g(u) > 0 \quad \text{for } u \text{ large},$$

whereas the one in this paper allows

$$\lim_{u\to\infty} \frac{g(u)}{u^{p-1}} = -\infty \quad \text{and} \quad g(u) \to -\infty \quad \text{as } u \to \infty.$$

Let λ_1 be the first eigenvalue of (3) with a positive, normalized corresponding eigenfunction ϕ_1, i.e., $||\phi_1||_\infty = 1$. It is well known that $\lambda_1 > 0, \phi_1 \in C^1(\bar{\Omega}), \frac{\partial\phi_1}{\partial n} < 0$ on $\partial\Omega$, where n denotes the outer unit normal vector on $\partial\Omega$ (see [1]).

By a positive solution of (1) we mean a function $u \in C^{1,\beta}(\bar{\Omega})$ for some $\beta \in (0,1)$ with $u = 0$ and $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\xi \, dx = \int_\Omega \left(g(x,u) - \frac{h(x)}{u^\alpha}\right)\xi \, dx$$

for all $\xi \in W_0^{1,p}(\Omega)$. Here n denotes the outer unit normal vector. Our main result is

Theorem 1.1. *Let $h \in L^\infty(\Omega)$ and suppose* (A1)-(A3) *hold. Then there exists a constant $\eta > 0$such that Problem* (1) *has a positive solution when* $\sup_\Omega h < \eta$. *Moreover, if h is a constant, then there exists a positive number h^* such that* (1) *has a positive solution for $h < h^*$ and no positive solutions for $h > h^*$.*

2. Preliminary results

We shall denote the norms in $L^p(\Omega)$, $W_0^{1,p}(\Omega)$, $C^1(\bar{\Omega})$ and $C^{1,\alpha}(\bar{\Omega})$ by $\|\cdot\|_p$, $\|.\|_{1,p}$, $|.|_1$ and $|.|_{1,\alpha}$, respectively.

For $x \in \Omega$, let $d(x)$ denote the distance from x to $\partial\Omega$. The following regularity result in [6, Lemma 3.1] plays a key role in the proof of the main results:

Lemma 2.1. *Let $h \in L^\infty_{loc}(\Omega)$ and suppose there exist numbers $\alpha \in (0,1)$ and $C > 0$ such that*

$$|h(x)| \le \frac{C}{d^\alpha(x)} \tag{8}$$

for a.e. $x \in \Omega$. Let $u \in W_0^{1,p}(\Omega)$ be the solution of

$$\begin{cases} -\Delta_p u = h & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Then there exist constants $\beta \in (0,1)$ and $M > 0$ depending only on C, α, Ω such that $u \in C^{1,\beta}(\bar{\Omega})$ and $|u|_{1,\beta} < M$.

Remark 2.2. (i) Since $\frac{\partial \phi_1}{\partial n} < 0$ on $\partial\Omega$, there exists a constant $k > 0$ such that $\phi_1(x) \ge k d(x)$ for $x \in \Omega$. Hence Lemma 2.1 holds if (8) is replaced by

$$|h(x)| \le \frac{C}{\phi_1^\alpha(x)}$$

for a.e. $x \in \Omega$.

(ii) Note that under the assumptions of Lemma 2.1, (9) has a unique solution $u \in W_0^{1,p}(\Omega)$. Indeed, define $A\colon W_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$ and $\hat{h} \in W_0^{-1,p'}(\Omega)$, where $p' = \frac{p}{p-1}$, by

$$\langle Au, \xi\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \xi \, dx, \quad \hat{h}(\xi) = \int_\Omega h\xi \, dx.$$

By Hardy's inequality (see e.g. [2, p. 194]), we obtain

$$|\hat{h}(\xi)| \le C \int_\Omega \left|\frac{\xi}{d^\alpha}\right| dx \le C||d||_\infty^{1-\alpha} \int_\Omega \left|\frac{\xi}{d}\right| dx \le \tilde{C}||\xi||_{1,p}$$

for all $\xi \in W_0^{1,p}(\Omega)$, where $\tilde{C}$ is a constant independent of ξ. Thus $\hat{h} \in W_0^{-1,p'}(\Omega)$. Since A is continuous, coercive, and strictly monotone, it follows from the Minty-Browder Theorem (see [2, p. 88]) that there exists a unique $u \in W_0^{1,p}(\Omega)$ such that $Au = \hat{h}$.

Lemma 2.3. *Let* $\varepsilon > 0$ *and let* $h, h_\varepsilon \in L^\infty_{loc}(\Omega)$ *satisfy* (8). *Let* $u, u_\varepsilon \in W_0^{1,p}(\Omega)$ *be the solutions of*

$$\begin{cases} -\Delta_p u = h & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta_p u_\varepsilon = h_\varepsilon & in\ \Omega \\ u_\varepsilon = 0 & on\ \partial\Omega, \end{cases}$$

respectively. Suppose $||h_\varepsilon - h||_1 \to 0$ *as* $\varepsilon \to 0$. *Then*

$$|u_\varepsilon - u|_1 \to 0$$

as $\varepsilon \to 0$.

Proof. By Lemma 2.1, there exist $\beta \in (0,1)$, $M > 0$ such that $u, u_\varepsilon \in C^{1,\beta}(\bar{\Omega})$ and

$$|u|_{1,\beta},\ |u_\varepsilon|_{1,\beta} < M. \tag{10}$$

Multiplying the equation $-\Delta_p u_\varepsilon - (-\Delta_p u) = h_\varepsilon - h$ in Ω by $u_\varepsilon - u$ and integrating, we obtain

$$\int_\Omega (|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla u|^{p-2}\nabla u)\cdot(\nabla u_\varepsilon - \nabla u)\ dx \le 2M||h_\varepsilon - h||_1. \tag{11}$$

By [9, Lemma 30.1], for $x, y \in \mathbb{R}^n$,

$$(|x| + |y|)^{2-\min(p,2)}(|x|^{p-2}x - |y|^{p-2}y)\cdot(x-y) \ge C_0|x-y|^{\max(p,2)}, \tag{12}$$

where $C_0 = (\frac{1}{2})^{p-1}$ if $p \ge 2$, $C_0 = p-1$ if $p < 2$.

Using (12) with $x = \nabla u_\varepsilon, y = \nabla u$ and the fact that $|x| + |y| < 2M$, we obtain from (11) that

$$C_1 \int_\Omega |\nabla(u_\varepsilon - u)|^q dx \le 2M||h_\varepsilon - h||_1,$$

where $C_1 = C_0 \cdot (2M)^{\min(p,2)-2}$ and $q = \max(p,2)$.

Hence, by Poincaré's inequality,

$$||u_\varepsilon - u||_q \to 0 \tag{13}$$

as $\varepsilon \to 0$. Suppose $|u_\varepsilon - u|_1 \not\to 0$ as $\varepsilon \to 0$. Then there exists a sequence (ε_n) which converges to 0 such that

$$|u_{\varepsilon_n} - u|_1 \not\to 0 \quad \text{as } n \to \infty \tag{14}$$

By (10), (u_{ε_n}) is bounded in $C^{1,\beta}(\bar{\Omega})$, and since $C^{1,\beta}(\bar{\Omega})$ is compactly embedded in $C^1(\bar{\Omega})$, there exist $v \in C^1(\bar{\Omega})$ and a subsequence $(u_{\varepsilon_{n_k}})$ of (u_{ε_n}) such that

$$|u_{\varepsilon_{n_k}} - v|_1 \to 0 \quad \text{as } k \to \infty. \tag{15}$$

From (13) and (15), we see that $u = v$ and so $|u_{\varepsilon_{n_k}} - u|_1 \to 0$ as $k \to \infty$, a contradiction with (14). This completes the proof of Lemma 2.3. □

Next, we recall some results in sub- and supersolutions method for singular boundary value problems in [6, Appendix A]. Related results can be found in [3]. Consider the problem

$$\begin{cases} -\Delta_p u = h(x,u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

where $h : \Omega \times (0,\infty) \to \mathbb{R}$ is continuous.

Let $\phi, \psi \in C^1(\bar{\Omega})$. Suppose there exist constants $c_0, C, \alpha > 0$ with $\alpha < 1$ such that $\phi(x), \psi(x) \geq c_0 d(x)$ for $x \in \Omega$ and

$$|h(x,w)| \leq \frac{C}{d^\alpha(x)} \tag{17}$$

for a.e. $x \in \Omega$ and all $w \in C(\bar{\Omega})$ with $\phi \leq w \leq \psi$ in Ω. Suppose ϕ, ψ are sub- and supersolutions of (16) respectively, i.e., for all $\xi \in W_0^{1,p}(\Omega)$ with $\xi \geq 0$,

$$\int_\Omega |\nabla\phi|^{p-2}\nabla\phi.\nabla\xi \; dx \leq \int_\Omega h(x,\phi)\xi \; dx \;,$$
$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi.\nabla\xi \; dx \geq \int_\Omega h(x,\psi)\xi \; dx,$$

and $\phi \leq 0 \leq \psi$ on $\partial\Omega$. Note that the integrals on the right-hand side are defined by virtue of Hardy's inequality.

Lemma 2.4. *Under the above assumptions, there exists a constant $\beta \in (0,1)$ such that* (16) *has a solution $u \in C^{1,\beta}(\bar{\Omega})$ with $\phi \leq u \leq \psi$ in Ω.*

3. Proof of the main result

Now, we are ready to give the proof of the main result.

Proof of Theorem 1.1. By (A2), there exists $\lambda_0 > \lambda_1$ and $\delta_0 > 0$ such that

$$g(x,u) \geq \lambda_0 m(x) u^{p-1} \tag{18}$$

for $x \in \Omega$ and $u \in (0, \delta_0]$. Choose $\delta \in (0,1)$ so that $\lambda_0 \delta^{p-1} > \lambda_1$.

For $\varepsilon > 0$, let $z_\varepsilon > 0$ be the solution of

$$-\Delta_p z_\varepsilon = h_\varepsilon \equiv \begin{cases} \lambda_1 m(x)(\delta_0\phi_1)^{p-1} & \text{in } \{\phi_1 > \varepsilon\} \\ -\,\phi_1^{-\alpha} & \text{in } \{\phi_1 < \varepsilon\} \end{cases}, \quad z_\varepsilon = 0 \quad \text{on } \partial\Omega.$$

Note that the existence of z_ε follows from Lemma 2.1 and Remark 2.2. Since

$$-\Delta_p(\delta_0\phi_1) = h \equiv \lambda_1 m(x)(\delta_0\phi_1)^{p-1} \quad \text{in } \Omega,$$

the weak maximum principle [10, Lemma A.2] implies $z_\varepsilon \le \delta_0\phi_1 \le \delta_0$ in Ω.

Next,

$$||h_\varepsilon - h||_1 = \int_{\phi_1<\varepsilon} |\lambda_1 m(x)(\delta_0\phi_1)^{p-1} + \phi_1^{-\alpha}|\ dx \le C_0 \int_{\phi_1<\varepsilon} \phi_1^{-\alpha}\ dx$$

and since $\int_\Omega \phi_1^{-\alpha}\ dx < \infty$ (see [7, p. 726], it follows that $||h_\varepsilon - h||_1 \to 0$ as $\varepsilon \to 0$. By Lemma 2.3, $|z_\varepsilon - \delta_0\phi_1|_1 \to 0$ as $\varepsilon \to 0$. Hence $|z_\varepsilon - \delta_0\phi_1|_1 < \frac{\delta_0(1-\delta)}{k}$, if ε is sufficiently small, where $k > 0$ is such that $\frac{d}{\phi_1} \le k$ in Ω.

By the Mean Value Theorem,

$$|z_\varepsilon(x) - \delta_0\phi_1(x)| \le \frac{\delta_0(1-\delta)}{k}\, d(x) \le \delta_0(1-\delta)\phi_1(x)$$

for $x \in \Omega$, which implies

$$z_\varepsilon \ge \delta\delta_0\phi_1 \quad \text{in } \Omega \tag{19}$$

if ε is sufficiently small, which we assume.

Suppose $\sup_\Omega h < \eta$, where

$$\eta = \min\left\{(\lambda_0\delta^{p-1} - \lambda_1)m_0\delta^\alpha(\delta_0\varepsilon)^{p-1+\alpha}, (\delta\delta_0)^\alpha\right\}.$$

We shall verify that z_ε is a subsolution of (1). Let $\xi \in W_0^{1,p}(\Omega)$ with $\xi \ge 0$. Then

$$\begin{aligned}\int_\Omega |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \cdot \nabla\xi\ dx &= -\int_\Omega (\Delta_p z_\varepsilon)\xi\ dx \\ &= \lambda_1 \int_{\phi_1>\varepsilon} m(x)(\delta_0\phi_1)^{p-1}\xi\ dx - \int_{\phi_1<\varepsilon} \frac{\xi}{\phi_1^\alpha}\ dx.\end{aligned} \tag{20}$$

In the set $\{\phi_1 > \varepsilon\}$, we have

$$(\lambda_0\delta^{p-1} - \lambda_1)m(x)(\delta_0\phi_1)^{p-1} \ge (\lambda_0\delta^{p-1} - \lambda_1)m_0(\delta_0\varepsilon)^{p-1} \ge \frac{\eta}{(\delta\delta_0\varepsilon)^\alpha},$$

which, together with (18), (19), implies

$$\begin{aligned} g(x, z_\varepsilon) - \frac{h(x)}{z_\varepsilon^\alpha} &\ge \lambda_0 m(x) z_\varepsilon^{p-1} - \frac{1}{z_\varepsilon^\alpha}\sup_\Omega h \\ &\ge \lambda_0\delta^{p-1}m(x)(\delta_0\phi_1)^{p-1} - \frac{\eta}{(\delta\delta_0\varepsilon)^\alpha} \\ &\ge \lambda_1 m(x)(\delta_0\phi_1)^{p-1}\end{aligned} \tag{21}$$

in $\{\phi_1 > \varepsilon\}$. On the other hand, since $\eta \le (\delta\delta_0)^\alpha$,

$$g(x, z_\varepsilon) - \frac{h(x)}{z_\varepsilon^\alpha} \ge -\frac{1}{z_\varepsilon^\alpha}\sup_\Omega h \ge -\frac{\eta}{(\delta\delta_0\phi_1)^\alpha} \ge -\frac{1}{\phi_1^\alpha} \quad \text{in } \Omega. \tag{22}$$

Combining (20)–(22), we obtain

$$\int_\Omega |\nabla z_\varepsilon|^{p-2}\nabla z_\varepsilon \cdot \nabla\xi \, dx \le \int_\Omega \left(g(x,z_\varepsilon) - \frac{h(x)}{z_\varepsilon^\alpha}\right)\xi \, dx,$$

i.e., z_ε is a subsolution of (1).

Next, in view of (A2) and (A3), there exist constants $b \in (0,\lambda_1)$ and $d_0 > 0$ such that

$$g(x,u) \le bm(x)u^{p-1} + \frac{d_0}{u^\alpha} \tag{23}$$

for all $u > 0$ and a.e. $x \in \Omega$. Choose $\gamma \in (0,1)$ and $\tilde\lambda_1, M_0 > 0$ so that

$$(1+\gamma)^{p-1}\left(b + \frac{d_0}{m_0 M_0^{p-1+\alpha}}\right) < \tilde\lambda_1 < \lambda_1, \tag{24}$$

and

$$\frac{(1+\gamma)^{p-1}||h||_\infty}{m_0 M_0^{p-1+\alpha}} < \lambda_1 - \tilde\lambda_1. \tag{25}$$

Let ψ_ε be the solution of

$$-\Delta_p\psi_\varepsilon = \begin{cases} \lambda_1 m(x){\phi_1}^{p-1} & \text{in } \{\phi_1 > \varepsilon\} \\ \lambda_1 m(x) + \phi_1^{-\alpha} & \text{in } \{\phi_1 < \varepsilon\} \end{cases}, \quad \psi_\varepsilon = 0 \quad \text{on } \partial\Omega.$$

Then, since $-\Delta_p\phi_1 = \lambda_1 m(x){\phi_1}^{p-1}$ in Ω, it follows from Lemma 2.3 that $|\psi_\varepsilon - \phi_1|_1 \to 0$ as $\varepsilon \to 0$. Hence, if ε is small enough,

$$(1-\gamma)\phi_1 \le \psi_\varepsilon \le (1+\gamma)\phi_1 \quad \text{in } \Omega, \tag{26}$$

which we assume. We shall verify that $Z_\varepsilon = M\psi_\varepsilon$ is a supersolution for (1) with $Z_\varepsilon \ge z_\varepsilon$ in Ω if M is large enough. Let $\xi \in W_0^{1,p}(\Omega)$ with $\xi \ge 0$. Then we have

$$\begin{aligned}\int_\Omega |\nabla Z_\varepsilon|^{p-2}\nabla Z_\varepsilon \cdot \nabla\xi \, dx = \lambda_1 &\int_{\phi_1>\varepsilon} m(x)(M\phi_1)^{p-1}\xi \, dx \\ &+ M^{p-1}\int_{\phi_1<\varepsilon} (\lambda_1 m(x) + {\phi_1}^{-\alpha})\xi \, dx.\end{aligned} \tag{27}$$

Suppose $M > \frac{M_0}{(1-\gamma)\varepsilon}$. Then

$$Z_\varepsilon \ge M(1-\gamma)\varepsilon > M_0 \tag{28}$$

in $\{\phi_1 > \varepsilon\}$. Since $M\phi_1 \ge (1+\gamma)^{-1}Z_\varepsilon$ in Ω, it follows from (23)-(25) that in $\{\phi_1 > \varepsilon\}$, $\frac{g(x,Z_\varepsilon)}{m(x)(M\phi_1)^{p-1}} \le \frac{(1+\gamma)^{p-1}g(x,Z_\varepsilon)}{m(x)Z_\varepsilon^{p-1}} \le (1+\gamma)^{p-1}\left(b + \frac{d_0}{m(x)Z_\varepsilon^{p-1+\alpha}}\right) \le (1+\gamma)^{p-1}\left(b + \frac{d_0}{m_0 M_0^{p-1+\alpha}}\right) < \tilde\lambda_1$, and

$$\frac{||h||_\infty}{m(x)(M\phi_1)^{p-1}Z_\varepsilon^\alpha} \le \frac{(1+\gamma)^{p-1}||h||_\infty}{m(x)Z_\varepsilon^{p-1+\alpha}} \le \frac{(1+\gamma)^{p-1}||h||_\infty}{m_0 M_0^{p-1+\alpha}} < \lambda_1 - \tilde\lambda_1.$$

Hence

$$g(x, Z_\varepsilon) - \frac{h(x)}{Z_\varepsilon^\alpha} \le g(x, Z_\varepsilon) + \frac{||h||_\infty}{Z_\varepsilon^\alpha} \le \lambda_1 m(x)(M\phi_1)^{p-1} \tag{29}$$

in $\{\phi_1 > \varepsilon\}$. From (23), (24), and (26), we get

$$\begin{aligned} g(x, Z_\varepsilon) - \frac{h(x)}{Z_\varepsilon^\alpha} &\le bm(x)Z_\varepsilon^{p-1} + \frac{d_0 + ||h||_\infty}{Z_\varepsilon^\alpha} \\ &\le b(1+\gamma)^{p-1}m(x)(M\phi_1)^{p-1} + \frac{d_0 + ||h||_\infty}{(M(1-\gamma))^\alpha}\phi_1^{-\alpha} \\ &\le \lambda_1 m(x)M^{p-1} + M^{p-1}\phi_1^{-\alpha}, \end{aligned} \tag{30}$$

if M is large enough so that $M^{p-1+\alpha} > (d_0 + ||h||_\infty)(1-\gamma)^{-\alpha}$, which we assume.

Combining (27), (29), and (30), we get

$$\int_\Omega |\nabla Z_\varepsilon|^{p-2}\nabla Z_\varepsilon \cdot \nabla \xi \ dx \ge \int_\Omega \left(g(x, Z_\varepsilon) - \frac{h(x)}{Z_\varepsilon^\alpha} \right) \xi \ dx,$$

i.e., Z_ε is a supersolution of (1) with $Z_\varepsilon \ge z_\varepsilon$ for large M.

Finally, it follows from (A3) and (19) that there exists a constant $K > 0$ depending on $||Z_\varepsilon||_\infty$ such that

$$|g(x, w)| \le \frac{K}{w^\alpha} \le \frac{K}{z_\varepsilon^\alpha} \le \frac{K}{(\delta\delta_0\phi_1)^\alpha}$$

for all $w \in C(\bar{\Omega})$ with $z_\varepsilon \le w \le Z_\varepsilon$ in Ω. The existence of a positive solution for (1) now follows from Lemma 2.4.

Next, suppose that h is a constant. Then, as in the above, we see that there exists a constant $h_0 > 0$ such that (1) has a positive solution for $h < h_0$. We claim that (1) has no positive solutions for large h. Indeed, let u be a positive solution of (1) with $h > 0$. Multiplying the equation $-\Delta_p u = g(x, u) - \frac{h}{u^\alpha}$ in Ω by u and integrating, we obtain, by (23),

$$\begin{aligned} \int_\Omega |\nabla u|^p dx &= \int_\Omega g(x, u)u \ dx - h\int_\Omega u^{1-\alpha} \ dx \\ &\le b\int_\Omega m(x)u^p \ dx + (d_0 - h)\int_\Omega u^{1-\alpha} \ dx \\ &\le b\int_\Omega m(x)u^p \ dx \end{aligned}$$

for $h \ge d_0$. Since

$$\lambda_1 = \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \ne 0}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega m(x)u^p \ dx},$$

it follows that $\left(1-\frac{b}{\lambda_1}\right)\int_\Omega |\nabla u|^p\,dx \le 0$, which implies $u\equiv 0$, a contradiction. Hence the claim is proved.

Define $h^* = \sup\{h>0 : (1) \text{ has a positive solution}\}$. Then $h^*\in(0,\infty)$ and (1) has no positive solutions for $h>h^*$. Let $h<h^*$. Then there exists $\tilde{h}>h$ such that (1) with $h=\tilde{h}$ has a positive solution $u_{\tilde{h}}$. Since

$$g(x,u_{\tilde{h}})-\frac{\tilde{h}}{\tilde{u}_h^\alpha} \le g(x,u_{\tilde{h}})-\frac{h}{\tilde{u}_h^\alpha}\ ,$$

in Ω, it follows that $u_{\tilde{h}}$ is a subsolution for (1). As above, we obtain a supersolution Z_ε for (1) with $Z_\varepsilon \ge u_{\tilde{h}}$ in Ω, and the existence of a positive solution to (1) follows. This completes the proof of Theorem 1.1. □

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